Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat. (2024) 118:93
https://doi.org/10.1007/s13398-024-01599-8

ORIGINAL PAPER

On the full range of Zippin and inclusion indices of rearrangement-invariant spaces

Guillermo P. Curbera[1] · **Oleksiy Karlovych**[2] · **Eugene Shargorodsky**[3]

Received: 3 November 2023 / Accepted: 29 March 2024

Abstract
Let X be a rearrangement-invariant space on $[0, 1]$. It is known that its Zippin indices $\underline{\beta}_X$, $\overline{\beta}_X$ and its inclusion indices γ_X, δ_X are related as follows: $0 \le \underline{\beta}_X \le 1/\gamma_X \le 1/\delta_X \le \overline{\beta}_X \le 1$. We show that given $\underline{\beta}, \overline{\beta} \in [0, 1]$ and $\gamma, \delta \in [1, \infty]$ satisfying $\underline{\beta} \le 1/\gamma \le 1/\delta \le \overline{\beta}$, there exists a rearrangement-invariant space X such that $\underline{\beta}_X = \underline{\beta}$, $\overline{\beta}_X = \overline{\beta}$ and $\gamma_X = \gamma$, $\delta_X = \delta$.

Keywords Rearrangement-invariant Banach function space · Lorentz spaces · Zippin indices · Inclusion indices · Embedding

Mathematics Subject Classification 46E30

1 Introduction

Indices associated to quasiconcave functions are an important tool for studying rearrangement-invariant (r.i. in short) spaces and the operators acting on them. An uppermost example are the Boyd indices of a r.i. space X, $\underline{\alpha}_X$ and $\overline{\alpha}_X$, which in general satisfy $0 \le \underline{\alpha}_X \le \overline{\alpha}_X \le 1$ and characterize the boundedness of the Hilbert transform acting on X, i.e., when $0 < \underline{\alpha}_X \le \overline{\alpha}_X < 1$; see [2, Ch. 3, Section 5]. Related, and simpler, indices are the Zippin indices, $\underline{\beta}_X$ and $\overline{\beta}_X$ (see bellow for the definition), which satisfy

$$0 \le \underline{\alpha}_X \le \underline{\beta}_X \le \overline{\beta}_X \le \overline{\alpha}_X \le 1.$$

✉ Oleksiy Karlovych
oyk@fct.unl.pt

Guillermo P. Curbera
curbera@us.es

Eugene Shargorodsky
eugene.shargorodsky@kcl.ac.uk

1 Facultad de Matemáticas & IMUS, Universidad de Sevilla, Calle Tarfia s/n, Sevilla 41012, Spain

2 Centro de Matemática e Aplicações, Departamento de Matemática, Faculdade de Ciências e Tecnologia, Universidade Nova de Lisboa, Quinta da Torre, 2829–516 Caparica, Portugal

3 Department of Mathematics, King's College London, Strand, London WC2R 2LS, UK

Published online: 17 April 2024 Springer

R.i. spaces X satisfying $\underline{\alpha}_X = \underline{\beta}_X$ and $\overline{\beta}_X = \overline{\alpha}_X$ are known as spaces of fundamental type. The class of r.i. spaces of fundamental type include most of the classical r.i. spaces; see [5].

The study of the fine spectra of the finite Hilbert transform acting on a r.i. space X over $(-1, 1)$ depends in a relevant way on the following inclusion indices

$$\gamma_X := \inf \left\{ p \in [1, \infty] : L^p \hookrightarrow X \right\}, \quad \delta_X := \sup \left\{ p \in [1, \infty] : X \hookrightarrow L^p \right\}.$$

In particular, the condition that the Boyd indices, the Zippin indices, and the inverse of the inclusion indices all coincide allows giving a full description of the fine spectra, see [4, Theorem 7.2]. The Zippin indices and the inverse of the inclusion indices satisfy

$$0 \le \underline{\beta}_X \le 1/\gamma_X \le 1/\delta_X \le \overline{\beta}_X \le 1, \tag{1}$$

see [4, Lemma 6.1(a)].

The inclusion indices appear in the study by Hernández and Rodríguez-Salinas of Orlicz spaces having a sublattice lattice isomorphic to L^p, see [9, p. 185], [10, p. 192], [11, p. 11]. García del Amo, Hernández, Sánchez, and Semenov have used them to study disjoint strict singularity of inclusions between r.i. spaces [8, p. 249]. The incluson indices have been specifically studied by Fernández-Cabrera [6, 7]; Fernández-Cabrera, Cobos, Hernández and Sánchez [7]; Cobos, Fernández-Cabrera, Manzano and Martínez [3].

The aim of this paper is to discuss the distribution of values in inequalities (1). Hernández and Rodríguez-Salinas in [9, Theorem A] proved that given a triple α, β, p satisfying $0 < \alpha < p \le \beta < \infty$ there exist an Orlicz space having the Zippin indices $1/\beta$ and $1/\alpha$ and a sublattice which is lattice isomorphic to L^p. Further, they proved in [10, Theorem 1] that for a triple α, β, γ satisfying $0 < \alpha \le \gamma \le \beta < \infty$, there exists an Orlicz space with the upper inclusion index γ and the Zippin indices $1/\beta$ and $1/\alpha$. This implies that the second inequality in (1) may be strict. By a duality argument based on formulae (2) and (4) below, this fact yields that the penultimate inequality in (1) also may be strict.

In this regard we establish the following result.

Theorem 1 *Given $\underline{\beta}, \overline{\beta} \in [0, 1]$ and $\gamma, \delta \in [1, \infty]$ satisfying*

$$\underline{\beta} \le 1/\gamma \le 1/\delta \le \overline{\beta},$$

there exists a quasiconcave function $\varphi : [0, 1] \to [0, \infty)$ such that for every r.i. space X with the fundamental function equivalent to φ one has

$$\underline{\beta}_X = \underline{\beta}, \quad \overline{\beta}_X = \overline{\beta}, \quad \gamma_X = \gamma, \quad \delta_X = \delta.$$

Note that, by [12, Ch. II, Theorem 4.2], a function $\varphi : [0, 1] \to [0, \infty)$ is a fundamental function of an r.i. space if and only if it is quasiconcave. If φ is quasiconcave, then the Marcinkiewicz space M_φ is an r.i. space whose fundamental function coincides with φ. Further, let $\widetilde{\varphi}$ be the least concave majorant of the quasiconcave function φ. Then the fundamental function of the Lorentz space $\Lambda_{\widetilde{\varphi}}$ is equal to $\widetilde{\varphi}$ and $\widetilde{\varphi}/2 \le \varphi \le \widetilde{\varphi}$ (see Sect. 2 below).

A consequence of the above theorem is the following.

Corollary 2 *If $p \in (1, \infty)$, then there exists an r.i. space X such that for every $\varepsilon \in (0, p-1)$,*

$$L^{p+\varepsilon,\infty} \hookrightarrow X \hookrightarrow L^{p-\varepsilon,1},$$

and its Zippin indices are trivial, that is, $\underline{\beta}_X = 0$ and $\overline{\beta}_X = 1$.

The paper is organised as follows. In Sect. 2, we collect necessary definitions. Section 3 contains the proofs of Theorem 1 and its Corollary 2.

2 Preliminaries

Let L^0 be the space of all equivalence classes of complex-valued Lebesgue measurable functions on [0, 1] and let m denote the Lebesgue measure on [0, 1]. The distribution function of $f \in L^0$ is defined by

$$d_f(y) := m\{t \in [0,1] : |f(t)| > y\}, \quad y > 0.$$

Functions $f, g \in L^0$ are called equimeasurable if $d_f = d_g$. The nonincreasing rearrangement of f is given by

$$f^*(t) = \sup\{y > 0 : d_f(y) > t\} = \inf\{y > 0 : d_f(y) \le t\}.$$

Following Semenov [15], a Banach subspace X of L^0 is called a symmetric space if

(a) for any $g \in L^0$ equimeasurable to $f \in X$, one has $g \in X$ and $\|g\|_X = \|f\|_X$;
(b) for every $g \in L^0$ and $f \in X$ the inequality $|f| \le |g|$ a.e. implies that $g \in X$ and $\|g\|_X \le \|f\|_X$.

Lebesgue spaces L^p with $p \in [1, \infty]$, Orlicz spaces L^Φ, and Lorentz spaces $L^{p,q}$ (see below) are the most widely used examples of symmetric spaces.

If X is symmetric, then the function $\varphi_X(t) := \|\chi_E\|_X$, where $E \subset [0,1]$ is a measurable set with $m(E) = t$, is well defined and is called the fundamental function of X.

The associate space X' of X consists of all functions $g \in L^0$ satisfying

$$\int_0^1 |f(x)g(x)|\,dx < \infty$$

for all $f \in X$. It is equipped with the norm

$$\|g\|_{X'} := \sup\left\{\int_0^1 |f(x)g(x)|\,dx : \|f\|_X \le 1\right\}.$$

Semenov proved [15, Theorem 2] (see also [12, Ch. II, Theorem 4.1]), that if X is symmetric, then $L^\infty \hookrightarrow X \hookrightarrow L^1$, where $\hookrightarrow$ denotes a continuous embedding. A symmetric space X is said to have the Fatou property if for every sequence $\{f_n\}$ in X such that $0 \le f_n \uparrow f$ a.e., one has either $f \in X$ and $\|f_n\|_X \uparrow \|f\|_X$, or $\|f_n\|_X \uparrow \infty$. Symmetric spaces with the Fatou property are usually called rearrangement-invariant Banach function spaces (or, shortly, r.i. spaces), see [2, Ch. 1–2].

A function $\varphi\colon [0,1] \to [0,\infty)$ is said to be quasiconcave if $\varphi(t) = 0$ precisely when $t = 0$, the function $\varphi(t)$ is increasing and the function $\varphi(t)/t$ is decreasing on (0, 1]. Following [14, Section 2], for a quasiconcave, and hence measurable, function $\varphi : [0,1] \to [0,\infty)$, one can define its dilation function

$$M_\varphi(x) := \sup_{s \in (0, \min\{1, 1/x\}]} \frac{\varphi(xs)}{\varphi(s)}.$$

It follows from [12, Ch. II, Theorem 1.3] (see also [14, Theorem 1.2]) that

$$\underline{\alpha}(\varphi) := \sup_{0<x<1} \frac{\log M_\varphi(x)}{\log x} = \lim_{x\to 0^+} \frac{\log M_\varphi(x)}{\log x},$$
$$\overline{\alpha}(\varphi) := \inf_{x>1} \frac{\log M_\varphi(x)}{\log x} = \lim_{x\to\infty} \frac{\log M_\varphi(x)}{\log x}.$$

If X is an r.i. space, then its fundamental function φ_X is quasiconcave (see, e.g., [15, Theorem 1], [2, Ch. 2, Corollary 5.3]). The numbers

$$\underline{\beta}_X := \underline{\alpha}(\varphi_X), \quad \overline{\beta}_X := \overline{\alpha}(\varphi_X)$$

are called the Zippin (fundamental) indices of the r.i. space X (see [14, p. 27], [16], and also [2, Ch. 3, Exercise 14]). It is well known that

$$0 \le \underline{\beta}_X \le \overline{\beta}_X \le 1, \quad \underline{\beta}_X + \overline{\beta}_{X'} = 1, \quad \underline{\beta}_{X'} + \overline{\beta}_X = 1 \tag{2}$$

(see, e.g., [14, formulae (4.14)–(4.15)]).

The inclusion indices γ_X and δ_X can be expressed as follows:

$$\delta_X = \liminf_{t\to 0^+} \frac{\log t}{\log \varphi_X(t)}, \quad \gamma_X = \limsup_{t\to 0^+} \frac{\log t}{\log \varphi_X(t)} \tag{3}$$

(see [8, p. 249] or [3, Theorems 1.1−1.2]). Since the fundamental functions of X and X' satisfy $\varphi_X(t)\varphi_{X'}(t) = 1$ (see [2, Ch. 2, Theorem 5.2]), the following relation between the inclusion indices of X and X' hold:

$$\frac{1}{\gamma_X} + \frac{1}{\delta_{X'}} = 1, \quad \frac{1}{\gamma_{X'}} + \frac{1}{\delta_X} = 1. \tag{4}$$

It follows from (3) that the middle inequality in (1) becomes the equality if and only if the limit

$$\lim_{t\to 0^+} \frac{\log t}{\log \varphi_X(t)}$$

exists (see, e.g., [6, p. 669] or [3, Corollary 1.3]).

Suppose $1 < p < \infty$ and $1 \le q \le \infty$. The Lorentz spaces $L^{p,q}$ consist of all measurable functions $f : [0, 1] \to \mathbb{C}$ such that $\|f\|_{(p,q)} < \infty$, where

$$\|f\|_{(p,q)} := \begin{cases} \left(\int_0^1 \left\{t^{1/p} f^{**}(t)\right\}^q \frac{dt}{t} \right)^{1/q}, & 1 \le q < \infty, \\ \sup\limits_{0<t\le 1} \left\{t^{1/p} f^{**}(t)\right\}, & q = \infty, \end{cases}$$

and

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(y)\, dy.$$

The spaces $L^{p,\infty}$ are frequently called weak L^p-spaces or Marcinkiewicz spaces. It follows from [2, Ch. 4, Theorem 4.6] that $L^{p,q}$ are rearrangement-invariant Banach function spaces. In view of [2, Ch. 2, Theorem 5.13], the Lorentz space $L^{p,1}$ and the Marcinkiewicz space $L^{p,\infty}$ are respectively the smallest and the largest of all r.i. spaces having the same fundamental function as L^p.

As usual, two functions $\phi, \psi : [0, 1] \to [0, \infty)$ are said to be equivalent if there exist constants $c, C \in (0, \infty)$ such that

$$c\phi(t) \le \psi(t) \le C\phi(t), \quad t \in [0, 1].$$

For a quasiconcave function $\varphi : [0, 1] \to [0, \infty)$, let M_φ be the Marcinkiewicz space consisting of measurable functions $f : [0, 1] \to \mathbb{C}$ satisfying

$$\|f\|_{M_\varphi} := \sup_{0<t\le 1} \left\{\varphi(t) f^{**}(t)\right\} < \infty.$$

Then M_φ is an r.i. space whose fundamental function coincides with φ (see, e.g., [2, Ch. 2, Proposition 5.8] or [12, formula (4.28)]).

For each quasiconcave function $\varphi : [0,1] \to [0,\infty)$, its least concave majorant $\widetilde{\varphi}$ satisfies

$$\widetilde{\varphi}(t)/2 \le \varphi(t) \le \widetilde{\varphi}(t), \quad t \in [0,1]$$

(see, e.g., [2, Ch. 2, Proposition 5.10]). The Lorentz space $\Lambda_{\widetilde{\varphi}}$ consists of all measurable functions $f : [0,1] \to \mathbb{C}$ such that

$$\|f\|_{\Lambda_{\widetilde{\varphi}}} := \int_0^1 f^*(t)\, d\widetilde{\varphi}(t) < \infty.$$

It is well known that $\Lambda_{\widetilde{\varphi}}$ is an r.i. space whose fundamental function is $\widetilde{\varphi}$ (see, e.g., [2, Ch. 2, Theorem 5.13] or [12, formula (4.28)]).

3 Proofs

Inequalities (1) were proved in [4, Lemma 6.1(a)] under the assumption that the Boyd indices of X are non-trivial. For completeness, we include a proof of them.

3.1 Proof of inequalities (1)

Inequality $1/\gamma_X \le 1/\delta_X$ follows immediately from equalities (3).

It follows from [13, Lemma 4.2] that if $\overline{\beta}_X < 1/p$, then $X \hookrightarrow L^p$. Hence

$$1/\overline{\beta}_X = \sup\{p \in [1,\infty] : \overline{\beta}_X < 1/p\} \le \sup\{p \in [1,\infty] : X \hookrightarrow L^p\} = \delta_X,$$

which implies that $1/\delta_X \le \overline{\beta}_X$.

Let $p' := p/(p-1)$. It follows from (2) that $\overline{\beta}_{X'} < 1/p'$ if and only if $\underline{\beta}_X > 1/p$. By [2, Ch. 1, Proposition 2.10], $X' \hookrightarrow L^{p'}$ if and only if $L^p \hookrightarrow X$. So, if $\underline{\beta}_X > 1/p$, then $L^p \hookrightarrow X$. Hence

$$1/\underline{\beta}_X = \inf\{p \in [1,\infty] : \underline{\beta}_X > 1/p\} \ge \inf\{p \in [1,\infty] : L^p \hookrightarrow X\} = \gamma_X.$$

Therefore, $\underline{\beta}_X \le 1/\gamma_X$.

□

3.2 The case of coinciding inclusion indices

In this subsection, we will prove Theorem 1 in the case where the inclusion indices coincide. The proof will follow from the theorem below.

Theorem 3 *Let $p \in [1,\infty]$ and $\underline{\beta}, \overline{\beta} \in [0,1]$ be such that $\underline{\beta} \le 1/p \le \overline{\beta}$, and let $\rho : [0,1] \to [0,1]$ be an increasing continuous function such that $\rho(t) = 0$ precisely when $t = 0$. Then there exists a quasiconcave function $\varphi : [0,1] \to [0,\infty)$ such that*

$$t^{1/p}\rho(t) \le \varphi(t) \le t^{1/p}/\rho(t), \quad t \in (0,1], \tag{5}$$

and

$$\underline{\alpha}(\varphi) = \underline{\beta}, \quad \overline{\alpha}(\varphi) = \overline{\beta}. \tag{6}$$

Proof Our construction is inspired by the construction presented in [1, p. 261], and it works as follows. It requires three sequences $\{a_k\}_{k\in\mathbb{N}}$, $\{b_k\}_{k\in\mathbb{N}}$, $\{c_k\}_{k\in\mathbb{N}}$, whose terms satisfy

$$0 < a_{k+1} < c_k < b_k < a_k \le 1, \quad k \in \mathbb{N},$$

and a continuous function $\varphi : [0, 1] \to [0, \infty)$ satisfying

$$\begin{aligned} &\frac{\varphi(t)}{t^{\underline{\beta}}} \text{ is constant on } [b_k, a_k)\,, \\ &\frac{\varphi(t)}{t^{\overline{\beta}}} \text{ is constant on } [c_k, b_k)\,, \\ &\frac{\varphi(t)}{t^{\frac{1}{p}}} \text{ is constant on } \left[a_{k+1}, c_k\right). \end{aligned} \tag{7}$$

The terms b_k should be defined so that the intervals $[b_k, a_k)$ are large enough for the first condition in (7) to ensure that the first equality in (6) holds. Similarly, the terms c_k should be defined so that the intervals $[c_k, b_k)$ are large enough for the second condition in (7) to ensure that the second equality in (6) holds. Finally, the sequence $\{a_k\}_{k\in\mathbb{N}}$ should converge to 0 sufficiently rapidly so that, via the third condition in (7), it is guaranteed that $\varphi(t)/t^{\frac{1}{p}}$ is constant “most of the time", which leads to (5).

Following the above, for any sequence $\{a_k\}_{k\in\mathbb{N}}$ with the terms in (0, 1], set

$$b_k := \frac{a_k}{k}, \quad c_k := \frac{b_k}{k} = \frac{a_k}{k^2}, \quad k \in \mathbb{N}.$$

Choose a sequence $\{a_k\}$ so that

$$a_{k+1} < c_k, \quad \rho(a_k) \le k^{-k(k+1)\left(\frac{1}{p}-\underline{\beta}\right)}, \quad \rho(b_k) \le k^{-k(k+1)\left(\overline{\beta}-\frac{1}{p}\right)}, \tag{8}$$

and set $a_0 := 1$. Let

$$\varphi(t) := t^{\frac{1}{p}}, \quad t \in [a_1, 1].$$

Suppose we have defined φ on $[a_k, 1]$ with $k \ge 1$ in such a way that

$$t^{\frac{1}{p}} \rho(t)^{1-\frac{1}{k}} \le \varphi(t) \le \frac{t^{\frac{1}{p}}}{\rho(t)^{1-\frac{1}{k}}}, \quad t \in [a_k, a_{k-1}). \tag{9}$$

Let us define φ on $[a_{k+1}, a_k)$. We start with

$$\varphi(t) := \varphi(a_k)\left(\frac{t}{a_k}\right)^{\underline{\beta}}, \quad t \in [b_k, a_k)\,. \tag{10}$$

It follows from (9) with $t = a_k$ and the second inequality in (8) that for all $t \in [b_k, a_k)$,

$$\begin{aligned} \varphi(t) &\le \frac{a_k^{\frac{1}{p}}}{\rho(a_k)^{1-\frac{1}{k}}}\left(\frac{t}{a_k}\right)^{\underline{\beta}} = \frac{a_k^{\frac{1}{p}}}{\rho(a_k)^{1-\frac{1}{k}}}\left(\frac{t}{a_k}\right)^{\underline{\beta}-\frac{1}{p}}\left(\frac{t}{a_k}\right)^{\frac{1}{p}} \\ &\le \frac{a_k^{\frac{1}{p}}}{\rho(a_k)^{1-\frac{1}{k}}}\left(\frac{b_k}{a_k}\right)^{\underline{\beta}-\frac{1}{p}}\left(\frac{t}{a_k}\right)^{\frac{1}{p}} = \frac{k^{\frac{1}{p}-\underline{\beta}}}{\rho(a_k)^{1-\frac{1}{k}}}\, t^{\frac{1}{p}} \le \frac{\rho(a_k)^{-\frac{1}{k(k+1)}}}{\rho(a_k)^{1-\frac{1}{k}}}\, t^{\frac{1}{p}} \\ &= \frac{t^{\frac{1}{p}}}{\rho(a_k)^{1-\frac{1}{k+1}}} \le \frac{t^{\frac{1}{p}}}{\rho(t)^{1-\frac{1}{k+1}}}. \end{aligned}$$

On the other hand, it follows from (9) with $t = a_k$ and (10) that for $t \in [b_k, a_k)$,

$$\varphi(t) \geq a_k^{\frac{1}{p}} \rho(a_k)^{1-\frac{1}{k}} \left(\frac{t}{a_k}\right)^{\underline{\beta}} \geq a_k^{\frac{1}{p}} \rho(a_k)^{1-\frac{1}{k}} \left(\frac{t}{a_k}\right)^{\frac{1}{p}} = \rho(a_k)^{1-\frac{1}{k}} t^{\frac{1}{p}} \geq t^{\frac{1}{p}} \rho(t)^{1-\frac{1}{k}}.$$

So,

$$t^{\frac{1}{p}} \rho(t)^{1-\frac{1}{k}} \leq \varphi(t) \leq \frac{t^{\frac{1}{p}}}{\rho(t)^{1-\frac{1}{k+1}}}, \quad t \in [b_k, a_k). \tag{11}$$

Now, take

$$\varphi(t) := \varphi(b_k) \left(\frac{t}{b_k}\right)^{\overline{\beta}}, \quad t \in [c_k, b_k). \tag{12}$$

It follows from (11) with $t = b_k$ and the third inequality in (8) that for all $t \in [c_k, b_k)$,

$$\begin{aligned} \varphi(t) &\geq b_k^{\frac{1}{p}} \rho(b_k)^{1-\frac{1}{k}} \left(\frac{t}{b_k}\right)^{\overline{\beta}} = b_k^{\frac{1}{p}} \rho(b_k)^{1-\frac{1}{k}} \left(\frac{t}{b_k}\right)^{\overline{\beta}-\frac{1}{p}} \left(\frac{t}{b_k}\right)^{\frac{1}{p}} \\ &\geq b_k^{\frac{1}{p}} \rho(b_k)^{1-\frac{1}{k}} \left(\frac{c_k}{b_k}\right)^{\overline{\beta}-\frac{1}{p}} \left(\frac{t}{b_k}\right)^{\frac{1}{p}} = \rho(b_k)^{1-\frac{1}{k}} k^{\frac{1}{p}-\overline{\beta}} t^{\frac{1}{p}} \\ &\geq \rho(b_k)^{1-\frac{1}{k}} \rho(b_k)^{\frac{1}{k(k+1)}} t^{\frac{1}{p}} = \rho(b_k)^{1-\frac{1}{k+1}} t^{\frac{1}{p}} \geq \rho(t)^{1-\frac{1}{k+1}} t^{\frac{1}{p}}. \end{aligned}$$

On the other hand, it follows from (11) with $t = b_k$ and (12) that for all $t \in [c_k, b_k)$,

$$\varphi(t) \leq \frac{b_k^{\frac{1}{p}}}{\rho(b_k)^{1-\frac{1}{k+1}}} \left(\frac{t}{b_k}\right)^{\overline{\beta}} \leq \frac{b_k^{\frac{1}{p}}}{\rho(b_k)^{1-\frac{1}{k+1}}} \left(\frac{t}{b_k}\right)^{\frac{1}{p}} = \frac{t^{\frac{1}{p}}}{\rho(b_k)^{1-\frac{1}{k+1}}} \leq \frac{t^{\frac{1}{p}}}{\rho(t)^{1-\frac{1}{k+1}}}.$$

So, combining the above two inequalities with inequality (11), we arrive at

$$t^{\frac{1}{p}} \rho(t)^{1-\frac{1}{k+1}} \leq \varphi(t) \leq \frac{t^{\frac{1}{p}}}{\rho(t)^{1-\frac{1}{k+1}}}, \quad t \in [c_k, a_k). \tag{13}$$

Finally, let

$$\varphi(t) := \varphi(c_k) \left(\frac{t}{c_k}\right)^{\frac{1}{p}}, \quad t \in \left[a_{k+1}, c_k\right). \tag{14}$$

Then it follows from (13) with $t = c_k$ that for all $t \in \left[a_{k+1}, c_k\right)$,

$$\begin{aligned} \varphi(t) &\leq \frac{c_k^{\frac{1}{p}}}{\rho(c_k)^{1-\frac{1}{k+1}}} \left(\frac{t}{c_k}\right)^{\frac{1}{p}} = \frac{t^{\frac{1}{p}}}{\rho(c_k)^{1-\frac{1}{k+1}}} \leq \frac{t^{\frac{1}{p}}}{\rho(t)^{1-\frac{1}{k+1}}}, \\ \varphi(t) &\geq c_k^{\frac{1}{p}} \rho(c_k)^{1-\frac{1}{k+1}} \left(\frac{t}{c_k}\right)^{\frac{1}{p}} = t^{\frac{1}{p}} \rho(c_k)^{1-\frac{1}{k+1}} \geq t^{\frac{1}{p}} \rho(t)^{1-\frac{1}{k+1}}. \end{aligned}$$

So, the above two inequalities and (13) imply that

$$t^{\frac{1}{p}} \rho(t)^{1-\frac{1}{k+1}} \leq \varphi(t) \leq \frac{t^{\frac{1}{p}}}{\rho(t)^{1-\frac{1}{k+1}}}, \quad t \in [a_{k+1}, a_k)$$

(cf. (9)). Since $\rho(t) \in (0, 1]$, this inequality implies (5).

For $k \in \mathbb{N}$, let

$$I_{1,k} := [b_k, a_k], \quad I_{2,k} := [c_k, b_k], \quad I_{3,k} := [a_{k+1}, c_k]$$

and

$$\beta_1 := \underline{\beta}, \quad \beta_2 := \overline{\beta}, \quad \beta_3 = \frac{1}{p}.$$

The above inductive argument produces a continuous function on (0, 1] satisfying (5) and such that $\varphi(t) = t^{1/p}$ for $t \in [a_1, 1]$,

$$\varphi(t) = \mathrm{const}_{j,k}\, t^{\beta_j}, \quad t \in I_{j,k}, \quad j = 1, 2, 3, \quad k \in \mathbb{N}. \tag{15}$$

Set $\varphi(0) = 0$. It is clear that φ is quasiconcave. Since $\log \varphi$ is continuous and piecewise smooth, it follows from (15) that if $x > 1$ and $0 < s \le 1/x$, then

$$\log \varphi(xs) - \log \varphi(s) = \int_s^{xs} (\log \varphi(t))'\, dt \le \int_s^{xs} \overline{\beta}\, \frac{dt}{t} = \overline{\beta}\, \log x,$$

and hence

$$\frac{\varphi(xs)}{\varphi(s)} \le x^{\overline{\beta}}. \tag{16}$$

Similarly, if $x, s \in (0, 1)$, then

$$\log \varphi(s) - \log \varphi(xs) = \int_{xs}^{s} (\log \varphi(t))'\, dt \ge \int_{xs}^{s} \underline{\beta}\, \frac{dt}{t} = -\underline{\beta}\, \log x,$$

and

$$\frac{\varphi(xs)}{\varphi(s)} \le x^{\underline{\beta}}. \tag{17}$$

Take any $x > 1$ and choose $k > x$, $k \in \mathbb{N}$. Then $xc_k \in [c_k, b_k)$ and it follows from (12) that

$$M_\varphi(x) = \sup_{0<s\le 1/x} \frac{\varphi(xs)}{\varphi(s)} \ge \frac{\varphi(xc_k)}{\varphi(c_k)} = x^{\overline{\beta}}.$$

Hence $M_\varphi(x) = x^{\overline{\beta}}$ (see (16)), and

$$\overline{\alpha}(\varphi) = \inf_{x>1} \frac{\log M_\varphi(x)}{\log x} = \overline{\beta}.$$

Similarly, take any $x < 1$ and choose $k \in \mathbb{N}$ such that $1/k < x$. Then $xa_k \in [b_k, a_k)$ and it follows from (10) that

$$M_\varphi(x) = \sup_{0<s\le 1} \frac{\varphi(xs)}{\varphi(s)} \ge \frac{\varphi(xa_k)}{\varphi(a_k)} = x^{\underline{\beta}}.$$

Hence $M_\varphi(x) = x^{\underline{\beta}}$ (see (17)), and

$$\underline{\alpha}(\varphi) = \sup_{0<x<1} \frac{\log M_\varphi(x)}{\log x} = \underline{\beta},$$

which completes the proof of (6). ◻

Corollary 4 *Given $p \in [1,\infty]$ and $\underline{\beta}, \overline{\beta}$ such that $0 \le \underline{\beta} \le 1/p \le \overline{\beta} \le 1$, there exists a quasiconcave function $\varphi : [0,1] \to [0,\infty)$ such that for every r.i. space X with the fundamental function equivalent to φ, its inclusion indices satisfy $\gamma_X = \delta_X = p$ and its Zippin indices satisfy $\underline{\beta}_X = \underline{\beta}$ and $\overline{\beta} = \overline{\beta}_X$.*

Proof Take $\rho(t) = 1/\log(e-1+1/t)$, $t \in (0,1]$, and $\rho(0) = 0$. It follows from Theorem 3 that there exists a quasiconcave function $\varphi : [0,1] \to [0,\infty)$ such that

$$t^{1/p}\frac{1}{\log(e-1+1/t)} \le \varphi(t) \le t^{1/p}\log(e-1+1/t), \quad t \in (0,1], \tag{18}$$

and (6) holds.

Let X be an r.i. space such that φ_X is equivalent to φ. Then there exist constants $c, C \in (0,\infty)$ such that

$$c\varphi(t) \le \varphi_X(t) \le C\varphi(t), \quad t \in [0,1].$$

Hence

$$\delta_X = \liminf_{t\to 0^+}\frac{\log t}{\log \varphi_X(t)} = \liminf_{t\to 0^+}\frac{\log t}{\log \varphi(t)},$$
$$\gamma_X = \limsup_{t\to 0^+}\frac{\log t}{\log \varphi_X(t)} = \limsup_{t\to 0^+}\frac{\log t}{\log \varphi(t)}.$$

Moreover, in view of (6), the Zippin indices of X satisfy

$$\underline{\beta}_X = \underline{\beta}, \quad \overline{\beta}_X = \overline{\beta}.$$

It follows from (18), for $t \in (0,1]$, that

$$\frac{1}{p}\log t - \log\log(e-1+1/t) \le \log\varphi(t) \le \frac{1}{p}\log t + \log\log(e-1+1/t).$$

Dividing by $\log t < 0$, we get

$$\frac{1}{p} + \frac{\log\log(e-1+1/t)}{\log t} \le \frac{\log\varphi(t)}{\log t} \le \frac{1}{p} - \frac{\log\log(e-1+1/t)}{\log t}.$$

Since

$$\lim_{t\to 0^+}\frac{\log\log(e-1+1/t)}{\log t} = \lim_{t\to 0^+}\frac{1}{1/t}\,\frac{-1/t^2}{(e-1+1/t)\log(e-1+1/t)}$$
$$= \lim_{t\to 0^+}\frac{-1}{((e-1)t+1)\log(e-1+1/t)} = 0, \tag{19}$$

we arrive at

$$\delta_X = \gamma_X = p$$

(see (3)), which completes the proof. □

3.3 The case of distinct inclusion indices

In this subsection, we prove Theorem 1 in the case where the inclusion indices are distinct. The proof will follow from the theorem below.

Theorem 5 *Let* $\underline{\beta}, \overline{\beta} \in [0, 1]$ *and* $\gamma, \delta \in [1, \infty]$ *be such that*

$$\underline{\beta} \le 1/\gamma < 1/\delta \le \overline{\beta}. \tag{20}$$

Then there exists a quasiconcave function $\psi : [0, 1] \to [0, \infty)$ *such that*

$$\underline{\alpha}(\psi) = \underline{\beta}, \quad \overline{\alpha}(\psi) = \overline{\beta}, \quad \limsup_{t\to 0^+} \frac{\log t}{\log \psi(t)} = \gamma, \quad \liminf_{t\to 0^+} \frac{\log t}{\log \psi(t)} = \delta. \tag{21}$$

Proof We first outline the proof. We will choose p and $\rho(t)$ such that $1/\gamma < 1/p < 1/\delta$ and $\rho(t)$ decays to 0 sufficiently slowly as $t \to 0$. Then Theorem 3 produces a quasiconcave function φ such that

$$\frac{t^{\frac{1}{\delta}}}{\rho(t)} \le \varphi(t) \le t^{\frac{1}{\gamma}} \rho(t), \quad t \in (0, 1],$$

and the first two equalities in (21) (with φ in place of ψ) are satisfied while the last two are not.

We need to modify φ on a part of [0, 1] to get the third and the fourth equalities in (21). To achieve this, we construct inductively a subsequence $\{a_{k(j)}\}_{j\in\mathbb{N}}$ of the sequence $\{a_k\}_{k\in\mathbb{N}}$ from the proof of Theorem 3. Take $\tau_j \in (0, a_{k(j)}]$ and set $\psi(t) := \text{const}\, t^{\underline{\beta}}$, $t \in [\tau_j, a_{k(j)}]$, where the constant is such that $\psi(a_{k(j)}) = \varphi(a_{k(j)})$. We can choose τ_j in such a way that

$$\psi(\tau_j) = \tau_j^{\frac{1}{\gamma}} \rho(\tau_j). \tag{22}$$

This is because $t^{\frac{1}{\gamma}} \rho(t)/t^{\underline{\beta}} \to 0$ as $t \to 0^+$. Next, take $\upsilon_j \in (0, \tau_j]$ and set $\psi(t) := \text{const}\, t^{\overline{\beta}}$, $t \in [\upsilon_j, \tau_j)$, where the constant is such that ψ is continuous at τ_j. We can choose υ_j in such a way that

$$\psi(\upsilon_j) = \frac{\upsilon_j^{\frac{1}{\delta}}}{\rho(\upsilon_j)}. \tag{23}$$

This is because

$$\frac{t^{\overline{\beta}}}{t^{\frac{1}{\delta}}/\rho(t)} \to 0 \quad \text{as} \quad t \to 0^+.$$

Finally, take $\varsigma_j \in (0, \upsilon_j]$ and set $\psi(t) := \text{const}\, t^{\underline{\beta}}$, $t \in [\varsigma_j, \upsilon_j)$, where the constant is such that ψ is continuous at υ_j. We can choose ς_j in such a way that

$$\psi(\varsigma_j) = \varphi(\varsigma_j).$$

This is because $0 \le \varphi(t)/t^{\underline{\beta}} \le t^{\frac{1}{\gamma}} \rho(t)/t^{\underline{\beta}} \to 0$ as $t \to 0^+$. We then choose $k(j+1)$ in such a way that $a_{k(j+1)-1} \le \varsigma_j$ and keep

$$\psi(t) = \varphi(t) \text{ on } [a_{k(j+1)}, \varsigma_j] \supseteq [a_{k(j+1)}, a_{k(j+1)-1}].$$

The last equality ensures that the first two equalities in (21) hold, while (22) and (23) allow one to prove the last two equalities in (21). Details of the above argument are given below.

Let

$$\frac{1}{p} := \frac{1}{2}\left(\frac{1}{\delta} + \frac{1}{\gamma}\right), \quad \kappa := \frac{1}{4}\left(\frac{1}{\delta} - \frac{1}{\gamma}\right) = \frac{1}{2}\left(\frac{1}{\delta} - \frac{1}{p}\right) = \frac{1}{2}\left(\frac{1}{p} - \frac{1}{\gamma}\right) > 0,$$

and

$$\rho(t) := \frac{1}{\log^{\kappa}(e-1+1/t)}, \quad t \in (0,1], \quad \rho(0) := 0. \tag{24}$$

Integrating the inequality

$$\frac{1}{s^2(e-1+1/s)} < \frac{1}{s^2}, \quad 0 < s \le 1$$

between $t \in (0,1]$ and 1, one gets $\log(e-1+1/t) - 1 \le 1/t - 1$. Therefore

$$\frac{1}{\log(e-1+1/t)} \ge t,$$

which implies that

$$1 \ge \rho^2(t) \ge t^{2\kappa} = t^{\frac{1}{\delta}-\frac{1}{p}} = t^{\frac{1}{p}-\frac{1}{\gamma}}, \quad t \in (0,1].$$

Hence

$$\frac{t^{\frac{1}{\delta}}}{\rho(t)} \le t^{\frac{1}{p}}\rho(t) \le \frac{t^{\frac{1}{p}}}{\rho(t)} \le t^{\frac{1}{\gamma}}\rho(t), \quad t \in (0,1].$$

Let $\varphi : [0,1] \to [0,\infty)$ be the quasiconcave function constructed in the proof of Theorem 3, which satisfies

$$\frac{t^{\frac{1}{\delta}}}{\rho(t)} \le t^{\frac{1}{p}}\rho(t) \le \varphi(t) \le \frac{t^{\frac{1}{p}}}{\rho(t)} \le t^{\frac{1}{\gamma}}\rho(t), \quad t \in (0,1]. \tag{25}$$

Below, we also use the notation a_k from the proof of Theorem 3. Note that the sequence $\{a_k\}$ is decreasing and $a_k \to 0$ as $k \to \infty$ (see the first inequality in (8)).

Let

$$\psi(t) := \varphi(t) = t^{\frac{1}{p}}, \quad t \in [a_1, 1],$$

and $k(1) := 1$. In the next step, $j = 1$, but it is convenient for future reference to write it up for a general $j \in \mathbb{N}$.

It follows from the last two inequalities in (25) with $t = a_{k(j)}$ that

$$\varphi(a_{k(j)}) \le \frac{a_{k(j)}^{\frac{1}{p}}}{\rho(a_{k(j)})} \le a_{k(j)}^{\frac{1}{\gamma}}\rho(a_{k(j)}), \quad j \in \mathbb{N},$$

which implies that

$$1 \le \frac{a_{k(j)}^{\frac{1}{\gamma}}\rho(a_{k(j)})}{\varphi(a_{k(j)})\left(\frac{a_{k(j)}}{a_{k(j)}}\right)^{\underline{\beta}}}, \quad j \in \mathbb{N}. \tag{26}$$

Since $1/\gamma - \underline{\beta} \ge 0$ in view of (20), we have

$$t^{\frac{1}{\gamma}-\underline{\beta}}\rho(t) \to 0 \quad \text{as} \quad t \to 0^+. \tag{27}$$

Then

$$\frac{t^{\frac{1}{\gamma}}\rho(t)}{\varphi(a_{k(j)})\left(\frac{t}{a_{k(j)}}\right)^{\underline{\beta}}} \to 0 \quad \text{as} \quad t \to 0^+, \quad j \in \mathbb{N}. \tag{28}$$

It follows from (26) and (28) that there exists $\tau_j \in (0, a_{k(j)}]$ such that

$$\frac{\tau_j^{\frac{1}{\gamma}}\rho(\tau_j)}{\varphi(a_{k(j)})\left(\frac{\tau_j}{a_{k(j)}}\right)^{\underline{\beta}}} = 1, \quad 1 \le \frac{t^{\frac{1}{\gamma}}\rho(t)}{\varphi(a_{k(j)})\left(\frac{t}{a_{k(j)}}\right)^{\underline{\beta}}}, \quad t \in [\tau_j, a_{k(j)}].$$

Hence

$$\varphi(a_{k(j)})\left(\frac{\tau_j}{a_{k(j)}}\right)^{\underline{\beta}} = \tau_j^{\frac{1}{\gamma}}\rho(\tau_j), \quad \varphi(a_{k(j)})\left(\frac{t}{a_{k(j)}}\right)^{\underline{\beta}} \le t^{\frac{1}{\gamma}}\rho(t), \ t \in [\tau_j, a_{k(j)}]. \tag{29}$$

Let

$$\psi(t) := \varphi(a_{k(j)})\left(\frac{t}{a_{k(j)}}\right)^{\underline{\beta}}, \quad t \in [\tau_j, a_{k(j)}). \tag{30}$$

Then ψ is continuous at $a_{k(j)}$. It follows from (17) with $s = a_{k(j)}$ and $x = t/a_{k(j)}$ that

$$\frac{\varphi(t)}{\varphi(a_{k(j)})} \le \left(\frac{t}{a_{k(j)}}\right)^{\underline{\beta}}, \quad t \in [\tau_j, a_{k(j)}).$$

Hence $\varphi(t) \le \psi(t)$ for $t \in [\tau_j, a_{k(j)}]$. Combining inequalities (25) with the above inequality, the inequality in (29) and definition (30), we get

$$\frac{t^{\frac{1}{\delta}}}{\rho(t)} \le \psi(t) \le t^{\frac{1}{\gamma}}\rho(t), \quad t \in [\tau_j, a_{k(j)}]. \tag{31}$$

It follows from the first inequality in (31) with $t = \tau_j$ that

$$1 \le \frac{\psi(\tau_j)\left(\frac{\tau_j}{\tau_j}\right)^{\overline{\beta}}}{\tau_j^{\frac{1}{\delta}}/\rho(\tau_j)}, \quad j \in \mathbb{N}. \tag{32}$$

Since $\overline{\beta} - 1/\delta \ge 0$ in view of (20), we have

$$t^{\overline{\beta}-\frac{1}{\delta}}\rho(t) \to 0 \quad \text{as} \quad t \to 0^+.$$

Then

$$\frac{\psi(\tau_j)\left(\frac{t}{\tau_j}\right)^{\overline{\beta}}}{t^{\frac{1}{\delta}}/\rho(t)} \to 0 \quad \text{as} \quad t \to 0^+, \quad j \in \mathbb{N}. \tag{33}$$

It follows from (32) and (33) that there exists $\upsilon_j \in (0, \tau_j]$ such that

$$\frac{\psi(\tau_j)\left(\frac{\upsilon_j}{\tau_j}\right)^{\overline{\beta}}}{\upsilon_j^{\frac{1}{\delta}}/\rho(\upsilon_j)} = 1, \quad 1 \le \frac{\psi(\tau_j)\left(\frac{t}{\tau_j}\right)^{\overline{\beta}}}{t^{\frac{1}{\delta}}/\rho(t)}, \quad t \in [\upsilon_j, \tau_j].$$

Hence

$$\psi(\tau_j)\left(\frac{\upsilon_j}{\tau_j}\right)^{\overline{\beta}} = \frac{\upsilon_j^{\frac{1}{\delta}}}{\rho(\upsilon_j)}, \quad \frac{t^{\frac{1}{\delta}}}{\rho(t)} \le \psi(\tau_j)\left(\frac{t}{\tau_j}\right)^{\overline{\beta}}, \quad t \in [\upsilon_j, \tau_j]. \tag{34}$$

Let

$$\psi(t) := \psi(\tau_j)\left(\frac{t}{\tau_j}\right)^{\overline{\beta}}, \quad t \in [\upsilon_j, \tau_j). \tag{35}$$

Then ψ is continuous at τ_j. It follows from the above definition and the second inequality in (31) with $t = \tau_j$ that

$$\psi(t) \le \tau_j^{\frac{1}{\gamma}} \rho(\tau_j)\left(\frac{t}{\tau_j}\right)^{\overline{\beta}} \le \tau_j^{\frac{1}{\gamma}} \rho(\tau_j)\left(\frac{t}{\tau_j}\right)^{\frac{1}{\gamma}} = \rho(\tau_j) t^{\frac{1}{\gamma}} \le t^{\frac{1}{\gamma}}, \quad t \in [\upsilon_j, \tau_j].$$

Combining inequalities (31) with the above inequality, the inequality in (34) and definition (35), we get

$$\frac{t^{\frac{1}{\delta}}}{\rho(t)} \le \psi(t) \le t^{\frac{1}{\gamma}}, \quad t \in [\upsilon_j, a_{k(j)}]. \tag{36}$$

It follows from definition (35), the equality in (34) and the first two inequalities in (25) that

$$\psi(\upsilon_j) = \frac{\upsilon_j^{\frac{1}{\delta}}}{\rho(\upsilon_j)} \le \upsilon_j^{\frac{1}{p}} \rho(\upsilon_j) \le \varphi(\upsilon_j),$$

which implies that

$$1 \le \frac{\varphi(\upsilon_j)}{\psi(\upsilon_j)\left(\frac{\upsilon_j}{\upsilon_j}\right)^{\underline{\beta}}}, \quad j \in \mathbb{N}. \tag{37}$$

The last two inequalities in (25) and the asymptotic relation in (27) imply that

$$0 < \frac{\varphi(t)}{\psi(\upsilon_j)\left(\frac{t}{\upsilon_j}\right)^{\underline{\beta}}} \le \frac{t^{\frac{1}{\gamma}} \rho(t)}{\psi(\upsilon_j)\left(\frac{t}{\upsilon_j}\right)^{\underline{\beta}}} \to 0 \quad \text{as} \quad t \to 0^+, \quad j \in \mathbb{N}. \tag{38}$$

It follows from (37) and (38) that there exists $\varsigma_j \in (0, \upsilon_j]$ such that

$$\frac{\varphi(\varsigma_j)}{\psi(\upsilon_j)\left(\frac{\varsigma_j}{\upsilon_j}\right)^{\underline{\beta}}} = 1, \quad 1 \le \frac{\varphi(t)}{\psi(\upsilon_j)\left(\frac{t}{\upsilon_j}\right)^{\underline{\beta}}}, \quad t \in [\varsigma_j, \upsilon_j].$$

Hence

$$\psi(\upsilon_j)\left(\frac{\varsigma_j}{\upsilon_j}\right)^{\underline{\beta}} = \varphi(\varsigma_j), \quad \psi(\upsilon_j)\left(\frac{t}{\upsilon_j}\right)^{\underline{\beta}} \le \varphi(t), \quad t \in [\varsigma_j, \upsilon_j]. \tag{39}$$

Let

$$\psi(t) := \psi(\upsilon_j)\left(\frac{t}{\upsilon_j}\right)^{\underline{\beta}}, \quad t \in [\varsigma_j, \upsilon_j). \tag{40}$$

Then ψ is continuous at υ_j. It follows from the first inequality in (36) with $t = \upsilon_j$ that

$$\psi(t) \ge \frac{\upsilon_j^{\frac{1}{\delta}}}{\rho(\upsilon_j)}\left(\frac{t}{\upsilon_j}\right)^{\underline{\beta}} \ge \frac{\upsilon_j^{\frac{1}{\delta}}}{\rho(\upsilon_j)}\left(\frac{t}{\upsilon_j}\right)^{\frac{1}{\delta}} = \frac{t^{\frac{1}{\delta}}}{\rho(\upsilon_j)} \ge t^{\frac{1}{\delta}}, \quad t \in [\varsigma_j, \upsilon_j].$$

The above inequality, definition (40), inequality in (39), and the last two inequalities in (25) imply that

$$t^{\frac{1}{\delta}} \le \psi(t) \le \varphi(t) \le t^{\frac{1}{\gamma}} \rho(t), \quad t \in [\varsigma_j, \upsilon_j].$$

Combining these inequalities with inequalities (36), we get

$$t^{\frac{1}{\delta}} \le \psi(t) \le t^{\frac{1}{\gamma}}, \quad t \in [\varsigma_j, a_{k(j)}]. \tag{41}$$

Let

$$k(j+1) := 1 + \min \left\{k \in \mathbb{N} : \ a_k \le \varsigma_j\right\}$$

and

$$\psi(t) := \varphi(t), \quad t \in [a_{k(j+1)}, \varsigma_j). \tag{42}$$

It follows from the equality in (39) and definition (40) that $\psi(\varsigma_j) = \varphi(\varsigma_j)$. Hence ψ is continuous at ς_j. Inequalities (25), (41), and definition (42) imply that

$$t^{\frac{1}{\delta}} \le \psi(t) \le t^{\frac{1}{\gamma}}, \quad t \in [a_{k(j+1)}, a_{k(j)}].$$

Above, we had $j = 1$. Repeating the same procedure for $j = 2, 3, \ldots$, we get a continuous function $\psi : (0, 1] \to (0, \infty)$ such that

$$\psi(t) = \varphi(t), \quad t \in [a_{k(j)}, a_{k(j)-1}], \quad j \in \mathbb{N} \tag{43}$$

(note that $[a_{k(j)}, a_{k(j)-1}] \subset [a_{k(j)}, \varsigma_{j-1}]$), and

$$t^{\frac{1}{\delta}} \le \psi(t) \le t^{\frac{1}{\gamma}}, \quad t \in (0, 1]. \tag{44}$$

It follows from definition (30) and the equality in (29) that

$$\psi(\tau_j) = \tau_j^{1/\gamma} \rho(\tau_j), \quad j \in \mathbb{N}. \tag{45}$$

Analogously, it follows from definition (35) and the equality in (34) that

$$\psi(\upsilon_j) = \frac{\upsilon_j^{1/\delta}}{\rho(\upsilon_j)}, \quad j \in \mathbb{N}. \tag{46}$$

Finally, there is a partition $\cup_{l\in\mathbb{N}}[\eta_{l+1}, \eta_l) = (0, 1)$ such that

$$\psi(t) = \text{const}_l\, t^{\beta_l}, \quad t \in [\eta_{l+1}, \eta_l), \quad \beta_l \in \left\{\underline{\beta}, \overline{\beta}, 1/p\right\}, \quad l \in \mathbb{N}. \tag{47}$$

Set $\psi(0) = 0$. It follows from (47) and the continuity of ψ that ψ is quasiconcave and (16) and (17) remain true with ψ in place of φ. Hence

$$M_\psi(x) = \sup_{0<s\le 1/x} \frac{\psi(xs)}{\psi(s)} \le x^{\overline{\beta}}, \quad x \in (1, \infty), \tag{48}$$

$$M_\psi(x) = \sup_{0<s\le 1} \frac{\psi(xs)}{\psi(s)} \le x^{\underline{\beta}}, \quad x \in (0, 1). \tag{49}$$

Let the sequences $b_k = a_k/k$ and $c_k = b_k/k$ be as in the proof of Theorem 3. Take any $x > 1$ and choose $j \in \mathbb{N}$ so that $k(j) - 1 > x$. Then $xc_{k(j)-1} \in [c_{k(j)-1}, b_{k(j)-1})$ and it follows from (43) and (12) that

$$M_\psi(x) = \sup_{0<s\le 1/x} \frac{\psi(xs)}{\psi(s)} \ge \frac{\psi(xc_{k(j)-1})}{\psi(c_{k(j)-1})} = x^{\overline{\beta}}.$$

Hence $M_\psi(x) = x^{\overline{\beta}}$ (see (48)), and

$$\overline{\alpha}(\psi) = \inf_{x>1} \frac{\log M_\psi(x)}{\log x} = \overline{\beta}.$$

Similarly, take any $x < 1$ and choose $j \in \mathbb{N}$ so that $1/(k(j)-1) < x$. Then $xa_{k(j)-1} \in [b_{k(j)-1}, a_{k(j)-1})$ and it follows from (43) and (10) that

$$M_\psi(x) = \sup_{0<s\le 1} \frac{\psi(xs)}{\psi(s)} \ge \frac{\psi(xa_{k(j)-1})}{\psi(a_{k(j)-1})} = x^{\underline{\beta}}.$$

Hence $M_\psi(x) = x^{\underline{\beta}}$ (see (49)), and

$$\underline{\alpha}(\psi) = \sup_{0<x<1} \frac{\log M_\psi(x)}{\log x} = \underline{\beta}.$$

So, the first two equalities in (21) hold.

Estimates (44) imply

$$\limsup_{t\to 0^+} \frac{\log t}{\log \psi(t)} \le \gamma, \quad \liminf_{t\to 0^+} \frac{\log t}{\log \psi(t)} \ge \delta. \tag{50}$$

Since $0 < a_{k(j+1)} < \upsilon_j \le \tau_j \le a_{k(j)}$ for all $j \in \mathbb{N}$ and $a_k \to 0$ as $k \to \infty$, we conclude that $\upsilon_j \to 0$ and $\tau_j \to 0$ as $j \to \infty$. Finally, it follows from (45)–(46), (24), and (19) that

$$\begin{aligned}
\limsup_{t\to 0^+} \frac{\log t}{\log \psi(t)} &\ge \lim_{j\to\infty} \frac{\log \tau_j}{\log \psi(\tau_j)} = \lim_{j\to\infty} \frac{\log \tau_j}{\frac{1}{\gamma}\log \tau_j + \log \rho(\tau_j)} \\
&= \lim_{j\to\infty} \frac{1}{\frac{1}{\gamma} - \kappa \frac{\log\log(e-1+1/\tau_j)}{\log \tau_j}} = \gamma, \\
\liminf_{t\to 0^+} \frac{\log t}{\log \psi(t)} &\le \lim_{j\to\infty} \frac{\log \upsilon_j}{\log \psi(\upsilon_j)} = \lim_{j\to\infty} \frac{\log \upsilon_j}{\frac{1}{\delta}\log \upsilon_j + \log \rho(\upsilon_j)} \\
&= \lim_{j\to\infty} \frac{1}{\frac{1}{\delta} - \kappa \frac{\log\log(e-1+1/\upsilon_j)}{\log \upsilon_j}} = \delta.
\end{aligned}$$

Combining the above inequalities with (50), we arrive at

$$\limsup_{t\to 0^+} \frac{\log t}{\log \psi(t)} = \gamma, \quad \liminf_{t\to 0^+} \frac{\log t}{\log \psi(t)} = \delta,$$

which completes the proof of the last two equalities in (21). □

Corollary 6 *Let $\underline{\beta}, \overline{\beta} \in [0,1]$ and $\gamma, \delta \in [1,\infty]$ satisfy $\underline{\beta} \le 1/\gamma < 1/\delta \le \overline{\beta}$. Then there exists a quasiconcave function $\psi : [0,1] \to [0,\infty)$ such that for every r.i. space X with the fundamental function equivalent to ψ one has*

$$\underline{\beta}_X = \underline{\beta}, \quad \overline{\beta}_X = \overline{\beta}, \quad \gamma_X = \gamma, \quad \delta_X = \delta.$$

This result follows from Theorem 5 and (3).

Theorem 1 immediately follows form Corollaries 4 and 6.

3.4 Inclusion indices in terms of Lorentz and Marcinkiewicz spaces

The following fact was observed in the case of r.i. spaces with nontrivial Boyd indices in [4, formulae (3.5)–(3.6) and (6.1)–(6.2)].

Lemma 7 *Let X be an r.i. space. If $1 < \delta_X \le \gamma_X < \infty$, then*

$$\gamma_X = \inf\left\{p \in (1,\infty) : L^{p,\infty} \hookrightarrow X\right\}, \quad \delta_X = \sup\left\{p \in (1,\infty) : X \hookrightarrow L^{p,1}\right\}.$$

Proof Since $1 < \delta_X \le \gamma_X < \infty$, we have

$$\gamma_X = \inf\left\{p \in (1,\infty) : L^p \hookrightarrow X\right\}, \quad \delta_X = \sup\left\{p \in (1,\infty) : X \hookrightarrow L^p\right\}.$$

Set

$$\gamma_X^\infty := \inf\left\{p \in (1,\infty) : L^{p,\infty} \hookrightarrow X\right\}, \quad \delta_X^1 := \sup\left\{p \in (1,\infty) : X \hookrightarrow L^{p,1}\right\}.$$

Since $L^{p,1} \hookrightarrow L^p \hookrightarrow L^{p,\infty}$ for every $p \in (1,\infty)$, we have

$$\gamma_X \le \gamma_X^\infty, \quad \delta_X^1 \le \delta_X.$$

If $\gamma_X < \gamma_X^\infty$, then there exist p_1, p_2 such that $\gamma_X < p_1 < p_2 < \gamma_X^\infty$. It follows from the definitions of γ_X and γ_X^∞ that $L^{p_1} \hookrightarrow X$ and $L^{p_2,\infty} \not\hookrightarrow X$. Since $L^{p_2,\infty} \hookrightarrow L^{p_1}$ (see, e.g., [2, p. 217]), the absence of inclusion of $L^{p_2,\infty}$ into X is impossible. Then $\gamma_X \ge \gamma_X^\infty$. The proof of $\delta_X^1 \ge \delta_X$ is similar. Thus $\gamma_X = \gamma_X^\infty$ and $\delta_X = \delta_X^1$. □

3.5 Proof of Corollary 2

It follows from Corollary 4 with $\underline{\beta} = 0$ and $\overline{\beta} = 1$ that there exists an r.i. space X such that its inclusion indices are $\gamma_X = \delta_X = p$ and its Zippin indices are trivial, that is, $\underline{\beta}_X = 0$ and $\overline{\beta}_X = 1$. Since $\gamma_X = \delta_X = p$, in view of Lemma 7, for every $\varepsilon \in (0, p-1)$, one has $L^{p+\varepsilon,\infty} \hookrightarrow X \hookrightarrow L^{p-\varepsilon,1}$, which completes the proof. □

Acknowledgements The authors thank Sergei V. Astashkin for useful discussions. They also thank the anonymous referee for useful remarks.

Funding Open access funding provided by FCT|FCCN (b-on). This work is funded by national funds through the FCT - Fundação para a Ciência e a Tecnologia, I.P., under the scope of the projects UIDB/00297/2020 (https://doi.org/10.54499/UIDB/00297/2020) and UIDP/00297/2020 (https://doi.org/10.54499/UIDP/00297/2020) (Center for Mathematics and Applications).

Data availability Not applicable.

Declarations

Conflict of interest The authors have no relevant financial or non-financial interests to disclose.

References

1. Astashkin, S.V.: On the interpolation of intersections by the real method. St. Petersburg Math. J. **17**(2), 239–265 (2006)
2. Bennett, C., Sharpley, R.: Interpolation of operators. Pure and Applied Mathematics, vol. 129. Academic Press Inc, Boston (1988)

3. Cobos, F., Fernández-Cabrera, L.M., Manzano, A., Martínez, A.: Inclusion indices of quasi-Banach spaces. Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8) **10**(1), 99–117 (2007)
4. Curbera, G.P., Okada, S., Ricker, W.J.: Fine spectra of the finite Hilbert transform in function spaces. Adv. Math. **380**:Paper No. 107597, 29, (2021)
5. Fehér, F.: Indices of Banach function spaces and spaces of fundamental type. J. Approx. Theory **37**(1), 12–28 (1983)
6. Fernández-Cabrera, L.M.: Inclusion indices of function spaces and applications. Math. Proc. Cambridge Philos. Soc. **136**(3), 665–674 (2004)
7. Fernández-Cabrera, L.M., Cobos, F., Hernández, F.L., Sánchez, V.M.: Indices defined by interpolation scales and applications. Proc. Roy. Soc. Edinburgh Sect. A **134**(4), 695–717 (2004)
8. García del Amo, A., Hernández, F.L., Sánchez, V.M., Semenov, E.M.: Disjointly strictly-singular inclusions between rearrangement invariant spaces. J. Lond. Math. Soc. (2) **62**(1), 239–252 (2000)
9. Hernández, F.L., Rodríguez-Salinas, B.: Lattice-embedding L^p into Orlicz spaces. Israel J. Math. **90**(1–3), 167–188 (1995)
10. Hernández, F.L., Rodriguez-Salinas, B.: Lattice-embedding scales of L^p spaces into Orlicz spaces. Israel J. Math. **104**, 191–220 (1998)
11. Hernández, F.L., Rodríguez-Salinas, B.: Lattice-universal Orlicz spaces on probability spaces. Israel J. Math. **133**, 9–28 (2003)
12. Kreĭn, S.G., Petunīn, Y.I., Semënov, E.M.: *Interpolation of linear operators*, Translations of Mathematical Monographs, vol. 54. American Mathematical Society, Providence, R.I., Translated from the Russian by J. Szűcs (1982)
13. Maligranda, L.: A generalization of the Shimogaki theorem. Studia Math. **71**(1):69–83, (1981/82)
14. Maligranda, L.: Indices and interpolation. Dissertationes Math. (Rozprawy Mat.) **234**, 49 (1985)
15. Semenov, E.M.: Embedding theorems for Banach spaces of measurable functions. Dokl. Akad. Nauk SSSR **156**, 1292–1295 (1964). (**in Russian**)
16. Zippin, M.: Interpolation of operators of weak type between rearrangement invariant function spaces. J. Funct. Anal. **7**, 267–284 (1971)