



Bernstein–Kantorovich operators, approximation and shape preserving properties

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Abstract

Inspired by the construction of Bernstein and Kantorovich operators, we introduce a family of positive linear operators \mathcal{K}_n preserving the affine functions. Their approximation properties are investigated and compared with similar properties of other operators. We determine the central moments of all orders of \mathcal{K}_n and use them in order to establish Voronovskaja type formulas. A special attention is paid to the shape preserving properties. The operators \mathcal{K}_n preserve monotonicity, convexity, strong convexity and approximate concavity. They have also the property of monotonic convergence under convexity. All the established inequalities involving convex functions can be naturally interpreted in the framework of convex stochastic ordering.

Keyword Bernstein–Kantorovich operators · Shape preserving properties · Inequalities for convex functions

Mathematics Subject Classification 41A36 · 26A51

1 Introduction

The classical Bernstein operators are the prototypical positive linear operators used in Approximation Theory. They approximate the continuous functions, preserve the functions $e_0(x) = 1$ and $e_1(x) = x$ and have remarkable shape preserving properties. The classical Kantorovich operators can be used to approximate integrable functions and preserve e_0 , but not e_1 . Their properties were intensively investigated from several points of view (see, e.g., [2, 3, 7, 8, 14–16] and the references therein). Some papers (see, e.g., [6, 28]) are devoted to modifying the Kantorovich operators in order to preserve e_0 and e_1 . Beside their importance

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in classical Approximation Theory, the Kantorovich polynomials play an important role in the theory of generalized sampling operators. The classical sampling operators are of discrete type and use point evaluations of the approximated function f . The Kantorovich type generalized operators use mean values of f on suitable intervals and consequently perform better than the classical ones from several points of view (see, e.g., [11–13] and the references therein).

In this paper we introduce a family of Bernstein–Kantorovich type operators \mathcal{K}_n preserving the affine functions. Their structure is inspired by the construction of the Kantorovich and Bernstein operators. The family depends on several parameters. In a limiting case the operators \mathcal{K}_n reduce to Bernstein operators. For special parameters one obtains the operators investigated in [28]. As mentioned above, the classical Kantorovich operators were the starting point for new developments in the theory of generalized sampling operators, leading to important results in digital image processing with medical and industrial applications. Our modified operators \mathcal{K}_n could also be used in this context.

The operators \mathcal{K}_n are defined as follows. For a given integer $n \geq 2$ let $a_{n,1}, \dots, a_{n,n-1}$ be real numbers such that

$$0 < a_{n,k} \leq \frac{1}{n}, \quad k = 1, \dots, n - 1. \tag{1.1}$$

Define the functionals $F_{n,k} : C[0, 1] \rightarrow \mathbb{R}$ as

$$F_{n,0}(f) := f(0), \quad F_{n,n}(f) := f(1),$$

$$F_{n,k}(f) := \frac{1}{2a_{n,k}} \int_{\frac{k}{n}-a_{n,k}}^{\frac{k}{n}+a_{n,k}} f(t)dt, \quad k = 1, \dots, n - 1. \tag{1.2}$$

Let $p_{n,k}(x) := \binom{n}{k}x^k(1-x)^{n-k}$, $k = 0, 1, \dots, n$, $x \in [0, 1]$.

Consider the operators \mathcal{K}_n defined by

$$\mathcal{K}_n f(x) := \sum_{k=0}^n F_{n,k}(f)p_{n,k}(x), \quad f \in C[0, 1], \quad x \in [0, 1]. \tag{1.3}$$

They are positive linear operators on $C[0, 1]$ and

$$\mathcal{K}_n e_0 = e_0, \quad \mathcal{K}_n e_1 = e_1, \tag{1.4}$$

where $e_j(x) := x^j$, $j = 0, 1, \dots$, $x \in [0, 1]$.

Section 2 contains the Voronovskaja type formula for the sequence \mathcal{K}_n , which coincides with the corresponding formula for the sequence of Bernstein operators B_n . Proposition 2.1 presents an example of a function for which the approximation in the sense of Voronovskaja’s formula, provided by \mathcal{K}_n , is better than that provided by the classical Kantorovich operators. The central moments of \mathcal{K}_n are described in Proposition 2.2 and used to present the Voronovskaja formula of order two for the sequence \mathcal{K}_n . The operators \mathcal{K}_n have useful shape preserving properties. Theorem 3.1 shows that they preserve the monotonicity. The images of a convex function f under B_n, \mathcal{K}_n and the genuine Bernstein–Durrmeyer operators U_n are compared in Sect. 4. The proof of the main result Theorem 4.1 is based on Ohlin’s Lemma, a result from the theory of convex stochastic ordering (see, e.g., [22, 23]). In fact, all the inequalities presented in this paper and involving convex functions have natural interpretations in the framework of convex stochastic ordering. Theorem 5.1 deals with the preservation of convexity. Theorems 5.2 and 5.3 describe the behavior of the operators \mathcal{K}_n with respect to the parameters in their structure. The monotonic convergence under convexity is an important

property of the Bernstein operators. Theorem 6.1 shows that the operators \mathcal{K}_n have also this property. Strongly convex functions and approximately concave functions were investigated in several papers. The preservation of the corresponding properties under the Bernstein operators was studied in [18]. The preservation of these properties under \mathcal{K}_n is presented in Sect. 7. Section 8 is devoted to conclusions and further work.

2 Voronovskaja type results

Let $B_n, n \geq 1$, be the classical Bernstein operators,

$$B_n f(x) := \sum_{k=0}^n f\left(\frac{k}{n}\right) p_{n,k}(x), \quad f \in C[0, 1], \quad x \in [0, 1].$$

According to Voronovskaja’s formula,

$$\lim_{n \rightarrow \infty} n(B_n f(x) - f(x)) = \frac{x(1-x)}{2} f''(x), \quad f \in C^2[0, 1], \tag{2.1}$$

uniformly on $[0, 1]$.

Theorem 2.1 *For each $f \in C^2[0, 1]$ we have*

$$\lim_{n \rightarrow \infty} n(\mathcal{K}_n f(x) - f(x)) = \frac{x(1-x)}{2} f''(x), \tag{2.2}$$

uniformly on $[0, 1]$.

Proof Let us remark that for $k = 1, \dots, n - 1$,

$$F_{n,k}(e_2) = \left(\frac{k}{n}\right)^2 + \frac{1}{3} a_{n,k}^2.$$

On the other hand, for $f \in C^2[0, 1]$ we have (see, e.g., [4, Lemma 4.1])

$$F_{n,k}(f) - f\left(\frac{k}{n}\right) = \left(F_{n,k}(e_2) - \left(\frac{k}{n}\right)^2\right) \frac{f''(\xi_{n,k})}{2}$$

with a certain $\xi_{n,k} \in [0, 1]$. Therefore,

$$F_{n,k}(f) - f\left(\frac{k}{n}\right) = \frac{1}{6} a_{n,k}^2 f''(\xi_{n,k}), \quad k = 1, \dots, n - 1. \tag{2.3}$$

Using (2.3) we can write

$$\begin{aligned} n(\mathcal{K}_n f(x) - f(x)) &= n(\mathcal{K}_n f(x) - B_n f(x)) + n(B_n f(x) - f(x)) \\ &= \frac{1}{6} n \sum_{k=1}^{n-1} p_{n,k}(x) a_{n,k}^2 f''(\xi_{n,k}) + n(B_n f(x) - f(x)). \end{aligned}$$

Due to (1.1) we have

$$\left| \frac{1}{6} n \sum_{k=1}^{n-1} p_{n,k}(x) a_{n,k}^2 f''(\xi_{n,k}) \right| \leq \frac{1}{6n} \|f''\|_\infty (1 - (1-x)^n - x^n) \rightarrow 0$$

uniformly on $[0, 1]$. Combined with (2.1), this leads to (2.2) and the proof is finished. \square

Let $K_n, n \geq 1$, be the classical Kantorovich operators,

$$K_n f(x) := (n + 1) \sum_{k=0}^n p_{n,k}(x) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t) dt, \quad f \in C[0, 1].$$

It is known (see [10]) that

$$\begin{aligned} & \lim_{n \rightarrow \infty} n(K_n f(x) - f(x)) \\ &= \frac{x(1-x)}{2} f''(x) + \left(\frac{1}{2} - x\right) f'(x) =: Vf(x), \quad f \in C^2[0, 1], \quad x \in [0, 1]. \end{aligned}$$

Set $Wf(x) := \frac{x(1-x)}{2} f''(x)$ (see (2.2)).

Proposition 2.1 *Let $f \in C^2[0, 1]$ be decreasing on $[0, 1]$, concave on $\left[0, \frac{1}{2}\right]$ and convex on $\left[\frac{1}{2}, 1\right]$.
Then,*

$$|Vf(x)| \geq |Wf(x)|, \quad x \in [0, 1]. \tag{2.4}$$

Proof It is easy to verify that for $x \in \left[0, \frac{1}{2}\right]$,

$$|Vf(x)| = |Wf(x)| - \left(\frac{1}{2} - x\right) f'(x) \geq |Wf(x)|,$$

while for $x \in \left[\frac{1}{2}, 1\right]$,

$$|Vf(x)| = |Wf(x)| + \left(\frac{1}{2} - x\right) f'(x) \geq |Wf(x)|.$$

This concludes the proof. □

The inequality (2.4) shows that, from the point of view of Voronovskaja’s formula, the approximation of f furnished by \mathcal{K}_n is better than that provided by K_n . For other results of this type see [5]. In fact, the following figures show that for the functions $f(x) = \cos(\pi x)$ and $f(x) = \arctan(x - 1/2), x \in [0, 1]$,

$$|\mathcal{K}_n f(x) - f(x)| \leq |K_n f(x) - f(x)|.$$

Remark 2.1 If $a_{n,k} := \frac{1}{n} \frac{k}{2k+1}, k = 1, \dots, n - 1$, the operators \mathcal{K}_n reduce to the operators K_n^* from [28]. A Voronovskaja type formula for the sequence (K_n^*) was proved in [28, p. 6194].

In the rest of this section we take $a_{n,k} = \frac{\theta}{n}, k = 1, \dots, n - 1$, for a given $\theta \in (0, 1]$ and we denote by \mathcal{K}_n^θ the corresponding operators \mathcal{K}_n . The next result presents the relation between central moments of \mathcal{K}_n^θ and those of B_n .

Fig. 1 Graph of $K_n f$, $\mathcal{K}_n f$ and f for $n = 10$ and $a_{n,k} = 1/2n$

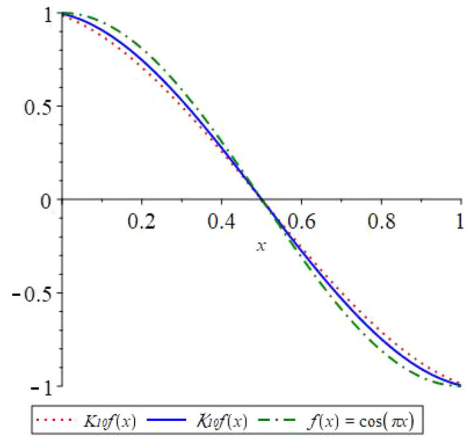


Fig. 2 Graph of $|K_n f - f|$ and $|\mathcal{K}_n f - f|$ for $n = 10$, $a_{n,k} = 1/2n$

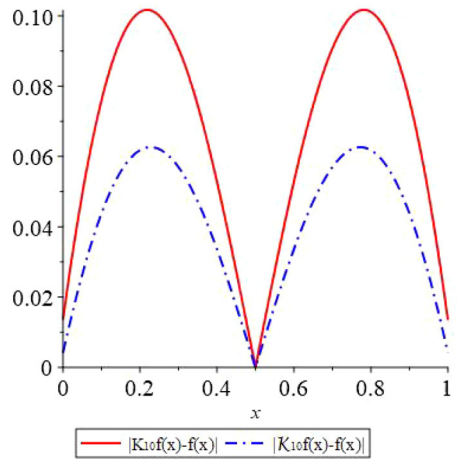


Fig. 3 Graph of $K_n f$, $\mathcal{K}_n f$ and f for $n = 4$ and $a_{n,k} = 1/2n$

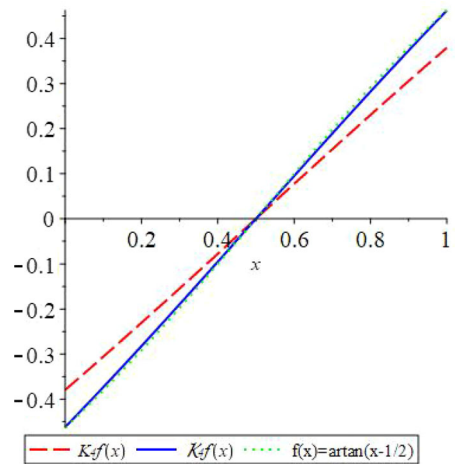
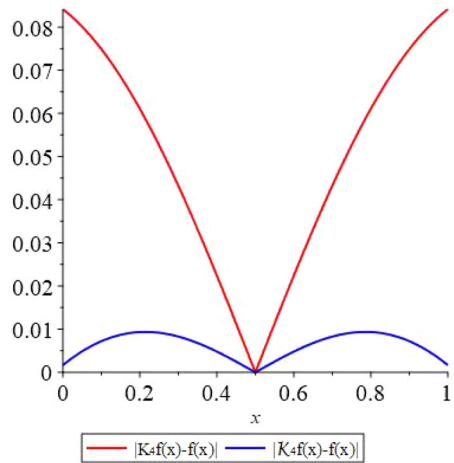


Fig. 4 Graph of $|\mathcal{K}_n f - f|$ and $|\mathcal{K}_n f - f|$ for $n = 4$, $a_{n,k} = 1/2n$



Proposition 2.2 For $k \geq 1$ one has

$$\begin{aligned}
 (i) \quad \mathcal{K}_n^\theta((t-x)^{2k}; x) &= B_n((t-x)^{2k}; x) + \sum_{j=1}^k \frac{1}{2j+1} \binom{2k}{2j} \\
 &\quad \times \left(\frac{\theta}{n}\right)^{2j} [B_n((t-x)^{2k-2j}; x) - (1-x)^n x^{2k-2j} - x^n (1-x)^{2k-2j}]; \\
 (ii) \quad \mathcal{K}_n^\theta((t-x)^{2k-1}; x) &= B_n((t-x)^{2k-1}; x) + \sum_{j=1}^{k-1} \frac{1}{2j+1} \binom{2k-1}{2j} \\
 &\quad \times \left(\frac{\theta}{n}\right)^{2j} [B_n((t-x)^{2k-1-2j}; x) + (1-x)^n x^{2k-1-2j} - x^n (1-x)^{2k-1-2j}].
 \end{aligned}$$

Proof The proof is based on straightforward calculation and we omit it. □

Remark 2.2 The central moments of Bernstein operators are investigated in detail in [9, Sect. 2.9]. In particular it is known that

$$B_n((t-x)^s; x) = \mathcal{O}\left(n^{-\left[\frac{s+1}{2}\right]}\right),$$

where $[a]$ is the integer part of a .

Combined with Proposition 2.2 this leads to

$$\mathcal{K}_n^\theta((t-x)^s; x) = \mathcal{O}\left(n^{-\left[\frac{s+1}{2}\right]}\right).$$

The important conclusion is that the classical result of Sikkema [26] can be applied to the sequence (\mathcal{K}_n^θ) . So, in this case we have another proof of Theorem 2.1 and moreover, for $f \in C^4[0, 1]$, $x \in [0, 1]$,

$$\begin{aligned}
 &\lim_{n \rightarrow \infty} n \left[n(\mathcal{K}_n^\theta f(x) - f(x)) - \frac{x(1-x)}{2} f^{(2)}(x) \right] \\
 &= \begin{cases} 0, & x \in \{0, 1\}, \\ \frac{1}{6} f^{(2)}(x) + \frac{1}{6} x(1-x)(1-2x) f^{(3)}(x) + \frac{1}{8} x^2(1-x)^2 f^{(4)}(x), & 0 < x < 1. \end{cases}
 \end{aligned}$$

It is well known that (see, e.g., [1])

$$\begin{aligned} & \lim_{n \rightarrow \infty} n \left[n (B_n f(x) - f(x)) - \frac{x(1-x)}{2} f^{(2)}(x) \right] \\ &= \frac{1}{6} x(1-x)(1-2x) f^{(3)}(x) + \frac{1}{8} x^2(1-x)^2 f^{(4)}(x), \quad 0 \leq x \leq 1. \end{aligned}$$

We conclude that

$$\lim_{n \rightarrow \infty} n^2 (\mathcal{K}_n^\theta f(x) - B_n f(x)) = \begin{cases} 0, & x \in \{0, 1\}, \\ \frac{1}{6} f^{(2)}(x), & 0 < x < 1. \end{cases}$$

3 \mathcal{K}_n and increasing functions

Consider the general case, i.e., $0 < a_{n,k} \leq 1/n$.

Theorem 3.1 *If $f \in C[0, 1]$ is increasing, then $\mathcal{K}_n f$ is increasing.*

Proof Using $p'_{n,k}(x) = n (p_{n-1,k-1}(x) - p_{n-1,k}(x))$, $n \geq 1$, we get

$$(\mathcal{K}_n f)'(x) = n \sum_{k=0}^{n-1} (F_{n,k+1}(f) - F_{n,k}(f)) p_{n-1,k}(x).$$

Clearly

$$F_{n,1}(f) - F_{n,0}(f) = \frac{1}{2a_{n,1}} \int_{\frac{1}{n}-a_{n,1}}^{\frac{1}{n}+a_{n,1}} f(t) dt - f(0) \geq 0$$

and

$$F_{n,n}(f) - F_{n,n-1}(f) \geq 0.$$

So, let $k \in \{1, \dots, n-1\}$. Then

$$\begin{aligned} F_{n,k+1}(f) - F_{n,k}(f) &= \frac{1}{2a_{n,k+1}} \int_{\frac{k+1}{n}-a_{n,k+1}}^{\frac{k+1}{n}+a_{n,k+1}} f(t) dt - \frac{1}{2a_{n,k}} \int_{\frac{k}{n}-a_{n,k}}^{\frac{k}{n}+a_{n,k}} f(t) dt \\ &= \frac{1}{2a_{n,k}} \int_{\frac{k}{n}-a_{n,k}}^{\frac{k}{n}+a_{n,k}} \left[f \left(\frac{a_{n,k+1}}{a_{n,k}} \left(s - \frac{k}{n} \right) + \frac{k+1}{n} \right) - f(s) \right] ds. \end{aligned}$$

Moreover, for $s \in \left[\frac{k}{n} - a_{n,k}, \frac{k}{n} + a_{n,k} \right]$ we have

$$\frac{a_{n,k+1}}{a_{n,k}} \left(s - \frac{k}{n} \right) + \frac{k+1}{n} \geq s$$

and so

$$F_{n,k+1}(f) - F_{n,k}(f) \geq 0.$$

Summing-up, we see that $(\mathcal{K}_n f)' \geq 0$. □

4 Approximating convex functions

A well known consequence of (1.4) is

Proposition 4.1 *If $f \in C[0, 1]$ is convex, then*

$$\mathcal{K}_n f \geq f, \quad n \geq 1. \tag{4.1}$$

In order to prove the main result, Theorem 4.1, we need two lemmas.

Lemma 4.1 *Let $i \geq 1, j \geq 1$ be integers. Then*

$$\gamma_{i,j} := \frac{(i+j)!}{(i+j)^{i+j}} \frac{i^i}{i!} \frac{j^j}{j!} \leq \frac{1}{2}. \tag{4.2}$$

Proof First, we have

$$\frac{\gamma_{i,j+1}}{\gamma_{i,j}} = \left(1 + \frac{1}{j}\right)^j \left(1 + \frac{1}{i+j}\right)^{-(i+j)} < 1.$$

Then $\gamma_{i,j} \leq \gamma_{i,1} = \left(1 + \frac{1}{i}\right)^{-i} \leq \frac{1}{2}$, and the proof is completed. □

Lemma 4.2 (Ohlin’s Lemma) [22] *Let X, Y be two random variables such that $\mathbb{E}X = \mathbb{E}Y$. If the distribution functions F_X, F_Y cross exactly one time, i.e., for some x_0 holds*

$$F_X(x) \leq F_Y(x) \text{ if } x < x_0 \text{ and } F_X(x) \geq F_Y(x) \text{ if } x > x_0,$$

then $\mathbb{E}f(X) \leq \mathbb{E}f(Y)$, for all convex functions $f : \mathbb{R} \rightarrow \mathbb{R}$.

Remark 4.1 Szostok noticed in [27] that if the measures μ_X, μ_Y corresponding to X, Y , respectively, are concentrated on the interval $[a, b]$, then, in fact, the relation $\mathbb{E}f(X) \leq \mathbb{E}f(Y)$ holds for all convex functions $f : \mathbb{R} \rightarrow \mathbb{R}$ if and only if this inequality is satisfied for all continuous convex functions $f : [a, b] \rightarrow \mathbb{R}$.

Consider the genuine Bernstein–Durrmeyer operators U_n defined by

$$U_n f(x) := f(0)p_{n,0}(x) + f(1)p_{n,n}(x) + (n-1) \sum_{k=1}^{n-1} \left(\int_0^1 p_{n-2,k-1}(t)f(t)dt \right) p_{n,k}(x).$$

Now we are in a position to state

Theorem 4.1 *If $a_{n,k} := \frac{1}{n}, 1 \leq k \leq n-1$, and $f \in C[0, 1]$ is convex, then*

$$f \leq B_n f \leq \mathcal{K}_n f \leq U_n f. \tag{4.3}$$

Proof The first inequality is well known, as a consequence of the fact that B_n preserves the affine functions. According to the Hermite–Hadamard inequality we have

$$f\left(\frac{k}{n}\right) \leq \frac{n}{2} \int_{\frac{k-1}{n}}^{\frac{k+1}{n}} f(t)dt, \quad k = 1, \dots, n-1,$$

and this implies the second inequality in (4.3). So, it remains to prove that

$$\frac{n}{2} \int_{\frac{k-1}{n}}^{\frac{k+1}{n}} f(t)dt \leq (n-1) \int_0^1 p_{n-2,k-1}(t)f(t)dt, \quad k = 1, \dots, n-1. \tag{4.4}$$

To this end, fix $k \in \{1, \dots, n-1\}$ and consider a random variable X uniformly distributed on $\left[\frac{k-1}{n}, \frac{k+1}{n}\right]$ and a Beta-type random variable Y having the density $(n-1)p_{n-2,k-1}(t)$, $t \in [0, 1]$. The distribution function of X is

$$F_X(x) = \begin{cases} 0, & x \leq \frac{k-1}{n}, \\ \frac{1}{2}(nx - k + 1), & \frac{k-1}{n} < x \leq \frac{k+1}{n}, \\ 1, & x > \frac{k+1}{n}, \end{cases}$$

and the distribution function of Y is

$$F_Y(x) = \begin{cases} 0, & x \leq 0, \\ \int_0^x (n-1)p_{n-2,k-1}(t)dt, & x \in (0, 1], \\ 1, & x > 1. \end{cases}$$

We have $\mathbb{E}X = \mathbb{E}Y = \frac{k}{n}, k = 1, \dots, n-1$.

(1) Let $k = 1$. Then

$$F_X(x) = \begin{cases} 0, & x \leq 0, \\ \frac{1}{2}nx, & 0 < x \leq \frac{2}{n}, \\ 1, & x > \frac{2}{n}, \end{cases}$$

and

$$F_Y(x) = \begin{cases} 0, & x \leq 0, \\ 1 - (1-x)^{n-1}, & 0 < x \leq 1, \\ 1, & x \geq 1. \end{cases}$$

It is easy to prove the existence of $x_0 \in (0, 1)$ from Ohlin’s Lemma.

(2) The case $k = n-1$ can be treated similarly.

(3) It remains to consider the case $2 \leq k \leq n-2$. Define $H(x) := F_Y(x) - F_X(x), x \in \mathbb{R}$. Clearly

$$H(x) \geq 0, \quad x \in \left(-\infty, \frac{k-1}{n}\right) \text{ and } H(x) \leq 0, \quad x \in \left(\frac{k+1}{n}, \infty\right).$$

Moreover, $H\left(\frac{k-1}{n}\right) > 0$ and $H\left(\frac{k+1}{n}\right) < 0$. For $\frac{k-1}{n} < x < \frac{k+1}{n}$ we have

$$\begin{aligned} H'(x) &= (n-1)p_{n-2,k-1}(x) - \frac{n}{2} \\ &\leq (n-1)p_{n-2,k-1}\left(\frac{k-1}{n-2}\right) - \frac{n}{2} \end{aligned}$$

$$= (n - 1) \frac{(n - 2)!}{(n - 2)^{n-2}} \frac{(k - 1)^{k-1}}{(k - 1)!} \frac{(n - k - 1)^{n-k-1}}{(n - k - 1)!} - \frac{n}{2}.$$

Using Lemma 4.1 we get

$$H'(x) \leq \frac{n - 1}{2} - \frac{n}{2} = -\frac{1}{2}.$$

Therefore H is strictly decreasing on $\left[\frac{k - 1}{n}, \frac{k + 1}{n}\right]$ and the existence of x_0 from Ohlin’s Lemma is proved. Summing-up, according to Ohlin’s Lemma we have $\mathbb{E}f(X) \leq \mathbb{E}f(Y)$ and this is (4.4). \square

Returning to the general case with $0 < a_{n,k} \leq \frac{1}{n}$ we have the following two remarks

Remark 4.2 Recall the operators K_n^* from [28] (see also Remark 2.1). Using the technique based on Ohlin’s lemma, it is not difficult to prove that for each convex function $f \in C[0, 1]$ we have

- (i) If $0 < a_{n,k} \leq \frac{1}{3n}$, $k = 1, \dots, n - 1$, then $f \leq \mathcal{K}_n f \leq K_n^* f$;
- (ii) If $\frac{n - 1}{2n - 1} \frac{1}{n} \leq a_{n,k} \leq \frac{1}{n}$, $k = 1, \dots, n - 1$, then $f \leq K_n^* f \leq \mathcal{K}_n f$.

Remark 4.3 We have

$$\begin{aligned} \mathcal{K}_n e_2(x) - e_2(x) &= \frac{x(1 - x)}{n} + \frac{1}{3} a_{n,k}^2 [1 - (1 - x)^n - x^n] \leq \frac{4}{3} \frac{x(1 - x)}{n}, \\ U_n e_2(x) - e_2(x) &= \frac{2x(1 - x)}{n + 1}. \end{aligned}$$

Therefore, in light of the classical result of Shisha–Mond [25],

$$\begin{aligned} |\mathcal{K}_n f(x) - f(x)| &\leq 2\omega \left(f; \sqrt{\frac{4}{3} \frac{x(1 - x)}{n}} \right), \\ |U_n f(x) - f(x)| &\leq 2\omega \left(f; \sqrt{\frac{2x(1 - x)}{n + 1}} \right). \end{aligned}$$

So, from this point of view, the approximation of a function $f \in C[0, 1]$ provided by \mathcal{K}_n is better than that provided by U_n .

5 \mathcal{K}_n and convex functions

In this section we take $a_{n,k} := \frac{\theta}{n}$, $k = 1, \dots, n - 1$, for a given $\theta \in (0, 1]$. Consequently, we denote the functionals by $F_{n,k}^\theta$ and the operators by \mathcal{K}_n^θ . So, we have

$$F_{n,k}^\theta(f) := \frac{n}{2\theta} \int_{\frac{k-\theta}{n}}^{\frac{k+\theta}{n}} f(t) dt, \quad k = 1, \dots, n - 1.$$

Theorem 5.1 *If $f \in C[0, 1]$ is convex, then $\mathcal{K}_n^\theta f$ is convex.*

Proof First, we have

$$(\mathcal{K}_n^\theta f)''(x) = n(n-1) \sum_{k=0}^{n-2} (F_{n,k+2}^\theta(f) - 2F_{n,k+1}^\theta(f) + F_{n,k}^\theta(f)) p_{n-2,k}(x)$$

- (i) In order to prove that $F_{n,2}^\theta(f) - 2F_{n,1}^\theta(f) + F_{n,0}^\theta(f) \geq 0$, it suffices to show that it holds for the functions $\varphi(t) = 1, \varphi(t) = t, \varphi(t) = \max\{t - x, 0\}, t, x \in [0, 1]$ (see [19, p. 645, B.4. Proposition and B.4.a. Proposition]). This can be done by elementary calculations.
- (ii) The proof of the inequality

$$F_{n,n}^\theta(f) - 2F_{n,n-1}^\theta(f) + F_{n,n-2}^\theta(f) \geq 0$$

is similar.

- (iii) It remains to prove that

$$F_{n,k+2}^\theta(f) - 2F_{n,k+1}^\theta(f) + F_{n,k}^\theta(f) \geq 0, \quad k = 1, \dots, n - 3.$$

This follows by integrating

$$f\left(t - \frac{1}{n}\right) + f\left(t + \frac{1}{n}\right) \geq 2f(t), \quad \frac{1}{n} \leq t \leq \frac{n-1}{n}$$

on the interval $\left[\frac{k+1-\theta}{n}, \frac{k+1+\theta}{n}\right]$.

□

Theorem 5.2 *If $f \in C[0, 1]$ is convex, then $\mathcal{K}_n^\sigma f \leq \mathcal{K}_n^\tau f$ whenever $0 < \sigma < \tau \leq 1$.*

Proof It suffices to prove that $F_{n,k}^\sigma(f) \leq F_{n,k}^\tau(f), k = 1, \dots, n - 1$. In fact, we will prove that

$$\frac{d}{d\theta} F_{n,k}^\theta(f) \geq 0, \quad 0 < \theta \leq 1.$$

Indeed,

$$\begin{aligned} \frac{d}{d\theta} F_{n,k}^\theta(f) &= -\frac{n}{2\theta^2} \int_{\frac{k-\theta}{n}}^{\frac{k+\theta}{n}} f(t)dt + \frac{n}{2\theta} \left[\frac{1}{n} f\left(\frac{k+\theta}{n}\right) + \frac{1}{n} f\left(\frac{k-\theta}{n}\right) \right] \\ &= \frac{1}{\theta} \left[\frac{f\left(\frac{k-\theta}{n}\right) + f\left(\frac{k+\theta}{n}\right)}{2} - \frac{n}{2\theta} \int_{\frac{k-\theta}{n}}^{\frac{k+\theta}{n}} f(t)dt \right] \geq 0, \end{aligned} \tag{5.1}$$

where the last inequality follows from the Hermite–Hadamard inequality. □

In the next result we estimate the difference between $\mathcal{K}_n^\tau f$ and $\mathcal{K}_n^\sigma f$.

Theorem 5.3 *Let $0 < \sigma < \tau \leq 1, f \in C^2[0, 1]$ and $x \in [0, 1]$. Then*

$$|\mathcal{K}_n^\tau f(x) - \mathcal{K}_n^\sigma f(x)| \leq \tau(\tau - \sigma) \frac{1}{3n^2} \|f''\|_\infty (1 - (1-x)^n - x^n),$$

where $\|\cdot\|_\infty$ is the supremum norm.

Proof Denote $\psi_{n,k}(\theta) := F_{n,k}^\theta(f)$, $\theta \in (0, 1]$. We have, for a suitable $\theta \in (\sigma, \tau)$,

$$|\psi_{n,k}(\tau) - \psi_{n,k}(\sigma)| = (\tau - \sigma)|\psi'_{n,k}(\theta)|. \tag{5.2}$$

Using (5.1) we can write

$$\frac{d}{d\theta} \psi_{n,k}(\theta) = \frac{1}{\theta} \left[\frac{f\left(\frac{k-\theta}{n}\right) + f\left(\frac{k+\theta}{n}\right)}{2} - \frac{n}{2\theta} \int_{\frac{k-\theta}{n}}^{\frac{k+\theta}{n}} f(t)dt \right]. \tag{5.3}$$

Applying the trapezoidal rule we get

$$\left| \frac{f\left(\frac{k-\theta}{n}\right) + f\left(\frac{k+\theta}{n}\right)}{2} - \frac{n}{2\theta} \int_{\frac{k-\theta}{n}}^{\frac{k+\theta}{n}} f(t)dt \right| \leq \frac{\theta^2}{3n^2} \|f''\|_\infty. \tag{5.4}$$

From (5.3) and (5.4) it follows that

$$|\psi'_{n,k}(\theta)| \leq \frac{\tau}{3n^2} \|f''\|_\infty.$$

Combined with (5.2) this leads to

$$|F_{n,k}^\tau(f) - F_{n,k}^\sigma(f)| \leq \tau(\tau - \sigma) \frac{1}{3n^2} \|f''\|_\infty.$$

Therefore,

$$\begin{aligned} |\mathcal{K}_n^\tau f(x) - \mathcal{K}_n^\sigma f(x)| &\leq \sum_{k=1}^{n-1} p_{n,k}(x) |F_{n,k}^\tau(f) - F_{n,k}^\sigma(f)| \\ &\leq \tau(\tau - \sigma) \frac{1}{3n^2} \|f''\|_\infty (1 - (1-x)^n - x^n) \end{aligned}$$

and this concludes the proof. □

Remark 5.1 For $\sigma \rightarrow 0$, from Theorem 5.3 we get

$$|\mathcal{K}_n^\tau f(x) - B_n f(x)| \leq \frac{\tau^2}{3n^2} \|f''\|_\infty (1 - (1-x)^n - x^n).$$

Estimating directly with (2.3) we obtain

$$\left| F_{n,k}^\tau(f) - f\left(\frac{k}{n}\right) \right| = \left(F_{n,k}^\tau(e_2) - \left(\frac{k}{n}\right)^2 \right) \frac{|f''(\xi)|}{2} \leq \frac{1}{6} \left(\frac{\tau}{n}\right)^2 \|f''\|_\infty.$$

This produces the better result

$$|\mathcal{K}_n^\tau f(x) - B_n f(x)| \leq \frac{\tau^2}{6n^2} \|f''\|_\infty (1 - (1-x)^n - x^n).$$

6 Monotonic convergence under convexity

In this section we consider again the case $a_{n,k} = \theta/n, n \geq 1, k = 1, \dots, n - 1$, for a given $\theta \in (0, 1]$.

Theorem 6.1 *If $f \in C[0, 1]$ is convex, then $\mathcal{K}_n^\theta f \geq \mathcal{K}_{n+1}^\theta f, n \geq 2$.*

Proof First, we have

$$\begin{aligned} & \mathcal{K}_n^\theta f(x) - \mathcal{K}_{n+1}^\theta f(x) \\ &= \sum_{k=1}^n \binom{n+1}{k} \left[\frac{k}{n+1} F_{n,k-1}^\theta(f) + \frac{n+1-k}{n+1} F_{n,k}^\theta(f) - F_{n+1,k}^\theta(f) \right] x^k(1-x)^{n+1-k}. \end{aligned} \tag{6.1}$$

The proof is similar to that of Proposition 2.10 from [9] and we omit it. It follows that

$$\begin{aligned} & \mathcal{K}_n^\theta f(x) - \mathcal{K}_{n+1}^\theta f(x) \\ &= \frac{n(n+1)}{2\theta} x(1-x)^n \int_{\frac{1-\theta}{n}}^{\frac{1+\theta}{n}} \left[\frac{1}{n+1} f(0) + \frac{n}{n+1} f(s) - f\left(\frac{n}{n+1}s\right) \right] ds \\ &+ \frac{n(n+1)}{2\theta} x^n(1-x) \int_{\frac{n-1-\theta}{n}}^{\frac{n-1+\theta}{n}} \left[\frac{n}{n+1} f(s) + \frac{1}{n+1} f(1) - f\left(\frac{n}{n+1}s + \frac{1}{n+1}\right) \right] ds \\ &+ \frac{n}{2\theta} \sum_{k=2}^{n-1} x^k(1-x)^{n+1-k} \binom{n+1}{k} \\ &\times \int_{\frac{k-1-\theta}{n}}^{\frac{k-1+\theta}{n}} \left[\frac{k}{n+1} f(s) + \frac{n+1-k}{n+1} f\left(s + \frac{1}{n}\right) - f\left(\frac{n}{n+1}s + \frac{1}{n+1}\right) \right] ds \\ &= I_1 + I_2 + I_3. \end{aligned}$$

Since f is convex, by Jensen inequality we have

$$\begin{aligned} & \frac{1}{n+1} f(0) + \frac{n}{n+1} f(s) - f\left(\frac{n}{n+1}s\right) \geq 0, \\ & \frac{n}{n+1} f(s) + \frac{1}{n+1} f(1) - f\left(\frac{n}{n+1}s + \frac{1}{n+1}\right) \geq 0, \end{aligned}$$

and so $I_1 \geq 0, I_2 \geq 0$. Concerning I_3 , we have

$$\begin{aligned} \tilde{I}_3 &:= \int_{\frac{k-1-\theta}{n}}^{\frac{k-1+\theta}{n}} \left[\frac{k}{n+1} f(s) + \frac{n+1-k}{n+1} f\left(s + \frac{1}{n}\right) - f\left(\frac{n}{n+1}s + \frac{1}{n+1}\right) \right] ds \\ &\geq \int_{\frac{k-1-\theta}{n}}^{\frac{k-1+\theta}{n}} \left[f\left(s + \frac{n+1-k}{n(n+1)}\right) - f\left(\frac{n}{n+1}s + \frac{1}{n+1}\right) \right] ds \\ &= \int_{\frac{k}{n+1} - \frac{\theta}{n}}^{\frac{k}{n+1} + \frac{\theta}{n}} f(s) ds - \frac{n+1}{n} \int_{\frac{k-\theta}{n+1}}^{\frac{k+\theta}{n+1}} f(s) ds. \end{aligned}$$

Let $y(s)$ be a polynomial function of degree at most 1 whose graph is a support line to the graph of $f(s)$ at $s = \frac{k}{n+1}$. Then

$$\int_{\frac{k}{n+1} - \frac{\theta}{n}}^{\frac{k}{n+1} + \frac{\theta}{n}} y(s) ds = \frac{n+1}{n} \int_{\frac{k-\theta}{n+1}}^{\frac{k+\theta}{n+1}} y(s) ds,$$

so that

$$\tilde{I}_3 \geq \int_{\frac{k}{n+1} - \frac{\theta}{n}}^{\frac{k}{n+1} + \frac{\theta}{n}} (f(s) - y(s)) ds - \frac{n+1}{n} \int_{\frac{k-\theta}{n+1}}^{\frac{k+\theta}{n+1}} (f(s) - y(s)) ds.$$

Setting $g(s) := f(s) - y(s)$ we have $g(s) \geq 0$ and the above inequality can be written as

$$\begin{aligned} \tilde{I}_3 \geq & \left(\int_{\frac{k}{n+1} - \frac{\theta}{n}}^{\frac{k-\theta}{n+1}} g(s) ds - \frac{1}{n} \int_{\frac{k-\theta}{n+1}}^{\frac{k}{n+1}} g(s) ds \right) \\ & + \left(\int_{\frac{k}{n+1} + \frac{\theta}{n}}^{\frac{k+\theta}{n+1}} g(s) ds - \frac{1}{n} \int_{\frac{k}{n+1}}^{\frac{k+\theta}{n+1}} g(s) ds \right) = J_1 + J_2. \end{aligned}$$

Let $h := g\left(\frac{k+\theta}{n+1}\right)$. With notation from Fig. 5, using the convexity of g and elementary geometric considerations we find that $H = \frac{n+1}{n}h$,

$$\begin{aligned} \int_{\frac{k}{n+1}}^{\frac{k}{n+1} + \frac{\theta}{n}} g(s) ds & \geq \frac{h+H}{2} \frac{\theta}{n(n+1)} = \frac{(2n+1)h\theta}{2n^2(n+1)}, \\ \int_{\frac{k}{n+1}}^{\frac{k+\theta}{n+1}} g(s) ds & \leq \frac{h\theta}{2(n+1)}. \end{aligned}$$

Therefore, $J_2 \geq 0$, and similarly $J_1 \geq 0$. We conclude that $\tilde{I}_3 \geq 0$, hence $I_3 \geq 0$, and the proof is complete. □

Remark 6.1 Let μ, ν be probability distributions (Borel measures on \mathbb{R} , $\mu(\mathbb{R}) = \nu(\mathbb{R}) = 1$). If $\int \varphi(x) d\nu(x) \leq \int \varphi(x) d\mu(x)$, for each $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ convex, ν is said to be smaller than μ in the convex stochastic order. One uses the notation $\nu \leq_{cx} \mu$ (see, e.g., [23, 24]).

The operators in this paper can be represented under the form

$$\mathcal{L}_n f = \int f d\mu_n, \quad \mathcal{M}_n f = \int f d\nu_n,$$

with suitable probability distributions μ_n, ν_n .

Moreover, they satisfy inequalities of the form (see Theorems 4.1, 5.2, 6.1)

$$\mathcal{M}_n f \leq \mathcal{L}_n f, \text{ for all convex functions } f.$$

This is equivalent to

$$\int f d\nu_n \leq \int f d\mu_n, \text{ for all convex functions } f,$$

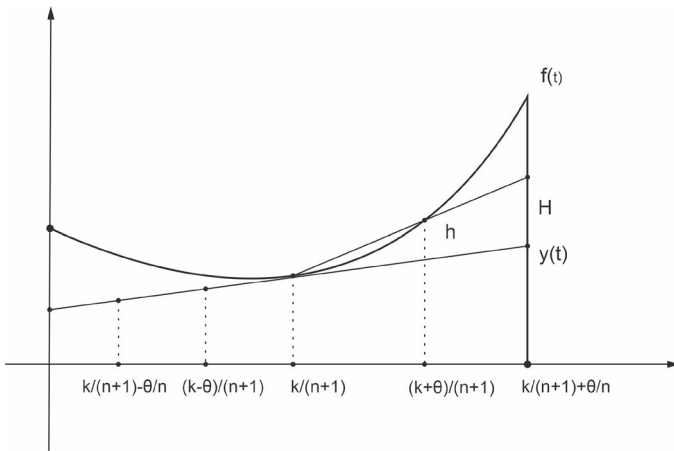


Fig. 5 Support for the proof of Theorem 6.1

and hence to

$$v_n \leq c x \mu_n.$$

Therefore, Theorems 4.1, 5.2, 6.1 have natural interpretations in the theory of convex stochastic ordering.

7 \mathcal{K}_n^θ and strongly convex functions with modulus c

For the definitions of strongly convex functions and approximately concave functions with modulus c see, e.g., [17, 18, 20, 21] and the references therein. We need the following characterizations of these functions.

Lemma 7.1 (i) A function $f : I \rightarrow \mathbb{R}$ is strongly convex with modulus $c > 0$ if and only if the function $g : I \rightarrow \mathbb{R}$ defined by $g = f - ce_2$ is convex.

(ii) A function $f : I \rightarrow \mathbb{R}$ is approximately concave with modulus $c > 0$ if and only if the function $g : I \rightarrow \mathbb{R}$ defined by $g = f - ce_2$ is concave.

Theorem 7.1 (i) If $f \in C[0, 1]$ is strongly convex with modulus c , then \mathcal{K}_n^θ is strongly convex with modulus $c \left(1 - \frac{\theta^2}{6}\right) \frac{n-1}{n}$.

(ii) Let $f \in C[0, 1]$ be approximately concave with modulus c , then \mathcal{K}_n^θ is approximately concave with modulus $c \left(1 - \frac{\theta^2}{3 \cdot 2^{n-2}}\right) \frac{n-1}{n}$.

Proof (i) Let $f \in C[0, 1]$ be strongly convex with modulus c . Then $f - ce_2$ is convex. According to Theorem 5.1, $\mathcal{K}_n^\theta f - c\mathcal{K}_n^\theta e_2$ is convex. Therefore,

$$(\mathcal{K}_n^\theta f)'' \geq c(\mathcal{K}_n^\theta e_2)''.$$

We have

$$\mathcal{K}_n^\theta e_2(x) = \frac{n-1}{n}x^2 + \frac{1}{n}x + \frac{\theta^2}{3n^2} [1 - (1-x)^n - x^n],$$

and so

$$\begin{aligned}
 (\mathcal{K}_n^\theta e_2)''(x) &= 2 \frac{n-1}{n} - \frac{\theta^2}{3} \frac{n-1}{n} [(1-x)^{n-2} + x^{n-2}] \\
 &\geq \left(2 - \frac{\theta^2}{3}\right) \frac{n-1}{n}.
 \end{aligned}$$

It follows that

$$(\mathcal{K}_n^\theta e_2)''(x) \geq c \left(2 - \frac{\theta^2}{3}\right) \frac{n-1}{n},$$

i.e., $\mathcal{K}_n^\theta f - c \left(1 - \frac{\theta^2}{6}\right) \frac{n-1}{n} e_2$ is convex. This shows that $\mathcal{K}_n^\theta f$ is strongly convex with modulus $c \left(1 - \frac{\theta^2}{6}\right) \frac{n-1}{n}$.

(ii) The proof is similar to the previous one and we omit it. □

8 Conclusions and further work

Let $c \in \mathbb{R}$, $n \in \mathbb{R}$, $n > c$ for $c \geq 0$ and $-n/c = l \in \mathbb{N}$ for $c < 0$. Furthermore, let $I_c = [0, \infty)$ for $c \geq 0$ and $I_c = [0, -1/c]$ for $c < 0$. Take $f : I_c \rightarrow \mathbb{R}$ given in such a way that the corresponding integrals and series are convergent.

Let $0 < a_{n,k} < \frac{1}{n}$. Define

$$\mathcal{K}_n^{[c]}(f; x) = f(0) p_{n,0}^{[c]}(x) + \frac{n}{2} \sum_{k=1}^{\infty} p_{n,k}^{[c]}(x) \int_{\frac{k}{n}-a_{n,k}}^{\frac{k}{n}+a_{n,k}} f(t) dt, \quad c \geq 0,$$

$$\mathcal{K}_n^{[c]}(f; x) = f(0) p_{n,0}^{[c]}(x) + f\left(\frac{1}{n}\right) p_{n,l}^{[c]}(x) + \frac{n}{2} \sum_{k=1}^{l-1} p_{n,k}^{[c]}(x) \int_{\frac{k}{n}-a_{n,k}}^{\frac{k}{n}+a_{n,k}} f(t) dt, \quad c < 0,$$

with the corresponding basis functions

$$p_{n,k}^{[c]}(x) = \begin{cases} \frac{n^k}{k!} x^k e^{-nx} & , c = 0, \\ \frac{n^{c,\bar{k}}}{k!} x^k (1 + cx)^{-\left(\frac{n}{c}+k\right)} & , c \neq 0, \end{cases} \tag{8.1}$$

and $a^{c,\bar{k}} := \prod_{l=0}^{k-1} (a + cl)$, $a^{c,\bar{0}} := 1$. As further work we propose to investigate the operators $\mathcal{K}_n^{[c]}$.

We intend to consider all these modified Kantorovich operators in the framework of generalized sampling operators with applications in medical and industrial domains.

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Data availability This manuscript has no associated data.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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References

1. Abel, U., Ivan, M.: Asymptotic expansion of the multivariate Bernstein polynomials on a simplex. *Approx. Theory Appl.* **16**, 85–93 (2000)
2. Abel, U., Gupta, V.: An estimate of the rate of convergence of a Bézier variant of the Baskakov–Kantorovich operators for bounded variation functions. *Demonstr. Math.* **36**(1), 123–136 (2003)
3. Acu, A.M., Gonska, H.: Classical Kantorovich operators revisited. *Ukr. Math. J.* **71**, 843–852 (2019)
4. Acu, A.M., Raşa, I.: Estimates for the differences of positive linear operators and their derivatives. *Numer. Algorithms* **85**, 191–208 (2020)
5. Acu, A.M., Măduţa, A.I., Raşa, I.: Voronovskaya type results and operators fixing two functions. *Math. Model. Anal.* **26**(3), 395–410 (2021)
6. Agratini, O.: Kantorovich-type operators preserving affine functions. *Hacet. J. Math. Stat.* **45**(6), 1657–1663 (2016)
7. Altomare, F., Cappelletti Montano, M., Leonessa, V.: Iterates of multidimensional Kantorovich-type operators and their associated positive C_0 -semigroups. *Stud. Univ. Babeş–Bolyai Math.* **56**(2), 219–235 (2011)
8. Altomare, F., Cappelletti Montano, M., Leonessa, V., Raşa, I.: Elliptic differential operators and positive semigroups associated with generalized Kantorovich operators. *J. Math. Anal. Appl.* **458**, 153–173 (2018)
9. Bustamante, J.: *Bernstein operators and their properties*, Springer (2017)
10. Butzer, P.L.: Summability of generalized Bernstein polynomials, I. *Duke Math. J.* **22**, 617–623 (1955)
11. Costarelli, D., Seracini, M., Vinti, G.: A comparison between the sampling Kantorovich algorithm for digital image processing with some interpolation and quasi-interpolation methods. *Appl. Math. Comput.* **374**, 125046 (2020)
12. Costarelli, D., Vinti, G.: An inverse result of approximation by sampling Kantorovich series. *Proc. Edinburgh Math. Soc.* **62**(1), 265–280 (2019)
13. Costarelli, D., Vinti, G.: Inverse results of approximation and saturation order for the sampling Kantorovich series. *J. Approx. Theory* **242**, 64–82 (2019)
14. Gonska, H., Heilmann, M., Raşa, I.: Kantorovich operators of order k . *Numer. Funct. Anal. Optim.* **32**, 717–738 (2011)
15. Gupta, V., Anjali: Higher order Kantorovich operators based on inverse Pólya–Eggenberger distribution. *RACSAM* **116**, 31 (2022)
16. Gupta, V., Tachev, G., Acu, A.M.: Modified Kantorovich operators with better approximation properties. *Numer. Algorithms* **81**, 125–149 (2019)
17. Hiriart-Urruty, J.-B., Lemaréchal, C.: *Fundamentals of Convex Analysis*. Springer, Berlin (2001)
18. Hodiş, S., Mesaroş, L., Raşa, I.: Smoothness and shape preserving properties of Bernstein semigroup. *Mediterr. J. Math.* **15**, 96 (2018)
19. Marshall, A. W., Olkin, I., Arnold, B. C.: *Inequalities: Theory of Majorization and Its Applications*, Second Edition, Springer, (2011)
20. Merentes, N., Nikodem, K.: Remarks on strongly convex functions. *Aequ. Math.* **80**, 193–199 (2010)
21. Nikodem, K., Rajba, T.: Ohlin and Levin–Stečkin-type results for strongly convex functions. *Ann. Math. Sil.* **34**(1), 123–132 (2020)
22. Ohlin, J.: On a class of measures of dispersion with application to optimal reinsurance. *Astin Bull.* **5**(2), 249–266 (1969)
23. Rajba, T.: On some recent applications of stochastic convex ordering theorems to some functional inequalities for convex functions: a survey, In: Brzdęk, J., Cieplinski, K., Rassias, T. (eds.) *Developments in*

- Functional Equations and Related Topics, Springer Optimization and Its Applications, vol 124, Springer, Cham (2017)
24. Shaked, M., Shanthikumar, J.G.: Stochastic Orders, Springer (2007)
 25. Shisha, O., Mond, B.: The degree of convergence of linear positive operators. Proc. Nat. Acad. Sci. USA **60**, 1196–1200 (1968)
 26. Sikkema, P.C.: On some linear positive operators. Indag. Math. **32**, 327–337 (1970)
 27. Szostok, T.: Ohlin's Lemma and some inequalities of the Hermite–Hadamard type. Aequ. Math. **89**, 915–926 (2015)
 28. Zhang, B., Yu, D., Wang, F.: Modified Bernstein–Kantorovich operators reproducing affine functions. Filomat **36**(18), 6187–6195 (2022)

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