Fractional Calculus and Applied Analysis (2024) 27:1–21
https://doi.org/10.1007/s13540-023-00230-8

Fractional Calculus & Applied Analysis
An International Journal for Theory and Applications

ORIGINAL PAPER

On the Filippov-Ważewski relaxation theorem for a certain class of fractional differential inclusions

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Dedicated to the memory of professor Andrzej Fryszkowski

Received: 10 March 2022 / Revised: 16 November 2023 / Accepted: 17 November 2023 /
Published online: 21 December 2023

Abstract

The purpose of this text is to propose an attempt of an extension of the Filippov-Ważewski Relaxation Theorem for a certain class of fractional differential inclusions. The classical result devoted to ordinary differential inclusions is a part of the qualitative theory: a description of the relationship between the solutions to the differential inclusion and the convexified differential inclusion was given. There exist several generalizations of that result and in this note a method is proposed so that the range of some parameters is extended. An example of a possible application may arise in control theory and the question is whether it is possible to have the same reachable set economizing the set of controls.

Keywords Filippov-Ważewski (primary) · Relaxation · Differential inclusion · Fractional derivative · Riemann-Liouville

Mathematics Subject Classification 34A08 (primary) · 26E25 · 49J21 · 34A60 · 28B20

1 Introduction

The aim of the text is to propose an attempt of a generalization of the Filippov-Ważewski Relaxation Theorem ([2], p.124) for the fractional differential inclusion

$$\begin{cases} D^{\alpha} y(t) \in F(t, y(t)) \\ (I^{1-\alpha} y)(0) = \tilde{y}_0 \, , \end{cases} \tag{1.1}$$

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where $F : [0, T] \times \mathbb{R}^n \mapsto 2^{R^n}$ is a set-valued mapping, $D^\alpha y$ stands for the fractional derivative

$$D^\alpha y(t) = \frac{d}{dt}(I^{1-\alpha} y)(t) = \frac{d}{dt} \int_0^t \frac{y(s)}{\Gamma(1-\alpha) \cdot (t-s)^\alpha} ds \tag{1.2}$$

and $I^{1-\alpha} y$ is the Riemann-Liouville integral. A solution to (1.1) is a function $y : (0, T] \mapsto \mathbb{R}^n$ given by the formula

$$y(t) = (I^{1-\alpha} y)(0) \cdot \frac{t^{\alpha-1}}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{v(s)}{(t-s)^{1-\alpha}} ds\ , \tag{1.3}$$

where $v(\cdot) : (0, T) \mapsto \mathbb{R}^n$ is a measurable function such that $v(s) \in F(s, y(s))$ a.e..

1.1 Definitions and the literature

There are different concepts of the fractional derivative (see [8], pp. 69-133) and we recall fractional derivatives in the sense of Riemann-Liouville, Caputo and Grünwald-Letnikov, for instance. In this paper the Riemann-Liouville derivative is considered. Such derivative is based on the convolution operator

$$t \mapsto (I^{1-\alpha} g)(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{g(s)}{(t-s)^\alpha} ds \tag{1.4}$$

and the function in (1.4) is supposed to be absolutely continuous (see [28], pp. 35-44). The formula of a solution to the Cauchy problem for a fractional differential equation $D^\alpha x = f(t, x)$ equipped with the initial condition $(I^{1-\alpha} x)(0)$ is

$$x(t) = \frac{(I^{1-\alpha} x)(0)}{\Gamma(\alpha)} \cdot t^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s, x(s))}{(t-s)^{1-\alpha}} ds\ ,$$

(see [28], p. 833). A solution to the differential inclusion

$$y'(t) \in F(t, y(t)), \quad y(0) = \tilde{y_0}, \tag{1.5}$$

where $F : [0, T] \times \mathbb{R}^n \mapsto 2^{\mathbb{R}^n}$ is a multifunction, is usually understood in the Caratheodory sense and the corresponding formula is $y(t) = \tilde{y_0} + \int_0^t v(s)\, ds$ with $v(\cdot) \in L^1([0, T])$ such that $v(t) \in F(t, y(t))$ a.e. (see [3], p. 384). The fractional analogue of (1.5) is given in (1.3).

Ordinary differential inclusions are closely related with control theory because Filippov and Ważewski proved that under very mild assumptions the control system

$$x' = f(t, x, u(t)), \quad u(t) \in U \text{ is measurable} \tag{1.6}$$

may be reduced to the differential inclusion $x'(t) \in F(t, x(t))$ (see [3], p. 384). Therefore some problems concerning system (1.6) might be investigated using the tools of set-valued analysis. For example the question to economize control U in (1.6) led to the Filippov-Ważewski relaxation theorem ([3], p. 402).

An attempt of an extension of (1.6) is the fractional differential system $D^\alpha x = f(t, x, u(t))$ with the Riemann-Liouville fractional derivative. The interest for the concept of the fractional derivative was not only purely theoretical, because fractional derivatives describe solutions of fractional integral equations, many times arising from physics, as done by Abel in 1823 to solve the brachistochrone problem. Recently applications of fractional calculus arise in applied sciences such as acoustic wave propagation in inhomogeneous porous material and diffusive transport (see [1], pp. 4-5). The reader is also referred to [21] and [25] for further applications of fractional calculus such as biophysics, thermodynamics, polymer physics, viscoelasticity and control theory.

The classical relaxation theorem by Filippov and Ważewski is concerned with differential inclusions on a finite dimensional space (see [2], [9] and [30]). Briefly speaking, the convexified system $x' \in co(F(x))$ is compared with the system $y' \in F(y)$ equipped with the same initial condition (and $co(A)$ stands for the closed convex hull of the set A). Under suitable assumptions, such as the boundedness of the right-hand side, theorem says that for every $\varepsilon > 0$ and every function $x(t)$, a solution to the convexified problem, there is $y(t)$, a solution to $y' \in F(y)$, such that $\|x(t) - y(t)\|_{L^\infty} \leq \varepsilon$.

There exist generalizations of the classical relaxation theorem in a few directions: for instance Ioffe [19] extended that result for a certain class of differential inclusions of the form $x' \in F(t, x)$, where the values of the set-valued mapping $F : [0, 1] \times \mathbb{R}^n \mapsto 2^{\mathbb{R}^n}$ might be unbounded. The converse statement of the relaxation theorem is proved in [20] and infinite dimensional versions were given by Górniewicz et al. [7], Frankowska [10], Papageorgiou [24], Polovinkin [26] and Tolstogonov [29]. Fryszkowski and Bartuzel proposed relaxation-type theorems concerned with differential inclusions of second order and fourth order (see [4] and [5], resp. - the corresponding differential inclusions are $y'' - ay \in F(t, y)$ and $y^{(4)} - (a^2 + b^2) \cdot y^{(2)} + (ab)^2 \cdot y \in F(t, y)$). Another extension of the relaxation theorem (see [12], [13] and the references therein) is devoted to the parametrized Cauchy problem

$$\begin{cases} \quad u' \in F(t, u, s) \\ u(0) = \xi(s), \end{cases}$$

where $F : I \times X \times S \mapsto cl(X)$ is a closed-valued multifunction, S is a separable metric space, X stands for the Banach space and I is the time interval. There is also a result [10] on the relaxation theorem concerned with the Cauchy problem equipped with some state constraints, i.e.

$$\begin{cases} \quad x' \in F(t, x)\ , \quad x(0) = x_0\ , \\ x(t) \in \overline{\Theta}\ , \end{cases}$$

where the state constraint $\Theta \in \mathbb{R}^n$ is an open subset and the relation $x(t) \in \overline{\Theta}$ is meant to hold for every t in the domain of $x(\cdot)$.

Existence and uniqueness results for fractional differential equations may be found in [8] and [28]. An extension for a class of fractional differential inclusions is given in [15], [16], [22] and [23]. Fractional differential inclusions with the Riemann-Liouville derivative are considered in [22]; [15] and [16] are devoted to Riemann-Liouville fractional differential inclusions with a delay; [23] is concerned with a semilinear differential inclusion with the Caputo fractional derivative.

Comparing to [2], where the distance between the solution and the quasisolution is estimated by an exponential series, in [15], [16], [22], [23] a geometric series is considered thus the smallness of some parameters is essential to guarantee the convergence of the geometric series.

This paper is an attempt to extend some recent results:

- A different procedure for the construction of a quasisolution is proposed to avoid the application of a geometric series: the estimate for the distance between the solution and the quasisolution is based on an exponential - type series thus the smallness of some parameters is no longer required.
- Comparing to [15] and [22], nonzero initial data (i.e. $(I^{1-\alpha}y)(0) \neq 0$) is considered and some modifications of the method of successive approximations are needed.
- A method is proposed to omit and exclude condition $(H3)$ given in [16] so that the range $0 < \alpha < 1$ is permitted.

The structure of this text is the following: a general existence result concerning solutions to (1.1) is given in Theorem 1; Theorem 2 is concerned with the existence of a particular solution being sufficiently close to a given function $r(t)$; Theorem 3 is devoted to the relaxation property.

1.2 Some examples on the relaxation theorem

Comparing the differential inclusion $y'(t) \in \{\pm 1\}$ with the convexified problem $x'(t) \in [-1, 1]$ equipped with the initial condition $x(0) = y(0) = 0$ it is clear that $\tilde{x}(t) \equiv 0$ is a solution to the convexified problem only. Function $y(t) = \int_0^t v_\varepsilon(s)\, ds$ with

$$v_\varepsilon(s) = \begin{cases} +1, & s \in (0, \varepsilon) \cup (3\varepsilon, 5\varepsilon) \cup \ldots \\ -1, & s \in (\varepsilon, 3\varepsilon) \cup (5\varepsilon, 7\varepsilon) \cup \ldots \end{cases} \tag{1.7}$$

is a solution to $y'(t) \in \{\pm 1\}$ and we have $\|y(\cdot) - \tilde{x}(\cdot)\|_{L^\infty} < \varepsilon$.

Take for instance $D^{3/4}y(t) \in \{\pm 1\}$ and $D^{3/4}x(t) \in [-1, 1]$ equipped with the initial condition $(I^{1/4}y)(0) = (I^{1/4}x)(0) = 0$. Function $\tilde{x}(t) \equiv 0$ is a solution to the convexified problem only and a solution to problem $D^{3/4}y(t) \in \{\pm 1\}$ satisfying the condition $\|\tilde{x}(\cdot) - y(\cdot)\|_{L^\infty} < \varepsilon$ takes the form $y(t) = \frac{1}{\Gamma(3/4)} \int\limits_0^t \frac{v(s)\, ds}{(t-s)^{1/4}}$ where

$$v(s) = \begin{cases} \;\;1\,, & s \in (0, \varepsilon_1) \cup (\varepsilon_2, \varepsilon_3) \cup \ldots \\ -1\,, & s \in (\varepsilon_1, \varepsilon_2) \cup \ldots\,, \end{cases} \tag{1.8}$$

with $\varepsilon_1 = (\varepsilon \cdot \Gamma(7/4))^{4/3}$, $\varepsilon_2 = 2^{4/3}/(2^{4/3}-1) \cdot \varepsilon_1 \approx 1.658 \cdot \varepsilon_1, \varepsilon_3 \approx 2.53 \cdot \varepsilon_1, \ldots .$

Remarks on the notation. In what follows, $\mathbb{R}^n$ is the n dimensional Euclidean space; $\Gamma(t)$ is the Gamma function and $B(a, b)$ is the Beta function; $d_H(A, B)$ is the Hausdorff distance between sets $A, B \subset \mathbb{R}^n$ and $d(x, A) = \inf\{\|x - a\| : a \in A\}$ is the distance between a point $x \in \mathbb{R}^n$ and a nonempty set $A \subset \mathbb{R}^n$; $\mathbb{B}(a, r) = \{y : \|y - a\| < r\}$ is the open ball in $\mathbb{R}^n$; $I^{1-\alpha}(g)(t)$ is the Riemann-Liouville fractional integral of order $1-\alpha$ given in (1.4). A set-valued mapping $F : \mathbb{R}^n \mapsto 2^{\mathbb{R}^n}$ is Lipschitz continuous with respect to the Hausdorff distance provided there is a constant k such that for every $x_1, x_2 \in \mathbb{R}^n$ is $d_H(F(x_1), F(x_2)) \le k \cdot \|x_1 - x_2\|$.

2 Existence of solutions

One of the possible way to achieve the task to derive the solutions to (1.1) is to employ fixed point theorems for multivalued mappings. This in turn requires the multivalued Niemytskij operator $K_0 : L^p([0, T], \mathbb{R}^n) \rightsquigarrow cl L^p([0, T], \mathbb{R}^n)$ defined by

$$K_0(u) = \Big\{ w \in L^p([0, T], \mathbb{R}^n) : \\ w(t) \in F\left(t, \frac{(I^{1-\alpha}y)(0)}{\Gamma(\alpha)} \cdot t^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{u(s)ds}{(t-s)^{1-\alpha}}\right)\Big\}. \tag{2.1}$$

In order to derive the existence of solutions on $[0, T]$ for arbitrary $T > 0$ we shall demonstrate that there is a Bielecki norm equivalent to the norm L_p such that $K_0(\cdot)$ is a contraction. The following hypothesis on multifunction $F : [0, T] \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ shall be assumed:

Hypothesis (H). *Multifunction F is nonempty compact valued; for each x mapping $F(\cdot, x)$ is measurable. There is a function $k(\cdot) \in L^p([0, T], \mathbb{R})$ such that mapping $F(t, \cdot)$ is $k(t)$-Lipschitz continuous with respect to the Hausdorff distance. There is a mapping $a(\cdot) \in L^p([0, T], \mathbb{R}_+)$ such that for eachx the estimate $\sup\{\|u\| : u \in F(t, x)\} \le a(t)$ holds a.e..*

Theorem 1 *Let $F : [0, T] \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ be a multifunction such that Hypothesis (H) is satisfied. Suppose that $\alpha p > 1$. Then there is a solution to the fractional differential inclusion (1.1) on $[0, T]$.*

Proof Let function $\phi : [0, T] \mapsto \mathbb{R}$ be given by the formula

$$\phi(t) = \exp\left(-2\int_0^t A(s)\, ds\right),$$

where $A(t) = t^{\alpha p-1} \cdot |k(t)|^p \cdot (\Gamma(\alpha))^{-p} \cdot ((\alpha - 1) \cdot \frac{p}{p-1} + 1)^{1-p}$. Then there are constants $0 < c_1 < c_2$ depending on T such that $c_1 < \phi(t) < c_2$ and the norm $\|f\|_{L_p([0,T],d\mu)} = (\int_0^T |f(s)|^p \phi(s)\, ds)^{1/p}$ is equivalent to the standard L_p-norm on $[0, T]$.

Fix $u_1, u_2 \in L^p([0,T], \mathbb{R}^n)$ and $w_1 \in K_0(u_1)$. Since $t \mapsto \frac{(I^{1-\alpha}y)(0)}{\Gamma(\alpha)} \cdot t^{\alpha-1} + I^{\alpha}(u_2)(t)$ is a measurable mapping then the composition

$$t \mapsto F\left(t, \frac{(I^{1-\alpha}y)(0)}{\Gamma(\alpha)} \cdot t^{\alpha-1} + I^{\alpha}(u_2)(t)\right)$$

is a measurable multifunction ([3], Thm. 8.2.8, p. 314). Then, according to [3] (Cor. 8.2.13, p. 317) there is a measurable selection $w_2 : [0,T] \mapsto \mathbb{R}^n$ such that

$$w_2(t) \in F\left(t, \frac{(I^{1-\alpha}y)(0)}{\Gamma(\alpha)} \cdot t^{\alpha-1} + I^{\alpha}(u_2)(t)\right) \tag{2.2}$$

for a.e. $t \in [0,T]$ and

$$\begin{aligned}
&|w_1(t) - w_2(t)| = \\
&= d\left(w_1(t), F\left(t, \frac{(I^{1-\alpha}y)(0)}{\Gamma(\alpha)} \cdot t^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{u_2(\tau)d\tau}{(t-\tau)^{1-\alpha}}\right)\right) \\
&\equiv d(w_1(t)\ ,\ P(t))\ .
\end{aligned}$$

It follows from (2.1) and (2.2) that $w_2 \in K_0(u_2)$. Then, employing the assumption that $w_1 \in K_0(u_1)$ and by the $k(t)$-Lipschitz continuity of $F(t,\cdot)$ for a.e. $t \in [0,T]$ we obtain the inequality

$$\begin{aligned}
&|w_1(t) - w_2(t)| = d(w_1(t), P(t)) \leq \\
&\leq d_H\left(F\left(t, \frac{(I^{1-\alpha}y)(0)}{\Gamma(\alpha)} \cdot t^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{u_1(s)\,ds}{(t-s)^{1-\alpha}}\right), P(t)\right) \\
&\leq k(t) \cdot \left|\int_0^t \frac{u_1(s)}{\Gamma(\alpha)(t-s)^{1-\alpha}}\,ds - \int_0^t \frac{u_2(s)}{\Gamma(\alpha)(t-s)^{1-\alpha}}\,ds\right|.
\end{aligned} \tag{2.3}$$

Keeping in mind that function ϕ is a solution to the differential equation $\phi'(t) = -2A(t) \cdot \phi(t)$, following (2.3) and employing the Hölder inequality we obtain

$$\begin{aligned}
\|w_1 - w_2\|^p_{L^p([0,T],d\mu)} &\leq \int_0^T \frac{|k(t)|^p \cdot \phi(t)}{|\Gamma(\alpha)|^p} \cdot \left|\int_0^t \frac{|u_1(s) - u_2(s)|}{(t-s)^{1-\alpha}}\,ds\right|^p dt \\
&\leq \int_0^T \left(A(t)\phi(t) \cdot \int_0^t |u_1(s) - u_2(s)|^p\,ds\right) dt \\
&= -\frac{1}{2}\int_0^T \phi'(t)\int_0^t |u_1(s) - u_2(s)|^p\,ds\,dt\ .
\end{aligned} \tag{2.4}$$

Then the integration of the r.h.s. of (2.4) by parts implies that

$$\|w_1 - w_2\|^p_{L^p([0,T],d\mu)} \leq \frac{1}{2} \cdot \|u_1 - u_2\|^p_{L^p([0,T],d\mu)}\,, \tag{2.5}$$

since we keep in mind the sign of the boundary terms. Estimate (2.5) implies that operator $K_0(\cdot)$ is a contraction, hence the Covitz-Nadler Jr. Theorem ([18], Thm. 1.11, p. 524) yields the existence of a fixed point $u_0 \in K_0(u_0)$. □

2.1 The method of successive approximations by Filippov

In comparison to the previous section concerned with a general existence result now we shall prove the existence of a certain solution $y(\cdot)$ such that $y(\cdot)$ is sufficiently close to a given function $r(\cdot)$. Conditions imposed on function $r(t)$ are described below:

Hypothesis (H2). *Let be given functionr* $: (0, T] \mapsto \mathbb{R}^n$ *of the form*

$$r(t) = (I^{1-\alpha} r)(0) \cdot t^{\alpha-1}/\Gamma(\alpha) + I^{\alpha}(v_0)(t) \tag{2.6}$$

with $v_0 \in L^p([0, T])$, $p > 1$. *Suppose that there is a function* $m(t) \in L^p([0, T])$ *such that for a.e.* $t \in [0, T]$ *there is* $d(v_0(t), F(t, r(t))) \leq m(t)$. Let $\xi = \frac{\alpha p-1}{p-1}$, *suppose that* $k \in L^p([0, T])$ *and let sequences* $\{A_j(t)\}_{j\geq 1}$ and $\{C_j(t)\}_{j\geq 0}$ *be given by formulas*

$$\begin{aligned} A_j(t) &= \frac{\left(\|k\|_{L^p([0,t])}\right)^{j-1}}{\left(\Gamma(\alpha)\right)^j} \cdot \left[\frac{(\Gamma(\xi))^j}{\Gamma(j\xi+1)}\right]^{1-\frac{1}{p}}\,, \\ C_j(t) &= \left(\frac{\|k\|_{L^p([0,t])}}{\Gamma(\alpha)}\right)^j \cdot \left[\frac{(\Gamma(\xi))^{j+1}}{\Gamma((j+1)\xi)}\right]^{1-\frac{1}{p}}. \end{aligned} \tag{2.7}$$

Keeping in mind(1.1) *and* (2.6) *we introduce symbols* $Y_0 = (I^{1-\alpha} y)(0)/\Gamma(\alpha)$ and $R_0 = (I^{1-\alpha} r)(0)/\Gamma(\alpha)$ *to simplify the notation.*

Theorem 2 *Let multifunction* $F : [0, T] \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ *fulfill Hypothesis* (H). *Assume that* $\alpha p > 1$. *Let be given a function* $r(t)$ *satisfying* $(H2)$. *Then there is a solution* $y(t)$ *to (1.1) such that for* $t \in (0, T]$ *is*

$$\begin{aligned} \|y(t) - r(t)\| \leq& \|m\|_{L^p([0,t])} \cdot \left(\sum_{n=1}^{\infty} A_n(t) \cdot t^{n\cdot(\alpha-\frac{1}{p})}\right) \\ &+ \|Y_0 - R_0\| \cdot \left(\sum_{n=0}^{\infty} C_n(t) \cdot t^{n(\alpha-\frac{1}{p})+\alpha-1}\right) \end{aligned} \tag{2.8}$$

and, for a.e. $t \in (0, T]$, *we have*

$$\|D^{\alpha} y(t) - D^{\alpha} r(t)\| \leq m(t)$$

$$+k(t)\cdot\left(\|m\|_{L^p([0,t])}\cdot\sum_{n=1}^{\infty}A_n(t)\cdot t^{n\cdot(\alpha-\frac{1}{p})}\right.$$
$$\left.+\|R_0-Y_0\|\cdot\sum_{n=0}^{\infty}C_n(t)\cdot t^{n\cdot(\alpha-\frac{1}{p})+\alpha-1}\right) \tag{2.9}$$

with $\{A_n\}$, $\{C_n\}$ *are given in (2.7).*

Proof The aim is to construct sequences $\{y_n\}$ and $\{v_n\}$. Functions $y=\lim y_n$ and $D^\alpha y=\lim v_n$ shall satisfy (2.8) and (2.9).

Let $y_0(t)=Y_0\cdot t^{\alpha-1}+I^\alpha(v_0)(t)$. Then the composition $t\mapsto F(t,y_0(t))$ is measurable ([3], Thm. 8.2.8, p. 314) and there is a measurable function $v_1(t):[0,T]\mapsto\mathbb{R}^n$ such that $v_1(t)\in F(t,y_0(t))$ and $\|v_1(t)-v_0(t)\|=d(v_0(t),F(t,y_0(t)))$ a.e. (see [3], Cor. 8.2.13). Hence by the triangle inequality for a.e. $t\in[0,T]$ we have

$$\begin{aligned}\|v_1(t)-v_0(t)\|&\le d(v_0(t),F(t,r(t)))+d_H(F(t,r(t)),F(t,y_0(t)))\\&\le m(t)+k(t)\cdot t^{\alpha-1}\cdot\|R_0-Y_0\|\ .\end{aligned} \tag{2.10}$$

Let us define function $y_1(t)=Y_0\cdot t^{\alpha-1}+I^\alpha(v_1)(t)$. Then $y_1(\cdot)$ is well defined since the fractional integral $I^\alpha(v_1)(t)$ converges because $\alpha p>1$ and $\sup_t\|I^\alpha(v_1)(t)\|\le c(\alpha,p,T)\cdot\|v_1\|_{L^p(0,T)}$. For completeness, $v_1\in L^p$ because $v_1(t)\in F(t,y_0(t))$ a.e. and the set-valued mapping F fulfills hypothesis (H), i.e. there is a function $a(\cdot)\in L^p$ such that for each x we have $\sup\{\|u\|:u\in F(t,x)\}\le a(t)$ a.e..

Employing (2.10) we derive an estimate between $y_0(\cdot)$ and $y_1(\cdot)$:

$$\begin{aligned}\|y_1(t)-y_0(t)\|&\le\frac{1}{\Gamma(\alpha)}\int_0^t\frac{m(s)+k(s)\cdot s^{\alpha-1}\cdot\|R_0-Y_0\|}{(t-s)^{1-\alpha}}\,ds\\&\le\frac{\|m\|_{L^p([0,t])}}{\Gamma(\alpha)}\cdot\left[\int_0^t(t-s)^{(\alpha-1)\frac{p}{p-1}}\,ds\right]^{1-\frac{1}{p}}\\&\quad+\frac{\|k\|_{L^p([0,t])}}{\Gamma(\alpha)}\cdot\|R_0-Y_0\|\\&\quad\cdot\left[\int_0^t\left[(t-s)^{(\alpha-1)\cdot\frac{p}{p-1}}\cdot s^{(\alpha-1)\cdot\frac{p}{p-1}}\right]ds\right]^{1-\frac{1}{p}}\ .\end{aligned} \tag{2.11}$$

Changing variable in (2.11): $s=t\tau$, $ds=td\tau$, leads to an estimate in terms of the Beta function depending on parameter $\xi=\frac{\alpha p-1}{p-1}$:

$$\begin{aligned}&\|y_1(t)-y_0(t)\|\\&\le\|m\|_{L^p([0,t])}\cdot\frac{[B(\xi,1)]^{1-\frac{1}{p}}}{\Gamma(\alpha)}\cdot t^{\alpha-\frac{1}{p}}\\&\quad+\|R_0-Y_0\|\cdot\frac{\|k\|_{L^p([0,t])}}{\Gamma(\alpha)}\cdot[B(\xi,\xi)]^{1-\frac{1}{p}}\cdot t^{2\alpha-1-\frac{1}{p}}\\&\equiv\|m\|_{L^p([0,t])}\cdot A_1(t)\cdot t^{\alpha-\frac{1}{p}}+\|R_0-Y_0\|\cdot C_1(t)\cdot t^{2\alpha-1-\frac{1}{p}}\end{aligned} \tag{2.12}$$

with $\{A_j\}$, $\{C_j\}$ given in (2.7). Proceeding step by step, suppose that functions $\{y_0, y_1, ..., y_n\}$ are already constructed and

$$y_j(t) = Y_0 \cdot t^{\alpha-1} + I^\alpha(v_j)(t), \quad v_j(t) \in F(t, y_{j-1}(t)) \text{ a.e. in } [0,T], \; j = 1, .., n,$$

as well as

$$\begin{aligned} &\|y_j(t) - y_{j-1}(t)\| \\ &\le \|m\|_{L^p([0,t])} \cdot A_j(t) \cdot t^{j\cdot(\alpha-\frac{1}{p})} + \|R_0 - Y_0\| \cdot C_j(t) \cdot t^{j\cdot(\alpha-\frac{1}{p})+\alpha-1}. \end{aligned} \tag{2.13}$$

The inductive step is to construct function y_{n+1}. Since $y_n(t) = Y_0 \cdot t^{\alpha-1} + I^\alpha(v_n)(t)$ fulfills $v_n(t) \in F(t, y_{n-1}(t))$ a.e. then there is a measurable function v_{n+1} such that

$$\begin{aligned} \|v_{n+1}(t) - v_n(t)\| &= d(v_n(t), F(t, y_n(t))) \\ &\le d_H(F(t, y_n(t)), F(t, y_{n-1}(t))) \le k(t) \cdot \|y_n(t) - y_{n-1}(t)\| \quad \text{a.e.,} \end{aligned} \tag{2.14}$$

hence we define $y_{n+1}(t) = Y_0 \cdot t^{\alpha-1} + I^\alpha(v_{n+1})(t)$ and compute

$$\begin{aligned} &\|y_{n+1}(t) - y_n(t)\| \\ &\le \int_0^t \frac{\|v_{n+1}(s) - v_n(s)\|}{\Gamma(\alpha)(t-s)^{1-\alpha}}\, ds \le \int_0^t \frac{k(s) \cdot \|y_n(s) - y_{n-1}(s)\|}{\Gamma(\alpha)(t-s)^{1-\alpha}}\, ds \\ &\le \frac{1}{\Gamma(\alpha)} \int_0^t \frac{k(s)}{(t-s)^{1-\alpha}} \Big[\|m\|_{L^p([0,s])} \cdot A_n(s) \cdot s^{n(\alpha-\frac{1}{p})} \\ &\qquad + \|R_0 - Y_0\| \cdot C_n(s) \cdot s^{n(\alpha-\frac{1}{p})+\alpha-1}\Big] ds \\ &\le \frac{\|k\|_{L^p([0,t])}}{\Gamma(\alpha)} \cdot \Bigg(\|m\|_{L^p([0,t])} \cdot A_n(t) \cdot \Big[\int_0^t (t-s)^{(\alpha-1)\frac{p}{p-1}} \cdot s^{n\cdot(\alpha-\frac{1}{p})\cdot\frac{p}{p-1}} ds\Big]^{1-\frac{1}{p}} \\ &\quad + \|R_0 - Y_0\| \cdot C_n(t) \cdot \Big[\int_0^t \Big[(t-s)^{(\alpha-1)\frac{p}{p-1}} \cdot s^{(n\cdot(\alpha-\frac{1}{p})+\alpha-1)\cdot\frac{p}{p-1}}\Big] ds\Big]^{1-\frac{1}{p}}\Bigg). \end{aligned}$$

Changing variables $s = t\tau$, $ds = t d\tau$ and recalling parameter ξ from Hypothesis (H2) we transform the above inequality into an estimate in terms of the Beta function:

$$\begin{aligned} &\|y_{n+1}(t) - y_n(t)\| \\ &\le \frac{\|k\|_{L^p([0,t])}}{\Gamma(\alpha)} \cdot \Bigg(\|m\|_{L^p([0,t])} \cdot A_n(t) \cdot \Big[B\Big(\xi, n\xi + 1\Big)\Big]^{1-\frac{1}{p}} \cdot t^{(n+1)(\alpha-\frac{1}{p})} \\ &\quad + \|R_0 - Y_0\| \cdot C_n(t) \cdot \Big[B\Big(\xi, (n+1)\xi\Big)\Big]^{1-\frac{1}{p}} \cdot t^{(n+1)(\alpha-\frac{1}{p})+\alpha-1}\Bigg). \end{aligned} \tag{2.15}$$

But (2.7) and the identity $B(a,b) = (\Gamma(a)\Gamma(b))/\Gamma(a+b)$ imply that

$$A_n \cdot \frac{\|k\|_{L^p([0,t])}}{\Gamma(\alpha)} \cdot \Big[B(\xi, n\xi + 1)\Big]^{1-\frac{1}{p}} \equiv A_{n+1}$$

and

$$C_n \cdot \frac{\|k\|_{L^p([0,t])}}{\Gamma(\alpha)} \cdot \Big[B\Big(\xi, (n+1)\xi\Big)\Big]^{1-\frac{1}{p}} \equiv C_{n+1}\ ,$$

therefore (2.15) takes the form

$$\begin{aligned} \|y_{n+1}(t) - y_n(t)\| &\le \|m\|_{L^p([0,t])} \cdot A_{n+1}(t) \cdot t^{(n+1)(\alpha - \frac{1}{p})} \\ &\quad + \|R_0 - Y_0\| \cdot C_{n+1}(t) \cdot t^{(n+1)(\alpha-\frac{1}{p})+\alpha-1}. \end{aligned} \tag{2.16}$$

Since $\alpha p > 1$ then the sign of the exponents in (2.16) is the following: $(n+1)\cdot(\alpha - \frac{1}{p}) \ge 0$ for $n = 0, 1, 2, ...$ and $(n+1)\cdot(\alpha - \frac{1}{p}) + \alpha - 1 \ge 0$ provided $n \ge \max\{0, (1 + \frac{1}{p} - 2\alpha)/(\alpha - \frac{1}{p})\}$. Therefore if $\alpha \in [\frac{1}{2} + \frac{1}{2p}, 1) \subset (\frac{1}{p}, 1)$ then $n \ge 0$ and the range $\alpha \in (\frac{1}{p}, \frac{1}{2} + \frac{1}{2p}) \subset (\frac{1}{p}, 1)$ implies that $n \ge (1 + \frac{1}{p} - 2\alpha)/(\alpha - \frac{1}{p})$, i.e. the exponents in (2.16) are positive for each n large enough.

Let N_0 be the smallest positive integer greater than $(1 + \frac{1}{p} - 2\alpha)/(\alpha - \frac{1}{p})$. Then, keeping in mind estimate (2.16), we shall prove that the series

$$\sum_{n=N_0}^{\infty} \|y_{n+1}(t) - y_n(t)\| \tag{2.17}$$

converges uniformly. This will be done by applying the Weierstrass M-test combined with the ratio test and the Gautschi's inequality [14] on the asympthotic behaviour of the Gamma function.

If follows from the definition in (2.7) that the mappings $t \mapsto A_{n+1}(t)$ and $t \mapsto C_{n+1}(t)$ are nondecreasing, hence (keeping in mind (2.16)) we obtain

$$\begin{aligned} &\sum_{n=N_0}^{\infty} \sup_t \|y_{n+1}(t) - y_n(t)\| \\ &\le \|m\|_{L^p([0,T])} \sum_{n=N_0}^{\infty} A_{n+1}(T) \cdot T^{(n+1)(\alpha-\frac{1}{p})} \\ &\quad + \|R_0 - Y_0\| \sum_{n=N_0}^{\infty} C_{n+1}(T) \cdot T^{(n+1)(\alpha-\frac{1}{p})+\alpha-1}. \end{aligned} \tag{2.18}$$

Employing the ratio test to the r.h.s. of (2.18) and recalling (2.7) we get

$$\begin{aligned}
&\lim_{n\to\infty}\frac{A_{n+2}(T)}{A_{n+1}(T)}\cdot\frac{T^{(n+2)(\alpha-\frac{1}{p})}}{T^{(n+1)(\alpha-\frac{1}{p})}}\\
&=\frac{T^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)}\cdot\|k\|_{L^p([0,T])}\cdot\lim_{n\to\infty}\left(\frac{(\Gamma(\xi))^{n+2}}{\Gamma((n+2)\xi+1)}\cdot\frac{\Gamma((n+1)\xi+1)}{(\Gamma(\xi))^{n+1}}\right)^{1-\frac{1}{p}} \qquad (2.19)\\
&=\frac{T^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)}\cdot\|k\|_{L^p([0,T])}\cdot(\Gamma(\xi))^{1-1/p}\cdot\lim_{n\to\infty}\left(\frac{\Gamma\Big((n+1)\xi+1\Big)}{\Gamma\Big((n+2)\xi+1\Big)}\right)^{1-\frac{1}{p}}\equiv I_1
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{n\to\infty}\frac{C_{n+2}(T)}{C_{n+1}(T)}\cdot T^{\alpha-\frac{1}{p}}\\
&=T^{\alpha-\frac{1}{p}}\cdot\frac{\|k\|_{L^p([0,T])}}{\Gamma(\alpha)}\cdot\lim_{n\to\infty}\left(\frac{(\Gamma(\xi))^{n+3}}{\Gamma((n+3)\xi)}\cdot\frac{\Gamma((n+2)\xi)}{(\Gamma(\xi))^{n+2}}\right)^{1-\frac{1}{p}} \qquad (2.20)\\
&=T^{\alpha-\frac{1}{p}}\cdot\frac{\|k\|_{L^p([0,T])}}{\Gamma(\alpha)}\cdot(\Gamma(\xi))^{1-\frac{1}{p}}\cdot\lim_{n\to\infty}\left(\frac{\Gamma\Big((n+2)\xi\Big)}{\Gamma\Big((n+3)\xi\Big)}\right)^{1-\frac{1}{p}}\equiv I_2
\end{aligned}$$

respectively.

In order to compute limits (2.19) and (2.20) we recall the Gautschi's inequality [14] on the asympthotic behaviour of the Gamma function, i.e.

$$\lim_{x\to\infty}\frac{\Gamma(x+s)\cdot x^{1-s}}{\Gamma(x+1)}=1 \qquad (2.21)$$

for $0<s<1$.

Suppose that $x=(n+2)\xi$ and $s=1-\xi\in(0,1)$ in (2.21). Then (2.21) implies that the limit (2.19) takes the form

$$\begin{aligned}
I_1&=c\cdot\lim_{n\to\infty}\left(\frac{\Gamma\Big((n+1)\xi+1\Big)\cdot\Big((n+2)\xi\Big)^{1-s}}{\Gamma\Big((n+2)\xi+1\Big)}\cdot\frac{1}{((n+2)\xi)^{1-s}}\right)^{1-\frac{1}{p}}\\
&=c\cdot\xi^{-\alpha+1/p}\cdot\lim_{n\to\infty}\left(\frac{1}{n+2}\right)^{\alpha-1/p}=0.
\end{aligned}$$

Similarly, the substitution $x=(n+3)\xi-1$ and $s=1-\xi$ in (2.21) implies that the limit (2.20) takes the form:

$$I_2 = c \cdot \lim_{n\to\infty} \left(\frac{1}{\left((n+3)\xi - 1\right)^{\xi}} \right)^{1-1/p} = c \cdot \lim_{n\to\infty} \frac{1}{\left((n+3)\xi - 1\right)^{\alpha - \frac{1}{p}}} = 0.$$

The latter implies that the series (2.17) converges uniformly. Since $\{y_n\}$ is a Cauchy sequence then there is a limit $y = \lim y_n$.

Observe that $\|v_{n+1}(t) - v_n(t)\| \leq k(t) \cdot \|y_n(t) - y_{n-1}(t)\|$ a.e. for $n \geq 1$, therefore $\{v_n\}$ is a Cauchy sequence in L^p and

$$\begin{aligned}
&\sum_{n=N_0+1}^{\infty} \|v_{n+1} - v_n\|_{L_p([0,T])} \leq \\
&\leq \|k\|_{L^p([0,T])} \cdot \left(\|m\|_{L^p([0,T])} \cdot \sum_{n=N_0}^{\infty} A_n(T) \cdot T^{n(\alpha - \frac{1}{p})} + \right. \\
&\left. + \|R_0 - Y_0\| \cdot \sum_{n=N_0}^{\infty} C_n(T) \cdot T^{n(\alpha - \frac{1}{p}) + \alpha - 1} \right).
\end{aligned}$$

Denote $v = \lim v_n$ in L^p and recall that $\alpha - \frac{1}{p} > 0$. Then by the Hölder inequality we get the convergence of the fractional integrals, i.e.

$$\lim_{n\to\infty} \sup_t \|I^\alpha(v_n)(t) - I^\alpha(v)(t)\| \leq c \cdot \lim_n \|v_n - v\|_{L^p([0,T])} = 0.$$

Hence, in particular, the limit of the approximate solutions $y_n(t) = Y_0 \cdot t^{\alpha-1} + I^\alpha(v_n)(t)$ is the function $y(t) = Y_0 \cdot t^{\alpha-1} + I^\alpha(v)(t)$.

We shall prove that $v(s) \in F(s, y(s))$ a.e.. Since $v_n \to v$ in L^p then there is a subsequence (still denoted as v_n) converging pointwise almost everywhere to v (Rudin [27], Thm. 3.12). Suppose that s is such that $v_n(s) \to v(s)$. Then by the triangle inequality and by the Lipschitz continuity of the set-valued mapping $F(t, \cdot)$ we obtain an estimate for the distance between $v(s)$ and $F(s, y(s))$:

$$\begin{aligned}
&d(v(s), F(s, y(s))) \\
&\leq \|v(s) - v_n(s)\| + d(v_n(s), F(s, y_{n-1}(s))) + d_H(F(s, y_{n-1}(s)), F(s, y(s))) \\
&\leq \|v(s) - v_n(s)\| + 0 + k(s) \cdot \|y_{n-1}(s) - y(s)\|.
\end{aligned}$$

If $n \to \infty$ then $d(v(s), F(s, y(s))) = 0$ almost everywhere. Since the multifunction F is supposed to be closed valued then $v(s) \in F(s, y(s))$ a.e..

Combining the above considerations we obtain estimates for $\|y(t) - r(t)\|$ and $\|D^\alpha y(t) - D^\alpha r(t)\|$ given in (2.8) and (2.9), respectively. Employing (2.16) implies that for arbitrary positive integer N there is an inequality:

$$
\begin{aligned}
&\|r(t)-y(t)\| \\
&\leq \|r(t)-y_0(t)\| + \left(\sum_{n=0}^{N} \|y_{n+1}(t)-y_n(t)\|\right) + \|y(t)-y_{N+1}(t)\| \\
&\leq \|R_0-Y_0\| \cdot t^{\alpha-1} + \sum_{n=0}^{N} \Big(\|m\|_{L^p(0,t)} \cdot A_{n+1}(t) \cdot t^{(n+1)(\alpha-\frac{1}{p})} \\
&\quad + \|R_0-Y_0\| \cdot C_{n+1}(t) \cdot t^{(n+1)(\alpha-\frac{1}{p})+\alpha-1} \Big) + \|y(t)-y_{N+1}(t)\|.
\end{aligned}
\tag{2.22}
$$

But $C_0(t) \equiv 1$ (by the definition (2.7)) and $\sup_t \|y_{N+1}(t)-y(t)\| \to 0$ as $N \to \infty$ therefore (2.22) takes the form given in (2.8).

Let $\{v_n\}$ be the sequence convergent to v in L^p and suppose that $\{v_{N_k}\}$ is the subsequence convergent to v pointwise almost everywhere. Assume that t is such that $v_{N_k}(t) \to v(t)$. Then for $N_1 < N_2 < \ldots$ there is an estimate between $D^\alpha y(t)$ and $D^\alpha r(t)$:

$$
\begin{aligned}
&\|D^\alpha y(t) - D^\alpha r(t)\| \leq \|D^\alpha r(t) - D^\alpha y_0(t)\| + \|D^\alpha y_{N_k}(t) - D^\alpha y(t)\| \\
&\quad + \sum_{j=0}^{N_k-1} \|D^\alpha y_{j+1}(t) - D^\alpha y_j(t)\| \\
&\equiv 0 + \|v_{N_k}(t) - v(t)\| + \|v_1(t)-v_0(t)\| + \sum_{j=1}^{N_k-1} \|v_{j+1}(t)-v_j(t)\| \\
&\leq \|v_1(t)-v_0(t)\| + \|v_{N_k}(t)-v(t)\| + k(t) \cdot \sum_{j=1}^{N_k-1} \|y_j(t)-y_{j-1}(t)\|.
\end{aligned}
\tag{2.23}
$$

The application of (2.10) and (2.13) to (2.23) and letting $N_k \to \infty$ yields estimate (2.9). This ends the proof of Theorem 2. □

3 Relaxation property

Following [2], below we would like to apply the Filippov-type approximations and propose an attempt of an extension of the relaxation property for a certain class of fractional differential inclusions. Recall that in Theorem 2 the conditions imposed on (α, p) are: $p > 1$ and $\alpha = \alpha(p) \in (\frac{1}{p}, 1)$. Below in Theorem 3 it is supposed that $p = +\infty$, therefore the range of α is $\alpha \in (0, 1)$.

Theorem 3 *Let $F : \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ be a set-valued mapping such that F bounded, nonempty compact-valued and Lipschitzean with constant k. Suppose that $\alpha \in (0,1)$ and let $x(\cdot)$ be a solution to problem*

$$
D^\alpha x(t) \in co(F(x(t)))\,, \quad (I^{1-\alpha}x)(0) = \xi_0. \tag{3.1}
$$

Then for every $\varepsilon > 0$ *there is* $y(\cdot)$*, a solution to problem*

$$D^{\alpha} y(t) \in F(y(t)) , \quad (I^{1-\alpha} y)(0) = \xi_0 \tag{3.2}$$

such that for $t \in [0, T]$*,* $\|y(t) - x(t)\| \leq \varepsilon$*.*

Proof Let $\varepsilon > 0$ be fixed and suppose that a function $x(t)$, a solution to (3.1), is given. Then the idea of the proof is to construct a sufficiently regular function $r(t)$ such that $\|r(\cdot) - x(\cdot)\|_{L_\infty} \leq \varepsilon/2$ and the distance between $D^{\alpha} r(t)$ and $F(r(t))$ shall be small enough, i.e. $d\big(D^{\alpha} r(t), F(r(t))\big) \leq m(t)$ a.e. for a function $m(\cdot) \in L^p$ satisfying

$$\|m\|_{L^p([0,T])} \leq \varepsilon / \Big(2 \cdot \Big\{ \sum_{n=1}^{\infty} A_n(T) \cdot T^{n(\alpha - \frac{1}{p})} \Big\}\Big) \equiv \varepsilon \cdot \overline{c}, \tag{3.3}$$

where $A_n(\cdot)$ is given in (2.7). If the desired function $r(t)$ exists then Theorem 2 implies the existence of a function $y(\cdot)$, a solution to (3.2), such that $\|y(t) - r(t)\| < \varepsilon/2$ for each $t > 0$ (in other words we shall apply (2.8) with $Y_0 = R_0$ and $m(t)$ as above). Combining these estimates with the triangle inequality we shall get $\|x(t) - y(t)\| < \varepsilon$.

In what follows function $r(\cdot)$ is constructed: the formula for $r(t)$ is given in (3.13) and further considerations are concerned with estimates justifying that function $r(t)$ is sufficiently close to the given trajectory $x(t)$.

Suppose that $M = \sup\{\|u\| : u \in F(x),\ x \in \mathbb{R}^n\}$ is the upper bound of multifunction F. Let be given the trajectory

$$x(t) = \frac{\xi_0}{\Gamma(\alpha)} \cdot t^{\alpha - 1} + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{v(s)\, ds}{(t-s)^{1-\alpha}} , \tag{3.4}$$

a solution to the convexified problem (3.1) and let $0 < \varepsilon < 1$ be fixed. Suppose that

$$\delta_1 = (\varepsilon \cdot \Gamma(\alpha + 1)/(8M))^{1/\alpha}$$

and let $y(t) = (\xi_0 \cdot t^{\alpha-1})/(\Gamma(\alpha)) + I^{\alpha}(u)(t)$ be a solution to (3.2). Take $r(t) = y(t)$ for $t < \delta_1$. Then for $t < \delta_1$ we have

$$\|r(t) - x(t)\| = \|(I^{\alpha} u)(t) - (I^{\alpha} v)(t)\| \leq \varepsilon/4.$$

Observe that function $[\delta_1, T] \ni t \to x(t)$ is Hölder continuous with exponent α and constant

$$\begin{aligned} c_1 &= \frac{3M}{\Gamma(\alpha+1)} + \frac{2T\|\xi_0\|}{\Gamma(\alpha) \cdot \delta_1^2} \\ &= \frac{3M}{\Gamma(\alpha+1)} + \frac{2T\|\xi_0\|}{\Gamma(\alpha)} \cdot \left(\frac{8M}{\Gamma(\alpha+1)}\right)^{2/\alpha} \cdot \varepsilon^{-2/\alpha} \\ &\leq \left(\frac{3M}{\Gamma(\alpha+1)} + \frac{2T\|\xi_0\|}{\Gamma(\alpha)} \cdot \left(\frac{8M}{\Gamma(\alpha+1)}\right)^{2/\alpha}\right) \cdot \varepsilon^{-2/\alpha} \end{aligned}$$

$$\equiv c_0 \cdot \varepsilon^{-2/\alpha}, \tag{3.5}$$

i.e. $\|x(t) - x(\tau)\| \leq c_1 \cdot |t - \tau|^\alpha$ for every $t, \tau \in [\delta_1, T]$. The Hölder continuity follows straightforward from (3.4) and from the boundedness of set-valued mapping F. Hence, in particular, multifunction $[\delta_1, T] \ni t \mapsto coF(x(t))$ is Hölder continuous with respect to the Hausdorff metric with exponent α and constant $c_2 = kc_1$, where k is the Lipschitz constant of multifunction $x \mapsto F(x)$.

Take a positive integer number $N > \varepsilon^{-3/\alpha^2}$ and consider the partition of the interval $[\delta_1, T]$ into subintervals of a sufficiently small length, i.e. let $[\delta_1, T] = \bigcup_{i=0}^{N-1} [t_i, t_{i+1}]$ with $|t_{i+1} - t_i| < T/N < T\varepsilon^{3/\alpha^2}$. Then, keeping in mind the Hölder continuity with respect to the Hausdorff metric of the multifunction $t \mapsto coF(x(t))$, for a.e. $t \in [t_i, t_{i+1}]$ we derive an estimate

$$\begin{aligned} d_H(coF(x(t)), coF(x(t_i))) &\leq k \cdot \|x(t) - x(t_i)\| \leq kc_1 \cdot |t - t_i|^\alpha \\ &\leq kc_0 \cdot \varepsilon^{-2/\alpha} \cdot \left(T\varepsilon^{3/\alpha^2}\right)^\alpha \equiv c_3 \cdot \varepsilon^{1/\alpha}, \end{aligned} \tag{3.6}$$

in other words,

$$D^\alpha x(t) \in coF(x(t)) \subset coF(x(t_i)) + \mathbb{B}\big(0, c_3 \cdot \varepsilon^{1/\alpha}\big).$$

Consider the partition of the set $\{coF(x(t)) : t \in [t_i, t_{i+1}]\}$ into a finite number of borel subsets S_j^i having the diameter less than ε. Set the preimage $E_j^i = (D^\alpha x)^{-1}(S_j^i)$, let $1_{E_j^i}(\cdot)$ be the characteristic function of E_j^i and take a point $\xi_j^i \in S_j^i$. Then the function

$$\xi(t) = \begin{cases} D^\alpha x(t), \ t < \delta_1, \\ \sum_j \xi_j^i \cdot \mathbf{1}_{E_j^i}(t), \ t \in [t_i, t_{i+1}] \subset [\delta_1, \ T], \ i = 0, 1, ..., N-1. \end{cases} \tag{3.7}$$

has the property that

$$\|\xi(t) - v(t)\| \equiv \|\xi(t) - D^\alpha x(t)\| \leq \varepsilon \quad a.e. \ on \ [0, T]. \tag{3.8}$$

The proof of (3.8) is that $\xi(t) = D^\alpha x(t)$ for almost every $t < \delta_1$. Moreover, for almost every $t > \delta_1$ we have: if $t \in E_j^i \in [t_i, t_{i+1}]$ for some fixed (i, j) then $D^\alpha x(t) \in S_j^i$ and $\|\xi(t) - D^\alpha x(t)\| = \|\xi_j^i - D^\alpha x(t)\| \leq diam(S_j) \leq \varepsilon$.

Another feature of function $\xi(\cdot)$ introduced in (3.7) is that point ξ_j^i remains sufficiently close to $coF(x(t_i))$: since $\xi_j^i \in S_j^i \subset \bigcup_{t \in [t_i, t_{i+1}]} coF(x(t))$ then there is $\tau \in [t_i, t_{i+1}]$ such that $\xi_j^i \in coF(x(\tau))$ and

$$\begin{aligned} d\Big(\xi_j^i, coF(x(t_i))\Big) &\le d_H\big(coF(x(\tau)), coF(x(t_i))\big) \le k \cdot \|x(\tau) - x(t_i)\| \\ &\le kc_0\varepsilon^{-2/\alpha} \cdot |\tau - t_i|^\alpha \\ &\le kc_0\varepsilon^{-2/\alpha} \cdot |t_{i+1} - t_i|^\alpha \le kc_0\varepsilon^{-2/\alpha} \cdot \Big(T\varepsilon^{3/\alpha^2}\Big)^\alpha \\ &= kc_0T^\alpha \cdot \varepsilon^{1/\alpha} \equiv c_3 \cdot \varepsilon^{1/\alpha}. \end{aligned} \tag{3.9}$$

Since $coF(x(t_i))$ is a closed subset of $\mathbb{R}^n$ then there is a point $\eta \in coF(x(t_i))$ such that $\|\xi_j^i - \eta\| = d(\xi_j^i, coF(x(t_i)))$. Then the definition of the convex hull and (3.9) imply that there are finitely many points $z_{jk}^i \in F(x(t_i))$ and positive constants α_{jk}^i satisfying

$$\sum_k \alpha_{jk}^i = 1 \quad \text{and} \quad \|\xi_j^i - \eta\| = \Big\|\xi_j^i - \sum_k \alpha_{jk}^i z_{jk}^i\Big\| \le c_3 \cdot \varepsilon^{1/\alpha}. \tag{3.10}$$

Moreover, there is a partition of the set E_j^i into measurable subsets $\{E_{j,k}^i\}_k$ such that

$$\int_{E_{j,k}^i} 1ds = \alpha_{j,k}^i \cdot \int_{E_j^i} 1ds, \tag{3.11}$$

where $\{\alpha_{j,k}^i\}_{k=1}^{N_1}$ is the convex combination introduced in (3.10). In order to derive the existence of partition $\{E_{j,k}^i\}_k$ we introduce a mapping $\psi : [t_i, t_{i+1}] \mapsto \mathbb{R}$ defined by $\psi(t) = \int_{t_i}^t 1_{E_j^i}(s)\, ds$, where $1_A(\cdot)$ is the characteristic function of a set $A \subset \mathbb{R}$. Function $\psi \ge 0$ is nondecreasing and continuous, therefore the sequence $\{\tau_k\}_{k=1}^{N_1}$ defined as

$$\tau_k = \sup\Big\{t \in [t_i, t_{i+1}] \;\Big|\; \psi(t) \le (\alpha_{j1}^i + .. + \alpha_{jk}^i) \cdot (\text{the measure of } E_j^i)\Big\}$$

leads to the desired partition $\{E_{jk}^i\}_k$ of E_j^i of the form: $E_{j1}^i = [t_i, \tau_1] \cap E_j^i$ and $E_{jk}^i = E_j^i \cap (\tau_{k-1}, \tau_k]$ for $2 \le k \le N_1$.

Consider function

$$\varrho(t) = \begin{cases} D^\alpha x(t)\,, \text{ a.e. in } t \in [0, \delta_1) \\ \sum_k z_{jk}^i \cdot 1_{E_{jk}^i}(t)\,,\ t \in [t_i, t_{i+1}] \end{cases} \tag{3.12}$$

and let

$$r(t) = \xi_0 \cdot \frac{t^{\alpha-1}}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{\varrho(s)}{(t-s)^{1-\alpha}}\, ds. \tag{3.13}$$

Function $r(\cdot)$ depends on ε and we shall denote $r(t) = r_\varepsilon(t)$ depending on the context. We will prove that the distance between the fractional integrals $\|(I^{1-\alpha}x)(t) -$

$(I^{1-\alpha}r_\varepsilon)(t)\|$ and the distance between the trajectories $\|x(t) - r_\varepsilon(t)\|$ shall be small enough.

If $0 < t < \delta_1$ then it follows from (3.4) and (3.12) that

$$\|(I^{1-\alpha}x)(t) - (I^{1-\alpha}r_\varepsilon)(t)\| = \left\| \int_0^t (v(s) - \varrho(s))\, ds \right\| = 0 \,,$$

while for $t > \delta_1$ there is an integer l such that $t \in [t_l, t_{l+1}]$. Therefore, recalling function $\xi(\cdot)$ from (3.7) and (3.13), we estimate:

$$\begin{aligned}
&\left\|(I^{1-\alpha}x)(t) - (I^{1-\alpha}r_\varepsilon)(t)\right\| \leq \\
&\leq \left\|(I^{1-\alpha}x)(t) - (I^{1-\alpha}x)(t_{l+1})\right\| + \left\|(I^{1-\alpha}r)(t) - (I^{1-\alpha}r)(t_{l+1})\right\| + \\
&\quad + \|(I^{1-\alpha}r_\varepsilon)(t_{l+1}) - (I^1\xi)(t_{l+1})\| + \|(I^1\xi)(t_{l+1}) - (I^{1-\alpha}x)(t_{l+1})\| \\
&\equiv \left\| \int_t^{t_{l+1}} v(s)\, ds \right\| + \left\| \int_t^{t_{l+1}} \varrho(s)\, ds \right\| + \left\| \int_0^{t_{l+1}} (\varrho(s) - \xi(s))\, ds \right\| + \\
&\quad + \left\| \int_0^{t_{l+1}} (\xi(s) - v(s))\, ds \right\| .
\end{aligned} \tag{3.14}$$

But $\|v(s)\| \leq M$ and $\|\varrho(s)\| \leq M$ a.e., where M is the uniform bound of multifunction F. Moreover, function $\xi(\cdot)$ is constructed in such a way that the estimate (3.8) holds, therefore (3.14) takes the form

$$\left\|(I^{1-\alpha}x)(t) - (I^{1-\alpha}r_\varepsilon)(t)\right\| \leq 2TM\varepsilon^{3/\alpha^2} + T\varepsilon + \left\| \int_0^{t_{l+1}} (\varrho(s) - \xi(s))\, ds \right\| .$$

It remains to estimate the last term above: recalling (3.7), (3.11), (3.12) and applying the partition $E^i_{jk} \subset E^i_j \subset [t_i, t_{i+1}]$ we transform

$$\begin{aligned}
&\left\| \int_0^{t_{l+1}} (\xi(s) - \varrho(s))\, ds \right\| = \left\| \int_{\delta_1}^{t_{l+1}} (\xi(s) - \varrho(s))\, ds \right\| \\
&\quad = \left\| \sum_{i=0}^{l} \int_{t_i}^{t_{i+1}} (\xi(s) - \varrho(s))\ ds \right\| = \left\| \sum_{i,j} \int_{E^i_j} (\xi(s) - \varrho(s))\ ds \right\| \\
&\quad = \left\| \sum_{i,j} \int_{E^i_j} \left(\xi^i_j - \sum_k z^i_{jk} \cdot 1_{E^i_{jk}}(s) \right) ds \right\| \\
&\quad = \left\| \sum_{i,j} \int_{E^i_j} \left(\xi^i_j - \sum_k \alpha^i_{jk} z^i_{jk} \right) ds \right\| .
\end{aligned} \tag{3.15}$$

But (3.10) combined with (3.15) implies that

$$\left\| \int_0^{t_{l+1}} (\xi(s) - \varrho(s)) ds \right\| \le c_3 \cdot \varepsilon^{1/\alpha} \cdot \sum_{i,j} \int_{E_j^i} 1\, ds \le c_3 \cdot T \cdot \varepsilon^{1/\alpha} ,$$

and, in particular, we have

$$\begin{aligned} \sup_{0<t<T} \left\| (I^{1-\alpha}x)(t) - (I^{1-\alpha}r_\varepsilon)(t) \right\| &\le 2TM\varepsilon^{3/\alpha^2} + T\varepsilon + c_3 \cdot T \cdot \varepsilon^{1/\alpha} \\ &\le c_6(k, M, \|\xi_0\|, \alpha, T) \cdot \varepsilon \end{aligned} \tag{3.16}$$

because $1 < 1/\alpha < 3/\alpha^2$ for $\alpha \in (0, 1)$ and $\varepsilon \in (0, 1)$. Combining the semigroup property and the linearity of the fractional integral with (3.16) implies that

$$\begin{aligned} &\sup_t \left\| \int_0^t (x(s) - r_\varepsilon(s))\, ds \right\| = \sup_t \left\| I^\alpha(I^{1-\alpha}x)(t) - I^\alpha(I^{1-\alpha}r_\varepsilon)(t) \right\| \\ &= \sup_t \| I^\alpha(I^{1-\alpha}x - I^{1-\alpha}r_\varepsilon)(t) \| \\ &\le \frac{1}{\Gamma(\alpha)} \sup_t \int_0^t \frac{\|(I^{1-\alpha}x)(s) - (I^{1-\alpha}r_\varepsilon)(s)\|}{(t-s)^{1-\alpha}}\, ds \le \frac{c_6 \cdot \varepsilon}{\Gamma(\alpha+1)} \cdot T^\alpha , \end{aligned} \tag{3.17}$$

therefore there is an estimate

$$\sup_{0\le\tau\le t\le T} \left\| \int_\tau^t (x(s) - r_\varepsilon(s))\, ds \right\| \le \frac{2c_6 T^\alpha}{\Gamma(\alpha+1)} \cdot \varepsilon \equiv c_7 \cdot \varepsilon. \tag{3.18}$$

Observe that the family $\{x(t) - r_\varepsilon(t)\}_{\varepsilon>0}$ is uniformly bounded and uniformly equicontinuous, i.e. $\|x(t) - r_\varepsilon(t)\| \le (2MT^\alpha)/(\Gamma(\alpha+1))$ and

$$\left\| \big(x(t) - r_\varepsilon(t)\big) - \big(x(\tau) - r_\varepsilon(\tau)\big) \right\| \le \frac{6M}{\Gamma(\alpha+1)} \cdot |t-\tau|^\alpha \equiv c_8 \cdot |t-\tau|^\alpha \tag{3.19}$$

for every $0 \le t, \tau \le T$ (the proof of (3.19) is to combine (3.4) and (3.13) with the boundedness of multifunction F).

The family $\{g_\varepsilon\}_{\varepsilon>0}$, where $g_\varepsilon(t) = x(t) - r_\varepsilon(t)$, is uniformly bounded and uniformly equicontinuous. Therefore by the Arzelá-Ascoli Theorem there is a subsequence $\{g_k\}_{k\ge1}$ convergent uniformly to a function g and g inherits the regularity of g_ε, i.e. g is Hölder continuous with constant c_8 and exponent α. Let us also introduce the notation for the components of the vector-valued function $g_k = (g_k^1, ..., g_k^n) \rightrightarrows g = (g^1, ..., g^n)$.

If we suppose that $g(s) \ne 0$ for some $s \in [0, T]$ then there is a nonzero component $g^i(s)$ of $g(s)$. Then the scalar-valued function $g^i(\cdot)$ is Hölder continuous with constant c_8 and exponent α and the Hölder continuity implies that there is a line segment $[a, b] \ni s$ such that $|g^i(t)| > \frac{1}{2}|g^i(s)| > 0$ for $t \in [a, b]$. Moreover, since $g^i(\cdot)$ does not change the sign on $[a, b]$ then $\int_a^b |g^i(t)|dt = |\int_a^b g^i(t)\,dt|$. Hence, in particular, the vector-valued functions $g_k \rightrightarrows g$ satisfy

$$
\begin{aligned}
0 < \frac{b-a}{2} \cdot |g^i(s)| &= \int_a^b \frac{|g^i(s)|}{2} dt < \int_a^b |g^i(t)|dt = \left| \int_a^b g^i(t)\,dt \right| \\
&\le \left| \int_a^b g_k^i(t)\,dt \right| + \int_a^b \left| g^i(t) - g_k^i(t) \right| dt\,,
\end{aligned} \tag{3.20}
$$

but the r.h.s. of (3.20) tends to 0 as $k \to \infty$ due to (3.18) and by the uniform convergence $g_k \rightrightarrows g$. Then the contradiction implies that there is a subsequence $\{x(t) - r_k(t)\}_{k \ge 1}$ convergent uniformly to 0. Choosing a subsubsequence, if necessary, we obtain that for every $\varepsilon > 0$ there is a function $r_\varepsilon(\cdot)$ satisfying

$$
\|x(\cdot) - r_\varepsilon(\cdot)\|_{L^\infty} \le \varepsilon/2 \tag{3.21}
$$

and the fractional derivative $D^\alpha r_\varepsilon(t)$ shall be sufficiently close to $F(r_\varepsilon(t))$ almost everywhere. In order to estimate the distance between $D^\alpha r_\varepsilon(t)$ and $F(r_\varepsilon(t))$ we recall that $(D^\alpha r_\varepsilon)(t) = (D^\alpha x)(t) \in coF(x(t))$ a.e. for $t < \delta_1$, therefore

$$
d(D^\alpha r_\varepsilon(t), F(r_\varepsilon(t))) \le k \cdot \|x(t) - r_\varepsilon(t)\| \le k \cdot \varepsilon/2 \tag{3.22}
$$

a.e. on $(0, \delta_1)$ due to (3.21) and by the Lipschitz continuity of multifunction F. On the other hand, for almost every $t > \delta_1$ the derivative $D^\alpha r_\varepsilon(t)$ exists and there are positive integers (i, j, k) such that $t \in E^i_{j,k} \subset [t_i, t_{i+1}]$, where $\{E^i_{j,k}\}_k$ is the partition introduced in (3.11). If $t \in E^i_{j,k}$ then $D^\alpha r_\varepsilon(t) = z^i_{j,k} \in F(x(t_i))$ due to (3.12) and (3.13). Hence by the Hölder continuity of $x(\cdot)$ on $[\delta_1, T]$ there is an estimate for a.e. $t > \delta_1$:

$$
\begin{aligned}
d(D^\alpha r_\varepsilon(t), F(r_\varepsilon(t))) &\le d_H(F(x(t_i)), F(r_\varepsilon(t))) \le k \cdot \|x(t_i) - r_\varepsilon(t)\| \\
&\le k \cdot (\|x(t_i) - x(t)\| + \|x(t) - r_\varepsilon(t)\|) \\
&\le k \cdot \left(c_0 \cdot \varepsilon^{-2/\alpha} \cdot |t - t_i|^\alpha + \varepsilon/2 \right) \\
&\le k \cdot (c_0 \cdot \varepsilon^{-2/\alpha} \cdot (T\varepsilon^{3/\alpha^2})^\alpha + \varepsilon/2) \le k \cdot (c_0 T^\alpha \varepsilon^{1/\alpha} + 0.5\varepsilon) \\
&\le (kc_0 T^\alpha + 0.5k) \cdot \varepsilon \equiv c_9 \cdot \varepsilon,
\end{aligned} \tag{3.23}
$$

and the latter follows because of the fact that $\varepsilon^{1/\alpha} < \varepsilon$ for $\varepsilon \in (0, 1)$.

Combining (3.22) with (3.23) we obtain that for a.e. $t \in [0, T]$ there is an estimate

$$
d(D^\alpha r_\varepsilon(t), F(r_\varepsilon(t))) \le m(t) = \max\{k/2, c_9\} \cdot \varepsilon \equiv c_{10} \cdot \varepsilon. \tag{3.24}
$$

Employing Theorem 2 with $R_0 = Y_0 = \xi_0/\Gamma(\alpha)$, $p = \infty$ and function $m(t)$ given in (3.24) implies the existence of $y(t)$, a solution to (3.2), and (2.8) takes the form:

$$\|y(t) - r_\varepsilon(t)\| \le c_{10} \cdot t^{1/p} \cdot \varepsilon \cdot \left(\sum_{n=1}^{\infty} A_n(t) \cdot t^{n\cdot(\alpha - \frac{1}{p})} \right) \equiv \varepsilon \cdot c_{11}(t) \le \varepsilon \cdot c_{11}(T).$$

As a conclusion, letting $\varepsilon_1 := \varepsilon/2 \cdot \min\{1, (c_{11}(T))^{-1}\} \le \varepsilon/2$ we obtain that for a given function $x(t)$, a solution to (3.1), there are functions $r_{\varepsilon_1}(t)$ and $y(t)$ such that $y(t)$ a solution to (3.2) and these functions satisfy the estimate

$$\begin{aligned} \|x(t) - y(t)\| &\le \|x(t) - r_{\varepsilon_1}(t)\| + \|r_{\varepsilon_1}(t) - y(t)\| \\ &\le \varepsilon_1/2 + c_{11}(T) \cdot \varepsilon_1 \\ &\le (1/2 + c_{11}(T)) \cdot \min\left\{1, \frac{1}{c_{11}(T)}\right\} \cdot \frac{\varepsilon}{2} \le \varepsilon \end{aligned}$$

for every $t \in [0, T]$. This ends the proof of Theorem 3. □

Declarations

Conflict of interest The author declares that he has no conflict of interest.

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