



# Marginal Continuation odds Ratio Model and Decomposition of Marginal Homogeneity Model for Multi-way Contingency Tables

Satoru Shinoda

*Taisho Pharmaceutical Co., Ltd., Toshima-ku, Japan*

Satoru Shinoda, Kouji Tahata and Sadao Tomizawa

*Tokyo University of Science, Noda City, Japan*

Kouji Yamamoto

*Yokohama City University, Yokohama City, Japan*

---

## Abstract

For square contingency tables with ordered categories, the marginal homogeneity model is represented by various expressions, and some extensions of the marginal homogeneity model were proposed. Herein we consider the marginal continuation-ratio to examine a new expression of the marginal homogeneity model. We also propose an extension of the marginal homogeneity model using the ratio of marginal continuation-ratios; namely, the marginal continuation odds ratio. The proposed model can be interpreted in various ways. Additionally, we propose a generalization of it, and decompose the marginal homogeneity model using the generalized model. Furthermore, we extend the models and decompositions into multi-way contingency tables.

*AMS (2000) subject classification.* Primary 62H17; Secondary 62H15.

*Keywords and phrases.* Complementary log-log transformation, Continuation-ratio, Hazards, Logit transformation, Marginal inhomogeneity, Probit transformation

---

## 1 Introduction

Consider an  $R \times R$  square contingency table with the same row and column ordinal classifications. Let  $X$  and  $Y$  denote the row and column variables, respectively, and let  $\Pr(X = i, Y = j) = p_{ij}$  for  $i = 1, \dots, R; j = 1, \dots, R$ . The marginal homogeneity (MH) model is defined by

$$p_{i.} = p_{.i} \quad \text{for } i = 1, \dots, R, \quad (1.1)$$

where  $p_{i\cdot} = \sum_{t=1}^R p_{it}$  and  $p_{\cdot i} = \sum_{s=1}^R p_{si}$ . See e.g., Stuart (1955) and Bishop et al. (1975, p.294). This indicates that the row marginal distribution is identical to the column marginal distribution.

Using the marginal cumulative probability, this model can be expressed as

$$F_i^X = F_i^Y \quad \text{for } i = 1, \dots, R-1, \quad (1.2)$$

where  $F_i^X = \sum_{s=1}^i p_{s\cdot} = \Pr(X \leq i)$  and  $F_i^Y = \sum_{t=1}^i p_{\cdot t} = \Pr(Y \leq i)$ . The MH model can also be expressed as

$$G_{1(i)} = G_{2(i)} \quad \text{for } i = 1, \dots, R-1, \quad (1.3)$$

where  $G_{1(i)} = \sum_{s=1}^i \sum_{t=i+1}^R p_{st} = \Pr(X \leq i, Y > i)$  and  $G_{2(i)} = \sum_{s=i+1}^R \sum_{t=1}^i p_{st} = \Pr(X > i, Y \leq i)$  (see e.g., Tomizawa 1993; Tahata and Tomizawa 2008). Furthermore, Tahata et al. (2006) expressed the MH model using marginal ridits (see e.g., Bross 1958; Fleiss et al. 2003, pp.198-205; Agresti 2010, p.10). Moreover, the MH model can be expressed with other formulas (see e.g., Iki et al. 2010; Altun and Aktaş 2018).

When the MH model does not fit for the data, we are interested in applying a model with weaker restrictions. One example is an extension based on expression (1.1) proposed by Miyamoto et al. (2006) for a square contingency table with nominal classifications. For a square contingency table with ordinal classifications, the marginal cumulative logistic (ML) model is defined by

$$\log \left( \frac{F_i^X}{1 - F_i^X} \right) = \log \left( \frac{F_i^Y}{1 - F_i^Y} \right) + \Delta \quad \text{for } i = 1, \dots, R-1. \quad (1.4)$$

See e.g., McCullagh (1977), Agresti (2010, p.241), and Kurakami et al. (2013). Saigusa et al. (2018) proposed the marginal cumulative complementary log-log (MCL) model, defined by

$$\log(-\log(1 - F_i^X)) = \log(-\log(1 - F_i^Y)) + \Delta \quad \text{for } i = 1, \dots, R-1. \quad (1.5)$$

The ML (MCL) model indicates that one marginal distribution is a location shift of another marginal distribution on a logistic (complementary log-log) scale. Each special case of the ML and MCL model obtained by setting  $\Delta = 0$  is the MH model. These models are extensions of the MH model based on expression (1.2). Furthermore, Tahata and Tomizawa (2008) proposed extensions of the MH model based on expressions (1.3).

Herein we examine a new expression of the MH model using the continuation-ratio (see e.g., Fienberg 1980, pp.110-111; Agresti 2010, p.45). The MH model can be expressed as

$$c_i^X = c_i^Y \quad \text{for } i = 1, \dots, R-1, \quad (1.6)$$

where

$$c_i^X = \frac{p_{i\cdot}}{1 - F_i^X}, \quad c_i^Y = \frac{p_{\cdot i}}{1 - F_i^Y}.$$

This states that the row marginal continuation-ratio is identical to the column marginal continuation-ratio. Note that there are various research focusing on the continuation-ratio (see e.g., Thompson 1977; McCullagh 1980; Läärä and Matthews 1985; Tutz 1991; Greenland 1994). As an example, Thompson (1977) used the continuation-ratio in modeling discrete survival time data. When the lengths of time intervals approach zero, his model converges to the Cox proportional hazards model.

For the square contingency table analysis, much research on the marginal homogeneity have been studied. However, research on the framework of the continuation-ratio, which is an important concept in categorical analysis, are not enough. As an example, the ML model cannot be interpreted under the continuation-ratio. The purpose of this study is to provide a new insight for the square contingency table analysis by studying the continuation-ratio. This paper can also further understand the previous research by considering the properties of the continuation-ratio. The plan of the paper is as follows. Section 2 extends the MH model based on expression (1.6). Section 3 decomposes the MH model. Section 4 extends the model into multi-way tables. Section 5 gives a test for the goodness-of-fit for the models. Section 6 provides some examples, and Section 7 discusses this paper in the context of related works.

## 2 Models

*2.1. The Marginal Continuation Odds Ratio Model* The ratio of marginal continuation-ratios is

$$\psi_i = \frac{c_i^X}{c_i^Y} = \frac{p_{i\cdot} / (1 - F_i^X)}{p_{\cdot i} / (1 - F_i^Y)} = \frac{\Pr(X = i) / \Pr(X > i)}{\Pr(Y = i) / \Pr(Y > i)},$$

for  $i = 1, \dots, R-1$ . We refer to the ratio of marginal continuation-ratios as the marginal continuation odds ratio. Note that this is different from the continuation odds ratio (Agresti 2010, p.24), and the quasi-symmetry model based on the continuation odds ratio was presented by Kateri et al. (2017).

We propose a new model defined by

$$\log \psi_i = \Delta \quad \text{for } i = 1, \dots, R-1, \quad (2.1)$$

where the parameter  $\Delta$  is unspecified. This model indicates that the ratios of marginal continuation-ratios are equal to  $\exp(\Delta)$ . A special case of this model obtained by setting  $\Delta = 0$  is the MH model. We shall refer to model (2.1) as the marginal continuation odds ratio (MCOR) model.

Let

$$\omega_i^X = \frac{p_{i\cdot}}{1 - F_{i-1}^X} = \Pr(X = i \mid X \geq i),$$

and

$$\omega_i^Y = \frac{p_{\cdot i}}{1 - F_{i-1}^Y} = \Pr(Y = i \mid Y \geq i),$$

for  $i = 1, \dots, R-1$  with  $F_0^X = F_0^Y = 0$ . For models based on these conditional probabilities, see e.g., Läärä and Matthews (1985) and McCullagh and Nelder (1983, pp.102-104). Then the marginal continuation odds ratio is also expressed as

$$\psi_i = \frac{\omega_i^X (1 - \omega_i^Y)}{\omega_i^Y (1 - \omega_i^X)} \quad \text{for } i = 1, \dots, R-1,$$

since

$$c_i^X = \frac{\omega_i^X}{1 - \omega_i^X},$$

and

$$c_i^Y = \frac{\omega_i^Y}{1 - \omega_i^Y}.$$

Then the MCOR model can be expressed as

$$\log \left( \frac{\omega_i^X}{1 - \omega_i^X} \right) = \log \left( \frac{\omega_i^Y}{1 - \omega_i^Y} \right) + \Delta \quad \text{for } i = 1, \dots, R-1. \quad (2.2)$$

Under this model,  $\Delta > 0$  is equivalent to  $\{\omega_i^X > \omega_i^Y\}$ .

The MCOR model can also be expressed as

$$\omega_i^X = \frac{\exp(\theta_i + \Delta)}{1 + \exp(\theta_i + \Delta)} \quad \text{for } i = 1, \dots, R-1,$$

where  $\theta_i = \log(\omega_i^Y / (1 - \omega_i^Y))$ . Therefore, the MCOR model indicates that the conditional probability  $\omega_i^X$  is a location shift of the conditional probability  $\omega_i^Y$  on a logistic scale. Thus, the MCOR model can also be called a marginal continuation-ratio logit model.

Interpretation of the proposed model will be described using the following examples. Consider the comparison of therapeutic effects when two drugs are administered to the same patient. The treatment effect is an ordinal score with  $R$  stages (larger scores indicate more severe symptoms). So, we obtain an  $R \times R$  contingency table with the same row and column classifications (row variable is drug A; column variable is drug B). We are now interested in the odds that an observation will fall in score category  $i$ , instead of score category  $i + 1$  or above for any  $i$ . From Eq. (2.1), under the MCOR model, the parameter  $\Delta$  indicates the odds ratio between drug A and B; if the  $\Delta$  is zero, the MH model holds, i.e., there is no difference between drug A and B; if the  $\Delta$  is positive, the odds ratio is  $\exp(\Delta)$  times higher, i.e., drug A is more therapeutic effect than drug B. We can also interpret the MCOR model in two ways. From Eq. (2.2), on condition that an observation will fall in score category  $i$  or above, the odds that the observation falls in score category  $i$  instead of not  $i$ , are  $\exp(\Delta)$  times higher for drug A than for drug B. Moreover, we can see that the conditional probability for drug A is a location shift of that for drug B on a logistic scale.

Note that model (1.4) can be transformed into model (2.2) by replacing the marginal cumulative probability with the corresponding marginal conditional probability. However, the meanings of these models completely differ, and the likelihood ratio chi-squared statistics for testing the goodness-of-fit of these models do not coincide.

*2.2. The Generalized Marginal Continuation-ratio Model* Model (2.2) is an extension of the MH model using the logit transformation. Hence, model (2.2) may be based on the idea of model (1.4). If we focus on the idea of model (1.5), a distinct extension can be derived using the complementary log-log transformation. Therefore, using a strictly increasing function such as a logit or a complementary log-log function, we propose a generalization of the MCOR model by

$$h^{-1}(\omega_i^X) = h^{-1}(\omega_i^Y) + \Delta \quad \text{for } i = 1, \dots, R - 1, \quad (2.3)$$

where the parameter  $\Delta$  is unspecified and  $h(\cdot)$  is a twice-differentiable and strictly increasing function with  $\lim_{x \rightarrow -\infty} h(x) = 0$  and  $\lim_{x \rightarrow \infty} h(x) = 1$ . We shall refer to model (2.3) as the generalized marginal continuation-ratio (GMC) model. A special case of this model obtained by setting  $\Delta = 0$  is the MH model.

By setting  $h^{-1}(\omega_i^Y) = \theta_i$ , the GMC model can be expressed as

$$\omega_i^X = h(\theta_i + \Delta) \quad \text{for } i = 1, \dots, R - 1.$$

Let

$$g(x) = \frac{h(x)}{1 - h(x)}.$$

Note that  $g(\cdot)$  is a strictly increasing function that gives  $\lim_{x \rightarrow -\infty} g(x) = 0$ ,  $\lim_{x \rightarrow \infty} g(x) = \infty$ , and  $g(\cdot) > 0$ . The GMC model can also be expressed as

$$\frac{\omega_i^X}{1 - \omega_i^X} = \frac{h(\theta_i + \Delta)}{1 - h(\theta_i + \Delta)} = g(\theta_i + \Delta) \quad \text{for } i = 1, \dots, R - 1.$$

Furthermore, since

$$\frac{\omega_i^X}{1 - \omega_i^X} = \frac{p_{i\cdot}}{1 - F_i^X},$$

the GMC model can be expressed as

$$\frac{p_{i\cdot}}{1 - F_i^X} = g(\theta_i + \Delta) \quad \text{for } i = 1, \dots, R - 1.$$

Especially, when  $h^{-1}(x) = \log(x/(1-x))$ , i.e.,  $g(x) = \exp(x)$ , the GMC model is equivalent to the MCOR model.

**2.3. Properties** In this section, we focus on the complementary log-log and probit transformation as the major transformations for the GMC model.

**2.3.1. The Marginal Continuation-ratio Complementary Log-log Model.** When  $h^{-1}(x) = \log(-\log(1-x))$ , the GMC model is expressed as

$$\log(-\log(1 - \omega_i^X)) = \log(-\log(1 - \omega_i^Y)) + \Delta \quad \text{for } i = 1, \dots, R - 1. \quad (2.4)$$

We shall refer to model (2.4) as the marginal continuation-ratio complementary log-log (MCC) model.

Läärä and Matthews (1985) noted that the complementary log-log transformation for the conditional probabilities is equivalent to the one using the same transformation but with the cumulative probabilities. This leads to the following property.

**Property 1.** *The MCC model is equivalent to the MCL model.*

We give the proof of Property 1 below: The conditional probabilities  $\omega_i^X$  can be expressed as

$$\omega_i^X = \frac{p_{i\cdot}}{1 - F_{i-1}^X} = 1 - \frac{1 - F_i^X}{1 - F_{i-1}^X},$$

then

$$1 - \omega_i^X = \frac{1 - F_i^X}{1 - F_{i-1}^X},$$

for  $i = 1, \dots, R - 1$ . Therefore, the MCC model is expressed as

$$\log(1 - F_i^X) - \log(1 - F_{i-1}^X) = \exp(\Delta) [\log(1 - F_i^Y) - \log(1 - F_{i-1}^Y)],$$

for  $i = 1, \dots, R - 1$ . When  $i = 1$ , we see

$$\log(1 - F_1^X) = \exp(\Delta) \log(1 - F_1^Y).$$

When  $i = 2$ , we see

$$\log(1 - F_2^X) - \log(1 - F_1^X) = \exp(\Delta) [\log(1 - F_2^Y) - \log(1 - F_1^Y)],$$

thus,

$$\log(1 - F_2^X) = \exp(\Delta) \log(1 - F_2^Y).$$

Hence, in a similar manner we see that the MCC model is expressed as

$$\log(1 - F_i^X) = \exp(\Delta) \log(1 - F_i^Y) \quad \text{for } i = 1, \dots, R - 1.$$

This expression represents the MCL model.

From above, the parameter  $\Delta$  in the MCC model can reflect the degree of inhomogeneity not only between  $\{\omega_i^X\}$  and  $\{\omega_i^Y\}$  but also between  $\{F_i^X\}$  and  $\{F_i^Y\}$ . Hence, the MCC model also states that one marginal distribution is a location shift of another marginal distribution on a complementary log-log scale.

*2.3.2. The Marginal Continuation-ratio Probit Model.* Using the probit transformation, the GMC model is expressed as

$$\Phi^{-1}(\omega_i^X) = \Phi^{-1}(\omega_i^Y) + \Delta \quad \text{for } i = 1, \dots, R - 1, \quad (2.5)$$

where  $\Phi(\cdot)$  is the cumulative distribution function of the standard normal distribution. We refer to model (2.5) as the marginal continuation-ratio probit (MCP) model.

### 3. Decompositions of the Marginal Homogeneity Model

Consider the marginal mean equality (ME) model defined by

$$E(X) = E(Y),$$

i.e.,

$$\sum_{i=1}^R ip_{i\cdot} = \sum_{i=1}^R ip_{\cdot i}.$$

Note that the MH model implies the ME model.

We obtain the following lemmas and theorem.

**Lemma 1.** *The GMC model can also be expressed as*

$$p_{i\cdot} = \frac{g(\theta_i + \Delta)}{\prod_{s=1}^i (1 + g(\theta_s + \Delta))}, \quad p_{\cdot i} = \frac{g(\theta_i)}{\prod_{s=1}^i (1 + g(\theta_s))},$$

for  $i = 1, \dots, R-1$ , and

$$p_{R\cdot} = \frac{1}{\prod_{s=1}^{R-1} (1 + g(\theta_s + \Delta))}, \quad p_{\cdot R} = \frac{1}{\prod_{s=1}^{R-1} (1 + g(\theta_s))}.$$

PROOF. The GMC model is expressed as

$$\frac{p_{i\cdot}}{1 - F_i^X} = g(\theta_i + \Delta) \quad \text{for } i = 1, \dots, R-1.$$

When  $i = 1$ ,

$$\frac{p_{1\cdot}}{1 - F_1^X} = g(\theta_1 + \Delta),$$

namely

$$p_{1\cdot} = \frac{g(\theta_1 + \Delta)}{1 + g(\theta_1 + \Delta)}.$$

When  $i = 2$ ,

$$\frac{p_{2\cdot}}{1 - (p_{1\cdot} + p_{2\cdot})} = g(\theta_2 + \Delta).$$

Namely

$$\begin{aligned} (1 + g(\theta_2 + \Delta)) p_{2\cdot} &= g(\theta_2 + \Delta) (1 - p_{1\cdot}) \\ &= g(\theta_2 + \Delta) \left( 1 - \frac{g(\theta_1 + \Delta)}{1 + g(\theta_1 + \Delta)} \right) \\ &= \frac{g(\theta_2 + \Delta)}{1 + g(\theta_1 + \Delta)}. \end{aligned}$$

Thus

$$p_{2\cdot} = \frac{g(\theta_2 + \Delta)}{\prod_{s=1}^2 (1 + g(\theta_s + \Delta))}.$$

When  $i = 3$ ,

$$\frac{p_{3\cdot}}{1 - (p_{1\cdot} + p_{2\cdot} + p_{3\cdot})} = g(\theta_3 + \Delta).$$

Namely

$$\begin{aligned}
 & (1 + g(\theta_3 + \Delta)) p_{3\cdot} \\
 = & g(\theta_3 + \Delta) (1 - p_{1\cdot} - p_{2\cdot}) \\
 = & g(\theta_3 + \Delta) \left( 1 - \frac{g(\theta_1 + \Delta)}{1 + g(\theta_1 + \Delta)} - \frac{g(\theta_2 + \Delta)}{\prod_{s=1}^2 (1 + g(\theta_s + \Delta))} \right) \\
 = & \frac{g(\theta_3 + \Delta)}{\prod_{s=1}^2 (1 + g(\theta_s + \Delta))}.
 \end{aligned}$$

Thus

$$p_{3\cdot} = \frac{g(\theta_3 + \Delta)}{\prod_{s=1}^3 (1 + g(\theta_s + \Delta))}.$$

By a similar manner, we obtain

$$p_{i\cdot} = \frac{g(\theta_i + \Delta)}{\prod_{s=1}^i (1 + g(\theta_s + \Delta))},$$

for  $i = 1, \dots, R-1$ . Moreover, we obtain

$$\begin{aligned}
 p_{R\cdot} &= 1 - \sum_{i=1}^{R-1} p_{i\cdot} \\
 &= 1 - \frac{g(\theta_1 + \Delta)}{1 + g(\theta_1 + \Delta)} - \sum_{i=2}^{R-1} \frac{g(\theta_i + \Delta)}{\prod_{s=1}^i (1 + g(\theta_s + \Delta))} \\
 &= \frac{1}{1 + g(\theta_1 + \Delta)} - \frac{g(\theta_2 + \Delta)}{\prod_{s=1}^2 (1 + g(\theta_s + \Delta))} - \sum_{i=3}^{R-1} \frac{g(\theta_i + \Delta)}{\prod_{s=1}^i (1 + g(\theta_s + \Delta))}.
 \end{aligned}$$

Since

$$\frac{1}{\prod_{s=1}^{k-1} (1 + g(\theta_s + \Delta))} - \frac{g(\theta_k + \Delta)}{\prod_{s=1}^k (1 + g(\theta_s + \Delta))} = \frac{1}{\prod_{s=1}^k (1 + g(\theta_s + \Delta))},$$

for  $k = 2, \dots, R-1$ , we see

$$p_{R\cdot} = \frac{1}{\prod_{s=1}^{R-1} (1 + g(\theta_s + \Delta))}.$$

In a similar manner, we obtain

$$p_{\cdot i} = \frac{g(\theta_i)}{\prod_{s=1}^i (1 + g(\theta_s))},$$

for  $i = 1, \dots, R-1$ , and

$$p \cdot R = \frac{1}{\prod_{s=1}^{R-1} (1 + g(\theta_s))}.$$

**Lemma 2.** *Under the GMC model, we have*

$$E(X) = 1 + \sum_{i=1}^{R-1} \frac{1}{\prod_{s=1}^i (1 + g(\theta_s + \Delta))}, \quad E(Y) = 1 + \sum_{i=1}^{R-1} \frac{1}{\prod_{s=1}^i (1 + g(\theta_s))}.$$

PROOF. Assume that the GMC model holds. From Lemma 1 we see

$$\begin{aligned} E(X) &= \sum_{i=1}^R ip_i. \\ &= \sum_{i=1}^{R-2} i \frac{g(\theta_i + \Delta)}{\prod_{s=1}^i (1 + g(\theta_s + \Delta))} \\ &\quad + \left[ (R-1) \frac{g(\theta_{R-1} + \Delta)}{\prod_{s=1}^{R-1} (1 + g(\theta_s + \Delta))} + R \frac{1}{\prod_{s=1}^{R-1} (1 + g(\theta_s + \Delta))} \right] \\ &= \sum_{i=1}^{R-2} i \frac{g(\theta_i + \Delta)}{\prod_{s=1}^i (1 + g(\theta_s + \Delta))} \\ &\quad + \left[ (R-1) \frac{1}{\prod_{s=1}^{R-2} (1 + g(\theta_s + \Delta))} + \frac{1}{\prod_{s=1}^{R-1} (1 + g(\theta_s + \Delta))} \right] \\ &= \sum_{i=1}^{R-3} i \frac{g(\theta_i + \Delta)}{\prod_{s=1}^i (1 + g(\theta_s + \Delta))} \\ &\quad + \left[ (R-2) \frac{g(\theta_{R-2} + \Delta)}{\prod_{s=1}^{R-2} (1 + g(\theta_s + \Delta))} + (R-1) \frac{1}{\prod_{s=1}^{R-2} (1 + g(\theta_s + \Delta))} \right] \\ &\quad + \frac{1}{\prod_{s=1}^{R-1} (1 + g(\theta_s + \Delta))}. \end{aligned}$$

Since

$$\begin{aligned} &(k-1) \frac{g(\theta_{k-1} + \Delta)}{\prod_{s=1}^{k-1} (1 + g(\theta_s + \Delta))} + k \frac{1}{\prod_{s=1}^{k-1} (1 + g(\theta_s + \Delta))} \\ &= (k-1) \frac{1}{\prod_{s=1}^{k-2} (1 + g(\theta_s + \Delta))} + \frac{1}{\prod_{s=1}^{k-1} (1 + g(\theta_s + \Delta))}, \end{aligned}$$

where  $\prod_{s=1}^0 (1 + g(\theta_s + \Delta)) = 1$  for  $k = 2, \dots, R$ , we obtain

$$E(X) = 1 + \sum_{i=1}^{R-1} \frac{1}{\prod_{s=1}^i (1 + g(\theta_s + \Delta))}.$$

In a similar manner, we obtain

$$E(Y) = 1 + \sum_{i=1}^{R-1} \frac{1}{\prod_{s=1}^i (1 + g(\theta_s))}.$$

**Theorem 1.** *The MH model holds if and only if both the GMC and ME models hold.*

PROOF. If the MH model holds, then the GMC and ME models hold. Assuming that the GMC and ME models hold, we shall show that the MH model holds.

Since the ME model holds,

$$E(X) - E(Y) = \sum_{i=1}^{R-1} \frac{\left[ \prod_{s=1}^i (1 + g(\theta_s)) \right] - \left[ \prod_{s=1}^i (1 + g(\theta_s + \Delta)) \right]}{\left[ \prod_{s=1}^i (1 + g(\theta_s + \Delta)) \right] \left[ \prod_{s=1}^i (1 + g(\theta_s)) \right]} = 0,$$

from Lemmas 1 and 2. When  $\Delta > 0$ ,  $E(X) - E(Y) < 0$ . When  $\Delta < 0$ ,  $E(X) - E(Y) > 0$ . Therefore,  $\Delta = 0$ . Consequently, the MH model holds. The proof is complete.

We can also describe the following decompositions of the MH model.

**Corollary 1.** *The MH model holds if and only if both the MCOR and ME models hold.*

**Corollary 2.** *The MH model holds if and only if both the MCC and ME models hold.*

**Corollary 3.** *The MH model holds if and only if both the MCP and ME models hold.*

#### 4. Extension into Multi-way Tables

We extend the models and decompositions in Sections 2 and 3 into multi-way contingency tables.

Note that we must consider extensions into multi-way tables not only theoretical aspects but also practical aspects since they are known to be sparse. Although application issues will be future research, we give theoretical extensions respecting the historical value of previous research.

4.1. *Models* Consider an  $R^T$  table ( $T \geq 2$ ) with ordered categories. Let  $X_t$  denote the  $t$ -th random variable for  $t = 1, \dots, T$ , and let  $\Pr(X_1 = i_1, \dots, X_T = i_T) = p_{i_1 \dots i_T}$  for  $i_t = 1, \dots, R$ . The  $MH^T$  model can be expressed as

$$p_i^{(1)} = p_i^{(2)} = \dots = p_i^{(T)} \quad \text{for } i = 1, \dots, R,$$

where  $p_i^{(t)} = \Pr(X_t = i)$ . See e.g., Bhapkar and Darroch (1990) and Agresti (2013, p.439).

Let  $\omega_i^{(t)} = \Pr(X_t = i \mid X_t \geq i)$  for  $i = 1, \dots, R-1$ ;  $t = 1, \dots, T$ . Then we propose a model defined by

$$h^{-1}(\omega_i^{(k)}) = h^{-1}(\omega_i^{(1)}) + \Delta_k \quad \text{for } i = 1, \dots, R-1; k = 2, \dots, T, \quad (4.1)$$

where the parameters  $\Delta_k$  are unspecified. A special case of this model obtained by setting  $\Delta_2 = \dots = \Delta_T = 0$  is the  $MH^T$  model. We refer to model (4.1) as the  $GMC^T$  model. Under the  $GMC^T$  model,  $\Delta_k > 0$  ( $k = 2, \dots, T$ ) is equivalent to  $\omega_i^{(k)} > \omega_i^{(1)}$  for  $i = 1, \dots, R-1$ . Therefore, the parameters  $\Delta_k$  in the  $GMC^T$  model reflect the degree of inhomogeneity between  $\{\omega_i^{(k)}\}$  and  $\{\omega_i^{(1)}\}$ . Incidentally, by setting  $h^{-1}(\omega_i^{(1)}) = \theta_i$ , the  $GMC^T$  model can be expressed as

$$\omega_i^{(t)} = h(\theta_i + \Delta_t) \quad \text{for } i = 1, \dots, R-1; t = 1, \dots, T,$$

where  $\Delta_1 = 0$ . Hence, under the  $GMC^T$  model, the conditional probability  $\omega_i^{(k)}$  is a location shift of the conditional probability  $\omega_i^{(1)}$  in terms of the above equation for  $k = 2, \dots, T$ .

Especially, when  $h^{-1}(x) = \log(x/(1-x))$ , the  $GMC^T$  model is expressed as

$$\log\left(\frac{\omega_i^{(k)}}{1 - \omega_i^{(k)}}\right) = \log\left(\frac{\omega_i^{(1)}}{1 - \omega_i^{(1)}}\right) + \Delta_k, \quad (4.2)$$

for  $i = 1, \dots, R-1$ ;  $k = 2, \dots, T$ . We shall refer to model (4.2) as the  $MCOR^T$  model. Note that

$$\frac{\omega_i^{(t)}}{1 - \omega_i^{(t)}} = \frac{p_i^{(t)}}{1 - F_i^{(t)}} \quad \text{for } i = 1, \dots, R-1; t = 1, \dots, T,$$

where  $F_i^{(t)} = \sum_{s=1}^i p_s^{(t)} = \Pr(X_t \leq i)$ . Using the marginal continuation odds ratio, the  $MCOR^T$  model can also be expressed as

$$\log \psi_i^{(k)} = \Delta_k \quad \text{for } i = 1, \dots, R-1; k = 2, \dots, T,$$

where

$$\psi_i^{(k)} = \frac{\omega_i^{(k)} (1 - \omega_i^{(1)})}{\omega_i^{(1)} (1 - \omega_i^{(k)})} = \frac{p_i^{(k)} (1 - F_i^{(1)})}{p_i^{(1)} (1 - F_i^{(k)})}.$$

Using the complementary log-log transformation, the  $\text{GMC}^T$  model is expressed as

$$\log \left( -\log \left( 1 - \omega_i^{(k)} \right) \right) = \log \left( -\log \left( 1 - \omega_i^{(1)} \right) \right) + \Delta_k, \quad (4.3)$$

for  $i = 1, \dots, R-1$ ;  $k = 2, \dots, T$ . We shall refer to model (4.3) as the  $\text{MCC}^T$  model.

Using the probit transformation, the  $\text{GMC}^T$  model is expressed as

$$\Phi^{-1} \left( \omega_i^{(k)} \right) = \Phi^{-1} \left( \omega_i^{(1)} \right) + \Delta_k \quad \text{for } i = 1, \dots, R-1; k = 2, \dots, T, \quad (4.4)$$

where  $\Phi(\cdot)$  is the cumulative distribution function of the standard normal distribution. We refer to model (4.4) as the  $\text{MCP}^T$  model.

**4.2. Decompositions of the Marginal Homogeneity Model** Consider the  $\text{ME}^T$  model defined by

$$\text{E}(X_1) = \dots = \text{E}(X_T),$$

i.e.,

$$\sum_{i=1}^R ip_i^{(1)} = \dots = \sum_{i=1}^R ip_i^{(T)}.$$

Note that the  $\text{MH}^T$  model implies the  $\text{ME}^T$  model.

We obtain the following theorem.

**Theorem 2.** *For the  $R^T$  table, the  $\text{MH}^T$  model holds if and only if both the  $\text{GMC}^T$  and  $\text{ME}^T$  models hold.*

The proof is omitted because it is obtained in a similar manner as the proof of Theorem 1. We also obtain the following corollaries.

**Corollary 4.** *For the  $R^T$  table, the  $\text{MH}^T$  model holds if and only if both the  $\text{MCOR}^T$  and  $\text{ME}^T$  models hold.*

**Corollary 5.** *For the  $R^T$  table, the  $\text{MH}^T$  model holds if and only if both the  $\text{MCC}^T$  and  $\text{ME}^T$  models hold.*

**Corollary 6.** *For the  $R^T$  table, the  $\text{MH}^T$  model holds if and only if both the  $\text{MCP}^T$  and  $\text{ME}^T$  models hold.*

## 5 Goodness-of-fit Test

Let  $n_{i_1 \dots i_T}$  denote the observed frequency in the  $(i_1, \dots, i_T)$  cell of the  $R^T$  table with  $n = \sum \dots \sum n_{i_1 \dots i_T}$ , and let  $m_{i_1 \dots i_T}$  denote the corresponding expected frequency. Assume that  $\{n_{i_1 \dots i_T}\}$  has a multinomial distribution. The maximum likelihood estimates (MLEs) of the expected frequencies under each model can be obtained using the Newton-Raphson method to solve the likelihood equations.

The likelihood ratio chi-squared statistic to test the goodness-of-fit of model  $M$  is given by

$$G^2(M) = 2 \sum_{i_1=1}^R \dots \sum_{i_T=1}^R n_{i_1 \dots i_T} \log \left( \frac{n_{i_1 \dots i_T}}{\hat{m}_{i_1 \dots i_T}} \right),$$

where  $\hat{m}_{i_1 \dots i_T}$  is the MLEs of  $m_{i_1 \dots i_T}$  under the model. The numbers of degrees of freedom (df) of statistics for testing the goodness-of-fit of the MH, GMC, and ME models are  $(T-1)(R-1)$ ,  $(T-1)(R-2)$ , and  $T-1$ , respectively. Consider two nested models, say  $M_1$  and  $M_2$ , such that if model  $M_1$  holds, then model  $M_2$  holds. To test the goodness-of-fit of model  $M_1$  assuming that model  $M_2$  holds, the conditional likelihood ratio statistic is given by  $G^2(M_1 | M_2) = G^2(M_1) - G^2(M_2)$ . The number of df for the conditional test is the difference between the numbers of df for models  $M_1$  and  $M_2$ .

## 6 Examples

*6.1. Example 1* We focus on the contingency table grouping the time scale into ordered categories such as the sleep-onset time. As an example, we used the research data of Marqueze et al. (2015a, b), which was found in the Dryad Digital Repository. We created a square contingency table by grouping the sleep-onset time scale between work days and days-off (Table 1). We used the pair sleep-onset time data of work days and days-off from the original data set, and combined two variables at once. Incidentally, the variable names of the dataset are “Bedtimew” and “Bedtimef”. Then we calculated the first quartile and the third quartile from the combined data to create a square contingency table using these quartiles as the cut points. Namely, we classified the continuous bedtime at three levels: (1) below the first quartile, (2) the first quartile or more but less than the third quartile, and (3) the third quartile or more.

We shall analyze the data in Table 1 using Corollary 1. The MCOR model fits these data well since  $G^2(\text{MCOR}) = 0.73$  with 1 df. However, the

Table 1: Marqueze’s data expressing the bedtime for work days and days-off using three levels: (1) below the first quartile, (2) the first quartile or more but less than the third quartile, and (3) the third quartile or more (Marqueze et al. 2015a, b). Parenthesized values are the MLEs of the expected frequencies under the MCOR model

Work days	Days-off			Totals
	(1)	(2)	(3)	
(1)	292 (293.74)	143 (137.70)	9 (8.92)	444 (440.36)
(2)	22 (23.03)	566 (566.13)	283 (291.79)	871 (880.95)
(3)	26 (26.22)	157 (151.55)	180 (178.92)	363 (356.69)
Totals	340 (342.99)	866 (855.38)	472 (479.63)	1678 (1678)

Source: <http://doi.org/10.5061/dryad.73f69>

MH and ME models do not fit these data well since  $G^2(\text{MH}) = 68.86$  with 2 df and  $G^2(\text{ME}) = 58.48$  with 1 df.

We shall consider the hypothesis that the MH model holds under the assumption that the MCOR model holds; namely, the hypothesis that  $\Delta = 0$  holds. Since  $G^2(\text{MH}|\text{MCOR}) = G^2(\text{MH}) - G^2(\text{MCOR}) = 68.13$  with 1 df, we reject this hypothesis at the 0.05 level. This shows  $\Delta \neq 0$  in the MCOR model. Therefore, the MCOR model is preferable to the MH model for the data in Table 1. Under the MCOR model, the MLEs of  $\exp(\Delta)$  are  $\exp(\hat{\Delta}) = 1.38$ . Noting that  $\omega_1^X / (1 - \omega_1^X) = p_{1\cdot} / (p_{2\cdot} + p_{3\cdot})$ ,  $\omega_2^X / (1 - \omega_2^X) = p_{2\cdot} / p_{3\cdot}$ ,  $\omega_1^Y / (1 - \omega_1^Y) = p_{1\cdot} / (p_{2\cdot} + p_{3\cdot})$ , and  $\omega_2^Y / (1 - \omega_2^Y) = p_{2\cdot} / p_{3\cdot}$ , we see under the MCOR model that (i) the odds that the sleep-onset time is (1) below the first quartile, instead of (2) or (3), i.e., the first quartile or more, is estimated to be  $\exp(\hat{\Delta}) = 1.38$  times higher for work days than for days-off, and (ii) the odds that it is (2) the first quartile or more but less than the third quartile, instead of (3) the third quartile or more is estimated to be 1.38 times higher for work days than for days-off.

Section 7 discusses the interpretation of this results from the viewpoint of time scales.

*6.2. Example 2* Consider the data in Table 2, which is obtained from the Meteorological Agency in Japan (Tahata et al., 2008). These are obtained from the daily atmospheric temperatures at Hiroshima, Tokyo, and

Table 2: Daily atmospheric temperatures at Hiroshima, Tokyo, and Sapporo in Japan in 2003, using three levels: (1) low, (2) normal, and (3) high (Tahata et al., 2008). Parenthesized values are the MLEs of the expected frequencies under the  $\text{MCOR}^T$  model

Hiroshima	Tokyo	Sapporo		
		(1)	(2)	(3)
(1)	(1)	37 (37.06)	13 (13.51)	3 (3.00)
(1)	(2)	21 (21.88)	17 (18.41)	5 (5.21)
(1)	(3)	4 (4.05)	4 (4.20)	5 (5.06)
(2)	(1)	19 (17.69)	15 (14.45)	5 (4.65)
(2)	(2)	20 (19.32)	29 (29.02)	8 (7.72)
(2)	(3)	20 (18.79)	20 (19.45)	12 (11.27)
(3)	(1)	2 (1.97)	8 (8.19)	4 (3.95)
(3)	(2)	8 (8.21)	15 (15.99)	14 (14.37)
(3)	(3)	7 (6.98)	21 (21.71)	29 (28.90)

Sapporo in Japan in 2003 using three levels: (1) low, (2) normal, and (3) high. Variables  $X_1$ ,  $X_2$ , and  $X_3$  mean the temperatures at Hiroshima, Tokyo, and Sapporo, respectively.

We shall analyze the data in Table 2 using Corollary 4. The  $\text{MCOR}^T$  model fits these data well since  $G^2(\text{MCOR}^T) = 0.61$  with 2 df, whereas the  $\text{MH}^T$  and  $\text{ME}^T$  models do not fit these data well since  $G^2(\text{MH}^T) = 16.80$  with 4 df and  $G^2(\text{ME}^T) = 16.39$  with 2 df.

We shall consider the hypothesis that the  $\text{MH}^T$  model holds under the assumption that the  $\text{MCOR}^T$  model holds; namely, the hypothesis that  $\Delta_2 = \Delta_3 = 0$  holds. Since  $G^2(\text{MH}^T | \text{MCOR}^T) = G^2(\text{MH}^T) - G^2(\text{MCOR}^T) = 16.19$  with 2 df, we reject this hypothesis at the 0.05 level. Therefore the  $\text{MCOR}^T$  model is preferable to the  $\text{MH}^T$  model for these data.

We see from Corollary 4 that the poor fit of the  $\text{MH}^T$  model is caused by the poor fit of the  $\text{ME}^T$  model rather than the  $\text{MCOR}^T$  model. That

is, the mean temperatures at Hiroshima, Tokyo, and Sapporo differ. Under the  $\text{MCOR}^T$  model, the MLEs of  $\{\exp(\Delta_k)\}$  are  $\exp(\hat{\Delta}_2) = 0.90$  and  $\exp(\hat{\Delta}_3) = 1.33$ . Noting that  $\omega_1^{(t)} / (1 - \omega_1^{(t)}) = p_1^{(t)} / (p_2^{(t)} + p_3^{(t)})$  and  $\omega_2^{(t)} / (1 - \omega_2^{(t)}) = p_2^{(t)} / p_3^{(t)}$ , we see under the  $\text{MCOR}^T$  model that the odds that the temperature is (1) Low instead of (2) Normal or (3) High is estimated to be  $\exp(\hat{\Delta}_2) = 0.90$  times higher in Tokyo than in Hiroshima, and the odds that it is (2) Normal instead of (3) High is estimated to be 0.90 times higher in Tokyo than in Hiroshima. Also we see that the odds that it is (1) Low instead of (2) Normal or (3) High is estimated to be  $\exp(\hat{\Delta}_3) = 1.33$  times higher in Sapporo than in Hiroshima, and the odds that it is (2) Normal instead of (3) High is estimated to be 1.33 times higher in Sapporo than in Hiroshima.

7 Discussion

7.1. *Comparison Between Models* Analyzing the data in Tables 1 and 2, the goodness-of-fits of the  $\text{MCOR}^T$ ,  $\text{MCC}^T$ , and  $\text{MCP}^T$  models are remarkably different (see Table 3). The  $\text{MCOR}^T$  and  $\text{MCP}^T$  models fit both the data in Tables 1 and 2 very well. However, the  $\text{MCC}^T$  model fits the data in Table 2 well, although it does not fit the data in Table 1 well. From above, considering special cases of the  $\text{GMC}^T$  model, the conditional probabilities of the  $\text{MCOR}^T$  and  $\text{MCP}^T$  models have a symmetric appearance. However, that of the  $\text{MCC}^T$  model is asymmetric,  $\log(-\log(1-x))$  approaches 0 fairly slowly but approaches 1 quite sharply.

The  $\text{MCOR}^T$  and  $\text{MCC}^T$  models may be useful because the parameter  $\exp(\Delta_k)$  of the  $\text{MCOR}^T$  model can be interpreted as the marginal continuation odds ratio and the parameter  $\Delta_k$  of the  $\text{MCC}^T$  model can be considered

Table 3: Likelihood ratio statistic  $G^2$  for models applied to the data in Tables 1 and 2

Models	Table 1		Table 2	
	df	$G^2$	df	$G^2$
$\text{MH}^T$	2	68.86*	4	16.80*
$\text{MCOR}^T$	1	0.73	2	0.61
$\text{MCC}^T$	1	5.17*	2	0.93
$\text{MCP}^T$	1	0.50	2	0.60
$\text{ME}^T$	1	58.48*	2	16.39*

Note: \* means significant at the 0.05 level

as a location shift between the marginal distributions. On the other hand, some models such as the MCP<sup>T</sup> model make it difficult to interpret the parameter  $\Delta_k$ . Therefore, the GMC<sup>T</sup> model may provide various strictly increasing functions to find the most applicable model to the data but the interpretation of  $\Delta_k$  may be difficult. Hence, it is important that an analyst decides what kind of model to employ for data analysis.

*7.2. Treating Conditional Probabilities as Discrete Time Hazards* Due to the different viewpoints, the conditional probability  $\omega_i^{(t)} = \Pr(X_t = i \mid X_t \geq i)$  may be considered as discrete time hazards. That is, it is the conditional probability of experiencing an event in the period  $i$  under the condition that has not experienced the event before the period  $i$ . Namely, if  $X_t$  represents a categorized survival time, the conditional probability represents the probability of survival to time level  $i$  given survival at least that long, which is the hazard rate. Hence, the MCOR<sup>T</sup> model can describe hazards functions for grouped survival data, and a certain model using the complementary log-log transformation is also useful for such data. When we consider the MCC<sup>T</sup> model, we can also consider the ratio of survival functions. For discretely measured survival, let  $S_i^{(t)} = 1 - F_{i-1}^{(t)} = \Pr(X_t \geq i)$  for  $i = 1, \dots, R-1$ ;  $t = 1, \dots, T$  (Agresti 2010, p.128). Namely,  $S_i^{(t)}$  denotes the discrete survival function of  $X_t$ . The conditional probabilities  $\omega_i^{(t)}$  can be expressed as

$$\omega_i^{(t)} = \frac{p_i^{(t)}}{1 - F_{i-1}^{(t)}} = 1 - \frac{1 - F_i^{(t)}}{1 - F_{i-1}^{(t)}} = 1 - \frac{S_{i+1}^{(t)}}{S_i^{(t)}},$$

for  $i = 1, \dots, R-1$ ;  $t = 1, \dots, T$ . Thus, the GMC<sup>T</sup> model can also be expressed as

$$h^{-1} \left( 1 - \frac{S_{i+1}^{(k)}}{S_i^{(k)}} \right) = h^{-1} \left( 1 - \frac{S_{i+1}^{(1)}}{S_i^{(1)}} \right) + \Delta_k,$$

for  $i = 1, \dots, R-1$ ;  $k = 2, \dots, T$ .

Consider the data in Table 1. Under the MCOR model, we can treat not only the marginal continuation odds ratio but also the discrete time hazards. The marginal continuation odds ratio is estimated to be  $\exp(\hat{\Delta}) = 1.38$  (see Example 1). Furthermore, the hazard of work days is estimated to be  $\hat{\Delta} = 0.32$  location shift of that of days-off on a logistic scale. Hence, the sleep-onset time for work days tends to be earlier than that for days-off at a constant hazard on a logistic scale.

When an analyst treats the contingency table by grouping the time scale such as studies of survival, the proposed models and decompositions may be useful from the viewpoint of discrete time hazards.

*Acknowledgements.* The authors would like to thank the editor and anonymous referees for the meaningful comments.

*Open Access.* This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

## References

- AGRESTI, A. (2010). *Analysis of Ordinal Categorical Data*, 2nd edn. Wiley, Hoboken.
- AGRESTI, A. (2013). *Categorical Data Analysis*, 3rd edn. Wiley, Hoboken.
- ALTUN, G. and AKTAŞ, S. (2018). Measures of departure from marginal homogeneity model in square contingency tables. *Journal of Statisticians: Statistics and Actuarial Sciences* **2**, 93–108.
- BHAPKAR, V.P. and DARROCH, J.N. (1990). Marginal symmetry and quasi symmetry of general order. *J. Multivar. Anal.* **34**, 173–184.
- BISHOP, Y.M.M., FIENBERG, S.E. and HOLLAND, P.W. (1975). *Discrete Multivariate Analysis: Theory and Practice* Cambridge. The MIT Press, Massachusetts.
- BROSS, I.D.J. (1958). How to use ridit analysis. *Biometrics* **14**, 18–38.
- FIENBERG, S.E. (1980). *The Analysis of Cross-Classified Categorical Data*, 2nd edn. The MIT Press, Cambridge.
- FLEISS, J.L., LEVIN, B. and PAIK, M.C. (2003). *Statistical Methods for Rates and Proportions*, 3rd edn. Wiley, Hoboken.
- GREENLAND, S. (1994). Alternative models for ordinal logistic regression. *Stat. Med.* **13**, 1665–1677.
- IKI, K., TAHATA, K. and TOMIZAWA, S. (2010). Decomposition of marginal homogeneity into logit and mean ridits equality for square contingency tables. *J. Math. Stat.* **6**, 64–67.
- KATERI, M., GOTTARD, A. and TARANTOLA, C. (2017). Generalised quasi-symmetry models for ordinal contingency tables. *Aust N Z J Stat* **59**, 239–253.
- KURAKAMI, H., TAHATA, K. and TOMIZAWA, S. (2013). Generalized marginal cumulative logistic model for multi-way contingency tables. *SUT J. Math.* **49**, 19–32.
- LÄÄRÄ, E. and MATTHEWS, J.N.S. (1985). The equivalence of two models for ordinal data. *Biometrika* **72**, 206–207.

- MARQUEZE, E.C., VASCONCELOS, S., GAREFELT, J., SKENE, D.J., MORENO, C.R. and LOWDEN, A. (2015a). Natural light exposure, sleep and depression among day workers and shiftworkers at arctic and equatorial latitudes. *PLOS ONE* **10**, e0122078. <https://doi.org/10.1371/journal.pone.0122078>.
- MARQUEZE, E.C., VASCONCELOS, S., GAREFELT, J., SKENE, D.J., MORENO, C.R. and LOWDEN, A. (2015b). Data from: Natural light exposure, sleep and depression among day workers and shiftworkers at arctic and equatorial latitudes. Dryad Digital Repository. <https://doi.org/10.5061/dryad.73f69>.
- MCCULLAGH, P. (1977). A logistic model for paired comparisons with ordered categorical data. *Biometrika* **64**, 449–453.
- MCCULLAGH, P. (1980). Regression models for ordinal data. *J. Royal Stat. Soc. Ser. B* **42**, 109–142.
- MCCULLAGH, P. and NELDER, J.A. (1983). *Generalized Linear Models*. Chapman & Hall, London.
- MIYAMOTO, N., TAHATA, K., EBIE, H. and TOMIZAWA, S. (2006). Marginal inhomogeneity models for square contingency tables with nominal categories. *J. Appl. Stat.* **33**, 203–215.
- SAIGUSA, Y., MARUYAMA, T., TAHATA, K. and TOMIZAWA, S. (2018). Extended marginal homogeneity model based on complementary log-log transform for square tables. *Int. J. Stat. Prob.* **7**, 27–31.
- STUART, A. (1955). A test for homogeneity of the marginal distributions in a two-way classification. *Biometrika* **42**, 412–416.
- TAHATA, K., KOBAYASHI, H. and TOMIZAWA, S. (2008). Conditional marginal cumulative logistic models and decomposition of marginal homogeneity model for multi-way tables. *J. Stat. Appl.* **3**, 239–252.
- TAHATA, K., TAJIMA, K. and TOMIZAWA, S. (2006). A measure of asymmetry of marginal ridits for square contingency tables with ordered categories. *JJSCS* **19**, 69–85.
- TAHATA, K. and TOMIZAWA, S. (2008). Generalized marginal homogeneity model and its relation to marginal equimoments for square contingency tables with ordered categories. *ADAC* **2**, 295–311.
- THOMPSON, W.A. (1977). On the treatment of grouped observations in life studies. *Biometrics* **33**, 463–470.
- TOMIZAWA, S. (1993). Diagonals-parameter symmetry model for cumulative probabilities in square contingency tables with ordered categories. *Biometrics* **49**, 883–887.
- TUTZ, G. (1991). Sequential models in categorical regression. *Comput. Stat. Data Anal.* **11**, 275–295.

*Publisher's Note.* Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

SATORU SHINODA  
CLINICAL DATA SCIENCE  
DEPARTMENT, DEVELOPMENT  
HEADQUARTERS, TAISHO  
PHARMACEUTICAL CO., LTD.,  
TOSHIMA-KU, TOKYO, 170-8633,  
JAPAN  
E-mail: sa-shinoda@taisho.co.jp

SATORU SHINODA  
KOUJI TAHATA  
SADAO TOMIZAWA  
DEPARTMENT OF INFORMATION SCIENCES,  
FACULTY OF SCIENCE AND TECHNOLOGY,  
TOKYO UNIVERSITY OF SCIENCE,  
NODA CITY, CHIBA, 278-8510, JAPAN

KOUJI YAMAMOTO  
DEPARTMENT OF BIostatISTICS, SCHOOL  
OF MEDICINE, YOKOHAMA CITY  
UNIVERSITY, YOKOHAMA CITY,  
KANAGAWA, 236-0004, JAPAN

Paper received: 7 June 2019.