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ORIGINAL PAPER

On generalized K-functionals in L_p for $0 < p < 1$

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Abstract
We show that the Peetre K-functional between the space L_p with $0 < p < 1$ and the corresponding smooth function space W_p^ψ generated by the Weyl-type differential operator $\psi(D)$, where ψ is a homogeneous function of any positive order, is identically zero. The proof of the main results is based on the properties of the de la Vallée Poussin kernels and the quadrature formulas for trigonometric polynomials and entire functions of exponential type.

Keywords K-functional · L_p with $0 < p < 1$ · Fractional derivatives · Homogeneous multipliers · Quadrature formula

Mathematics Subject Classification 26A33 · 46E35 · 42A10 · 42A45 · 41A30

1 Introduction

The classical Peetre K-functional is defined by

$$K(f, t; X, Y) := \inf_{g\in Y} (\|f - g\|_X + t|g|_Y),$$

where $(X, \|\cdot\|_X)$ is a (quasi)-Banach space and $Y \subset X$ is a complete subspace with semi-norm $|\cdot|_Y$. The K-functional is one of the main tool in the theory of interpolation spaces. Moreover, it has important applications in approximation theory. Namely, smoothness properties of a function as well as errors of various approximation

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methods can be efficiently expressed by means of K-functionals, especially when the classical moduli of smoothness cannot be applied, see, e.g., [6], [7], [11], [15], [16].

In this paper, we are interested in the case, where X is an L_p space and Y is a smooth function space W_p^ψ generated by the Weyl-type differential operator $\psi(D)$, where ψ is a homogeneous function. The class of such differential operators includes, for example, the classical partial derivatives, Weyl and Riesz derivatives, the Laplace-operator and its (fractional) powers. Let us consider the K-functional for the pair $(L_p(\mathbb{T}), W_p^\alpha(\mathbb{T}))$, where $\mathbb{T}$ is the circle and $W_p^\alpha(\mathbb{T})$ is the fractional Sobolev space defined via the Weyl derivative of order $\alpha > 0$, i.e.,

$$K(f, \delta^\alpha; L_p, W_p^\alpha) = \inf_{g \in W_p^\alpha(\mathbb{T})} (\|f - g\|_{L_p(\mathbb{T})} + \delta^\alpha \|g^{(\alpha)}\|_{L_p(\mathbb{T})}). \tag{1.1}$$

It is well-known that if $1 \le p \le \infty$, then this K-functional is equivalent to the classical modulus of smoothness of order α, see [8] for the case $\alpha \in \mathbb{N}$ and [4] for arbitrary $\alpha > 0$. A similar result for the Riesz derivative and special modulus of smoothness was established in [21]. Properties and applications of the K-functionals between the space L_p on the torus $\mathbb{T}^d$ or $\mathbb{R}^d$ and the corresponding smooth function space W_p^ψ with a particular homogeneous function ψ were studied in [2], [6], [14], [19], [25]. Also, there are many works dedicated to the study of K-functionals in different quasi-normed Hardy spaces H_p, $0 < p < 1$, see, e.g., [10], [11], [16], [17, Ch. 4]. In particular, as in the case of the Banach spaces L_p, the K-functional of type (1.1) in the quasi-normed Hardy spaces is equivalent to the corresponding modulus of smoothness of integer or fractional order, see, e.g., [10], [11], [17, Ch. 4].

In contrast to the case of Banach spaces and quasi-normed Hardy spaces, the K-functionals in L_p with $0 < p < 1$ are no longer relevant. Namely, it was shown in [5] that the K-functional (1.1) with $0 < p < 1$ and the derivative of integer order $\alpha \in \mathbb{N}$ is identically zero. In [19], exploiting the approach from [5], the same property was established for the K-functional between the space $L_p(\mathbb{T}^d)$ and the smooth function space $W_p^\psi(\mathbb{T}^d)$, where ψ is a homogeneous function of order $\alpha \ge 1$ if $d = 1$ and $\alpha \ge 2$ if $d \ge 2$. Note that the restriction on the parameter α is due to the fact that the proof of the above property in [19] is essentially based on the results in [5] obtained for the derivatives of integer orders. But it is well known that a solution of problems involving fractional smoothness in L_p with $0 < p < 1$ usually is more involved than its integer counterparts and very often requires development essentially new approaches, see, e.g., [3], [20], [12], [13].

In the papers [14] and [16], it was stated without the proof that the K-functional $K(f, t; L_p(\Omega), W_p^\alpha(\Omega))$, where $\Omega = \mathbb{T}^d$ or $\mathbb{R}^d$, is identically zero for any positive $\alpha > 0$ and $0 < p < 1$. But, as it was pointed by S. Artamonov, this fact has not yet been established anywhere. The purpose of the present paper is to improve this drawback by showing that in the case $0 < p < 1$, the K-functional is identically zero for various differential operators $\psi(D)$ generated by a homogeneous function ψ of any order $\alpha > 0$. Our approach is different from the one presented in [5] and [19] and is based on properties of the de la Vallée Poussin kernels and the quadrature formulas for trigonometric polynomials and entire functions of exponential type.

2 Notation and definitions

Let $\mathbb{R}^d$ be the d-dimensional Euclidean space with elements $x = (x_1, \dots, x_d)$, and $(x, y) = x_1 y_1 + \dots + x_d y_d$, $|x| = (x, x)^{1/2}$. Let $\mathbb{N}$ be the set of positive integers, $\mathbb{Z}^d$ be the integer lattice in $\mathbb{R}^d$, and $\mathbb{T}^d = \mathbb{R}^d / 2\pi\mathbb{Z}^d$. By $\{e_j\}_{j=1}^d$ we denote the standard basis in $\mathbb{R}^d$. For $n \in \mathbb{N}$, the space of trigonometric polynomials of degree at most than n is defined by

$$\mathcal{T}_n = \operatorname{span}\{e^{i(k,x)} : k \in [-n, n]^d\}.$$

As usual, the space $L_p(\Omega)$ consists of all measurable functions f such that $\|f\|_{L_p(\Omega)} < \infty$, where

$$\|f\|_{L_p(\Omega)} = \begin{cases} \left(\int_\Omega |f(x)|^p dx\right)^{\frac{1}{p}}, & 0 < p < \infty, \\ ess \sup\limits_{x \in \Omega} |f(x)|, & p = \infty. \end{cases}$$

Note that $\|f\|_{L_p(\Omega)}$ for $0 < p < 1$ is a quasi-norm satisfying $\|f + g\|_{L_p(\Omega)}^p \le \|f\|_{L_p(\Omega)}^p + \|g\|_{L_p(\Omega)}^p$. By $C_0(\mathbb{R}^d)$, we denote the set of all continuous functions f such that $\lim_{|x| \to \infty} f(x) = 0$. For any $q \in (0, \infty]$, we set

$$q_1 = \begin{cases} q, & 0 < q < 1, \\ 1, & 1 \le q \le \infty. \end{cases}$$

If $f \in L_1(\mathbb{T}^d)$, then its k-th Fourier coefficient is defined by

$$\widehat{f}(k) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(x) e^{-i(x,k)} dx.$$

By $\Delta_h^r f$, where $r \in \mathbb{N}$ and $h \in \mathbb{R}^d$, we denote the symmetric difference of the function f,

$$\Delta_h^r f(x) = \sum_{\nu=0}^{r} (-1)^\nu \binom{r}{\nu} f\left(x + (r - 2\nu)h\right).$$

We say that a function ψ belongs to the class $\mathcal{H}_\alpha$, $\alpha \in \mathbb{R}$, if $\psi(\xi) \ne 0$ for $\xi \in \mathbb{R}^d \setminus \{0\}$, $\psi \in C^\infty(\mathbb{R}^d \setminus \{0\})$, and ψ is a homogeneous function of order α, i.e.,

$$\psi(\tau\xi) = \tau^\alpha \psi(\xi), \quad \tau > 0, \quad \xi \in \mathbb{R}^d.$$

Any function ψ defined on $\mathbb{Z}^d \setminus \{0\}$ generates the Weyl-type differentiation operator as follows:

$$\psi(D) : \sum_{k \in \mathbb{Z}^d} c_k e^{i(k,x)} \to \sum_{k \in \mathbb{Z}^d \setminus \{0\}} \psi(k) c_k e^{i(k,x)}.$$

Important examples of the Weyl-type operators generated by homogeneous functions are the following:

- the linear differential operator

$$P_m(D) f = \sum_{\substack{k_1+\dots+k_d=m \\ k \in \mathbb{Z}_+^d}} a_k D^k f, \qquad D^k = \frac{\partial^{k_1+\dots+k_d}}{\partial x_1^{k_1} \cdots \partial x_d^{k_d}},$$

with

$$\psi(\xi) = \sum_{\substack{k_1+\dots+k_d=m \\ k \in \mathbb{Z}_+^d}} a_k (i\xi_1)^{k_1} \dots (i\xi_d)^{k_d};$$

- the fractional Laplacian $(-\Delta)^{\alpha/2} f$ with $\psi(\xi) = |\xi|^\alpha$, $\xi \in \mathbb{R}^d$;
- the classical Weyl derivative $f^{(\alpha)}$ with $\psi(\xi) = (i\xi)^\alpha$, $\xi \in \mathbb{R}$.

Let $\psi \in \mathcal{H}_\alpha$, $\alpha > 0$ and $0 < p \le 1$. By $W_p^\psi(\mathbb{T}^d)$ we denote the space of ψ-smooth functions in $L_p(\mathbb{T}^d)$, i.e.,

$$W_p^\psi(\mathbb{T}^d) = \Big\{ g \in L_1(\mathbb{T}^d) : \psi(D) g \in L_p(\mathbb{T}^d) \Big\}$$

with

$$|g|_{W_p^\psi} = \|\psi(D) g\|_{L_p(\mathbb{T}^d)}.$$

3 Main result in the periodic case

Theorem 1 *Let* $0 < p < 1$, $0 < q \le \infty$, $\alpha > \max\{0, d(1 - \frac{1}{q})\}$, *and* $\psi \in \mathcal{H}_\alpha$. *Then, for any* $f \in L_p(\mathbb{T}^d)$ *and* $\delta > 0$, *we have*

$$K\big(f, \delta, L_q(\mathbb{T}^d), W_p^\psi(\mathbb{T}^d)\big) = 0.$$

To prove this theorem, we need the following auxiliary results and notations. In what follows, the de la Vallée Poussin type kernel is defined by

$$V_n(x) := \sum_{k \in \mathbb{Z}^d} v\left(\frac{k}{n}\right) e^{i(k,x)},$$

where $v \in C^\infty(\mathbb{R}^d)$, $v(\xi) = 1$ for $\xi \in [-1, 1]^d$ and $v(\xi) = 0$ for $\xi \in \mathbb{R}^d \setminus [-2, 2]^d$.

Lemma 1 *(See [24, Ch. 4 and Ch. 9].) Let* $0 < p \le 1$ *and* $\varphi \in C^\infty(\mathbb{R}^d)$ *have a compact support. Then*

$$\sup_{\varepsilon>0} \varepsilon^{d(1-\frac{1}{p})} \left\| \sum_{k\in\mathbb{Z}^d} \varphi(\varepsilon k) e^{i(k,x)} \right\|_{L_p(\mathbb{T}^d)} < \infty.$$

In particular, $\|V_n\|_{L_p(\mathbb{T}^d)} \le c_p n^{d(1-\frac{1}{p})}$.

We will also use the following quadrature formula and the Marcinkiewicz–Zygmund inequality.

Lemma 2 *Let* $T_n \in \mathcal{T}_n$, $t_{k,n} = \frac{2\pi k}{2n+1}$, $k \in [0, 2n]^d$, *and* $0 < p < \infty$. *Then*

$$\frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} T_n(x)dx = \frac{1}{(2n+1)^d} \sum_{k\in[0,2n]^d} T_n\left(t_{k,n}\right) \tag{3.1}$$

and

$$\frac{1}{(2n+1)^d} \sum_{k\in[0,2n]^d} \left|T_n\left(t_{k,n}\right)\right|^p \le C_p \|T_n\|^p_{L_p(\mathbb{T}^d)}. \tag{3.2}$$

Proof Equality (3.1) can be obtained by applying the univariate quadrature formulas for trigonometric polynomials in [26, Ch. X, (2.5)] to each variable one after another. Similarly, using the univariate Marcinkiewicz–Zygmund inequality in [18, Theorem 2], we can prove (3.2). □

Proof of Theorem 1. In what follows, for simplicity, we write $\|f\|_p = \|f\|_{L_p(\mathbb{T}^d)}$. Note that in view of the obvious inequality

$$\begin{aligned} &K\big(f, t\delta, L_q(\mathbb{T}^d), W_p^\psi(\mathbb{T}^d)\big) \\ &\qquad \le \max\{1, t\} K\big(f, \delta, L_q(\mathbb{T}^d), W_p^\psi(\mathbb{T}^d)\big), \quad \delta, t > 0, \end{aligned} \tag{3.3}$$

it is enough to prove the theorem only for the case $\delta = 1$. Let $\varepsilon > 0$ be fixed and let $T_\mu \in \mathcal{T}_\mu$ be such that

$$\|f - T_\mu\|_q^{q_1} < \frac{\varepsilon}{3}.$$

It is clear that

$$K\big(f, 1, L_q(\mathbb{T}^d), W_p^\psi(\mathbb{T}^d)\big)^{q_1} < \frac{\varepsilon}{3} + K\big(T_\mu, 1, L_q(\mathbb{T}^d), W_p^\psi(\mathbb{T}^d)\big)^{q_1}. \tag{3.4}$$

Let $m > \mu$, $m \in \mathbb{N}$. We set

$$\mathcal{V}_{2^m}(x) = -\frac{1}{4}\sum_{j=1}^{d} \Delta_{e_j}^2 V_{2^m}(x) = \sum_{k\in\mathbb{Z}^d} (\sin^2 k_1 + \cdots + \sin^2 k_d) v\left(\frac{k}{2^m}\right) e^{i(k,x)}$$

and

$$\psi_1(\xi) = \frac{\psi(\xi)}{\sin^2 \xi_1 + \cdots + \sin^2 \xi_d}, \quad \xi \in \mathbb{Z}^d \setminus \{0\}.$$

Then, denoting

$$\tilde{\psi}(\xi) = \frac{1}{\psi(\xi)}, \quad \xi \in \mathbb{R}^d \setminus \{0\},$$

we see that equality (3.1) implies

$$\begin{aligned} T_\mu(x) &= \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} \psi_1(D)T_\mu(t)\cdot\tilde{\psi}(D)\mathcal{V}_{2^m}(x-t)dt + \widehat{T_\mu}(0) \\ &= \frac{1}{(2M+1)^d}\sum_{\ell\in[0,2M]^d} \psi_1(D)T_\mu(t_\ell)\cdot\tilde{\psi}(D)\mathcal{V}_{2^m}(x-t_\ell) + \widehat{T_\mu}(0), \end{aligned} \tag{3.5}$$

where $M = \mu + 2^{m+1}$ and $t_\ell = t_{\ell,M} = \frac{2\pi\ell}{2M+1}$.

Let $n > m$, $n \in \mathbb{N}$. From the definition of the K-functional, it follows that

$$\begin{aligned} &K\big(T_\mu, 1, L_q(\mathbb{T}^d), W_p^\psi(\mathbb{T}^d)\big) \\ &\le \left\| T_\mu - \frac{1}{(2M+1)^d}\sum_{\ell\in[0,2M]^d} \psi_1(D)T_\mu(t_\ell)\cdot\tilde{\psi}(D)\mathcal{V}_{2^n}(x-t_\ell) - \widehat{T_\mu}(0)\right\|_q \\ &\quad + \left\| \frac{1}{(2M+1)^d}\sum_{\ell\in[0,2M]^d} \psi_1(D)T_\mu(t_\ell)\cdot\mathcal{V}_{2^n}(x-t_\ell)\right\|_p = I_1 + I_2. \end{aligned} \tag{3.6}$$

Using (3.5), (3.2), and a telescopic sum, we obtain

$$
\begin{aligned}
I_1^{q_1} &= \left\| \frac{1}{(2M+1)^d} \sum_{\ell\in[0,2M]^d} \psi_1(D)T_\mu(t_\ell)\cdot\tilde{\psi}(D)\big(\mathcal{V}_{2^n}(x-t_\ell)-\mathcal{V}_{2^m}(x-t_\ell)\big)\right\|_q^{q_1} \\
&\le \frac{1}{(2M+1)^{dq_1}} \sum_{\ell\in[0,2M]^d} |\psi_1(D)T_\mu(t_\ell)|^{q_1} \\
&\qquad\qquad \times \left\|\tilde{\psi}(D)\big(\mathcal{V}_{2^n}(x-t_\ell)-\mathcal{V}_{2^m}(x-t_\ell)\big)\right\|_q^{q_1} \\
&\le C_{q_1}(2M+1)^{d(1-q_1)}\|\psi_1(D)T_\mu\|_{q_1}^{q_1}\left\|\tilde{\psi}(D)\left(\mathcal{V}_{2^n}-\mathcal{V}_{2^m}\right)\right\|_q^{q_1} \\
&\le C_{q_1}(2M+1)^{d(1-q_1)}\|\psi_1(D)T_\mu\|_{q_1}^{q_1}\sum_{\nu=m}^{n-1}\left\|\tilde{\psi}(D)\left(\mathcal{V}_{2^{\nu+1}}-\mathcal{V}_{2^\nu}\right)\right\|_q^{q_1}.
\end{aligned}
\tag{3.7}
$$

Next, denoting

$$
\mathcal{N}_{2^\nu}(x)=\sum_{k\in\mathbb{Z}^d}\eta\left(\frac{k}{2^\nu}\right)e^{i(k,x)} \quad \text{with} \quad \eta(\xi)=\begin{cases}\dfrac{v(\frac{\xi}{2})-v(\xi)}{\psi(\xi)}, & \xi\in\mathbb{R}^d\setminus\{0\},\\ 0, & \xi=0,\end{cases}
$$

we get

$$
\begin{aligned}
&\tilde{\psi}(D)\left(\mathcal{V}_{2^{\nu+1}}(x)-\mathcal{V}_{2^\nu}(x)\right) \\
&\qquad = \frac{1}{2^{\alpha\nu}}\sum_{k\in\mathbb{Z}^d}(\sin^2 k_1+\dots+\sin^2 k_d)\eta\left(\frac{k}{2^\nu}\right)e^{i(k,x)} \\
&\qquad = -\frac{1}{2^{\alpha\nu+2}}\sum_{j=1}^d \Delta_{e_j}^2\mathcal{N}_{2^\nu}(x)
\end{aligned}
$$

and hence

$$
\begin{aligned}
\left\|\tilde{\psi}(D)\left(\mathcal{V}_{2^{\nu+1}}-\mathcal{V}_{2^\nu}\right)\right\|_q^{q_1} &= \frac{1}{2^{(\alpha\nu+2)q_1}}\left\|\sum_{j=1}^d\Delta_{e_j}^2\mathcal{N}_{2^\nu}\right\|_q^{q_1} \\
&\le \frac{4^{1-q_1}d}{2^{\alpha\nu q_1}}\|\mathcal{N}_{2^\nu}\|_q^{q_1}.
\end{aligned}
\tag{3.8}
$$

If $0<q\le 1$, then with the help of Lemma 1, we obtain

$$
\|\mathcal{N}_{2^\nu}\|_q^q \le \frac{c_{q,\eta}}{2^{d(1-q)\nu}}, \tag{3.9}
$$

where we have used the fact that $\eta\in C^\infty(\mathbb{R}^d)$ and $\operatorname{supp}\eta$ is compact. Next, for $1<q\le\infty$, exploiting the Nikolskii inequality of different metrics (see, e.g., [24,

4.3.6]) and again Lemma 1, we get

$$\left\|\mathcal{N}_{2^\nu}\right\|_q \le 2^{d(1-\frac{1}{q})\nu}\left\|\mathcal{N}_{2^\nu}\right\|_1 \le \frac{c_{1,\eta}}{2^{d(\frac{1}{q}-1)\nu}}. \tag{3.10}$$

Thus, inequalities (3.7), (3.8), (3.9), (3.10), and the condition $\alpha > \max\{0, d(1-\frac{1}{q})\}$ imply that, for sufficiently large $m > m_0(T_\mu, \psi, q, \varepsilon)$,

$$\begin{aligned} I_1^{q_1} \le \frac{2^{q_1(\alpha+d(\frac{1}{q}-1))+2(1-q_1)} d C_{q_1} c_{q_1,\eta}}{2^{q_1(\alpha+d(\frac{1}{q}-1))}-1}\|\psi_1(D)T_\mu\|_{q_1}^{q_1} \\ \times \frac{(2^{m+2}+2\mu+1)^{d(1-q_1)}}{2^{q_1(\alpha+d(\frac{1}{q}-1))m}} < \frac{\varepsilon}{3}. \end{aligned} \tag{3.11}$$

Now we estimate I_2. An application of the Marcinkiewicz-Zygmund inequality (3.2) and Lemma 1 yields

$$\begin{aligned} I_2^p &\le \frac{1}{(2M+1)^{dp}} \sum_{\ell\in[0,2M]^d} |\psi_1(D)T_\mu(t_\ell)|^p \|\mathcal{V}_{2^n}\|_p^p \\ &\le 4dC_p(2M+1)^{d(1-p)}\|\psi_1(D)T_\mu\|_p^p\|V_{2^n}\|_p^p \\ &\le 4dc_pC_p(2M+1)^{d(1-p)}\|\psi_1(D)T_\mu\|_p^p \cdot 2^{d(p-1)n} < \left(\frac{\varepsilon}{3}\right)^{p/q_1} \end{aligned} \tag{3.12}$$

for sufficiently large $n > n_0(T_\mu, m, \psi, p, \varepsilon)$.

Finally, combining (3.4), (3.6), (3.11), and (3.12) for appropriate $n > n_0$ and $m > m_0$, we obtain that $K\big(f, 1, L_q(\mathbb{T}^d), W_p^\psi(\mathbb{T}^d)\big)^{q_1} < \varepsilon$. This proves the theorem. □

4 Main result in the non-periodic case

To formulate an analogue of Theorem 1 for non-periodic functions, we introduce additional notations. As usual, by $\mathcal{S}$ and $\mathcal{S}'$ we denote the Schwartz space of infinitely differentiable rapidly decreasing functions on $\mathbb{R}^d$ and its dual (the space of tempered distributions), respectively. The Fourier transform and the inverse Fourier transform of $f \in L_1(\mathbb{R}^d)$ are given by

$$\mathcal{F}f(\xi) = \widehat{f}(\xi) = \frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d} f(x)e^{-i(x,\xi)}dx$$

and

$$\mathcal{F}^{-1}f(\xi) = \frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d} f(x)e^{i(x,\xi)}dx.$$

The convolution of two appropriate functions f and g is defined by

$$f * g(x) = \int_{\mathbb{R}^d} f(y)g(x-y)dx.$$

For $f \in \mathcal{S}'$, we define the Fourier transform $\widehat{f}$ and the inverse Fourier transform $\mathcal{F}^{-1}f$ by $\langle \widehat{f}, \varphi\rangle = \langle f, \widehat{\varphi}\rangle$ and $\langle \mathcal{F}^{-1}f, \varphi\rangle = \langle f, \mathcal{F}^{-1}\varphi\rangle$, $\varphi \in \mathcal{S}$. Next, by $\mathcal{B}_{\sigma,p} = \mathcal{B}_{\sigma,p}(\mathbb{R}^d)$ with $\sigma > 0$ and $0 < p \le \infty$, we denote the Bernstein space of entire functions of exponential type σ. That is, $f \in \mathcal{B}_{\sigma,p}$ if $f \in L_p(\mathbb{R}^d) \cap \mathcal{S}'(\mathbb{R}^d)$ and $\operatorname{supp} \mathcal{F}f \subset [-\sigma,\sigma]^d$.

Similarly as in the periodic case, by $W_p^\psi(\mathbb{R}^d)$ we denote the space of ψ-smooth functions in $L_p(\mathbb{R}^d)$, that is,

$$W_p^\psi(\mathbb{R}^d) = \left\{ g \in \mathcal{S}(\mathbb{R}^d) : \psi(D)g \in L_p(\mathbb{R}^d) \right\}$$

with

$$|g|_{W_p^\psi(\mathbb{R}^d)} = \|\psi(D)g\|_{L_p(\mathbb{R}^d)}, \quad \text{where} \quad \psi(D)g = \mathcal{F}^{-1}(\psi \widehat{g}).$$

Theorem 2 *Let* $0 < p < 1$, $0 < q \le \infty$, $\alpha > \max\{d(\frac{1}{p}-1), d(1-\frac{1}{q})\}$, *and* $\psi \in \mathcal{H}_\alpha$. *Then, for any* $f \in L_p(\mathbb{R}^d)$ *(*$f \in C_0(\mathbb{R}^d)$ *if* $q = \infty$*) and* $\delta > 0$, *we have*

$$K\big(f, \delta, L_q(\mathbb{R}^d), W_p^\psi(\mathbb{R}^d)\big) = 0. \tag{4.1}$$

To prove Theorem 2, we will need the following analogue of Lemma 2 for entire functions of exponential type.

Lemma 3 1) *Let* $1 \le p \le \infty$, $1/p + 1/q = 1$, *and* $\sigma > 0$. *Then, for all* $g \in \mathcal{B}_{\pi\sigma,p}$ *and* $h \in \mathcal{B}_{\pi\sigma,q}$, *we have*

$$(g * h)(x) = \frac{1}{\sigma^d} \sum_{k \in \mathbb{Z}^d} g\left(\frac{k}{\sigma}\right) h\left(x - \frac{k}{\sigma}\right), \quad x \in \mathbb{R}^d. \tag{4.2}$$

The series on the right-hand side of (4.2) converges absolutely for all $x \in \mathbb{R}^d$ *and this converges is uniform on each compact subset of* $\mathbb{R}^d$.

2) *Let* $0 < p < \infty$ *and* $\sigma > 0$. *Then, for all* $g \in \mathcal{B}_{\pi\sigma,p}$, *we have*

$$\frac{1}{\sigma^d} \sum_{k \in \mathbb{Z}^d} \left| g\left(\frac{k}{\sigma}\right)\right|^p \le C_p \|g\|_{L_p(\mathbb{R}^d)}^p. \tag{4.3}$$

Equality (4.2) can be found, e.g., in [22, Lemma 6.2]. For the Plancherel–Polya-type inequality (4.3), see, e.g., [24, 4.3.1].

Recall also the following convolution inequality, see, e.g., [23, 1.5.3].

Lemma 4 *Let $0 < p \le 1$ and $\sigma > 0$. Then, for all $f, g \in \mathcal{B}_{\sigma,p}$, we have*

$$\|f * g\|_{L_p(\mathbb{R}^d)} \le c_p \sigma^{d(\frac{1}{p}-1)} \|f\|_{L_p(\mathbb{R}^d)} \|g\|_{L_p(\mathbb{R}^d)}.$$

Proof of Theorem 2. The proof of the theorem is similar to the one of Theorem 1. However, because several steps are different, we present a detailed proof.

In what follows, we denote $\|\cdot\|_p = \|\cdot\|_{L_p(\mathbb{R}^d)}$. By the same arguments as in (3.3), we can restrict ourselves to the case $\delta = 1$. Let $\varepsilon > 0$ be fixed and let $g_\mu \in \mathcal{S}$, $\mu > 1$, be such that $\operatorname{supp} \widehat{g_\mu} \subset [-2^\mu, 2^\mu]^d$ and

$$\|f - g_\mu\|_q^{q_1} < \frac{\varepsilon}{3}.$$

Then, as in the periodic case, we have

$$K\big(f, 1, L_q(\mathbb{R}^d), W_p^\psi(\mathbb{R}^d)\big)^{q_1} < \frac{\varepsilon}{3} + K\big(g_\mu, 1, L_q(\mathbb{R}^d), W_p^\psi(\mathbb{R}^d)\big)^{q_1}. \tag{4.4}$$

For $\lambda > \mu$ and $m > \mu$, $\lambda, m \in \mathbb{N}$, we introduce the following functions:

$$f_{\mu,\lambda}(x) = \mathcal{F}^{-1}\left(v(2^\lambda \xi)\widehat{g_\mu}(\xi)\right)(x),$$
$$g_{\mu,\lambda}(x) = g_\mu(x) - f_{\mu,\lambda}(x),$$

and

$$V_{2^m}(x) = \mathcal{F}^{-1}(v_m(\xi))(x) \quad \text{with} \quad v_m(\xi) = v\left(2^{-m}\xi\right).$$

Further we denote

$$r_2 = \begin{cases} \lceil \alpha \rceil, & d = 1, \\ 2\lceil \frac{\alpha}{2} \rceil, & d \ge 2, \end{cases}$$

where $\lceil \cdot \rceil$ is the ceil function, and consider the functions

$$\mathcal{V}_{2^m}(x) = \frac{1}{(2i)^{r_2}} \sum_{j=1}^{d} \Delta_{2^{-\mu} e_j}^{r_2} V_{2^m}(x)$$
$$= \mathcal{F}^{-1}\big((\sin^{r_2} 2^{-\mu}\xi_1 + \dots + \sin^{r_2} 2^{-\mu}\xi_d) v_m(\xi)\big)(x),$$

$$\psi_2(\xi) = \begin{cases} \dfrac{\psi(\xi)}{\sin^{r_2} 2^{-\mu}\xi_1 + \dots + \sin^{r_2} 2^{-\mu}\xi_d}, & \xi \in [-2^{\mu+1}, 2^{\mu+1}]^d \setminus \{0\}, \\ 0, & \text{otherwise}, \end{cases}$$

and

$$\tilde{\psi}(\xi) = \begin{cases} \frac{1}{\psi(\xi)}, & \xi \in \mathbb{R}^d \setminus \{0\}, \\ 0, & \xi = 0. \end{cases}$$

Let us show that there exists $q > 1$ such that

$$\psi_2(D)g_{\mu,\lambda} \in L_q(\mathbb{R}^d) \tag{4.5}$$

and

$$\tilde{\psi}(D)\mathcal{V}_{2^m}(x) \in L_{q'}(\mathbb{R}^d), \tag{4.6}$$

where $1/q + 1/q' = 1$. Indeed, relation (4.5) is obvious since $\psi_2\widehat{g_{\mu,\lambda}} \in \mathcal{S}$. To verify (4.6), we use the following representation

$$\tilde{\psi}(\xi)\left(\sum_{j=1}^{d} \sin 2^{-\mu}\xi_j\right) v_m(\xi) = \sum_{j=1}^{d} h_j(\xi)\varphi_j(\xi), \tag{4.7}$$

where

$$h_j(\xi) = \frac{\xi_j^{r_2}}{\psi(\xi)} \in C^\infty(\mathbb{R}^d \setminus \{0\}) \quad \text{and} \quad \varphi_j(\xi) = \left(\frac{\sin 2^{-\mu}\xi_j}{\xi_j}\right)^{r_2} v_m(\xi) \in \mathcal{S}.$$

Since h_j is a homogeneous function of order $r_2 - \alpha \ge 0$, we have that $\widehat{h_j}$ belongs to $C^\infty(\mathbb{R}^d \setminus \{0\})$ and it is homogeneous of order $-(r_2 - \alpha + d)$, see, e.g. [9, Theorems 7.1.16 and 7.1.18]. Thus, applying the properties of convolution and choosing σ_m such that $\operatorname{supp} v_m \subset \{|\xi| < \sigma_m\}$, we obtain

$$\begin{aligned}
|\mathcal{F}(h_j\varphi_j)(\xi)| = |\langle \widehat{h_j}, \widehat{\varphi_j}(\xi - \cdot)\rangle| &= \left|\int_{\mathbb{R}^d} \widehat{h_j}(y)\widehat{\varphi_j}(\xi - y)dy\right| \\
&\le \int_{|\xi - y| \le \sigma_m} |\widehat{h_j}(y)\widehat{\varphi_j}(\xi - y)|dy \\
&\le \max_{|\xi - y| \le \sigma_m} |\widehat{h_j}(y)| \int_{\mathbb{R}^d} |\widehat{\varphi_j}(y)|dy \\
&\le c'_m \max_{|\xi - y| \le \sigma_m} |y|^{-(r_2 - \alpha + d)} \le c''_m |\xi|^{-\gamma}
\end{aligned} \tag{4.8}$$

for $|\xi| > 2\sigma_m$ and $\gamma = r_2 - \alpha + d \ge d$. Moreover, since $h_j\varphi_j \in L_1(\mathbb{R}^d)$, it follows from the standard properties of the Fourier transform, that $\mathcal{F}(h_j\varphi_j) \in C_0(\mathbb{R}^d)$. Therefore, $\mathcal{F}(h_j\varphi_j) \in L_s(\mathbb{R}^d)$ for all $s > 1$. In particular, we have that $\mathcal{F}^{-1}(h_j\varphi_j) \in L_{q'}(\mathbb{R}^d)$, which together with equality (4.7) implies (4.6).

Now, taking into account (4.5)–(4.6), we can apply Lemma 3, which yields

$$\begin{aligned}
g_{\mu,\lambda}(x) &= g_\mu(x) - f_{\mu,\lambda}(x) \\
&= \int_{\mathbb{R}^d} \psi_2(D)g_{\mu,\lambda}(y)\tilde{\psi}(D)\mathcal{V}_{2^m}(x - y)dy \\
&= M^{-d} \sum_{\ell \in \mathbb{Z}^d} \psi_2(D)g_{\mu,\lambda}(t_\ell)\tilde{\psi}(D)\mathcal{V}_{2^m}(x - t_\ell),
\end{aligned} \tag{4.9}$$

where $M = \mu + 2^{m+1}$ and $t_\ell = \frac{\ell}{M}$.

Let $n > m$, $n \in \mathbb{N}$. We have

$$
\begin{aligned}
&K\big(g_\mu, 1, L_q(\mathbb{R}^d), W_p^{\psi}(\mathbb{R}^d)\big)\\
&\quad\le \Big\| g_\mu - M^{-d} \sum_{\ell\in\mathbb{Z}^d} \psi_2(D) g_{\mu,\lambda}(t_\ell) \tilde{\psi}(D) \mathcal{V}_{2^n}(x - t_\ell) - f_{\mu,\lambda} \Big\|_q\\
&\quad+ \Big\| M^{-d} \sum_{\ell\in\mathbb{Z}^d} \psi_2(D) g_{\mu,\lambda}(t_\ell) \mathcal{V}_{2^n}(x - t_\ell) + \psi(D) f_{\mu,\lambda} \Big\|_p = I_1 + I_2.
\end{aligned}
\tag{4.10}
$$

Using (4.9) and the Plancherel–Polya-type inequality (4.3), we obtain

$$
\begin{aligned}
I_1^{q_1} &= \Big\| M^{-d} \sum_{\ell\in\mathbb{Z}^d} \psi_2(D) g_{\mu,\lambda}(t_\ell) \cdot \tilde{\psi}(D)\big(\mathcal{V}_{2^n}(x - t_\ell) - \mathcal{V}_{2^m}(x - t_\ell)\big) \Big\|_q^{q_1}\\
&\le M^{-dq_1} \sum_{\ell\in\mathbb{Z}^d} |\psi_2(D) g_{\mu,\lambda}(t_\ell)|^{q_1}\\
&\qquad\qquad \times \big\| \tilde{\psi}(D)\big(\mathcal{V}_{2^n}(x - t_\ell) - \mathcal{V}_{2^m}(x - t_\ell)\big) \big\|_q^{q_1}\\
&\le C_{q_1} M^{d(1-q_1)} \|\psi_2(D) g_{\mu,\lambda}\|_{q_1}^{q_1} \big\| \tilde{\psi}(D)\left(\mathcal{V}_{2^n} - \mathcal{V}_{2^m}\right) \big\|_q^{q_1}\\
&\le C_{q_1} M^{d(1-q_1)} \|\psi_2(D) g_{\mu,\lambda}\|_{q_1}^{q_1} \sum_{\nu=m}^{n-1} \big\| \tilde{\psi}(D)\left(\mathcal{V}_{2^{\nu+1}} - \mathcal{V}_{2^\nu}\right) \big\|_q^{q_1}.
\end{aligned}
\tag{4.11}
$$

Note that in the above relations $\psi_2(D) g_{\mu,\lambda} \in L_{q_1}(\mathbb{R}^d)$ because $\psi_2 \widehat{g_{\mu,\lambda}} \in \mathcal{S}$. Next, denoting

$$
\mathcal{N}_{2^\nu}(x) = \mathcal{F}^{-1}\left(\eta\left(2^{-\nu}\xi\right)\right)(x) \quad \text{with} \quad \eta(\xi) = \begin{cases} \dfrac{v(\frac{\xi}{2}) - v(\xi)}{\psi(\xi)}, & \xi \in \mathbb{R}^d \setminus \{0\},\\ 0, & \xi = 0, \end{cases}
$$

we get

$$
\begin{aligned}
&\tilde{\psi}(D)\left(\mathcal{V}_{2^{\nu+1}}(x) - \mathcal{V}_{2^\nu}(x)\right)\\
&\qquad= \frac{1}{2^{\alpha\nu}} \mathcal{F}^{-1}\left((\sin^{r_2} 2^{-\mu}\xi_1 + \cdots + \sin^{r_2} 2^{-\mu}\xi_d)\eta\left(2^{-\nu}\xi\right)\right)(x)\\
&\qquad= \frac{1}{2^{\alpha\nu}(2i)^{r_2}} \sum_{j=1}^{d} \Delta_{2^{-\mu} e_j}^{r_2} \mathcal{N}_{2^\nu}(x).
\end{aligned}
$$

Thus, taking into account that $\eta \in C^\infty(\mathbb{R}^d)$ and $\operatorname{supp}\eta$ is compact, we obtain

$$\begin{aligned}\left\|\tilde\psi(D)\left(\mathcal{V}_{2^{\nu+1}}-\mathcal{V}_{2^\nu}\right)\right\|_q^{q_1}&=\frac{1}{2^{(\alpha\nu+r_2)q_1}}\left\|\sum_{j=1}^d \Delta_{2^{-\mu}e_j}^{r_2}\mathcal{N}_{2^\nu}\right\|_q^{q_1}\\ &\le\frac{2^{(1-q_1)r_2}d}{2^{\alpha\nu q_1}}\|\mathcal{N}_{2^\nu}\|_q^{q_1}=\frac{2^{(1-q_1)r_2}d\|\widehat{\eta}\|_q^{q_1}}{2^{q_1(\alpha+d(\frac1q-1))\nu}}=\frac{c_{q,\eta}}{2^{q_1(\alpha+d(\frac1q-1))\nu}}.\end{aligned}\tag{4.12}$$

Then, combining (4.11) and (4.12), it is easy to see that

$$\begin{aligned}I_1^{q_1}\le&\frac{2^{q_1(\alpha+d(\frac1q-1))}C_{q_1}c_{q_1,\eta}}{2^{q_1(\alpha+d(\frac1q-1))}-1}\|\psi_2(D)g_{\mu,\lambda}\|_{q_1}^{q_1}\\ &\times\frac{(2^{m+2}+2\mu+1)^{d(1-q_1)}}{2^{q_1(\alpha+d(\frac1q-1))m}}<\frac{\varepsilon}{3}\end{aligned}\tag{4.13}$$

for sufficiently large $m>m_0(g_{\mu,\lambda},\psi,q,\varepsilon)$.

Further we find

$$\begin{aligned}I_2^p&\le\left\|M^{-d}\sum_{\ell\in\mathbb{Z}^d}\psi_2(D)g_{\mu,\lambda}(t_\ell)\mathcal{V}_{2^n}(x-t_\ell)\right\|_p^p+\|\psi(D)f_{\mu,\lambda}\|_p^p\\ &=J_1^p+J_2^p.\end{aligned}\tag{4.14}$$

First we estimate J_2. Taking into account that $\mathcal{F}^{-1}(\psi v)\in\mathcal{B}_{2,p}(\mathbb{R}^d)$ for $\alpha>d(1/p-1)$, see, e.g., [20] (this can also be verified as (4.8)) and applying Lemma 4, we obtain

$$\begin{aligned}J_2&=2^{-\alpha\lambda}\left\|\mathcal{F}^{-1}\left(\psi(2^\lambda\cdot)v(2^\lambda\cdot)\widehat{g_\mu}\right)\right\|_p=2^{-\alpha\lambda}\|\mathcal{F}^{-1}(\psi(2^\lambda\cdot)v(2^\lambda\cdot))*g_\mu\|_p\\ &\le c_p2^{-\alpha\lambda+d(\frac1p-1)(\mu+1)}\|\mathcal{F}^{-1}(\psi(2^\lambda\cdot)v(2^\lambda\cdot))\|_p\|g_\mu\|_p\\ &=c_p2^{-\alpha\lambda+d(\frac1p-1)(\mu+1)+d(\frac1p-1)\lambda}\|\mathcal{F}^{-1}(\psi v)\|_p\|g_\mu\|_p.\end{aligned}$$

Thus, for sufficiently large $\lambda>\lambda_0(g_\mu,\psi,p,\varepsilon)$, we get

$$J_2^p<\frac12\left(\frac{\varepsilon}{3}\right)^{p/q_1}.\tag{4.15}$$

Next, the Plancherel–Polya-type inequality (4.3) yields

$$\begin{aligned}J_1^p&\le M^{-dp}\sum_{\ell\in\mathbb{Z}^d}|\psi_2(D)g_{\mu,\lambda}(t_\ell)|^p\|\mathcal{V}_{2^n}\|_p^p\\ &\le 2^{r_2}dC_pM^{d(1-p)}\|\psi_2(D)g_{\mu,\lambda}\|_p^p\|V_{2^n}\|_p^p\\ &=2^{r_2}dC_pc_p(\mu+2^{m+1})^{d(1-p)}\|\psi_2(D)g_{\mu,\lambda}\|_p^p\|\widehat{v}\|_p^p\cdot 2^{d(p-1)n}\\ &<\frac12\left(\frac{\varepsilon}{3}\right)^{p/q_1}\end{aligned}\tag{4.16}$$

for sufficiently large $n > n_0(g_{\mu,\lambda}, m, \psi, p, \varepsilon)$, which together with (4.14), (4.15), and (4.16) implies that

$$I_2^{q_1} < \frac{\varepsilon}{3}. \tag{4.17}$$

Finally, combining (4.4), (4.10), (4.13), and (4.17) for appropriate $\lambda > \lambda_0$, $n > n_0$, and $m > m_0$, we obtain that $K\big(f, 1, L_q(\mathbb{R}^d), W_p^{\psi}(\mathbb{R}^d)\big)^{q_1} < \varepsilon$, which proves the theorem. □

Remark 1 If we suppose that $0 < q \le 1$ in Theorem 2, then (4.1) holds for any $\alpha > 0$. Indeed, according to [1, Theorem 5.1 and Corollary 2.2], there exists $g_\mu \in \mathcal{S}$ such that $\operatorname{supp} \widehat{g_\mu} \subset [-2^\mu, 2^\mu]^d \setminus [-1, 1]^d$ and $\|f - g_\mu\|_q^q < \frac{\varepsilon}{3}$. Thus, in the proof of Theorem 2, we can put $\lambda = 1$, $g_\mu = g_{\mu,1}$, and $f_{\mu,\lambda} = 0$, which implies that we do not need to estimate the term J_2, where the restriction $\alpha > d(1/p - 1)$ appears.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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