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ORIGINAL PAPER

Existence, uniqueness and regularity for a semilinear stochastic subdiffusion with integrated multiplicative noise

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Abstract

We investigate a semilinear stochastic time-space fractional subdiffusion equation driven by fractionally integrated multiplicative noise. The equation involves the ψ-Caputo derivative of order $\alpha \in (0, 1)$ and the spectral fractional Laplacian of order $\beta \in (\frac{1}{2}, 1]$. The existence and uniqueness of the mild solution are proved in a suitable Banach space by using the Banach contraction mapping principle. The spatial and temporal regularities of the mild solution are established in terms of the smoothing properties of the solution operators.

Keywords ψ-Caputo derivative · Spectral fractional Laplacian · Fractionally integrated multiplicative noise · existence and uniqueness · Regularity

Mathematics Subject Classification 35R11 · 60J65 · 35A01 · 35A02 · 35B65

1 Introduction

In this paper, we consider the following semilinear stochastic time-space subdiffusion equation driven by fractionally integrated multiplicative noise, with $a \in \mathbb{R}$,

$$
\begin{aligned}
&{}_{C\psi}\mathrm{D}_{a,t}^{\alpha} u(t) + A^{\beta} u(t) = f(t, u(t)) + {}_{\psi}\mathrm{D}_{a,t}^{-\gamma}\, g(t, u(t)) \frac{\mathrm{d}W(t)}{\mathrm{d}t}, \quad a < t \le T, \\
&u(a) = u_a,
\end{aligned}
\tag{1.1}
$$

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where ${}_{C\psi}\mathrm{D}_{a,t}^{\alpha}u(t)$ and ${}_{\psi}\mathrm{D}_{a,t}^{-\gamma}u(t)$ denote the ψ-Caputo fractional derivative of order $\alpha \in (0, 1)$ and the ψ-fractional integral of order $\gamma \in [0, 1]$ defined in Definitions 2.3 and 2.1, respectively.

Let $H = L^2(\mathcal{D})$, $H_0^1(\mathcal{D})$ and $H^2(\mathcal{D})$ be the standard Sobolev spaces. Let $A = -\Delta$ with $D(A) = H_0^1(\mathcal{D}) \cap H^2(\mathcal{D})$ where $\mathcal{D} \subset \mathbb{R}^d$, $d = 1, 2, 3$ is a bounded domain with the smooth boundary and Δ denotes the Laplacian. Let A^{β}, $\beta \in (1/2, 1]$ be the fractional power of A. Further, W denotes an H-valued Wiener process with covariance operator Q. The nonlinear functions f and g satisfy some properties in Assumptions 2.2, 2.4, 5.1, 5.2. The initial value $u_a \in H$ is a random variable.

Fractional calculus, which includes fractional integrals and derivatives, has various practical and theoretical applications in fields such as physics, chemistry, biology, automatic control systems, anomalous diffusion, stochastic processes, and contaminant transport in underground water flows, among others [4, 8, 13, 18, 34, 35, 37, 41]. The Riemann-Liouville-type fractional calculus, which includes the Riemann-Liouville integral, Riemann-Liouville derivative, and Caputo derivative, has been extensively studied [14, 18, 37, 38]. Some researchers have also focused on the Hadamard-type fractional calculus [15, 17, 27–29], which includes the Hadamard integral, Hadamard derivative, and Caputo-Hadamard derivative. These two types of fractional integrals and derivatives have the different convolutional kernels. The Riemann-Liouville fractional integral has a power kernel and the Hadamard fractional integral has a logarithmic kernel. Using a general function ψ as a convolutional kernel instead of a specific function has significant implications, leading to the development of fractional integrals and derivatives of a function with respect to another function, commonly known as ψ-fractional calculus (ψ-fractional integral, ψ-Riemann-Liouville derivative, and ψ-Caputo derivative). Recently, some authors have investigated these types of fractional integrals and derivatives [2, 3, 11, 16, 30, 31, 33].

Deterministic fractional differential equations (FDEs) and fractional partial differential equations (FPDEs) have been extensively studied in the literature, with numerous theoretical and numerical analyses [9, 12, 14, 18, 24–26]. However, in many practical problems, stochastic perturbations from natural sources cannot be ignored, requiring the consideration of stochastic fractional differential equations (SFDEs) and stochastic fractional partial differential equations (SFPDEs). In this regard, several theoretical results have been achieved for SFPDEs. Cui and Yan [5] discussed the existence of solutions for fractional neutral stochastic integro-differential equations with infinite delay. Sakthivel et al. [39] derived the existence of mild solutions for a class of fractional stochastic differential equations with impulses in Hilbert spaces, utilizing fixed point techniques. In [36], the authors proved the existence and uniqueness of mild solutions for space-time fractional stochastic partial differential equations. Liu et al. [23] demonstrated the existence and uniqueness of solutions to stochastic quasi-linear partial differential equations with time-fractional derivatives. For additional references on this subject, we refer to [1, 6, 7, 20] and the references therein.

In [19], the author studied the following semilinear stochastic evolution equation

$$\begin{aligned} &\mathrm{d}u(t) + [Au(t) + f(t, u(t))]\mathrm{d}t = g(t, u(t))\mathrm{d}W(t), \quad 0 < t \le T, \\ &u(0) = u_0, \end{aligned} \tag{1.2}$$

and proved the existence and uniqueness of the mild solution to Eq. (1.2) and derived spatial and temporal regularity results. The authors in [20] extended the model (1.2) and considered the following stochastic semilinear space and time fractional subdiffusion problem driven by fractionally integrated additive noise with $0 < \alpha < 1$, $\frac{1}{2} < \beta \leq 1$, $0 \leq \gamma \leq 1$,

$$\begin{aligned} &{}_0^C\mathrm{D}_t^\alpha u(t) + A^\beta u(t) = f(t, u(t) + {}_0^R\mathrm{D}_t^{-\gamma} \frac{dW(t)}{dt}, \quad 0 < t \leq T, \\ &u(0) = u_0, \end{aligned} \tag{1.3}$$

where ${}_0^C\mathrm{D}_t^\alpha$, ${}_0^R\mathrm{D}_t^{-\gamma}$ and A^β denote the Caputo fractional derivative operator, the Riemann-Liouville fractional integral operator and the spectral fractional Laplace operator, respectively. They shown the existence and uniqueness of mild solution to Eq. (1.3) by the Banach contraction mapping principle and further obtained some regularity results of the mild solution. Yang [42] investigated the following stochastic evolution equations with ψ-Caputo derivative and varying-time delay driven by fractional Brownian motion, with $\tau > 0$,

$$\begin{aligned} &{}_{C\psi}\mathrm{D}_{a,t}^\alpha u(t) + Au(t) = f(t, u(t - r(t))) + \sigma(t)\frac{dW^{H_1}(t)}{dt}, \quad 0 < t \leq T, \\ &u(t) = \phi(t),\ t \in [-\tau, 0], \end{aligned} \tag{1.4}$$

where ${}_{C\psi}\mathrm{D}_{a,t}^\alpha u(t)$ is the ψ-Caputo fractional derivative operator of order $\frac{1}{2} < \alpha \leq 1$ and $W^{H_1}(t)$ is a H-valued Wiener process with Hurst parameter $H_1 \in (\frac{1}{2}, 1)$. The existence and uniqueness of the mild solution of (1.4) is proved by using the Banach fixed point theorem.

In this paper, we expand upon the equations presented in (1.2)- (1.4) by further exploring the equation in (1.1), which includes the ψ-Caputo fractional derivative and fractionally integrated multiplicative noise. This equation has significant mathematical and practical applications. Our methodology follows the same idea as Chapter 2 in [19]. We establish a $L_p(\Omega)$, $p \geq 2$ integrable mild solution to Eq. (1.1) using the well-known Mittag-Leffler functions as solution operators. We investigate the properties of the solution operators and demonstrate the existence and uniqueness of the mild solution to Eq. (1.1) using the Banach contraction mapping principle. Furthermore, we present the spatial and temporal regularity of the mild solution to Eq. (1.1). The challenge of this paper is to develop various smoothing properties associated with Mittag-Leffler operators that involve the general function ψ and understand the relationships between the parameters α, β, γ and the spatial as well as temporal regularities inherent in the mild solution of (1.1).

The paper is organized as follows. In Section 2, we provide some background information, including notations, definitions, and lemmas. In Section 3, we establish the existence, uniqueness of the mild solution to Eq. (1.1). In Section 4, we consider the spatial and temporal regularities of the mild solution. In Section 5, we discuss the further spatial and temporal regularities of the mild solution under strong assumptions for f and g. The conclusion is presented in Section 6 and the Appendix is in final Section 1.

Throughout the paper, we use C to denote a generic positive constant, which may vary at different occurrences and may depend on a and T.

2 Preliminaries

In this section, we shall introduce some notations, definitions, assumptions, and lemmas.

Denote by $(\Omega, \mathcal{F}, \mathbb{P})$ the complete probability space with filtration $(\mathcal{F}_t)_{t\in[a,T]}$ and by $W : [a, T]\times \to H$ a Q-Wiener process on H. Let $\mathcal{L}(H) = \mathcal{L}(H, H)$ denote the space of bounded linear operators from H to H. Let $Q \in \mathcal{L}(H)$. The Hilbert space of all Hilbert-Schmidt operators from $Q^{\frac{1}{2}}(H)$ to H is defined by

$$\mathcal{L}_2^0 = \{\Psi : Q^{\frac{1}{2}}(H) \to H,\ \|\Psi\|^2_{\mathcal{L}_2^0} := \sum_{j=1}^{\infty} \|\Psi Q^{\frac{1}{2}}\varphi_j\|^2 < \infty\},$$

where $\{\varphi_j\}_{j=1}^{\infty}$ is the orthonormal basis in H.

Assume that $(\lambda_k, \varphi_k)_{k=1}^{\infty}$ is a sequence of the eigenpairs of $A : D(A) = H_0^1(\mathcal{D}) \cap H^2(\mathcal{D}) \to H$. Let $r \in \mathbb{R}$ and denote $\dot{H}^r := D(A^{\frac{r}{2}})$. The norm for $\upsilon \in \dot{H}^r$ is defined by

$$|\upsilon|_r^2 = \sum_{k=1}^{\infty} \lambda_k^r (\upsilon, \varphi_k)^2 < \infty. \tag{2.1}$$

Let $L^p(\Omega; \dot{H}^r),\ p \geq 2$ be the Hilbert space of $L^p(\Omega)$-integrable random variables with values in $\dot{H}^r$ such that $||v||^p_{L^p(\Omega;\dot{H}^r)} = \mathbb{E}|v|_r^p < \infty$, where $\mathbb{E}$ denotes the expectation.

Let

$$\mathcal{H}_p = \{u : [a, T] \to H,\ \sup_{t\in[a,T]} \mathbb{E}[||u(t)||^p] < \infty\}$$

endowed with the norm

$$||u||^p_{\mathcal{H}_p} := \sup_{t\in[a,T]} \mathbb{E}[||u(t)||^p].$$

It is obvious that $(\mathcal{H}_p, ||\cdot||_{\mathcal{H}_p})$ is a Banach space.

Next we introduce the ψ-fractional integral and derivatives.

Definition 2.1 [31] Let (a, b) $(-\infty \leq a < b \leq \infty)$ be a finite or infinite interval. Suppose that $\psi(t)$ is a strictly monotone increasing function with a continuous derivative

$\psi'(t)$ and $\psi(t) \to +\infty$ as $t \to +\infty$. Assume that $f \in L^1(a, b)$. The ψ-fractional integral of order $\alpha > 0$ is defined by

$$ {}_{\psi}\mathrm{D}_{a,t}^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (\psi(t) - \psi(\tau))^{\alpha-1} f(\tau)\psi'(\tau)\mathrm{d}\tau, \quad t > a. \tag{2.2} $$

In particular, if $\psi(t) = t$, $\log t$, then the ψ-fractional integral is reduced to the Riemann-Liouville and Hadamard fractional integrals, respectively [18]; If $\psi(t) = e^t$, then it is reduced to the exponential fractional integral [32].

Let $AC[a, b]$ be the space of absolutely continuous function defined on $[a, b]$. Denote $AC^n_{\delta_\psi}[a, b] = \{f : [a, b] \to \mathbb{R}, \delta_\psi^{n-1} f(t) \in AC[a, b]\}$ with $n = 1, 2, \ldots$, where $\delta_\psi f(t) = \left(\frac{1}{\psi'(t)}\frac{\mathrm{d}}{\mathrm{d}t}\right) f(t)$, $\delta_\psi^n f(t) = \delta_\psi(\delta_\psi^{n-1} f(t))$, and $\delta_\psi^0 f(t) = f(t)$.

Definition 2.2 [31] Let (a, b) $(-\infty \le a < b \le \infty)$ be a finite or infinite interval and $n - 1 < \alpha < n \in \mathbb{N}$. Suppose that $\psi(t)$ is a strictly monotone increasing function with $\psi'(t) \neq 0$ and $\psi(t) \to +\infty$ as $t \to +\infty$. Assume that $f \in AC^n_{\delta_\psi}[a, b]$. The ψ-Riemann–Liouville fractional derivative of order α is defined by

$$ {}_{\psi}\mathrm{D}_{a,t}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \delta_\psi^n \int_a^t (\psi(t) - \psi(\tau))^{n-\alpha-1} f(\tau)\psi'(\tau)\mathrm{d}\tau, \quad t > a. \tag{2.3} $$

Definition 2.3 [31] Let (a, b) $(-\infty \le a < b \le \infty)$ be a finite or infinite interval and $n - 1 < \alpha < n \in \mathbb{N}$. Suppose that $\psi(t)$ is a strictly monotone increasing function with $\psi'(t) \neq 0$ and $\psi(t) \to +\infty$ as $t \to +\infty$. Assume that $f \in AC^n_{\delta_\psi}[a, b]$. The ψ-Caputo fractional derivative of order α is defined as

$$ {}_{C\psi}\mathrm{D}_{a,t}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (\psi(t) - \psi(\tau))^{n-\alpha-1} \delta_\psi^n f(\tau)\psi'(\tau)\mathrm{d}\tau, \quad t > a. \tag{2.4} $$

Now we introduce some assumptions. All the parameters α, β, γ, the functions ψ, f and the operators A, g in the Assumptions below are related to Eq. (1.1).

Assumption 2.1 The linear operator $A : D(A) \subset H \to H$ is densely defined, self-adjoint and positive definite with compact inverse.

We introduce two parameters κ and κ_1 which are related to the spatial and temporal regularities of the mind solution to Eq. (1.1).

Let $\alpha \in (0, 1)$, $\beta \in (\frac{1}{2}, 1]$, $\gamma \in [0, 1]$, $\alpha + \gamma > \frac{1}{2}$ and denote

$$ \kappa = \min\left\{2\beta - 1, \frac{2\beta}{\alpha}(\alpha + \gamma - \frac{1}{2})\right\}, \tag{2.5} $$

and

$$ \kappa_1 = \min\{\frac{\alpha\kappa}{2\beta}, \alpha - \frac{\alpha}{2\beta}, \alpha + \gamma - \frac{1}{2}\}. \tag{2.6} $$

Remark 2.1 We shall prove the solution $u \in L^p(\Omega; \dot{H}^s)$ for $s \in [0, \kappa)$ in Theorem 4.1 where κ is defined by (2.5). When $\alpha = 1,\ \beta = 1,\ \gamma = 0$, the value κ is reduced to $\kappa = 1$ which is the value used in [19, Theorem 2.25] for the stochastic heat equation.

Remark 2.2 We shall prove the solution u in Theorem 4.2 has the following temporal regularity

$$\|u(t_1) - u(t_2)\|_{L^p(\Omega;H)} \leq C(\psi(t_1) - \psi(t_2))^{\kappa_1},$$

for all $t_1, t_2 \in [a, T]$ where κ_1 is defined by (2.6). When $\alpha = 1,\ \beta = 1,\ \gamma = 0$ and $\kappa = 1$, the value κ_1 is reduced to $\kappa_1 = \frac{1}{2}$ which is the value used in [19, Theorem 2.25] for the stochastic heat equation.

Assumption 2.2 Let $\alpha \in (0, 1)$, $\beta \in (\frac{1}{2}, 1]$, $\gamma \in [0, 1]$ and $\alpha + \gamma > \frac{1}{2}$. The nonlinear function $f : [a, T] \times H \to \dot{H}^{-1}$ satisfies $|f(a, 0)|_{-1} \leq C$ and

$$|f(t, h_1) - f(t, h_2)|_{-1} \leq C||h_1 - h_2|| \tag{2.7}$$

for any $t \in [a, T], h_1, h_2 \in H$. Moreover,

$$|f(t_1, h) - f(t_2, h)|_{-1} \leq C(1 + ||h||)|\psi(t_1) - \psi(t_2)|^{\kappa_1} \tag{2.8}$$

for any $t_1, t_2 \in [a, T],\ h \in H$, where κ_1 is defined by (2.6).

For example, we may choose $f(t, h) = h$ or $f(t, h) = \sin(h)$.

Remark 2.3 When $\alpha = 1,\ \beta = 1, \gamma = 0, \kappa = 1$ and $\psi(t) = t$, the power κ_1 in (2.8) is reduced to $\frac{1}{2}$ which is the power in [19, Assumption 2.14] for the stochastic heat equation. We may choose $f(t, h) = f_1(\psi(t), h)$ where $f_1(s, h)$ satisfies the Hölder condition with order κ_1 for the variable s. We need such kind of assumption for f in the proofs of existence, uniqueness and regularities of the mild solution to Eq. (1.1).

Assumption 2.3 The covariance operator $Q \in \mathcal{L}(H)$ is self-adjoint and positive semidefinite but not necessarily of finite trace.

Assumption 2.4 Let $\alpha \in (0, 1)$ and $\beta \in (\frac{1}{2}, 1]$. The nonlinear function $g : [a, T] \times H \to \mathcal{L}_2^0$ satisfies $||g(a, 0)||_{\mathcal{L}_2^0} \leq C$ and

$$||g(t, h_1) - g(t, h_2)||_{\mathcal{L}_2^0} \leq C||h_1 - h_2||, \tag{2.9}$$

for any $t \in [a, T], h_1, h_2 \in H$,.

Moreover,

$$||g(t_1, h) - g(t_2, h)||_{\mathcal{L}_2^0} \leq C(1 + ||h||)|\psi(t_1) - \psi(t_2)|^{\kappa_1} \tag{2.10}$$

for any $t_1, t_2 \in [a, T], h \in H$, where κ_1 is defined in (2.6).

For example, we may choose $g(t,h)=h$ or $g(t,h)=\sin(h)$.

Assumption 2.5 Let $\alpha \in (0,1)$, $\beta \in (\frac{1}{2},1]$, $\gamma \in [0,1]$, $p \in [2,\infty)$, and $\alpha+\gamma > \frac{1}{2}$. Let κ be defined by (2.5). The initial value $u_a : \Omega \to \dot{H}^\kappa$ is a random variable and satisfies

$$||u_a||_{L^p(\Omega;\dot{H}^\kappa)} := (\mathbb{E}[|u_a|_\kappa^p])^{\frac{1}{p}} < \infty. \tag{2.11}$$

We now introduce the definition and some properties of the Mittag-Leffler function.

Definition 2.4 [37] For $\sigma > 0$ and $\rho \in \mathbb{R}$, the Mittag-Leffler function is defined as

$$E_{\sigma,\rho}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\sigma k+\rho)}, \quad z \in \mathbb{C}.$$

Lemma 2.1 [37, Theorem 1.4] [40, Theorem 4] *For* $0 < \sigma < 2$, $\rho \in \mathbb{R}$, *and* $\frac{\sigma\pi}{2} < \theta < \min(\pi, \sigma\pi)$, *there exists a constant* $C > 0$ *such that, for* $\theta \le |\arg(z)| \le \pi$,

$$|E_{\sigma,\rho}(z)| \le \begin{cases} \dfrac{C}{1+|z|^2}, & \rho-\sigma \in \mathbb{Z}^- \cup \{0\}, \\ \dfrac{C}{1+|z|}, & \text{otherwise}. \end{cases}$$

Below is the definition of the mild solution for Eq. (1.1).

Definition 2.5 A predictable stochastic process $u : [a,T] \to H$ is called a $L_p(\Omega)$-integrable mild solution to Eq. (1.1) with $p \ge 2$ if

$$\sup_{t\in[a,T]} ||u(t)||_{L^p(\Omega;H)} < \infty \tag{2.12}$$

and, for any $t \in [a,T]$, it holds that $\mathbb{P}$-a.s.

$$\begin{aligned} u(t) =& \mathbf{E}_{\alpha,1}^{(\beta)}\big(\psi(t)-\psi(a)\big)\, u_a + \int_a^t \mathbf{E}_{\alpha,\alpha}^{(\beta)}\big(\psi(t)-\psi(\tau)\big) f(\tau,u(\tau))\psi'(\tau)\mathrm{d}\tau \\ &+ \int_a^t \mathbf{E}_{\alpha,\alpha+\gamma}^{(\beta)}\big(\psi(t)-\psi(\tau)\big) g(\tau,u(\tau))\psi'(\tau)\mathrm{d}W(\tau), \end{aligned} \tag{2.13}$$

where, for any $w \ge 0$,

$$\mathbf{E}_{\alpha,1}^{(\beta)}(w) = E_{\alpha,1}(-A^\beta w^\alpha),$$
$$\mathbf{E}_{\alpha,\alpha}^{(\beta)}(w) = w^{\alpha-1} E_{\alpha,\alpha}(-A^\beta w^\alpha), \quad \mathbf{E}_{\alpha,\alpha+\gamma}^{(\beta)}(w) = w^{\alpha+\gamma-1} E_{\alpha,\alpha+\gamma}(-A^\beta w^\alpha).$$

Here $E_{\alpha,1}(\cdot)$, $E_{\alpha,\alpha}(\cdot)$ and $E_{\alpha,\alpha+\gamma}(\cdot)$ are the Mittag-Leffler functions.

The following lemma examines the smoothing properties of the solution operators, which will be frequently utilized in the subsequent proofs of lemmas and theorems.

Lemma 2.2 *Let* $\alpha \in (0,1)$, $\beta \in (\frac{1}{2}, 1]$ *and* $\gamma \in [0,1]$. *Let* $|\cdot|_r$, $r > 0$ *be the norm defined in* (2.1). *Then there hold, for any* $\mu \in \mathbb{R}$, $\nu \in \mathbb{R}$ *and* $w \geq 0$,

$$\left|\mathbf{E}^{(\beta)}_{\alpha,1}(w)v\right|_{\mu} \leq Cw^{-\frac{\alpha}{2\beta}(\mu-\nu)}|v|_{\nu},\ 0 \leq \mu-\nu \leq 2\beta, \tag{2.14}$$

$$\left|\mathbf{E}^{(\beta)}_{\alpha,\alpha}(w)v\right|_{\mu} \leq Cw^{\alpha-1-\frac{\alpha}{2\beta}(\mu-\nu)}|v|_{\nu},\ 0 \leq \mu-\nu \leq 4\beta, \tag{2.15}$$

and

$$\left|\mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma}(w)v\right|_{\mu} \leq Cw^{\alpha+\gamma-1-\frac{\alpha}{2\beta}(\mu-\nu)}|v|_{\nu},\ 0 \leq \mu-\nu \leq 2\beta. \tag{2.16}$$

Moreover,

$$\left|\mathbf{E}^{(\beta)}_{\alpha,\alpha-1}(w)v\right|_{\mu} \leq Cw^{\alpha-2-\frac{\alpha}{2\beta}(\mu-\nu)}|v|_{\nu},\ 0 \leq \mu-\nu \leq 4\beta, \tag{2.17}$$

and

$$\left|\mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma-1}(w)v\right|_{\mu} \leq Cw^{\alpha+\gamma-2-\frac{\alpha}{2\beta}(\mu-\nu)}|v|_{\nu},\ 0 \leq \mu-\nu \leq 2\beta, \tag{2.18}$$

where

$$\mathbf{E}^{(\beta)}_{\alpha,\alpha-1}(w) = w^{\alpha-2}E_{\alpha,\alpha-1}(-A^{\beta}w^{\alpha}),$$

and

$$\mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma-1}(w) = w^{\alpha+\gamma-2}E_{\alpha,\alpha+\gamma-1}(-A^{\beta}w^{\alpha}).$$

Proof We only give the proofs for (2.14) and (2.17). The proofs of other inequalities are similar. We first prove (2.14). According to the definition of Mittag-Leffler function, we have

$$\begin{aligned}
|\mathbf{E}^{(\beta)}_{\alpha,1}(w)v|^2_{\mu} &= \sum_{k=1}^{\infty}\lambda_k^{\mu}\left(\mathbf{E}^{(\beta)}_{\alpha,1}(w)v,\varphi_k\right)^2 = \sum_{k=1}^{\infty}\lambda_k^{\mu}\left(E_{\alpha,1}(-A^{\beta}w^{\alpha})v,\varphi_k\right)^2 \\
&= \sum_{k=1}^{\infty}\lambda_k^{\mu}\left(\sum_{m=0}^{\infty}\frac{(-1)^m w^{m\alpha}}{\Gamma(m\alpha+1)}A^{m\beta}v,\varphi_k\right)^2 \\
&= \sum_{k=1}^{\infty}\lambda_k^{\mu}\left(\sum_{m=0}^{\infty}\frac{(-1)^m w^{m\alpha}}{\Gamma(m\alpha+1)}\sum_{l=1}^{\infty}\lambda_l^{m\beta}(v,\varphi_l)\varphi_l,\varphi_k\right)^2
\end{aligned}$$

$$=\sum_{k=1}^{\infty}\lambda_k^{\mu}\left(\sum_{m=0}^{\infty}\frac{(-1)^m w^{m\alpha}}{\Gamma(m\alpha+1)}\lambda_k^{m\beta}(v,\varphi_k)\right)^2$$
$$=\sum_{k=1}^{\infty}\lambda_k^{\mu}\big(E_{\alpha,1}(-\lambda_k^{\beta}w^{\alpha})\big)^2(v,\varphi_k)^2.$$

Applying Lemma 2.1 with $\sigma=\alpha$, $\rho=1$, we arrive at,

$$\begin{aligned}|\mathbf{E}_{\alpha,1}^{(\beta)}(w)v|_{\mu}^2 &\le \sum_{k=1}^{\infty}\lambda_k^{\mu}\frac{C}{(1+\lambda_k^{\beta}w^{\alpha})^2}(v,\varphi_k)^2\\ &\le C w^{-\frac{\alpha}{\beta}(\mu-\nu)}\sum_{k=1}^{\infty}\frac{(\lambda_k^{\beta}w^{\alpha})^{\frac{\mu-\nu}{\beta}}}{(1+\lambda_k^{\beta}w^{\alpha})^2}\lambda_k^{\nu}(v,\varphi_k)^2\\ &\le C w^{-\frac{\alpha}{\beta}(\mu-\nu)}\sum_{k=1}^{\infty}\lambda_k^{\nu}|(v,\varphi_k)|^2\le C w^{-\frac{\alpha}{\beta}(\mu-\nu)}|v|_{\nu}^2,\end{aligned}$$

which shows (2.14) where we use the condition $0\le\mu-\nu\le 2\beta$ in the third inequality.

We now turn to the proof of (2.17). Following the same argument as in the proof of (2.14), we have

$$|\mathbf{E}_{\alpha,\alpha-1}^{(\beta)}(w)v|_{\mu}^2=\sum_{k=1}^{\infty}\lambda_k^{\mu}\left|w^{\alpha-2}E_{\alpha,\alpha-1}\left(-\lambda_k^{\beta}w^{\alpha}\right)\right|^2(v,\varphi_k)^2.$$

Applying Lemma 2.1 with $\sigma=\alpha$, $\rho=\alpha-1$ which implies $\rho-\sigma=-1\in\mathbb{Z}^-$, we arrive at

$$\begin{aligned}|\mathbf{E}_{\alpha,\alpha-1}^{(\beta)}(w)v|_{\mu}^2 &\le C\sum_{k=1}^{\infty}\lambda_k^{\mu}\frac{w^{2(\alpha-2)}}{(1+\lambda_k^{2\beta}w^{2\alpha})^2}(v,\varphi_k)^2\\ &\le C w^{2(\alpha-2)-\frac{\alpha}{\beta}(\mu-\nu)}\sum_{k=1}^{\infty}\frac{(\lambda_k^{2\beta}w^{2\alpha})^{\frac{\mu-\nu}{2\beta}}}{(1+\lambda_k^{2\beta}w^{2\alpha})^2}\lambda_k^{\nu}(v,\varphi_k)^2\\ &\le C w^{2(\alpha-2)-\frac{\alpha}{\beta}(\mu-\nu)}\sum_{k=1}^{\infty}\lambda_k^{\nu}(v,\varphi_k)^2\le C w^{2(\alpha-2)-\frac{\alpha}{\beta}(\mu-\nu)}|v|_{\nu}^2,\end{aligned}$$

which completes the proof of (2.17) where we use the condition $0\le\mu-\nu\le 4\beta$ in the third inequality. The proof of Lemma 2.2 is now complete. □

We also need the following properties for the derivative of solution operators.

Lemma 2.3 *Let* $\alpha\in(0,1)$, $\beta\in(\frac{1}{2},1]$ *and* $\gamma\in[0,1]$. *We then have*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}_{\alpha,1}^{(\beta)}(\psi(t)-\psi(a))=-A^{\beta}\mathbf{E}_{\alpha,\alpha}^{(\beta)}(\psi(t)-\psi(a))\psi'(t), \tag{2.19}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma}(\psi(t)-\psi(a)) = \mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma-1}(\psi(t)-\psi(a))\psi'(t). \tag{2.20}$$

Proof We only prove (2.19) here. Similarly one may show (2.20).

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}^{(\beta)}_{\alpha,1}(\psi(t)-\psi(a)) &= \frac{\mathrm{d}}{\mathrm{d}t}E_{\alpha,1}(-A^{\beta}(\psi(t)-\psi(a))^{\alpha}) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left[\sum_{k=0}^{\infty}\frac{(-A^{\beta})^{k}(\psi(t)-\psi(a))^{\alpha k}}{\Gamma(\alpha k+1)}\right] \\
&= \sum_{k=1}^{\infty}\frac{(-A^{\beta})^{k}(\alpha k)(\psi(t)-\psi(a))^{\alpha k-1}}{\Gamma(\alpha k+1)}\psi'(t) \\
&= -A^{\beta}(\psi(t)-\psi(a))^{\alpha-1}\sum_{k=0}^{\infty}\frac{[(-A^{\beta})(\psi(t)-\psi(a))^{\alpha}]^{k}}{\Gamma(\alpha k+\alpha)}\psi'(t) \\
&= -A^{\beta}(\psi(t)-\psi(a))^{\alpha-1}E_{\alpha,\alpha}(-A^{\beta}(\psi(t)-\psi(a))^{\alpha})\psi'(t) \\
&= -A^{\beta}\mathbf{E}^{(\beta)}_{\alpha,\alpha}(\psi(t)-\psi(a))\psi'(t),
\end{aligned}$$

which shows (2.19). □

We close this section with a Burkholder-Davis-Gundy-type inequality.

Lemma 2.4 [10, Theorem 7.2] *For any $r \geq 1$ and for arbitrary $\mathcal{L}_2^0$-valued predictable process $\Phi(t)$, $t \in [0, T]$, we have*

$$\begin{aligned}
\mathbb{E}\left(\sup_{s\in[0,t]}\left|\int_0^s \Phi(\sigma)\mathrm{d}W(\sigma)\right|^{2r}\right) &\leq c_r \sup_{s\in[0,t]}\mathbb{E}\left(\left|\int_0^s \Phi(\sigma)\mathrm{d}W(\sigma)\right|^{2r}\right) \\
&\leq C_r\mathbb{E}\left(\int_0^t ||\Phi(s)||^2_{\mathcal{L}_2^0}\mathrm{d}s\right)^{r},
\end{aligned}$$

where

$$c_r = \left(\frac{2r}{2r-1}\right)^{2r}, \quad C_r = (r(2r-1))^{r}\left(\frac{2r}{2r-1}\right)^{2r^2}.$$

Proof For the convenience of the readers, we give the sketch of the proof here. By the martingale [10, ineqality (3.8), p. 79] and [10, Corollary 4.14, p. 103] the result is true for $r = 1$. Assume now that $r > 1$, set $Z(t) = \int_0^t \Phi(s)\mathrm{d}W(s)$, $t \geq 0$, and apply Ito's formula to $f(Z(\cdot))$ where $f(x) = |x|^{2r}$, $x \in H$. Since

$$f_{xx}(x) = 4r(r-1)|x|^{2(r-2)}x\otimes x + 2r|x|^{2(r-1)}I, x \in H,$$

and

$$||f_{xx}(x)|| \leq 2r(2r-1)|x|^{2(r-1)},$$

therefore

$$|Tr\Phi^*(t)f_{xx}(Z(t))\Phi(t)Q| \leq 2r(2r-1)|Z(t)|^{2(r-1)}||\Phi(t)||^2_{\mathcal{L}_2^0}.$$

By taking the expectation, we obtain

$$\begin{aligned}\mathbb{E}|Z(t)|^{2r} \leq& r(2r-1)\mathbb{E}\left(\int_0^t |Z(s)|^{2(r-1)}||\Phi(s)||^2_{\mathcal{L}_2^0}\mathrm{d}s\right)\\ \leq& r(2r-1)\mathbb{E}\left(\sup_{s\in[0,t]}|Z(s)|^{2(r-1)}\int_0^t ||\Phi(s)||^2_{\mathcal{L}_2^0}\mathrm{d}s\right).\end{aligned}$$

By the Hölder inequality with $p = r/(r-1)$,

$$\mathbb{E}|Z(t)|^{2r} \leq r(2r-1)\left[\mathbb{E}\left(\sup_{s\in[0,t]}|Z(s)|^{2(r-1)p}\right)\right]^{\frac{1}{p}}\left[\mathbb{E}\left(\int_0^t ||\Phi(s)||^2_{\mathcal{L}_2^0}\mathrm{d}s\right)^r\right]^{\frac{1}{r}}.$$

Using the martingales inequality (2.21) [10, p.50] one arrives at

$$\mathbb{E}|Z(t)|^{2r} \leq r(2r-1)\left[\left(\frac{2r}{2r-1}\right)^{2r}(\mathbb{E}|Z(t)|^{2r})\right]^{1-\frac{1}{r}}\left[\mathbb{E}\left(\int_0^t ||\Phi(s)||^2_{\mathcal{L}_2^0}\mathrm{d}s\right)^r\right]^{\frac{1}{r}}.$$

Dividing both sides by $(\mathbb{E}|Z(t)|^{2r})^{1-\frac{1}{r}}$ one gets

$$\mathbb{E}|Z(t)|^{2r} \leq (r(2r-1))^r\left(\frac{2r}{2r-1}\right)^{2r(r-1)}\mathbb{E}\left(\int_0^t ||\Phi(s)||^2_{\mathcal{L}_2^0}\mathrm{d}s\right)^r,$$

which completes the proof of Lemma 2.4. □

3 Existence, uniqueness of the solution of Eq. (1.1)

In this section, we shall prove the existence and uniqueness of the mild solution to the semilinear stochastic time-space fractional subdiffusion Eq. (1.1) driven by fractionally integrated multiplicative noise by using the Banach contraction mapping principle.

We first introduce the following lemma which are related to the properties of f and g.

Lemma 3.1 *Let* $\alpha \in (0, 1)$, $\beta \in (\frac{1}{2}, 1]$, *and* $p \in [2, \infty)$. *Let Assumptions 2.2 and 2.4 be fulfilled. Let* κ_1 *be defined by (2.6). Then, for any* $t_1, t_2 \in [a, T]$ *and* $X, Y \in \mathcal{H}_p$, *there exists a constant* $C > 0$ *such that*

$$||f(t_1, X(t_1)) - f(t_2, Y(t_2))||_{L^p(\Omega;\dot{H}^{-1})} + ||g(t_1, X(t_1)) - g(t_2, Y(t_2))||_{L^p(\Omega;\mathcal{L}_2^0)}$$

$$\leq C(1+||X(t_1)||_{L^p(\Omega;H)})|\psi(t_1)-\psi(t_2)|^{\kappa_1}+C||X(t_1)-Y(t_2)||_{L^p(\Omega;H)}. \tag{3.1}$$

In particular, there holds

$$||f(t,X(t))||_{L^p(\Omega;\dot{H}^{-1})}+||g(t,X(t))||_{L^p(\Omega;\mathcal{L}_2^0)}\leq C(1+||X(t)||_{L^p(\Omega;H)}) \tag{3.2}$$

for any $t\in[a,T]$ *and all* $X\in\mathcal{H}_p$.

Proof Note that

$$\begin{aligned}||f(t_1,X(t_1))-f(t_2,Y(t_2))||_{L^p(\Omega;\dot{H}^{-1})}\leq&||f(t_1,X(t_1))-f(t_2,X(t_1))||_{L^p(\Omega;\dot{H}^{-1})}\\&+||f(t_2,X(t_1))-f(t_2,Y(t_2))||_{L^p(\Omega;\dot{H}^{-1})}.\end{aligned}$$

Applying Assumption 2.2 it follows that

$$\begin{aligned}&||f(t_1,X(t_1))-f(t_2,X(t_1))||_{L^p(\Omega;\dot{H}^{-1})}\\=&\left(\int_\Omega|f(t_1,X(t_1))-f(t_2,X(t_1))|_{-1}^p\mathrm{d}\mathbb{P}(\omega)\right)^{\frac{1}{p}}\\\leq&C|\psi(t_1)-\psi(t_2)|^{\kappa_1}\left(\int_\Omega(1+||X(t_1,\omega)||)^p\mathrm{d}\mathbb{P}(\omega)\right)^{\frac{1}{p}}\\\leq&C(1+||X(t_1)||_{L^p(\Omega;H)})|\psi(t_1)-\psi(t_2)|^{\kappa_1}.\end{aligned} \tag{3.3}$$

Using Assumption 2.2 yields

$$\begin{aligned}&||f(t_2,X(t_1))-f(t_2,Y(t_2))||_{L^p(\Omega;\dot{H}^{-1})}\\=&\left(\int_\Omega|f(t_2,X(t_1))-f(t_2,Y(t_2))|_{-1}^p\mathrm{d}\mathbb{P}(\omega)\right)^{\frac{1}{p}}\\\leq&C\left(\int_\Omega||X(t_1,\omega)-Y(t_2,\omega)||^p\mathrm{d}\mathbb{P}(\omega)\right)^{\frac{1}{p}}\\\leq&C||X(t_1)-Y(t_2)||_{L^p(\Omega;H)}).\end{aligned} \tag{3.4}$$

Combining the estimates (3.3) and (3.4) gives the proof of the function f in (3.1). Further Assumption 2.2 implies $||f(a,0)||_{L^p(\Omega;\dot{H}^{-1})}<\infty$, we then obtain

$$\begin{aligned}||f(t,X(t))||_{L^p(\Omega;\dot{H}^{-1})}&\leq||f(t,X(t))-f(a,0)||_{L^p(\Omega;\dot{H}^{-1})}+||f(a,0)||_{L^p(\Omega;\dot{H}^{-1})}\\&\leq C(1+||X(t)||_{L^p(\Omega;H)}),\end{aligned}$$

which gives the estimate for the function f in (3.2). Similarly, we can derive the estimates with respect to the operator g under Assumption 2.4. The proof is completed. □

The following theorem establishes existence and uniqueness of the mild solution to Eq. (1.1).

Theorem 3.1 *Let $\alpha \in (0,1)$, $\beta \in (\frac{1}{2}, 1]$, $\gamma \in [0,1]$, and $\alpha+\gamma > \frac{1}{2}$. Let Assumptions 2.1-2.5 be satisfied and $p \in [2,\infty)$. Then there exists a unique $L^p(\Omega)$-integrable mild solution $u : [a,T] \to H$ to Eq. (1.1) such that $\mathbb{P}(u(t) \in H) = 1$ and*

$$\sup_{t\in[a,T]} ||u(t)||_{L^p(\Omega;H)} < \infty \tag{3.5}$$

for every $t \in [a,T]$.

Proof We shall apply the Banach contraction mapping principle to prove this theorem. We first construct a mapping on the space $\mathcal{H}_p$ and prove that it is well-defined. Then, we introduce a equivalent norm on the space $\mathcal{H}_p$ to show that the mapping is contractive.

Define a mapping $\mathcal{F} : \mathcal{H}_p \to \mathcal{H}_p$ given by

$$\begin{aligned}\mathcal{F}(u)(t) :=& \mathbf{E}^{(\beta)}_{\alpha,1}(\psi(t)-\psi(a))u_a + \int_a^t \mathbf{E}^{(\beta)}_{\alpha,\alpha}(\psi(t)-\psi(\tau))f(\tau,u(\tau))\psi'(\tau)\mathrm{d}\tau\\ &+ \int_a^t \mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma}(\psi(t)-\psi(\tau))g(\tau,u(\tau))\psi'(\tau)\mathrm{d}W(\tau)\\ =&\mathcal{F}_0(t)+\mathcal{F}_1(u)(t)+\mathcal{F}_2(u)(t)\end{aligned} \tag{3.6}$$

for $t \in [a,T]$ and $u \in \mathcal{H}_p$.

Step 1: Prove that the mapping $\mathcal{F} : \mathcal{H}_p \to \mathcal{H}_p$ is well-defined.

In view of (2.14) in Lemma 2.2 with $\mu = \nu = 0$ and Assumption 2.5, one has

$$\begin{aligned}||\mathcal{F}_0(t)||_{L^p(\Omega;H)} =& \left|\left|\mathbf{E}^{(\beta)}_{\alpha,1}(\psi(t)-\psi(a))u_a\right|\right|_{L^p(\Omega;H)}\\ =& \left(\mathbb{E}\left[\left\|\mathbf{E}^{(\beta)}_{\alpha,1}(\psi(t)-\psi(a))u_a\right\|^p\right]\right)^{\frac{1}{p}}\\ \le& C\left(\mathbb{E}\left[\|u_a\|^p\right]\right)^{\frac{1}{p}} = C||u_a||_{L^p(\Omega;H)}.\end{aligned} \tag{3.7}$$

This implies $\mathcal{F}_0 \in \mathcal{H}_p$ and $\mathcal{F}_0(t)$ takes almost surely values in H for any $t \in [a,T]$.

Next, we consider $\mathcal{F}_1(u)(t)$. It follows from (2.15) ($\mu = 1$ and $\nu = 0$) in Lemma 2.2 and Lemma 3.1 that, noting $\beta > \frac{1}{2}$,

$$\begin{aligned}&||\mathcal{F}_1(u)(t)||_{L^p(\Omega;H)} = \left(\mathbb{E}\left[\|\mathcal{F}_1(u)(t)\|^p\right]\right)^{\frac{1}{p}}\\ =& \left(\mathbb{E}\left[\left\|\int_a^t \mathbf{E}^{(\beta)}_{\alpha,\alpha}(\psi(t)-\psi(\tau))f(\tau,u(\tau))\psi'(\tau)\mathrm{d}\tau\right\|^p\right]\right)^{\frac{1}{p}}\\ \le& \left(\mathbb{E}\left[\left(\int_a^t \left|\mathbf{E}^{(\beta)}_{\alpha,\alpha}(\psi(t)-\psi(\tau))A^{-\frac{1}{2}}f(\tau,u(\tau))\right|_1 \psi'(\tau)\mathrm{d}\tau\right)^p\right]\right)^{\frac{1}{p}}\\ \le& \left(\mathbb{E}\left[\left(\int_a^t (\psi(t)-\psi(\tau))^{\alpha-1-\frac{\alpha}{2\beta}} \left|\left|A^{-\frac{1}{2}}f(\tau,u(\tau))\right|\right| \psi'(\tau)\mathrm{d}\tau\right)^p\right]\right)^{\frac{1}{p}}\end{aligned}$$

$$
\begin{aligned}
&\le \int_a^t (\psi(t)-\psi(\tau))^{\alpha-1-\frac{\alpha}{2\beta}} ||f(\tau,u(\tau))||_{L^p(\Omega;\dot{H}^{-1})}\psi'(\tau)\mathrm{d}\tau \\
&\le C\int_a^t (\psi(t)-\psi(\tau))^{\alpha-1-\frac{\alpha}{2\beta}}\psi'(\tau)\mathrm{d}\tau \left(1+\sup_{\tau\in[a,T]}||u(\tau)||_{L^p(\Omega;H)}\right) \\
&\le \frac{C}{\alpha-\frac{\alpha}{2\beta}}(\psi(t)-\psi(a))^{\alpha-\frac{\alpha}{2\beta}}(1+||u(\tau)||_{\mathcal{H}_p}).
\end{aligned}
$$

This states that $\mathcal{F}_1(u)$ is an adapted stochastic process such that

$$\mathbb{P}(\mathcal{F}_1(u)(t)\in H)=1$$

for any $u\in\mathcal{H}_p$ with $t\in[a,T]$. Therefore,

$$\sup_{t\in[a,T]} ||\mathcal{F}_1(u)(t)||_{L^p(\Omega;H)} \le C(1+||u||_{\mathcal{H}_p})<\infty. \tag{3.8}$$

We continue to investigate $\mathcal{F}_2(u)(t)$. Note that (2.16) ($\mu=0$ and $\nu=0$) in Lemma 2.2, Lemma 2.4, and Lemma 3.1 it follows that, noting $\alpha+\gamma>\frac{1}{2}$,

$$
\begin{aligned}
&||\mathcal{F}_2(u)(t)||_{L^p(\Omega;H)} = \left(\mathbb{E}\left[\|\mathcal{F}_2(u)(t)\|^p\right]\right)^{\frac{1}{p}} \\
&= \left(\mathbb{E}\left[\left|\left|\int_a^t \mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma}(\psi(t)-\psi(\tau))g(\tau,u(\tau))\psi'(\tau)\mathrm{d}W(\tau)\right|\right|^p\right]\right)^{\frac{1}{p}} \\
&\le C\left(\mathbb{E}\left[\left(\int_a^t \left|\left|\mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma}(\psi(t)-\psi(\tau))g(\tau,u(\tau))\right|\right|^2_{\mathcal{L}_2^0}\psi'(\tau)\mathrm{d}\tau\right)^{\frac{p}{2}}\right]\right)^{\frac{1}{p}} \\
&\le C\left(\int_a^t (\psi(t)-\psi(\tau))^{2(\alpha+\gamma-1)}||g(\tau,u(\tau))||^2_{L^p(\Omega;\mathcal{L}_2^0)}\psi'(\tau)\mathrm{d}\tau\right)^{\frac{1}{2}} \\
&\le \frac{C}{(2\alpha+2\gamma-1)^{1/2}}(\psi(t)-\psi(\tau))^{\alpha+\gamma-\frac{1}{2}}(1+||u(\tau)||_{\mathcal{H}_p}),
\end{aligned}
$$

which implies

$$\sup_{t\in[a,T]} ||\mathcal{F}_2(u)(t)||_{L^p(\Omega;H)} \le C(1+||u||_{\mathcal{H}_p})<\infty. \tag{3.9}$$

Hence, the random variable $\mathcal{F}_2(u)(t)$ takes almost surely values in H for any $t\in[a,T]$.

Step 2: Prove that the mapping $\mathcal{F}:\mathcal{H}_p\to\mathcal{H}_p$ is a contraction mapping.

We introduce the following norm, with $\rho\in\mathbb{R}$,

$$||u||_{\mathcal{H}_{p,\rho}} := \sup_{t\in[a,T]} e^{-\rho\psi(t)}||u||_{L^p(\Omega;H)}$$

on $\mathcal{H}_p$. It is easy to verify that the norm $||\cdot||_{\mathcal{H}_{p,\rho}}$ is equivalent to $||\cdot||_{\mathcal{H}_p}$ since $\psi(t)$ is a bounded function on $[a, T]$. Such equivalent norm was used in [20] when $\psi(t) = t$.

Note that, by Lemma 3.1,

$$\begin{aligned}&||f(\tau, X(\tau)) - f(\tau, Y(\tau))||_{L^p(\Omega;\dot{H}^{-1})} + ||g(\tau, X(\tau)) - g(\tau, Y(\tau))||_{L^p(\Omega;\mathcal{L}_2^0)}\\ \leq& C||X(\tau) - Y(\tau)||_{L^p(\Omega;H)}\end{aligned} \tag{3.10}$$

for all $\tau \in [a, T]$. Then, for any $u, v \in \mathcal{H}_p$ we obtain by (4.1)

$$\begin{aligned}&||\mathcal{F}(u)(t) - \mathcal{F}(v)(t)||_{L^p(\Omega;H)}\\ \leq& ||\mathcal{F}_1(u)(t) - \mathcal{F}_1(v)(t)||_{L^p(\Omega;H)} + ||\mathcal{F}_2(u)(t) - \mathcal{F}_2(v)(t)||_{L^p(\Omega;H)}.\end{aligned} \tag{3.11}$$

Thus the first summand in (3.11) is estimated by

$$\begin{aligned}&||\mathcal{F}_1(u)(t) - \mathcal{F}_1(v)(t)||_{L^p(\Omega;H)}\\ \leq& \int_a^t \left|\left|\mathbf{E}_{\alpha,\alpha}^{(\beta)}(\psi(t) - \psi(\tau))(f(\tau, u(\tau)) - f(\tau, v(\tau)))\right|\right|_{L^p(\Omega;H)} \psi'(\tau)\mathrm{d}\tau\\ \leq& C\int_a^t (\psi(t) - \psi(\tau))^{\alpha-1-\frac{\alpha}{2\beta}} ||f(\tau, u(\tau)) - f(\tau, v(\tau))||_{L^p(\Omega;\dot{H}^{-1})}\psi'(\tau)\mathrm{d}\tau\\ \leq& C\int_a^t (\psi(t) - \psi(\tau))^{\alpha-1-\frac{\alpha}{2\beta}} ||u(\tau) - v(\tau)||_{L^p(\Omega;H)}\psi'(\tau)\mathrm{d}\tau\\ \leq& C\int_a^t (\psi(t) - \psi(\tau))^{\alpha-1-\frac{\alpha}{2\beta}} e^{\rho\psi(\tau)}\psi'(\tau)\mathrm{d}\tau||u - v||_{\mathcal{H}_{p,\rho}},\end{aligned}$$

where we have used (2.15) ($\mu = 0$ and $\nu = -1$) in Lemma 2.2 and (3.10). Analogously, the second summand can also be estimated by applying Lemma 2.4, (2.16) ($\mu = 0$ and $\nu = 0$) in Lemma 2.2 and (3.10)

$$\begin{aligned}&||\mathcal{F}_2(u)(t) - \mathcal{F}_2(v)(t)||_{L^p(\Omega;H)}\\ \leq& C\left(\mathbb{E}\left[\left(\int_a^t \left|\left|\mathbf{E}_{\alpha,\alpha+\gamma}^{(\beta)}(\psi(t) - \psi(\tau))(g(\tau, u(\tau)) - g(\tau, v(\tau)))\right|\right|_{\mathcal{L}_2^0}^2 \psi'(\tau)\mathrm{d}\tau\right)^{\frac{p}{2}}\right]\right)^{\frac{1}{p}}\\ \leq& C\left(\int_a^t (\psi(t) - \psi(\tau))^{2\alpha+2\gamma-2}||g(\tau, u(\tau)) - g(\tau, v(\tau))||_{L^p(\Omega;\mathcal{L}_2^0)}^2\psi'(\tau)\mathrm{d}\tau\right)^{\frac{1}{2}}\\ \leq& C\left(\int_a^t (\psi(t) - \psi(\tau))^{2\alpha+2\gamma-2}e^{2\rho\psi(\tau)}\psi'(\tau)\mathrm{d}\tau\right)^{\frac{1}{2}} ||u - v||_{\mathcal{H}_{p,\rho}}.\end{aligned}$$

Therefore we derive

$$\begin{aligned}&||\mathcal{F}(u)(t) - \mathcal{F}(v)(t)||_{L^p(\Omega;H)}\\ \leq& C\Bigg(\int_a^t (\psi(t) - \psi(\tau))^{\alpha-1-\frac{\alpha}{2\beta}} e^{\rho\psi(\tau)}\psi'(\tau)\mathrm{d}\tau\end{aligned}$$

$$+\left(\int_a^t (\psi(t)-\psi(\tau))^{2\alpha+2\gamma-2}e^{2\rho\psi(\tau)}\psi'(\tau)\mathrm{d}\tau\right)^{\frac{1}{2}}\Bigg)||u-v||_{\mathcal{H}_{p,\rho}}.$$

For the first integral above, using Hölder inequality with $\frac{1}{\mu}+\frac{1}{\mu'}=1$ yields

$$\begin{aligned}
&\int_a^t (\psi(t)-\psi(\tau))^{\alpha-1-\frac{\alpha}{2\beta}}e^{\rho\psi(\tau)}\psi'(\tau)\mathrm{d}\tau\\
&\le \left(\int_a^t (\psi(t)-\psi(\tau))^{(\alpha-1-\frac{\alpha}{2\beta})\mu'}\psi'(\tau)\mathrm{d}\tau\right)^{\frac{1}{\mu'}}\left(\int_a^t e^{\mu\rho\psi(\tau)}\psi'(\tau)\mathrm{d}\tau\right)^{\frac{1}{\mu}}\\
&\le \left(\frac{(\psi(T)-\psi(a))^{(\alpha-1-\frac{\alpha}{2\beta})\mu'+1}}{(\alpha-1-\frac{\alpha}{2\beta})\mu'+1}\right)^{\frac{1}{\mu'}}\left(\frac{1}{\mu\rho}\left(e^{\mu\rho\psi(t)}-e^{\mu\rho\psi(a)}\right)\right)^{\frac{1}{\mu}},
\end{aligned}$$

where we can always choose $1<\mu'<\frac{1}{1+\frac{\alpha}{2\beta}-\alpha}$ such that $(\alpha-1-\frac{\alpha}{2\beta})\mu'>-1$ due to $\beta>\frac{1}{2}$. As the second integral, by Hölder inequality with $\frac{1}{\nu}+\frac{1}{\nu'}=1$, one has

$$\begin{aligned}
&\left(\int_a^t (\psi(t)-\psi(\tau))^{2\alpha+2\gamma-2}e^{2\rho\psi(\tau)}\psi'(\tau)\mathrm{d}\tau\right)^{\frac{1}{2}}\\
&\le \left(\int_a^t (\psi(t)-\psi(\tau))^{(2\alpha+2\gamma-2)\nu'}\psi'(\tau)\mathrm{d}\tau\right)^{\frac{1}{2\nu'}}\left(\int_a^t e^{2\mu\rho\psi(\tau)}\psi'(\tau)\mathrm{d}\tau\right)^{\frac{1}{2\nu}}\\
&\le \left(\frac{(\psi(T)-\psi(a))^{(2\alpha+2\gamma-2)\nu'+1}}{(2\alpha+2\gamma-2)\nu'+1}\right)^{\frac{1}{2\nu'}}\left(\frac{1}{2\nu\rho}\left(e^{2\nu\rho\psi(t)}-e^{2\nu\rho\psi(a)}\right)\right)^{\frac{1}{2\nu}},
\end{aligned}$$

where we can also take $\nu'>1$ due to $\alpha+\gamma>\frac{1}{2}$.

Hence we obtain for $\rho>0$

$$\begin{aligned}
&e^{-\rho\psi(t)}||\mathcal{F}(u)(t)-\mathcal{F}(v)(t)||_{L^p(\Omega;H)}\\
&\le C\Bigg(\left(\frac{(\psi(T)-\psi(a))^{(\alpha-1-\frac{\alpha}{2\beta})\mu'+1}}{(\alpha-1-\frac{\alpha}{2\beta})\mu'+1}\right)^{\frac{1}{\mu'}}\left(\frac{1}{\mu\rho}\left(1-e^{-\mu\rho(\psi(t)-\psi(a))}\right)\right)^{\frac{1}{\mu}}\\
&+\left(\frac{(\psi(T)-\psi(a))^{(2\alpha+2\gamma-2)\nu'+1}}{(2\alpha+2\gamma-2)\nu'+1}\right)^{\frac{1}{2\nu'}}\\
&\quad\cdot\left(\frac{1}{2\nu\rho}\left(1-e^{-2\nu\rho(\psi(t)-\psi(a))}\right)\right)^{\frac{1}{2\nu}}\Bigg)||u-v||_{\mathcal{H}_{p,\rho}}\\
&\le C\Bigg(\left(\frac{(\psi(T)-\psi(a))^{(\alpha-1-\frac{\alpha}{2\beta})\mu'+1}}{(\alpha-1-\frac{\alpha}{2\beta})\mu'+1}\right)^{\frac{1}{\mu'}}(\mu\rho)^{-\frac{1}{\mu}}
\end{aligned}$$

$$+\left(\frac{(\psi(T)-\psi(a))^{(2\alpha+2\gamma-2)\nu'+1}}{(2\alpha+2\gamma-2)\nu'+1}\right)^{\frac{1}{2\nu'}}(2\nu\rho)^{-\frac{1}{2\nu}}\Bigg)||u-v||_{\mathcal{H}_{p,\rho}}.$$

Choose $\rho > 0$ large enough, the mapping $\mathcal{F}$ is a contraction on the space $\mathcal{H}_p$ with respect to the norm $||\cdot||_{\mathcal{H}_{p,\rho}}$ and thus there exists a unique fixed point $u \in \mathcal{H}_p$, which is a unique mild solution to Eq. (1.1). The proof of this theorem is now completed. □

4 Regularities of the mild solution to Eq. (1.1)

In this section, we shall prove the spatial and temporal regularities for the mild solution to the semilinear stochastic time-space fractional subdiffusion Eq. (1.1) under Assumptions 2.1-2.5.

4.1 Spatial regularity

In this subsection, we shall prove the spatial regularity for the mild solution to Eq. (1.1) under Assumptions 2.1-2.5.

Theorem 4.1 *Let $\alpha \in (0,1)$, $\beta \in (\frac{1}{2},1]$, $\gamma \in [0,1]$, and $\alpha+\gamma > \frac{1}{2}$. Let Assumptions 2.1-2.5 be satisfied and $p \in [2,\infty)$. Let κ be defined in (2.5). Then for any $s \in [0,\kappa)$ the mild solution u of Eq. (1.1) determined in Theorem 3.1 satisfies*

$$\sup_{t\in[a,T]} \|u(t)\|_{L^p(\Omega;\dot{H}^s)} < \infty,$$

for any $t \in [a,T]$.

Proof By Theorem 3.1, Eq. (1.1) admits the following unique mild solution

$$\begin{aligned}u(t) =&\mathbf{E}^{(\beta)}_{\alpha,1}(\psi(t)-\psi(a))u_a + \int_a^t \mathbf{E}^{(\beta)}_{\alpha,\alpha}(\psi(t)-\psi(\tau))f(\tau,u(\tau))\psi'(\tau)\mathrm{d}\tau\\ &+\int_a^t \mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma}(\psi(t)-\psi(\tau))g(\tau,u(\tau))\psi'(\tau)\mathrm{d}W(\tau)\\ =&\mathcal{F}_0(t)+\mathcal{F}_1(u)(t)+\mathcal{F}_2(u)(t)\end{aligned}\tag{4.1}$$

for $t \in [a,T]$ and $u \in \mathcal{H}_p$.

In view of (2.14) in Lemma 2.2 with $\mu=\nu=s$ and Assumption 2.5, one has

$$\begin{aligned}||\mathcal{F}_0(t)||_{L^p(\Omega;\dot{H}^s)} &= \left|\left|\mathbf{E}^{(\beta)}_{\alpha,1}(\psi(t)-\psi(a))u_a\right|\right|_{L^p(\Omega;\dot{H}^s)}\\ &=\left(\mathbb{E}\left[\left|\mathbf{E}^{(\beta)}_{\alpha,1}(\psi(t)-\psi(a))u_a\right|_s^p\right]\right)^{\frac{1}{p}}\\ &\le C\left(\mathbb{E}\left[|u_a|_s^p\right]\right)^{\frac{1}{p}} = C||u_a||_{L^p(\Omega;\dot{H}^s)}.\end{aligned}\tag{4.2}$$

This implies $\mathcal{F}_0(t)$ takes almost surely values in $\dot{H}^s$ for any $s \in [0, \kappa)$ and $t \in [a, T]$.

Next, we consider $\mathcal{F}_1(u)(t)$. It follows from (2.15)($\mu = s+1$ and $\nu = 0$) in Lemma 2.2 and Lemma 3.1 that, for $s < 2\beta - 1$,

$$
\begin{aligned}
&||\mathcal{F}_1(u)(t)||_{L^p(\Omega;\dot{H}^s)} = \left(\mathbb{E}\left[|\mathcal{F}_1(u)(t)|_s^p\right]\right)^{\frac{1}{p}} \\
&= \left(\mathbb{E}\left[\left|\int_a^t \mathbf{E}_{\alpha,\alpha}^{(\beta)}(\psi(t)-\psi(\tau)) f(\tau,u(\tau))\psi'(\tau)\mathrm{d}\tau\right|_s^p\right]\right)^{\frac{1}{p}} \\
&\leq \left(\mathbb{E}\left[\left(\int_a^t \left|\mathbf{E}_{\alpha,\alpha}^{(\beta)}(\psi(t)-\psi(\tau)) A^{-\frac{1}{2}} f(\tau,u(\tau))\right|_{s+1} \psi'(\tau)\mathrm{d}\tau\right)^p\right]\right)^{\frac{1}{p}} \\
&\leq \left(\mathbb{E}\left[\left(\int_a^t (\psi(t)-\psi(\tau))^{\alpha-1-\frac{\alpha}{2\beta}(s+1)} \left|\left|A^{-\frac{1}{2}} f(\tau,u(\tau))\right|\right| \psi'(\tau)\mathrm{d}\tau\right)^p\right]\right)^{\frac{1}{p}} \\
&\leq \int_a^t (\psi(t)-\psi(\tau))^{\alpha-1-\frac{\alpha}{2\beta}(s+1)} ||f(\tau,u(\tau))||_{L^p(\Omega;\dot{H}^{-1})}\psi'(\tau)\mathrm{d}\tau \\
&\leq C\int_a^t (\psi(t)-\psi(\tau))^{\alpha-1-\frac{\alpha}{2\beta}(s+1)}\psi'(\tau)\mathrm{d}\tau \left(1+\sup_{\tau\in[a,T]}||u(\tau)||_{L^p(\Omega;H)}\right) \\
&\leq \frac{C}{\alpha-\frac{\alpha}{2\beta}(s+1)}(\psi(t)-\psi(a))^{\alpha-\frac{\alpha}{2\beta}(s+1)}(1+||u(\tau)||_{\mathcal{H}_p}).
\end{aligned}
$$

This states that $\mathcal{F}_1(u)$ is an adapted stochastic process such that

$$
\mathbb{P}(\mathcal{F}_1(u)(t) \in \dot{H}^s) = 1
$$

for any $s \in [0, \kappa)$ and $t \in [a, T]$. Therefore, for any $s \in [0, \kappa)$,

$$
\sup_{t\in[a,T]} ||\mathcal{F}_1(u)(t)||_{L^p(\Omega;\dot{H}^s)} \leq C(1+||u||_{\mathcal{H}_p}) < \infty. \tag{4.3}
$$

We continue to investigate $\mathcal{F}_2(u)(t)$. Note that (2.16) ($\mu = s$ and $\nu = 0$) in Lemma 2.2, Lemma 2.4, and Lemma 3.1 it follows that, for $s < \frac{2\beta}{\alpha}(\alpha+\gamma-\frac{1}{2})$,

$$
\begin{aligned}
&||\mathcal{F}_2(u)(t)||_{L^p(\Omega;\dot{H}^s)} = \left(\mathbb{E}\left[|\mathcal{F}_2(u)(t)|_s^p\right]\right)^{\frac{1}{p}} \\
&= \left(\mathbb{E}\left[\left|\left|\int_a^t A^{\frac{s}{2}}\mathbf{E}_{\alpha,\alpha+\gamma}^{(\beta)}(\psi(t)-\psi(\tau)) g(\tau,u(\tau))\psi'(\tau)\mathrm{d}W(\tau)\right|\right|^p\right]\right)^{\frac{1}{p}} \\
&\leq C\left(\mathbb{E}\left[\left(\int_a^t \left|\left|A^{\frac{s}{2}}\mathbf{E}_{\alpha,\alpha+\gamma}^{(\beta)}(\psi(t)-\psi(\tau)) g(\tau,u(\tau))\right|\right|_{\mathcal{L}_2^0}^2 \psi'(\tau)\mathrm{d}\tau\right)^{\frac{p}{2}}\right]\right)^{\frac{1}{p}} \\
&\leq C\left(\int_a^t (\psi(t)-\psi(\tau))^{2(\alpha+\gamma-1-\frac{\alpha s}{2\beta})} ||g(\tau,u(\tau))||_{L^p(\Omega;\mathcal{L}_2^0)}^2 \psi'(\tau)\mathrm{d}\tau\right)^{\frac{1}{2}} \\
&\leq \frac{C}{(2\alpha+2\gamma-1-\frac{\alpha s}{\beta})^{1/2}}(\psi(t)-\psi(\tau))^{\alpha+\gamma-\frac{1}{2}-\frac{\alpha s}{2\beta}}(1+||u(\tau)||_{\mathcal{H}_p}),
\end{aligned}
$$

which implies

$$\sup_{t\in[a,T]} ||\mathcal{F}_2(u)(t)||_{L^p(\Omega;\dot{H}^s)} \le C(1+||u||_{\mathcal{H}_p}) < \infty. \tag{4.4}$$

Hence, $\mathcal{F}_2(u)(t)$ takes almost surely values in $\dot{H}^s$ for $s \in [0,\kappa)$ and for any $t \in [a,T]$.
Together these estimates complete the proof of Theorem 4.1. □

4.2 Temporal regularity

In this subsection, we shall prove the temporal regularity for the mild solution to Eq. (1.1) under Assumptions 2.1-2.5.

Theorem 4.2 *Let $\alpha \in (0,1)$, $\beta \in (\frac{1}{2},1]$, $\gamma \in [0,1]$, and $\alpha+\gamma > \frac{1}{2}$. Let Assumptions 2.1-2.5 be satisfied and $p \in [2,\infty)$. Let κ_1 be defined in (2.6). Then the mild solution u determined in Theorem 3.1 satisfies*

$$||u(t_1)-u(t_2)||_{L^p(\Omega;H)} < C(\psi(t_1)-\psi(t_2))^{\kappa_1}, \tag{4.5}$$

for all $t_1, t_2 \in [a,T]$.

Proof We again use the mild solution formula defined in (4.1). Applying (2.19), (2.15) ($\mu = 2\beta$ and $\nu = \kappa$) and Assumption 2.5 gives

$$\begin{aligned}
&||\mathcal{F}_0(t_2)-\mathcal{F}_0(t_1)||_{L^p(\Omega;H)} && (4.6)\\
=&\left|\left|\mathbf{E}^{(\beta)}_{\alpha,1}(\psi(t_2)-\psi(a))u_a - \mathbf{E}^{(\beta)}_{\alpha,1}(\psi(t_1)-\psi(a))u_a\right|\right|_{L^p(\Omega;H)}\\
=&\left|\left|\int_{t_1}^{t_2} \frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{E}^{(\beta)}_{\alpha,1}(\psi(\tau)-\psi(a))\mathrm{d}\tau u_a\right|\right|_{L^p(\Omega;H)} && (4.7)\\
=&\left|\left|\int_{t_1}^{t_2} -A^\beta\mathbf{E}^{(\beta)}_{\alpha,\alpha}(\psi(\tau)-\psi(a))\psi'(\tau)\mathrm{d}\tau u_a\right|\right|_{L^p(\Omega;H)} && (4.8)\\
\le&\int_{t_1}^{t_2} \left|\left|-A^\beta\mathbf{E}^{(\beta)}_{\alpha,\alpha}(\psi(\tau)-\psi(a))u_a\right|\right|_{L^p(\Omega;H)}\psi'(\tau)\mathrm{d}\tau\\
\le& C\int_{t_1}^{t_2}(\psi(\tau)-\psi(a))^{\alpha-1-\frac{\alpha}{2\beta}(2\beta-\kappa)}\psi'(\tau)\mathrm{d}\tau||u_a||_{L^p(\Omega;\dot{H}^\kappa)}\\
\le& C(\psi(t_2)-\psi(t_1))^{\frac{\alpha\kappa}{2\beta}}||u_a||_{L^p(\Omega;\dot{H}^\kappa)}. && (4.9)
\end{aligned}$$

Further we have

$$\begin{aligned}
&||\mathcal{F}_1(u)(t_2)-\mathcal{F}_1(u)(t_1)||_{L^p(\Omega;H)}\\
=&\Big|\Big|\int_a^{t_2}\mathbf{E}^{(\beta)}_{\alpha,\alpha}(\psi(t_2)-\psi(\tau))f(\tau,u(\tau))\psi'(\tau)\mathrm{d}\tau\\
&-\int_a^{t_1}\mathbf{E}^{(\beta)}_{\alpha,\alpha}(\psi(t_1)-\psi(\tau))f(\tau,u(\tau))\psi'(\tau)\mathrm{d}\tau\Big|\Big|_{L^p(\Omega;H)}
\end{aligned}$$

$$\leq \left\|\int_{t_1}^{t_2} \mathbf{E}_{\alpha,\alpha}^{(\beta)}(\psi(t_2)-\psi(\tau)) f(\tau,u(\tau))\psi'(\tau)\mathrm{d}\tau\right\|_{L^p(\Omega;H)}$$
$$+\left\|\int_a^{t_1} \left(\mathbf{E}_{\alpha,\alpha}^{(\beta)}(\psi(t_2)-\psi(\tau)) - \mathbf{E}_{\alpha,\alpha}^{(\beta)}(\psi(t_1)-\psi(\tau))\right) f(\tau,u(\tau))\psi'(\tau)\mathrm{d}\tau\right\|_{L^p(\Omega;H)}.$$

By using (2.15) ($\mu = 0$ and $\nu = -1$) in Lemma 2.2 and Lemma 3.1 we obtain, for $\beta > \frac{1}{2}$,

$$\begin{aligned}
&\left\|\int_{t_1}^{t_2} \mathbf{E}_{\alpha,\alpha}^{(\beta)}(\psi(t_2)-\psi(\tau)) f(\tau,u(\tau))\psi'(\tau)\mathrm{d}\tau\right\|_{L^p(\Omega;H)} \\
\leq& \int_{t_1}^{t_2} \left\|\mathbf{E}_{\alpha,\alpha}^{(\beta)}(\psi(t_2)-\psi(\tau)) f(\tau,u(\tau))\right\|_{L^p(\Omega;H)} \psi'(\tau)\mathrm{d}\tau \\
\leq& C\int_{t_1}^{t_2} (\psi(t_2)-\psi(\tau))^{\alpha-1-\frac{\alpha}{2\beta}} ||f(\tau,u(\tau))||_{L^p(\Omega;\dot{H}^{-1})}\psi'(\tau)\mathrm{d}\tau \\
\leq& \frac{C}{\alpha-\frac{\alpha}{2\beta}} (\psi(t_2)-\psi(t_1))^{\alpha-\frac{\alpha}{2\beta}} \left(1+\sup_{\tau\in[a,T]} ||u(\tau)||_{L^p(\Omega;H)}\right) \\
\leq& \frac{C}{\alpha-\frac{\alpha}{2\beta}} (\psi(t_2)-\psi(t_1))^{\alpha-\frac{\alpha}{2\beta}} (1+||u||_{\mathcal{H}_p}).
\end{aligned}$$

From (2.20) for $\gamma = 0$, (2.17) ($\mu = 0$ and $\nu = -1$) in Lemma 2.2, and Lemma 3.1 we arrive at, for $\beta > \frac{1}{2}$,

$$\begin{aligned}
&\left\|\int_a^{t_1} \left(\mathbf{E}_{\alpha,\alpha}^{(\beta)}(\psi(t_2)-\psi(\tau)) - \mathbf{E}_{\alpha,\alpha}^{(\beta)}(\psi(t_1)-\psi(\tau))\right) f(\tau,u(\tau))\psi'(\tau)\mathrm{d}\tau\right\|_{L^p(\Omega;H)} \\
=& \left\|\int_a^{t_1}\int_{t_1}^{t_2} \mathbf{E}_{\alpha,\alpha-1}^{(\beta)}(\psi(\zeta)-\psi(\tau)) f(\tau,u(\tau))\psi'(\zeta)\mathrm{d}\zeta\,\psi'(\tau)\mathrm{d}\tau\right\|_{L^p(\Omega;H)} \\
\leq& \int_{t_1}^{t_2}\int_a^{t_1} \left\|\mathbf{E}_{\alpha,\alpha-1}^{(\beta)}(\psi(\zeta)-\psi(\tau)) f(\tau,u(\tau))\right\|_{L^p(\Omega;H)} \psi'(\tau)\mathrm{d}\tau\,\psi'(\zeta)\mathrm{d}\zeta \\
\leq& C\int_{t_1}^{t_2}\int_a^{t_1} (\psi(\zeta)-\psi(\tau))^{\alpha-2-\frac{\alpha}{2\beta}} ||f(\tau,u(\tau))||_{L^p(\Omega;\dot{H}^{-1})}\psi'(\tau)\mathrm{d}\tau\,\psi'(\zeta)\mathrm{d}\zeta \\
=& \frac{C}{1+\frac{\alpha}{2\beta}-\alpha}\int_{t_1}^{t_2} \left[(\psi(\zeta)-\psi(t_1))^{\alpha-1-\frac{\alpha}{2\beta}} - (\psi(\zeta)-\psi(a))^{\alpha-1-\frac{\alpha}{2\beta}}\right]\psi'(\zeta)\mathrm{d}\zeta \\
&\times\left(1+\sup_{\tau\in[a,T]} ||u(\tau)||_{L^p(\Omega;H)}\right) \\
\leq& \frac{C}{(1+\frac{\alpha}{2\beta}-\alpha)(\alpha-\frac{\alpha}{2\beta})} (\psi(t_2)-\psi(t_1))^{\alpha-\frac{\alpha}{2\beta}} (1+||u||_{\mathcal{H}_p}).
\end{aligned}$$

Together with the above two estimates yields, for $\beta > \frac{1}{2}$,

$$||\mathcal{F}_1(u)(t_2) - \mathcal{F}_1(u)(t_1)||_{L^p(\Omega;H)} \leq C(\psi(t_2) - \psi(t_1))^{\alpha - \frac{\alpha}{2\beta}} \tag{4.10}$$

for all $a \leq t_1 < t_2 \leq T$.

We continue to investigate $\mathcal{F}_2(u)(t)$ in $\|\cdot\|_{L^p(\Omega;H)}$ norm. Applying (2.20), (2.16) ($\mu = 0$ and $\nu = 0$) and (2.18) ($\mu = 0$ and $\nu = 0$), Lemma 2.4, and Lemma 3.1, it holds that, for $\alpha + \gamma > \frac{1}{2}$,

$$\begin{aligned}
&||\mathcal{F}_2(u)(t_2) - \mathcal{F}_2(u)(t_1)|| \\
&\leq \left|\left|\int_{t_1}^{t_2} \mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma}(\psi(t_2) - \psi(\tau))g(\tau, u(\tau))\psi'(\tau)\mathrm{d}W(\tau)\right|\right| \\
&\quad + \left|\left|\int_a^{t_1} \left(\mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma}(\psi(t_2) - \psi(\tau)) - \mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma}(\psi(t_1) - \psi(\tau))\right) g(\tau, u(\tau))\psi'(\tau)\mathrm{d}W(\tau)\right|\right| \\
&\leq C\left(\mathbb{E}\left[\left(\int_{t_1}^{t_2} \left|\left|\mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma}(\psi(t_2) - \psi(\tau))g(\tau, u(\tau))\right|\right|^2_{\mathcal{L}_2^0} \psi'(\tau)\mathrm{d}\tau\right)^{\frac{p}{2}}\right]\right)^{\frac{1}{p}} \\
&\quad + \left|\left|\int_a^{t_1}\int_{t_1}^{t_2} \mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma-1}(\psi(\zeta) - \psi(\tau))\psi'(\zeta)\mathrm{d}\zeta\, g(\tau, u(\tau))\psi'(\tau)\mathrm{d}W(\tau)\right|\right|_{L^p(\Omega;H)} \\
&\leq C\left(\int_{t_1}^{t_2} (\psi(t_2) - \psi(\tau))^{2(\alpha+\gamma-1)}||g(\tau, u(\tau))||^2_{L^p(\Omega;\mathcal{L}_2^0)}\psi'(\tau)\mathrm{d}\tau\right)^{\frac{1}{2}} \\
&\quad + \int_{t_1}^{t_2} C\left(\mathbb{E}\left[\left(\int_a^{t_1} \left|\left|\mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma-1}(\psi(\zeta) - \psi(\tau))g(\tau, u(\tau))\right|\right|^2_{\mathcal{L}_2^0} \psi'(\tau)\mathrm{d}\tau\right)^{\frac{p}{2}}\right]\right)^{\frac{1}{p}} \\
&\quad \cdot \psi'(\zeta)\mathrm{d}\zeta \\
&\leq \left(\frac{C}{(2(\alpha+\gamma-1)+1)^{\frac{1}{2}}} + \frac{C}{|2(\alpha+\gamma-2)+1|^{\frac{1}{2}}(\alpha+\gamma-\frac{1}{2})}\right) \\
&\quad \cdot (\psi(t_2) - \psi(t_1))^{\alpha+\gamma-\frac{1}{2}}(1 + ||u(\tau)||_{\mathcal{H}_p}),
\end{aligned}$$

that is,

$$||\mathcal{F}_2(u)(t_2) - \mathcal{F}_2(u)(t_1)||_{L^p(\Omega;H)} \leq C(\psi(t_2) - \psi(t_1))^{\alpha+\gamma-\frac{1}{2}}. \tag{4.11}$$

Combining (4.6), (4.10) with (4.11) completes the proof of Theorem 4.2. □

5 Further regularities of the mild solution to Eq. (1.1)

Under the following further assumptions for f, g and u_a, we may obtain better regularity results of the mild solution to Eq. (1.1) than those obtained in Section 4.

Assumption 5.1 Let $r \in [0, \kappa)$ where κ is defined by (2.5). The nonlinear function $f : [a, T] \times H \to \dot{H}^{-1}$ satisfies

$$|f(t, h)|_{-1+r} \leq C(1 + |h|_r) \tag{5.1}$$

for every $t \in [a, T],\ h \in \dot{H}^r$.

Assumption 5.2 Let $r \in [0, \kappa)$ where κ is defined by (2.5). The nonlinear operator $g : [a, T] \times H \to \mathcal{L}_2^0$ satisfies

$$\left|\left|A^{\frac{r}{2}} g(t, h)\right|\right|_{\mathcal{L}_2^0} \leq C(1 + |h|_r) \tag{5.2}$$

for every $t \in [a, T], h \in \dot{H}^r$.

Assumption 5.3 Let $\alpha \in (0, 1)$, $\beta \in (\frac{1}{2}, 1]$, $\gamma \in [0, 1]$, $p \in [2, \infty)$, and $\alpha + \gamma > \frac{1}{2}$. Let κ be defined by (2.5) and $r \in [0, \kappa]$. The initial value $u_a : \Omega \to \dot{H}^{\kappa+r}$ is a random variable and satisfies

$$||u_a||_{L^p(\Omega; \dot{H}^{\kappa+r})} := (\mathbb{E}[|u_a|_{\kappa+r}^p])^{\frac{1}{p}} < \infty. \tag{5.3}$$

We first give the following two technical lemmas.

Lemma 5.1 *Let $\alpha \in (0, 1)$, $\beta \in (\frac{1}{2}, 1]$, and $p \in [2, \infty)$. Let Assumptions 2.1-2.4, and Assumptions 5.1-5.3 hold. Let κ and κ_1 be defined by (2.5) and (2.6), respectively. Given a predictable stochastic process $X : [a, T] \times \Omega \to H$ with $\mathbb{P}(X(t) \in \dot{H}^r) = 1$ for any $t \in [a, T]$ and*

$$\sup_{t\in[a,T]} ||X(t)||_{L^p(\Omega; \dot{H}^r)} < \infty.$$

Then for all $s \in \left[0, (2\beta - 1) + \min(r, \frac{2\beta}{\alpha}\kappa_1)\right)$ there exists a constant $C > 0$ such that

$$\begin{aligned} &\left|\left|A^{\frac{s}{2}} \int_{t_1}^{t_2} \mathbf{E}_{\alpha,\alpha}^{(\beta)}(\psi(t_2) - \psi(\tau)) f(t_2, X(t_2))\psi'(\tau)\mathrm{d}\tau\right|\right|_{L^p(\Omega; H)} \\ &\leq C\left(1 + \sup_{\tau\in[a,T]} ||X(\tau)||_{L^p(\Omega; \dot{H}^r)}\right) (\psi(t_2) - \psi(t_1))^{\alpha - \frac{s+1}{2\beta}\alpha + \frac{\alpha}{2\beta}r}. \end{aligned} \tag{5.4}$$

for $a \leq t_1 < t_2 \leq T$. Moreover, if there exists a constant $C > 0$ such that

$$||X(t_1) - X(t_2)||_{L^p(\Omega; H)} \leq C|\psi(t_2) - \psi(t_1)|^{\kappa_1} \tag{5.5}$$

for all $t_1, t_2 \in [a, T]$, then one has

$$\left|\left|A^{\frac{s}{2}} \int_{t_1}^{t_2} \mathbf{E}_{\alpha,\alpha}^{(\beta)}(\psi(t_2) - \psi(\tau))(f(t_2, X(t_2)) - f(\tau, X(\tau)))\psi'(\tau)\mathrm{d}\tau\right|\right|_{L^p(\Omega; H)}$$

$$\leq C\left(1+\sup_{\tau\in[a,T]}||X(\tau)||_{L^p(\Omega;H)}\right)(\psi(t_2)-\psi(t_1))^{\alpha-\frac{s+1}{2\beta}\alpha+\kappa_1}. \tag{5.6}$$

Further we have

$$\begin{aligned}&\left|\left|A^{\frac{s}{2}}\int_{t_1}^{t_2}\mathbf{E}_{\alpha,\alpha}^{(\beta)}(\psi(t_2)-\psi(\tau))f(\tau,X(\tau))\psi'(\tau)\mathrm{d}\tau\right|\right|_{L^p(\Omega;H)}\\&\leq C\left(1+\sup_{\tau\in[a,T]}||X(\tau)||_{L^p(\Omega;\dot{H}^r)}\right)(\psi(t_2)-\psi(t_1))^{\alpha-\frac{s+1}{2\beta}\alpha+\min(\kappa_1,\frac{\alpha}{2\beta}r)}. \end{aligned}\tag{5.7}$$

Proof We first prove (5.4). By using (2.15) with $\mu=s+1-r$, $\nu=0$ (note that $\mu=s+1-r>0$ for any $s>0$), and Assumption 5.1, it follows that

$$\begin{aligned}&\left|\left|A^{\frac{s}{2}}\int_{t_1}^{t_2}\mathbf{E}_{\alpha,\alpha}^{(\beta)}(\psi(t_2)-\psi(\tau))f(t_2,X(t_2))\psi'(\tau)\mathrm{d}\tau\right|\right|_{L^p(\Omega;H)}\\&\leq\left|\left|A^{\frac{s+1-r}{2}}\int_{t_1}^{t_2}\mathbf{E}_{\alpha,\alpha}^{(\beta)}(\psi(t_2)-\psi(\tau))A^{\frac{-1+r}{2}}f(t_2,X(t_2))\psi'(\tau)\mathrm{d}\tau\right|\right|_{L^p(\Omega;H)}\\&\leq C(\psi(t_2)-\psi(t_1))^{(\alpha-1)-\frac{s+1-r}{2\beta}\alpha+1}\left|\left|A^{\frac{-1+r}{2}}f(t_2,X(t_2))\right|\right|_{L^p(\Omega;H)}\\&\leq C(\psi(t_2)-\psi(t_1))^{\alpha-\frac{s+1-r}{2\beta}\alpha}||f(t_2,X(t_2))||_{L^p(\Omega;\dot{H}^{-1+r})}\\&\leq C\left(1+\sup_{t\in[a,T]}||X(t)||_{L^p(\Omega;\dot{H}^r)}\right)(\psi(t_2)-\psi(t_1))^{\alpha-\frac{s+1}{2\beta}\alpha+\frac{\alpha}{2\beta}r},\end{aligned}$$

which is the desired form (5.4), where we require $\alpha-\frac{s+1}{2\beta}\alpha+\frac{\alpha}{2\beta}r>0$, that is, $s\in[0,(2\beta-1)+r)$.

We now show (5.6). In terms of (2.15) ($\mu=s+1$ and $\nu=0$) in Lemma 2.2, Lemma 3.1 and the condition (5.5) we derive

$$\begin{aligned}&\left|\left|A^{\frac{s}{2}}\int_{t_1}^{t_2}\mathbf{E}_{\alpha,\alpha}^{(\beta)}(\psi(t_2)-\psi(\tau))(f(t_2,X(t_2))-f(\tau,X(\tau)))\psi'(\tau)\mathrm{d}\tau\right|\right|_{L^p(\Omega;H)}\\&\leq\int_{t_1}^{t_2}\left|\left|A^{\frac{s+1}{2}}\mathbf{E}_{\alpha,\alpha}^{(\beta)}(\psi(t_2)-\psi(\tau))A^{-\frac{1}{2}}(f(t_2,X(t_2))-f(\tau,X(\tau)))\right|\right|_{L^p(\Omega;H)}\psi'(\tau)\mathrm{d}\tau\\&\leq C\int_{t_1}^{t_2}(\psi(t_2)-\psi(\tau))^{(\alpha-1)-\frac{s+1}{2\beta}\alpha+\kappa_1}\left(1+||X(\tau)||_{L^p(\Omega;H)}\right)\psi'(\tau)\mathrm{d}\tau\\&\leq\frac{C}{\alpha-\frac{s+1}{2\beta}\alpha+\kappa_1}\left(1+\sup_{\tau\in[a,T]}||X(\tau)||_{L^p(\Omega;H)}\right)(\psi(t_2)-\psi(t_1))^{\alpha-\frac{s+1}{2\beta}\alpha+\kappa_1},\end{aligned}$$

where we require $\alpha-\frac{s+1}{2\beta}\alpha+\kappa_1>0$, that is, $s\in\left[0,(2\beta-1)+\frac{2\beta}{\alpha}\kappa_1\right)$.

Finally, combining (5.4) with (5.6), we obtain

$$\begin{aligned}
&\left\|A^{\frac{s}{2}}\int_{t_1}^{t_2}\mathbf{E}^{(\beta)}_{\alpha,\alpha}(\psi(t_2)-\psi(\tau))f(\tau,X(\tau))\psi'(\tau)\mathrm{d}\tau\right\|_{L^p(\Omega;H)}\\
&\leq \left\|A^{\frac{s}{2}}\int_{t_1}^{t_2}\mathbf{E}^{(\beta)}_{\alpha,\alpha}(\psi(t_2)-\psi(\tau))f(t_2,X(t_2))\psi'(\tau)\mathrm{d}\tau\right\|_{L^p(\Omega;H)}\\
&\quad+\left\|A^{\frac{s}{2}}\int_{t_1}^{t_2}\mathbf{E}^{(\beta)}_{\alpha,\alpha}(\psi(t_2)-\psi(\tau))(f(t_2,X(t_2))-f(\tau,X(\tau)))\psi'(\tau)\mathrm{d}\tau\right\|_{L^p(\Omega;H)}\\
&\leq C\left(1+\sup_{t\in[a,T]}||X(t)||_{L^p(\Omega;\dot{H}^r)}\right)(\psi(t_2)-\psi(t_1))^{\alpha-\frac{s+1}{2\beta}\alpha+\min(\kappa_1,\frac{\alpha}{2\beta}r)},
\end{aligned}$$

where we require $s\in\left[0,(2\beta-1)+\min(\kappa_1,\frac{\alpha}{2\alpha}r\right)$. The proof of Lemma 5.1 is now comlete. □

Lemma 5.2 *Let $\alpha\in(0,1)$, $\beta\in(\frac{1}{2},1]$, and $p\in[2,\infty)$. Let Assumptions 2.1-2.4, and Assumptions 5.1-5.3 hold. Let κ and κ_1 be defined by (2.5) and (2.6), respectively. Given a predictable stochastic process $X:[a,T]\times\Omega\to H$ with $\mathbb{P}(X(t)\in\dot{H}^r)=1$ for any $t\in[a,T]$ and*

$$\sup_{t\in[a,T]}||X(t)||_{L^p(\Omega;\dot{H}^r)}<\infty.$$

Then for all $s\in\left[r,\frac{2\beta}{\alpha}(\alpha+\gamma-\frac{1}{2})+\min(r,\frac{2\beta}{\alpha}\kappa_1)\right)$ there exists a constant $C>0$ such that

$$\begin{aligned}
&\left(\mathbb{E}\left[\left(\int_{t_1}^{t_2}\left\|A^{\frac{s}{2}}\mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma}(\psi(t_2)-\psi(\tau))g(t_2,X(t_2))\right\|^2_{\mathcal{L}_2^0}\psi'(\tau)\mathrm{d}\tau\right)^{\frac{p}{2}}\right]\right)^{\frac{1}{p}}\\
&\quad\leq C\left(1+\sup_{\tau\in[a,T]}||X(\tau)||_{L^p(\Omega;\dot{H}^r)}\right)(\psi(t_2)-\psi(t_1))^{\alpha+\gamma-\frac{1}{2}-\frac{\alpha s}{2\beta}+\frac{\alpha r}{2\beta}}
\end{aligned}\tag{5.8}$$

for any $a\leq t_1<t_2\leq T$. Moreover, if the condition (5.5) is fulfilled for some κ_1 and for all $t_1,t_2\in[a,T]$, then it holds that

$$\begin{aligned}
&\left(\mathbb{E}\left[\left(\int_{t_1}^{t_2}\left\|A^{\frac{s}{2}}\mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma}(\psi(t_2)-\psi(\tau))(g(t_2,X(t_2))-g(\tau,X(\tau)))\right\|^2_{\mathcal{L}_2^0}\psi'(\tau)\mathrm{d}\tau\right)^{\frac{p}{2}}\right]\right)^{\frac{1}{p}}\\
&\leq C\left(1+\sup_{\tau\in[a,T]}||X(\tau)||_{L^p(\Omega;H)}\right)(\psi(t_2)-\psi(t_1))^{\alpha+\gamma-\frac{1}{2}-\frac{\alpha s}{2\beta}+\kappa_1}.
\end{aligned}\tag{5.9}$$

Further one has

$$\left\|\int_{t_1}^{t_2} A^{\frac{s}{2}}\mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma}(\psi(t_2)-\psi(\tau))g(\tau,X(\tau))\psi'(\tau)\mathrm{d}W(\tau)\right\|_{L^p(\Omega;H)}$$
$$\le C\left(1+\sup_{\tau\in[a,T]}||X(\tau)||_{L^p(\Omega;\dot H^r)}\right)(\psi(t_2)-\psi(t_1))^{\alpha+\gamma-\frac{1}{2}-\frac{\alpha}{2\beta}(s-r)}. \quad (5.10)$$

Proof First we show (5.8). For any $a\le t_1<t_2\le T$, by applying (2.16) with $\mu=s-r,\ \nu=0$ (note that $\mu=s-r\ge 0$ since $s\ge r$) and Assumption 5.2, it yields that

$$\int_{t_1}^{t_2}\left\|A^{\frac{s}{2}}\mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma}(\psi(t_2)-\psi(\tau))g(t_2,X(t_2))\right\|^2_{\mathcal{L}_2^0}\psi'(\tau)\mathrm{d}\tau$$
$$\le\int_{t_1}^{t_2}\left\|A^{\frac{s-r}{2}}\mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma}(\psi(t_2)-\psi(\tau))A^{\frac{r}{2}}g(t_2,X(t_2))\right\|^2_{\mathcal{L}_2^0}\psi'(\tau)\mathrm{d}\tau$$
$$\le C(\psi(t_2)-\psi(t_1))^{2[(\alpha+\gamma-1)-\frac{s-r}{2\beta}\alpha]+1}\left\|A^{\frac{r}{2}}g(t_2,X(t_2))\right\|^2_{\mathcal{L}_2^0},$$

where we require $2[(\alpha+\gamma-1)-\frac{s-r}{2\beta}\alpha]+1>0$, that is, $s\in[r,\frac{2\beta}{\alpha}(\alpha+\gamma-\frac{1}{2})+r)$.

Hence, the proof of (5.8) can be obtained from Assumption 5.2

$$\left(\mathbb{E}\left[\left(\int_{t_1}^{t_2}\left\|A^{\frac{s}{2}}\mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma}(\psi(t_2)-\psi(\tau))g(t_2,X(t_2))\right\|^2_{\mathcal{L}_2^0}\psi'(\tau)\mathrm{d}\tau\right)^{\frac{p}{2}}\right]\right)^{\frac{1}{p}}$$
$$=\left\|\left(\int_{t_1}^{t_2}\left\|A^{\frac{s}{2}}\mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma}(\psi(t_2)-\psi(\tau))g(t_2,X(t_2))\right\|^2_{\mathcal{L}_2^0}\psi'(\tau)\mathrm{d}\tau\right)^{\frac{1}{2}}\right\|_{L^p(\Omega;\mathbb{R})}$$
$$\le C\left(1+\sup_{\tau\in[a,T]}||X(\tau)||_{L^p(\Omega;\dot H^r)}\right)(\psi(t_2)-\psi(t_1))^{\alpha+\gamma-\frac{1}{2}-\frac{\alpha s}{2\beta}+\frac{\alpha r}{2\beta}},$$

where we require $\alpha+\gamma-\frac{1}{2}-\frac{\alpha s}{2\beta}+\frac{\alpha r}{2\beta}>0$, that is, $s\in[r,\frac{2\beta}{\alpha}(\alpha+\gamma-\frac{1}{2})+r)$.

We next estimate (5.9). Based on (2.16) ($\mu=s$ and $\nu=0$) in Lemma 2.2, Lemma 3.1 , the condition (5.5), and Assumption 5.2, we conclude that, with $g_2(\tau)=g(t_2,X(t_2))-g(\tau,X(\tau))$,

$$\left(\mathbb{E}\left[\left(\int_{t_1}^{t_2}\left\|A^{\frac{s}{2}}\mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma}(\psi(t_2)-\psi(\tau))g_2(\tau)\right\|^2_{\mathcal{L}_2^0}\psi'(\tau)\mathrm{d}\tau\right)^{\frac{p}{2}}\right]\right)^{\frac{1}{p}}$$
$$\le C\left\|\left(\int_{t_1}^{t_2}(\psi(t_2)-\psi(\tau))^{2[(\alpha+\gamma-1)-\frac{s}{2\beta}\alpha]}||g_2(\tau)||^2_{\mathcal{L}_2^0}\psi'(\tau)\mathrm{d}\tau\right)^{\frac{1}{2}}\right\|_{L^p(\Omega;\mathbb{R})}$$
$$\le C\left(\int_{t_1}^{t_2}(\psi(t_2)-\psi(\tau))^{2[(\alpha+\gamma-1)-\frac{s}{2\beta}\alpha]}||g_2(\tau)||^2_{L^p(\Omega;\mathcal{L}_2^0)}\psi'(\tau)\mathrm{d}\tau\right)^{\frac{1}{2}}$$

$$\leq \frac{C(\psi(t_2)-\psi(t_1))^{(\alpha+\gamma-\frac{1}{2})-\frac{\alpha s}{2\beta}+\kappa_1}}{\sqrt{2(\alpha+\gamma-1-\frac{\alpha s}{2\beta})+2\kappa_1+1}}\left(1+\sup_{t\in[a,T]}||X(t)||_{L^p(\Omega;H)}\right),$$

where we require $\alpha+\gamma-\frac{1}{2}-\frac{\alpha s}{2\beta}+\kappa_1>0$, that is, $s\in[r,\frac{2\beta}{\alpha}(\alpha+\gamma-\frac{1}{2})+\frac{2\beta}{\alpha}\kappa_1)$.

Finally, we can use (5.8) and (5.9) to obtain

$$\begin{aligned}
&\left|\left|\int_{t_1}^{t_2} A^{\frac{s}{2}}\mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma}(\psi(t_2)-\psi(\tau))g(\tau,X(\tau))\psi'(\tau)\mathrm{d}W(\tau)\right|\right|_{L^p(\Omega;H)}\\
&\leq C\left(\mathbb{E}\left[\left(\int_{t_1}^{t_2}\left|\left|A^{\frac{s}{2}}\mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma}(\psi(t_2)-\psi(\tau))g(\tau,X(\tau))\right|\right|^2_{\mathcal{L}_2^0}\psi'(\tau)\mathrm{d}\tau\right)^{\frac{p}{2}}\right]\right)^{\frac{1}{p}}\\
&\leq C\left(1+\sup_{t\in[a,T]}||X(t)||_{L^p(\Omega;\dot{H}^r)}\right)(\psi(t_2)-\psi(t_1))^{(\alpha+\gamma-\frac{1}{2})-\frac{\alpha s}{2\beta}+\min(\kappa_1,\frac{\alpha r}{2\beta})},
\end{aligned}$$

where we require $s\in\left[r,\frac{2\beta}{\alpha}(\alpha+\gamma-\frac{1}{2})+\min(r,\frac{2\beta}{\alpha}\kappa_1)\right)$. The proof is thus completed. □

Now we present the spatial and temporal regularity results of the mild solution (2.13) to Eq. (1.1). The proofs are included in the Appendix.

Theorem 5.1 *Let $\alpha\in(0,1)$, $\beta\in(\frac{1}{2},1]$, and $p\in[2,\infty)$. Let Assumptions 2.1-2.4, and Assumptions 5.1-5.3 hold. Let κ and κ_1 be defined by (2.5) and (2.6), respectively. Let $r_1=\min(r,\frac{2\beta}{\alpha}\kappa_1)$. Then the unique mild solution u to Eq. (1.1) satisfies*

$$\mathbb{P}(u(t)\in\dot{H}^{\kappa+r_1})=1$$

for any $t\in[a,T]$. Moreover, there exists a constant $C>0$ such that

$$\sup_{t\in[a,T]}||u(t)||_{L^p(\Omega;\dot{H}^{\kappa+r_1})}\leq C||u_a||_{L^p(\Omega;\dot{H}^{\kappa+r_1})}+C\left(1+\sup_{t\in[a,T]}||u(t)||_{L^p(\Omega;\dot{H}^{r_1})}\right). \tag{5.11}$$

Theorem 5.2 *Let $\alpha\in(0,1)$, $\beta\in(\frac{1}{2},1]$, and $p\in[2,\infty)$. Let Assumptions 2.1-2.4, and Assumptions 5.1-5.3 hold. Let κ and κ_1 be defined by (2.5) and (2.6), respectively. Then, for every $s\in[r,r+\kappa)$, the unique mild solution $u(t)$ to Eq. (1.1) is Hölder continuous with respect to the norm $||\cdot||_{L^p(\Omega;\dot{H}^s)}$ and satisfies*

$$\sup_{\substack{t_1,t_2\in[a,T]\\ t_1\neq t_2}}\frac{\left(\mathbb{E}[|u(t_1)-u(t_2)|_s^p]\right)^{\frac{1}{p}}}{|\psi(t_1)-\psi(t_2)|^{\min\{\frac{\alpha}{2\beta}(r+\kappa-s),\alpha-\frac{\alpha}{2\beta}(s+1-r),\alpha+\gamma-\frac{1}{2}-\frac{\alpha}{2\beta}(s-r)\}}}<\infty. \tag{5.12}$$

6 Conclusions

This paper deals with a semilinear stochastic time-space fractional evolution equation driven by fractionally integrated multiplicative noise. The equation involves a more general ψ-Caputo temporal derivative and a spectral fractional Laplacian. We establish the existence and uniqueness of the mild solution using the Banach contraction mapping principle under appropriate assumptions. Based on this, we derive spatial and temporal regularity results for the mild solution. When we set $\psi(t) = t$, $a = 0$, $\beta = 1$, $\gamma = 0$, and let $\alpha \to 1$ in Eq. (1.1), the results presented in this paper reduce to the results in Chapter 2 in [19] for the stochastic heat equation. The model (1.1) with ψ-Caputo temporal derivative has more applications, for example, describing subdiffusion in a medium having a structure evolving over time [21] or modeling diffusion in a complex system consisting of a matrix and channels [22].

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Declarations

Conflict of interest The authors declare no conflicts of interest.

Appendix

Proof of Theorem 5.1

According to the mild solution formula (2.13) to Eq. (1.1), we have

$$
\begin{aligned}
||u(t)||_{L^p(\Omega;\dot{H}^{\kappa+r_1})} &= \left|\left|A^{\frac{r_1+\kappa}{2}} u(t)\right|\right|_{L^p(\Omega;H)} \\
&\leq \left|\left|A^{\frac{r_1+\kappa}{2}} \mathbf{E}_{\alpha,1}^{(\beta)}(\psi(t)-\psi(a))u_a\right|\right|_{L^p(\Omega;H)} \\
&\quad + \left|\left|A^{\frac{r_1+\kappa}{2}} \int_a^t \mathbf{E}_{\alpha,\alpha}^{(\beta)}(\psi(t)-\psi(\tau)) f(\tau,u(\tau))\psi'(\tau)\mathrm{d}\tau\right|\right|_{L^p(\Omega;H)} \\
&\quad + \left|\left|A^{\frac{r_1+\kappa}{2}} \int_a^t \mathbf{E}_{\alpha,\alpha+\gamma}^{(\beta)}(\psi(t)-\psi(\tau)) g(\tau,u(\tau))\psi'(\tau)\mathrm{d}W(\tau)\right|\right|_{L^p(\Omega;H)}
\end{aligned}
$$

$$= I + II + III.$$

We now estimate these three terms, respectively. For the first term, we use (2.14) ($\mu = \nu = r_1 + \kappa$) in Lemma 2.2 and Assumption 5.3 to derive

$$I = \left\| A^{\frac{r_1+\kappa}{2}} \mathbf{E}_{\alpha,1}^{(\beta)}(\psi(t) - \psi(a)) u_a \right\|_{L^p(\Omega; H)} \leq C ||u_a||_{L^p(\Omega; \dot{H}^{r_1+\kappa})} < \infty.$$

Applying (5.7) in Lemma 5.1 with $t_1 = a, t_2 = t, s = r_1 + \kappa \leq r_1 + 2\beta - 1$ and $X = u$ we can obtain

$$II \leq \left(1 + \sup_{t \in [a,T]} ||u(t)||_{L^p(\Omega; \dot{H}^r)} \right) < \infty.$$

For the last term, since $s = r_1 + \kappa \leq r_1 + \frac{2\beta}{\alpha}(\alpha + \gamma - \frac{1}{2})$, we take $t_1 = a$, $t_2 = t$, $X = u$ in (5.10) of Lemma 5.2 and hence get

$$III \leq \left(1 + \sup_{t \in [a,T]} ||u(t)||_{L^p(\Omega; \dot{H}^r)} \right) < \infty.$$

Combining the above estimates for I, II and III yields (5.11) and the proof of the theorem is thus completed.

Proof of Theorem 5.2

Applying the mild solution formula (2.13) to Eq. (1.1), for $a \leq t_1 < t_2 \leq T$ and $s \in [r, r + \kappa)$, one has

$$\begin{aligned}
&||u(t_1) - u(t_2)||_{L^p(\Omega; \dot{H}^s)} \\
&\quad \leq \left\| \left(\mathbf{E}_{\alpha,1}^{(\beta)}(\psi(t_1) - \psi(a)) - \mathbf{E}_{\alpha,1}^{(\beta)}(\psi(t_2) - \psi(a)) \right) u_a \right\|_{L^p(\Omega; \dot{H}^s)} \\
&\qquad + \left\| \int_{t_1}^{t_2} \mathbf{E}_{\alpha,\alpha}^{(\beta)}(\psi(t_2) - \psi(\tau)) f(\tau, u(\tau)) \psi'(\tau) \mathrm{d}\tau \right\|_{L^p(\Omega; \dot{H}^s)} \\
&\qquad + \left\| \int_a^{t_1} \left(\mathbf{E}_{\alpha,\alpha}^{(\beta)}(\psi(t_2) - \psi(\tau)) - \mathbf{E}_{\alpha,\alpha}^{(\beta)}(\psi(t_1) - \psi(\tau)) \right) f(\tau, u(\tau)) \psi'(\tau) \mathrm{d}\tau \right\|_{L^p(\Omega; \dot{H}^s)} \\
&\qquad + \left\| \int_{t_1}^{t_2} \mathbf{E}_{\alpha,\alpha+\gamma}^{(\beta)}(\psi(t_2) - \psi(\tau)) g(\tau, u(\tau)) \psi'(\tau) \mathrm{d}W(\tau) \right\|_{L^p(\Omega; \dot{H}^s)} \\
&\qquad + \left\| \int_a^{t_1} \left(\mathbf{E}_{\alpha,\alpha+\gamma}^{(\beta)}(\psi(t_2) - \psi(\tau)) - \mathbf{E}_{\alpha,\alpha+\gamma}^{(\beta)}(\psi(t_1) - \psi(\tau)) \right) \right. \\
&\qquad \left. \cdot\, g(\tau, u(\tau)) \psi'(\tau) \mathrm{d}W(\tau) \right\|_{L^p(\Omega; \dot{H}^s)} \\
&\quad = I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned}$$

We now estimate these five terms respectively. From (2.15) ($\mu = s + 2\beta$ and $\nu = r + \kappa$) in Lemma 2.2, (2.19) in Lemma 2.3 and Assumption 5.3, the first term I_1 is estimated by

$$\begin{aligned} I_1 &= \left\| \left(\mathbf{E}_{\alpha,1}^{(\beta)}(\psi(t_1) - \psi(a)) - \mathbf{E}_{\alpha,1}^{(\beta)}(\psi(t_2) - \psi(a)) \right) u_a \right\|_{L^p(\Omega;\dot{H}^s)} \\ &= \left\| A^{\frac{s}{2}} \int_{t_1}^{t_2} A^{\beta} \mathbf{E}_{\alpha,\alpha}^{(\beta)}(\psi(\tau) - \psi(a)) u_a \psi'(\tau) \mathrm{d}\tau \right\|_{L^p(\Omega;H)} \\ &\leq \int_{t_1}^{t_2} \left\| A^{\frac{s}{2}+\beta} \mathbf{E}_{\alpha,\alpha}^{(\beta)}(\psi(\tau) - \psi(a)) u_a \right\|_{L^p(\Omega;H)} \psi'(\tau) \mathrm{d}\tau \\ &\leq C \int_{t_1}^{t_2} (\psi(\tau) - \psi(a))^{\alpha - 1 - \frac{\alpha}{2\beta}(s + 2\beta - r - \kappa)} ||u_a||_{L^p(\Omega;\dot{H}^{r+\kappa})} \psi'(\tau) \mathrm{d}\tau \\ &\leq C (\psi(t_2) - \psi(t_1))^{\frac{\alpha}{2\beta}(r + \kappa - s)} ||u_a||_{L^p(\Omega;\dot{H}^{r+\kappa})}, \end{aligned}$$

where we require $r + \kappa - s > 0$, that is, $s < r + \kappa$.

The second term I_2 follows from (5.7) of Lemma 5.1, that is,

$$I_2 \leq C \left(1 + \sup_{t \in [a,T]} ||u(t)||_{L^p(\Omega;\dot{H}^r)} \right) (\psi(t_2) - \psi(t_1))^{\alpha - \frac{\alpha}{2\beta}(s+1-r)},$$

where we require $\alpha - \frac{\alpha}{2\beta}(s + 1 - r) > 0$, that is, $s < r + 2\beta - 1$.

For the third term I_3, it follows from (2.20) in Lemma 2.3, (2.17) ($\mu = s$ and $\nu = -1 + r$) in Lemma 2.2 and Assumption (5.1) that

$$\begin{aligned} I_3 &= \left\| \int_a^{t_1} \left(\mathbf{E}_{\alpha,\alpha}^{(\beta)}(\psi(t_2) - \psi(\tau)) - \mathbf{E}_{\alpha,\alpha}^{(\beta)}(\psi(t_1) - \psi(\tau)) \right) f(\tau, u(\tau)) \psi'(\tau) \mathrm{d}\tau \right\|_{L^p(\Omega;\dot{H}^s)} \\ &= \left\| A^{\frac{s}{2}} \int_a^{t_1} \int_{t_1}^{t_2} \mathbf{E}_{\alpha,\alpha-1}^{(\beta)}(\psi(\zeta) - \psi(\tau)) f(\tau, u(\tau)) \psi'(\zeta) \mathrm{d}\zeta \, \psi'(\tau) \mathrm{d}\tau \right\|_{L^p(\Omega;H)} \\ &\leq \int_a^{t_1} \int_{t_1}^{t_2} \left\| A^{\frac{s}{2}} \mathbf{E}_{\alpha,\alpha-1}^{(\beta)}(\psi(\zeta) - \psi(\tau)) f(\tau, u(\tau)) \right\|_{L^p(\Omega;H)} \psi'(\zeta) \mathrm{d}\zeta \, \psi'(\tau) \mathrm{d}\tau \\ &\leq C \int_a^{t_1} \int_{t_1}^{t_2} (\psi(\zeta) - \psi(\tau))^{\alpha - 2 - \frac{\alpha}{2\beta}(s+1-r)} \\ &\quad \cdot ||f(\tau, u(\tau))||_{L^p(\Omega;\dot{H}^{-1+r})} \psi'(\zeta) \mathrm{d}\zeta \, \psi'(\tau) \mathrm{d}\tau \\ &\leq C \left(1 + \sup_{t \in [a,T]} ||u(t)||_{L^p(\Omega;\dot{H}^r)} \right) (\psi(t_2) - \psi(t_1))^{\alpha - \frac{\alpha}{2\beta}(s+1-r)}, \end{aligned}$$

where we require $s < r + (2\beta - 1)$.

A straightforward application of (5.10) in Lemma 5.2 gives

$$I_4 \leq C \left(1 + \sup_{t \in [a,T]} ||u(t)||_{L^p(\Omega;\dot{H}^r)} \right) (\psi(t_2) - \psi(t_1))^{\alpha + \gamma - \frac{1}{2} - \frac{\alpha}{2\beta}(s-r)},$$

where we require $s < r + \frac{2\beta}{\alpha}(\alpha + \gamma - \frac{1}{2})$.

To estimate the last term I_5, applying (2.20) in Lemma 2.3, (2.18) in Lemma 2.2 with $\mu = s - r,\ \nu = 0$ (note that $\mu = s - r \geq 0$), and Assumption (5.2), we arrive at

$$\begin{aligned}
I_5 =& \Big\| \int_a^{t_1} \Big(\mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma}(\psi(t_2) - \psi(\tau)) - \mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma}(\psi(t_1) - \psi(\tau)) \Big) \\
&\quad \cdot g(\tau, u(\tau))\psi'(\tau)\mathrm{d}W(\tau) \Big\|_{L^p(\Omega;\dot{H}^s)} \\
=& \Big\| \int_a^{t_1} \int_{t_1}^{t_2} A^{\frac{s}{2}} \mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma-1}(\psi(\zeta) - \psi(\tau))\psi'(\zeta)\mathrm{d}\zeta\, g(\tau, u(\tau))\psi'(\tau)\mathrm{d}W(\tau) \Big\|_{L^p(\Omega;H)} \\
\leq& \int_{t_1}^{t_2} \Big\| \int_a^{t_1} A^{\frac{s-r}{2}} \mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma-1}(\psi(\zeta) - \psi(\tau)) A^{\frac{r}{2}} g(\tau, u(\tau))\psi'(\tau)\mathrm{d}W(\tau) \Big\|_{L^p(\Omega;H)} \\
&\quad \cdot \psi'(\zeta)\mathrm{d}\zeta \\
\leq& C \int_{t_1}^{t_2} \left(\mathbb{E} \left(\int_a^{t_1} \Big\| A^{\frac{s-r}{2}} \mathbf{E}^{(\beta)}_{\alpha,\alpha+\gamma-1}(\psi(\zeta) - \psi(\tau)) A^{\frac{r}{2}} g(\tau, u(\tau)) \Big\|^2_{\mathcal{L}_2^0} \psi'(\tau)\mathrm{d}\tau \right)^{\frac{p}{2}} \right)^{\frac{1}{p}} \\
&\quad \cdot \psi'(\zeta)\mathrm{d}\zeta \\
\leq& C \int_{t_1}^{t_2} \left(\int_a^{t_1} (\psi(\zeta) - \psi(\tau))^{2[(\alpha+\gamma-2) - \frac{s-r}{2\beta}\alpha]} \Big\| A^{\frac{r}{2}} g(\tau, u(\tau)) \Big\|^2_{L^p(\Omega;\mathcal{L}_2^0)} \psi'(\tau)\mathrm{d}\tau \right)^{\frac{1}{2}} \\
&\quad \cdot \psi'(\zeta)\mathrm{d}\zeta \\
\leq& C \int_{t_1}^{t_2} \left(\int_a^{t_1} (\psi(\zeta) - \psi(\tau))^{2[(\alpha+\gamma-2) - \frac{s-r}{2\beta}\alpha]} \Big(1 + \|u(\tau)\|_{L^p(\Omega;\dot{H}^r)} \Big)^2 \psi'(\tau)\mathrm{d}\tau \right)^{\frac{1}{2}} \\
&\quad \cdot \psi'(\zeta)\mathrm{d}\zeta \\
\leq& C \left(1 + \sup_{t\in[a,T]} \|u(t)\|_{L^p(\Omega;\dot{H}^r)} \right) (\psi(t_2) - \psi(t_1))^{\alpha+\gamma-\frac{1}{2}-\frac{\alpha}{2\beta}(s-r)},
\end{aligned}$$

where we require $\alpha + \gamma - \frac{1}{2} - \frac{\alpha}{2\beta}(s - r) > 0$, that is, $s < r + \frac{2\beta}{\alpha}(\alpha + \gamma - \frac{1}{2})$.

Combining the estimates I_1-I_5, we obtain the desired result (5.12), and thus conclude the proof.

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