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ORIGINAL PAPER

Formulas for *p*-adic *q*-integrals including falling-rising factorials, combinatorial sums and special numbers

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Abstract

The main purpose of this paper is to provide a novel approach to deriving formulas for the p-adic q-Volkenborn integral including the Volkenborn integral and p-adic fermionic integral. By applying integral equations and these integral formulas to the falling factorials, the rising factorials and binomial coefficients, we derive some various identities, formulas and relations related to several combinatorial sums, well-known special numbers such as the Bernoulli and Euler numbers, the harmonic numbers, the Stirling numbers, the Lah numbers, the Harmonic numbers, the Fubini numbers, the Daehee numbers and the Changhee numbers. Applying these identities and formulas, we give some new combinatorial sums. Finally, by using integral equations, we derive generating functions for new families of special numbers and polynomials. By using generating functions, we give relations between the Lah numbers, the Bernoulli numbers, the Euler numbers and the Laguerre polynomials. We also give further comments and remarks on these functions, numbers and integral formulas related to q-type operators potentially used to solve problems in the areas such as physics, quantum mechanics, quantum systems and the others. In addition, we provide some tables containing some of the p-adic integral formulas obtained in this paper.

Keywords p-adic q-integrals · Generating function · Bernoulli numbers and polynomials · Euler numbers and polynomials · Stirling numbers · Lah numbers · Harmonic numbers · Fubini numbers · Combinatorial numbers and sums

Mathematics Subject Classification 11S80 · 11B68 · 05A15 · 05A19 · 11M35 · 30C15 · 26C05 · 12D10 · 33C45

1 Introduction

This paper deals with comprehensive study of analytic objects linked to theory of the Volkenborn integral, the fermionic p-adic integral, and the generating functions for special numbers and polynomials. The p-adic integral and generating functions have been used in mathematics, in mathematical physics and in others sciences. Especially the p-adic integral and p-adic

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numbers are used in the theory of ultrametric calculus, the p-adic quantum mechanics and the p-adic mechanics.

In this paper, by using Volkenborn and fermionic integrals with their integral equations, we present generating functions for special numbers and polynomials in terms of these integrals. By applying these integrals to the falling factorial and the rising factorial with their identities and relations, we derive both the standard and new p-adic integral formulas, identities and relations closely related to the Volkenborn integral, the fermionic integral, combinatorial sums and special numbers. These formulas in particular will allow us to solve and compute efficiently such those integrals including the falling and rising factorials, and other differentiable functions.

We believe that in the light of the studies in this paper so many applied areas especially mathematics and physics will have various new p-adic integral formulas, combinatorial sums and identities including special numbers and polynomials. In fact those formulas getting so much interest specially for researchers who work in ultrametric calculus and the theory of p-adic integrals. It is inevitable that integral formulas, combinatorial sums and identities elaborated by the p-adic integrals formulas will illuminate into the areas of combinatorial physics, quantum physics and mathematics and statistics. For the fundamentals of p-adic integrals in which some of them will be mentioned briefly in the next, we may refer the references [2, 20, 23, 24, 40, 50, 51]; and the references cited therein.

In this paper we use the following notations:

Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of natural numbers, the set of integers, the set of rational numbers, the set of real numbers and the set of complex numbers, respectively. $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $n, k \in \mathbb{Z}$. If $n < 0$ or $k > n$ or $k < 0$, then

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = 0$$

(cf. [1–52]).

The rising factorial is defined by

$$x^{(n)} = x(x+1)(x+2)\ldots(x+n-1), \tag{1}$$

where $n \in \mathbb{N}$ and

$$x^{(0)} = 1,$$

and the falling factorial is defined by

$$x_{(n)} = x(x-1)(x-2)\ldots(x-n+1),$$

where $n \in \mathbb{N}$ and

$$x_{(0)} = 1,$$

(cf. [11, 13, 14, 47]).

In order to give our results, we need the following properties and definitions for generating functions of the special numbers and polynomials. In addition, p-adic integrals and their properties, which have many valuable applications in almost all areas of mathematics as well as mathematical physics, engineering and other areas of science, are given. New p-adic integral formulas and generating functions are also given with the help of these integrals.

The Apostol-Bernoulli polynomials $\mathcal{B}_n(x;\lambda)$ are defined by

$$F_A(t,x;\lambda) = \frac{t}{\lambda e^t - 1} e^{tx} = \sum_{n=0}^{\infty} \mathcal{B}_n(x;\lambda) \frac{t^n}{n!}. \tag{2}$$

Substituting $x = 0$ into (2), we have

$$\lambda \mathcal{B}_1(1; \lambda) = 1 + \mathcal{B}_1(\lambda)$$

and for $n \geq 2$,

$$\lambda \mathcal{B}_n(1; \lambda) = \mathcal{B}_n(\lambda)$$

(cf. [3]).

Substituting $x = 1$ into (2), we have the following Apostol-Bernoulli numbers:

$$\mathcal{B}_n(1, \lambda) = \sum_{j=0}^{n} \binom{n}{j} \mathcal{B}_n(\lambda),$$

where

$$\mathcal{B}_n(\lambda) = \mathcal{B}_n(0, \lambda)$$

and

$$\mathcal{B}_0(\lambda) = 0$$

(cf. [3], for detail, see also [24, 35, 48]). When $\lambda = 1$ in (2), we have the Bernoulli polynomials of the first kind

$$B_n(x) = \mathcal{B}_n(x; 1)$$

and also $B_n = B_n(0)$ denotes the Bernoulli numbers of the first kind (cf. [4–52]; see also the references cited in each of these works).

The λ-Bernoulli polynomials (Apostol-type Bernoulli) polynomials $\mathfrak{B}_n(x; \lambda)$ are defined by means of the following generating function:

$$F_B(t, x; \lambda) = \frac{\log \lambda + t}{\lambda e^t - 1} e^{tx} = \sum_{n=0}^{\infty} \mathfrak{B}_n(x; \lambda) \frac{t^n}{n!}, \tag{3}$$

(cf. [30]; see also [46–48]).

The Apostol-Euler polynomials of first kind $\mathcal{E}_n(x, \lambda)$ are defined by means of the following generating function:

$$F_{P1}(t, x; k, \lambda) = \frac{2}{\lambda e^t + 1} e^{tx} = \sum_{n=0}^{\infty} \mathcal{E}_n(x, \lambda) \frac{t^n}{n!}, \tag{4}$$

(cf. [4–48]. Substituting $x = 0$ into (4), we have the Apostol-Euler numbers of the first kind:

$$\mathcal{E}_n(\lambda) = \mathcal{E}_n(0, \lambda).$$

Setting $\lambda = 1$ into (4), we have the Euler numbers of the first kind:

$$E_n = \mathcal{E}_n^{(1)}(1)$$

(cf. [4–48]; see also the references cited in each of these earlier works).

Let u be a complex numbers with $u \neq 1$. The Frobenius–Euler numbers $H_n(u)$ are defined by means of the following generating function:

$$F_f(t, u) = \frac{1 - u}{e^t - u} = \sum_{n=0}^{\infty} H_n(u) \frac{t^n}{n!}. \tag{5}$$

Substituting $u = -1$ into (5), we have

$$E_n = H_n(-1)$$

(cf. [30, Theorem 1, p. 439], [34, 48]; see also the references cited in each of these earlier works).

By using (2) and (4), we have the following well-know relation between the polynomials $\mathcal{B}_n(x;\lambda)$ and $\mathcal{E}_n(x,\lambda)$

$$\mathcal{B}_n(x;\lambda) = -\frac{n}{2}\mathcal{E}_{n-1}(x,-\lambda)$$

(cf. [47]).

The Euler numbers of the second kind E_n^* are given by

$$E_n^* = 2^n E_n\left(\frac{1}{2}\right)$$

(cf. [39, 47]; see also the references cited in each of these earlier works).

The Fubini numbers $w_g(n)$ are defined by means of the following generating functions:

$$F_{Fu}(t) = \frac{1}{2-e^t} = \sum_{n=0}^{\infty} w_g(n)\frac{t^n}{n!}, \tag{6}$$

(cf. [16]).

The Fubini numbers of order k are defined by the following generating function:

$$F_{Fu}(t,k) = \frac{1}{(2-e^t)^k} = \sum_{n=0}^{\infty} w_g^{(k)}(n)\frac{t^n}{n!}, \tag{7}$$

(cf. [21]).

The Stirling numbers of the first kind $S_1(n,k)$ the number of permutations of n letters which consist of k disjoint cycles, are defined by means of the following generating function:

$$F_{S1}(t,k) = \frac{(\log(1+t))^k}{k!} = \sum_{n=0}^{\infty} S_1(n,k)\frac{t^n}{n!}. \tag{8}$$

These numbers have the following properties:

$$\begin{aligned}
S_1(0,0) &= 1,\\
S_1(0,k) &= 0 \text{ if } k > 0,\\
S_1(n,0) &= 0 \text{ if } n > 0,\\
S_1(n,k) &= 0 \text{ if } k > n,
\end{aligned}$$

and

$$S_1(n+1,k) = -nS_1(n,k) + S_1(n,k-1) \tag{9}$$

(cf. [4, 10, 11, 39, 41, 43]; and see also the references cited in each of these earlier works).

A relation between falling factorial and Stirling numbers of the first kind is given by

$$x_{(n)} = \sum_{k=0}^{n} S_1(n,k)x^k \tag{10}$$

(cf. [11, 13, 14, 47]).

The unsigned Stirling numbers of the first kind are defined by

$$C(n,k) = |S_1(n,k)|$$

(cf. [11, 13, 14, 47]). The numbers $C(n,k)$ are also defined as follows:

$$x^{(n)} = \sum_{k=0}^{n} C(n,k)x^k \tag{11}$$

(cf. [12]).

The Bernoulli polynomials of the second kind $b_n(x)$ are defined by means of the following generating function:

$$F_{b2}(t,x) = \frac{t}{\log(1+t)}(1+t)^x = \sum_{n=0}^{\infty} b_n(x)\frac{t^n}{n!} \tag{12}$$

(cf. [39, pp. 113–117]; see also the references cited in each of these earlier works).

The Bernoulli numbers of the second kind $b_n(0)$ are defined by means of the following generating function:

$$F_{b2}(t) = \frac{t}{\log(1+t)} = \sum_{n=0}^{\infty} b_n(0)\frac{t^n}{n!}. \tag{13}$$

The Bernoulli polynomials of the second kind are defined by

$$b_n(x) = \int_x^{x+1} u_{(n)}du.$$

Substituting $x = 0$ into the above equation, one has

$$b_n(0) = \int_0^1 u_{(n)}du. \tag{14}$$

The numbers $b_n(0)$ are also so-called the Cauchy numbers (i.e. Bernoulli numbers of the second kind) (cf. [39, p. 116], [32, 44]; see also the references cited in each of these earlier works).

The λ-array polynomials $S_k^n(x;\lambda)$ are defined by the following generating function:

$$F_A(t,x,k;\lambda) = \frac{\left(\lambda e^t - 1\right)^k}{k!}e^{tx} = \sum_{n=0}^{\infty} S_k^n(x;\lambda)\frac{t^n}{n!}, \tag{15}$$

where $k \in \mathbb{N}_0$ and $\lambda \in \mathbb{C}$ (cf. [41], for detail see also [4, 10, 43]). Substituting $x = 0$ into (15), we have the λ-Stirling numbers $S_2(n,k;\lambda)$, which are defined by the following generating function:

$$F_S(t,k;\lambda) = \frac{\left(\lambda e^t - 1\right)^k}{k!} = \sum_{n=0}^{\infty} S_2(n,k;\lambda)\frac{t^n}{n!}, \tag{16}$$

where $k \in \mathbb{N}_0$ and $\lambda \in \mathbb{C}$ (cf. [35, 47], see also [41]).

Substituting $\lambda = 1$ into (16), then we get the Stirling numbers of the second kind, the number of partitions of a set of n elements into k nonempty subsets,

$$S_2(n,k) = S_2(n,k;1).$$

The Stirling numbers of the second kind are also given by the following generating function including falling factorial:

$$x^n = \sum_{k=0}^{n} S_2(n,k)x_{(k)}, \tag{17}$$

(cf. [2–48]; see also the references cited in each of these earlier works).

The Schlomilch formula, associated with the Stirling numbers of the first and the second kind, is given by

$$S_1(n,k) = \sum_{j=0}^{n-k}(-1)^j \binom{n+j-1}{k-1}\binom{2n-k}{n-k-j} S_2(n-k+j,j)$$

(cf. [12, p. 115], [11, p. 290, Eq. (8.21)]).

The associated Stirling numbers of the second kind are defined by means of the following generating function:

$$F_{S2}(t,k;\lambda) = \frac{\left(e^t-1-t\right)^k}{k!} = \sum_{n=0}^{\infty} S_{22}(n,k)\frac{t^n}{n!}, \tag{18}$$

where $k \in \mathbb{N}_0$. By using (18), we have

$$S_2(n,k) = \sum_{j=0}^{k}\binom{k}{j} S_{22}(n-j,k-j)$$

and $S_{22}(n,k) = 0$ if $k > \frac{n}{2}$ (cf. [12, pp. 123–127]). Using (18), we give the following functional equation:

$$F_{S2}(t,v;\lambda) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j} j! F_S(t,j;1)u^{k-j}.$$

Using the above functional equation, we have the following well-known identity:

$$S_{22}(n,k) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j} j! S_2(n,j)u^{k-j}$$

(cf. [12, pp. 123–127]).

The associated Stirling numbers of the first kind are defined by means of the following generating function:

$$F_{S12}(t,k;\lambda) = \frac{(\log(1+t)-t)^k}{k!} = \sum_{n=0}^{\infty} S_{12}(n,k)\frac{t^n}{n!},$$

where $k \in \mathbb{N}_0$,

$$S_1(n,k) = \sum_{j=0}^{k}\binom{k}{j} S_{12}(n-j,k-j)$$

and $S_{12}(n,k) = 0$ if $k > \frac{n}{2}$ (cf. [12, pp. 123–127]).

The Lah numbers was discovered by Ivo Lah in 1955 (cf. [11, 12, 14, 37, 38]). The unsigned Lah numbers have an interesting meaning especially in combinatorics. These numbers count the number of ways a set of n elements can be partitioned into k nonempty linearly ordered subsets. These numbers are related to some well-known special numbers such as the Stirling numbers of the first and the second kind, and the Laguerre polynomials.

The Lah numbers are defined by means of the following generating function:

$$F_L(t,k) = \frac{1}{k!}\left(\frac{t}{1-t}\right)^k = \sum_{n=k}^{\infty} L(n,k)\frac{t^n}{n!} \tag{19}$$

(cf. [38, p. 44], [5, 37], and the references cited therein). By using this equation, we have

$$L(n,k) = (-1)^n \frac{n!}{k!}\binom{n-1}{k-1}. \tag{20}$$

The unsigned Lah numbers are defined by

$$|L(n,k)| = \frac{n!}{k!}\binom{n-1}{k-1}, \tag{21}$$

where $n, k \in \mathbb{N}$ with $1 \leq k \leq n$.

Two recurrence relations of these numbers are given by

$$L(n+1,k) = -(n+k)L(n,k) - L(n,k-1)$$

with the initial conditions

$$L(n,0) = \delta_{n,0}$$

and

$$L(0,k) = \delta_{0,k},$$

for all $k, n \in \mathbb{N}$ and

$$L(n,k) = \sum_{j=0}^{n} (-1)^j S_1(n,j) S_2(j,k)$$

(cf. [38, p. 44], [37]).

Another definition of the Lah numbers are related to the falling factorial and the rising factorial. Let $n \in \mathbb{N}_0$. Since

$$(-1)^n\,(-x)_{(n)} = (x+n-1)_{(n)} = x^{(n)}, \tag{22}$$

we have the following well-known formulas:

$$(-x)_{(n)} = \sum_{k=1}^{n} L(n,k) x_{(k)} \tag{23}$$

so that

$$x_{(n)} = \sum_{k=1}^{n} L(n,k)\,(-x)_{(k)}$$

and

$$x^{(n)} = \sum_{k=1}^{n} |L(n,k)|\, x_{(k)}. \tag{24}$$

(cf. [11, 12, 14, 37, 38]).

The Daehee numbers of the first kind and the second kind are defined by means of the following generating functions, respectively:

$$\frac{\log(1+t)}{t} = \sum_{n=0}^{\infty} D_n \frac{t^n}{n!} \tag{25}$$

and

$$\frac{(1+t)\log(1+t)}{t} = \sum_{n=0}^{\infty} \widehat{D}_n \frac{t^n}{n!},$$

(cf. [38, p. 45], [15, 27]). Using (25), we have

$$D_n = (-1)^n \frac{n!}{n+1}$$

(cf. [27]).

The Changhee numbers of the first kind and the second kind are defined by means of the following generating functions:

$$\frac{2}{t+1} = \sum_{n=0}^{\infty} Ch_n \frac{t^n}{n!} \tag{26}$$

and

$$\frac{2(1+t)}{t+2} = \sum_{n=0}^{\infty} \widehat{Ch}_n \frac{t^n}{n!},$$

(cf. [31]). Using (26), we have

$$Ch_n = \sum_{k=0}^{n} S_1(n,k) E_k = (-1)^n \frac{n!}{2^n}$$

(cf. [31]).

The Peters polynomials $s_k(x;\lambda,\mu)$ are defined by means of the following generating function:

$$\frac{1}{\left(1+(1+t)^{\lambda}\right)^{\mu}}(1+t)^x = \sum_{n=0}^{\infty} s_k(x;\lambda,\mu) \frac{t^n}{n!} \tag{27}$$

(*cf.* [19, 39]).

If we substitute $\mu = 1$ into (27), then we have the Boole polynomials. If we substitute $\lambda = 1$ and $\mu = 1$ into (27), then we have the Changhee polynomials (*cf.* [28, 39]).

This paper have exactly 13 main sections including introduction. We summarize as follows:

In Sect. 2, we give some properties of the p-adic q-integrals and the p-adic fermionic integral with their integral equations. Using these equations, we give generating functions for special numbers and polynomials, some identities and formulas including combinatorial sums.

In Sect. 3, we give some applications of the Volkenborn integral to the falling and rising factorials. We define sequences of the Bernoulli numbers related to these applications. We give some integral formulas including the Bernoulli numbers and polynomials, the Euler numbers and polynomials, the Stirling numbers, the Lah numbers and the combinatorial sums.

In Sect. 4, we give some formulas for the sequence of the Bernoulli numbers. By using the Volkenborn integral and its integral equations, we give some formulas and identities of the Bernoulli numbers sequence. We also give some integral formulas including falling factorials.

In Sect. 5, we give some computation formulas for the sequence including the Bernoulli numbers. Using the Volkenborn integral, we also derive some formulas and identities of this sequence and some integral formulas related to rising factorials.

In Sect. 6, we give various integral formulas for the Volkenborn integral associated with the falling factorials, the combinatorial sums, the special numbers including the Bernoulli numbers, the Stirling numbers and the Lah numbers.

In Sect. 7, we give various integral formulas for the fermionic p-adic integral including the falling factorial and the rising factorial with their identities and relations, the combinatorial sums, the special numbers such as the Euler numbers, the Stirling numbers and the Lah numbers.

In Sect. 8, we give some applications of the p-adic fermionic integral associated with falling factorial and rising factorial. We define two kinds of sequences related to the Euler numbers and the Euler polynomials and also the Stirling numbers, the Lah numbers and the combinatorial sums.

In Sect. 9, by using the fermionic integral and its integral equations, we derive some formulas for the sequence of the Euler numbers and the p-adic fermionic integral formulas related to the falling factorials.

In Sect. 10, using the fermionic integral, we give some interesting formulas for the Euler numbers sequence and the p-adic fermionic integral including the raising factorial.

In Sect. 11, We give some novel identities for combinatorial sums including special numbers associated with the Bernoulli numbers, the Euler numbers, the Stirling numbers, the Eulerian numbers, the Fubini numbers and the Lah numbers.

In Sect. 12, we conclude this paper by providing some observations on our results.

In Section A, we finalize this paper along with an appendix of some tables containing p-adic integral formulas obtained in this paper.

2 Integral equations for p-adic q-integrals

In this section, we give some properties of p-adic q-integrals. We study integral equations for these integrals. By using these integral equations, we derive generating functions for special numbers and polynomials. Using these generating functions, some identities and formulas including these numbers and polynomials and also combinatorial sums are given.

To state the p-adic q-Volkenborn integral, it is useful to firstly introduce the following notations.

Let $\mathbb{Z}_p$ be a set of p-adic integers. Let $\mathbb{K}$ be a field with a complete valuation and $C^1(\mathbb{Z}_p \to \mathbb{K})$ be a set of functions which have continuous derivative (see, for detail, [40]).

By taking into account the set of p-adic rational numbers $\mathbb{Q}_p$ having the algebraic closure $\mathbb{C}_p$, Kim [23] defined the following p-adic q-integral:

Let $f \in C^1(\mathbb{Z}_p \to \mathbb{K})$ and $q \in \mathbb{C}_p$ with $|1-q|_p < 1$. Then we have

$$I_q(f(x)) = \int_{\mathbb{Z}_p} f(x)d\mu_q(x) = \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x)q^x, \tag{28}$$

where

$$[x] = [x:q] = \begin{cases} \frac{1-q^x}{1-q}, q \neq 1 \\ x, q = 1 \end{cases}$$

and

$$\mu_q(x) = \mu_q\left(x + p^N\mathbb{Z}_p\right)$$

denotes q-distribution on $\mathbb{Z}_p$, which is defined by

$$\mu_q\left(x + p^N\mathbb{Z}_p\right) = \frac{q^x}{\left[p^N\right]_q},$$

(cf. [23]). For a compact-open subset $\mathbb{X}$ of $\mathbb{Q}_p$, a p-adic distribution μ on $\mathbb{X}$ is a $\mathbb{Q}_p$-linear vector space homomorphism from the $\mathbb{Q}_p$ -vector space of locally constant functions on $\mathbb{X}$ to $\mathbb{Q}_p$ (cf. [40]).

Observe that if $q \to 1$, then (28) reduces to the following well-known Volkenborn integral (bosonic integral), which is denoted by $I_1(f(x))$:

$$\lim_{q\to 1} I_q(f(x)) = I_1(f(x)) = \int\limits_{\mathbb{Z}_p} f(x)\, d\mu_1(x) = \lim_{N\to\infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x), \tag{29}$$

where

$$\mu_1(x) = \mu_1\left(x + p^N\mathbb{Z}_p\right) = \frac{1}{p^N}$$

denotes the Haar distribution (cf. [2, 20, 40, 50, 51]); see also the references cited in each of these earlier works). The above integral has many applications not only in mathematics, but also in mathematical physics. By using this integral and its integral equations, various family of generating functions associated with Bernoulli-type numbers and polynomials have been constructed (cf. [2–52]).

If $q \to -1$, then (28) reduces to the following well-known fermionic p-adic integral, which is denoted by $I_{-1}(f(x))$:

$$\lim_{q\to -1} I_q(f(x)) = I_{-1}(f(x)) = \int\limits_{\mathbb{Z}_p} f(x)\, d\mu_{-1}(x) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} (-1)^x f(x) \tag{30}$$

(cf. [24]).

By using p-adic fermionic integral and its integral equations, various different generating functions including Euler-type numbers and polynomials and Genocchi-type numbers and polynomials have been constructed (cf. [2–52]).

We also note that p-adic q-integrals are related to the theory of the generating functions, ultrametric calculus, the quantum groups, cohomology groups, q-deformed oscillator and p-adic models (cf. [20, 50]).

2.1 Some properties of the Volkenborn integral

Here, we give some properties of the Volkenborn integral.

The Volkenborn integral is given in terms of the Mahler coefficients as follows:

$$\int_{\mathbb{Z}_p} f(x)\, d\mu_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} a_n,$$

where

$$f(x) = \sum_{n=0}^{\infty} a_n \binom{x}{j} \in C^1(\mathbb{Z}_p \to \mathbb{K}).$$

(cf. [40, p. 168-Proposition 55.3]).

Let $f : \mathbb{Z}_p \to \mathbb{K}$ be an analytic function and

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

with $x \in \mathbb{Z}_p$.

The Volkenborn integral of this analytic function is given by

$$\int_{\mathbb{Z}_p} \left(\sum_{n=0}^{\infty} a_n x^n \right) d\mu_1(x) = \sum_{n=0}^{\infty} a_n \int_{\mathbb{Z}_p} x^n d\mu_1(x)$$

and

$$\int_{\mathbb{Z}_p} \left(\sum_{n=0}^{\infty} a_n x^n \right) d\mu_1(x) = \sum_{n=0}^{\infty} a_n B_n$$

where

$$B_n = \int_{\mathbb{Z}_p} x^n d\mu_1(x),$$

which is known as the Witt's formula for the Bernoulli numbers (cf. [23, 24, 40]; see also the references cited in each of these earlier works).

Integral equation for the Volkenborn integral is given as follows:

$$\int_{\mathbb{Z}_p} E^m [f(x)]\, d\mu_1(x) = \int_{\mathbb{Z}_p} f(x) d\mu_1(x) + \sum_{x=0}^{m-1} \frac{d}{dx} \{f(x)\} \tag{31}$$

where

$$E^m [f(x)] = f(x+m)$$

(cf. [23, 24, 40]; see also the references cited in each of these earlier works).

Using (28), the following integral equation was given by Kim [26]:

$$q \int_{\mathbb{Z}_p} E[f(x)]\, d\mu_q(x) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) + \frac{q-1}{\log q} f'(0) + (q-1) f(0), \tag{32}$$

where

$$f^{'}(0)=\frac{d}{dx}\left\{f(x)\right\}|_{x=0}\,.$$

As usual, exponential function is defined as follows:

$$e^{t}=\sum_{n=0}^{\infty}\frac{t^{n}}{n!}.$$

The above series convergences in region E, which subset of field $\mathbb{K}$ with $char(\mathbb{K})=0$ (cf. [40, p. 70]). Let k be residue class field of $\mathbb{K}$. If $char(k)=p$, then

$$E=\left\{x\in\mathbb{K}:|x|<p^{\frac{1}{1-p}}\right\}$$

and if $char(k)=0$, then

$$E=\left\{x\in\mathbb{K}:|x|<1\right\}.$$

Let $f\in C^{1}(\mathbb{Z}_p\to\mathbb{K})$. Kim [26, Theorem 1] gave the following integral equation:

$$\begin{aligned}q^{n}\int\limits_{\mathbb{Z}_p}E^{n}\left[f\left(x\right)\right]d\mu_{q}\left(x\right)-\int\limits_{\mathbb{Z}_p}f\left(x\right)d\mu_{q}\left(x\right)\\=\frac{q-1}{\log q}\left(\sum_{j=0}^{n-1}q^{j}f^{'}(j)+\log q\sum_{j=0}^{n-1}q^{j}f(j)\right),\end{aligned}\tag{33}$$

where n is a positive integer and

$$f^{'}(j)=\frac{d}{dx}\left\{f(x)\right\}\Big|_{x=j}\,.$$

Observe that substituting $n=1$ into (33), we arrive at (32).

Theorem 1 *Let $n\in\mathbb{N}_0$. Then we have*

$$\int\limits_{\mathbb{Z}_p}\binom{x}{n}d\mu_{1}\left(x\right)=\frac{(-1)^{n}}{n+1}.\tag{34}$$

Theorem 1 was proved by Schikhof [40].
Combining

$$x_{(n)}=n!\binom{x}{n},$$

and (34) with (46), we have

$$\int\limits_{\mathbb{Z}_p}x_{(n)}d\mu_{1}\left(x\right)=\frac{(-1)^{n}n!}{n+1},\tag{35}$$

(cf. [27]).

By using (34), we have

$$\int_{\mathbb{Z}_p} \binom{x+n-1}{n} d\mu_1(x) = \sum_{m=0}^{n} \binom{n-1}{n-m} \int_{\mathbb{Z}_p} \binom{x}{m} d\mu_1(x)$$
$$= \sum_{m=1}^{n} (-1)^m \binom{n-1}{m-1} \frac{1}{m+1}$$
$$= \sum_{m=0}^{n} (-1)^m \binom{n-1}{n-m} \frac{1}{m+1} \tag{36}$$

(cf. [27, 31, 43]). By using (36), we obtain

$$\int_{\mathbb{Z}_p} (x+n-1)_{(n)}\, d\mu_1(x) = n! \sum_{m=0}^{n} (-1)^m \binom{n-1}{n-m} \frac{1}{m+1}. \tag{37}$$

2.2 Generating functions with help of integral equation of (32)

Here, we give generating functions with help of integral equation of (32). By using these functions, we give new families of special numbers and polynomials including Bernoulli-type numbers and polynomials.

Let $\lambda \in \mathbb{Z}_p$. We define

$$f(x) = \lambda^x a^{xt}. \tag{38}$$

Substituting (38) into (32), we get

$$\int_{\mathbb{Z}_p} \lambda^x a^{xt} d\mu_q(x) = \frac{q-1}{\log q} \left(\frac{t \log a + \log \lambda + \log q}{\lambda q a^t - 1} \right), \tag{39}$$

where

$$a \in \mathbb{C}_p^+ = \left\{ x \in \mathbb{C}_p : |1-x|_p < 1 \right\}$$

and $a \neq 1$.

Substituting $t = 1$ and $q \to 1$ into (39), we have

$$\int_{\mathbb{Z}_p} \lambda^x a^x d\mu_1(x) = \frac{\log \lambda + \log(a)}{\lambda a - 1}.$$

Substituting $\lambda = 1$ into the above equation, we arrive at the Exercise 55A-1 of [40, p. 170] as follows:

$$\int_{\mathbb{Z}_p} a^x d\mu_1(x) = \frac{\log(a)}{a-1},$$

where $a \in \mathbb{C}_p^+$ with $a \neq 1$.

Remark 1 Substituting $a = e$, $\lambda = 1$ and $q \to 1$ into (39), we arrive at the equation (3). Substituting $a = e$ and $q \to 1$ into (39), we get the Exercise 55A-2 of [40, p. 170], which gives us the generating function for the Bernoulli numbers B_n as follows:

$$\int_{\mathbb{Z}_p} e^{tx} d\mu_1(x) = \frac{t}{e^t - 1}$$

where $t \in E$ with $t \neq 0$.

By using (39), we define the following generating function for special numbers $\mathfrak{S}_n(a; \lambda, q)$:

$$H_1(t; \lambda; a, q) = \frac{q-1}{\log q}\left(\frac{t \log a + \log(\lambda q)}{\lambda q a^t - 1}\right) = \sum_{n=0}^{\infty} \mathfrak{S}_n(a; \lambda, q)\frac{t^n}{n!}. \tag{40}$$

By using (39), we define the following generating function for special numbers $\mathfrak{S}_n(a; \lambda, q)$:

$$H_2(t, x; \lambda; a, q) = a^{tx} H_1(t; \lambda; a, q) = \sum_{n=0}^{\infty} \mathfrak{S}_n(x, a; \lambda, q)\frac{t^n}{n!}. \tag{41}$$

Observe that

$$\mathfrak{S}_n(a; \lambda, q) = \mathfrak{S}_n(0, a; \lambda, q).$$

Combining (40) with (41), we get

$$\sum_{n=0}^{\infty} \mathfrak{S}_n(x, a; \lambda, q)\frac{t^n}{n!} = \sum_{n=0}^{\infty} (x \ln a)^n \frac{t^n}{n!} \sum_{n=0}^{\infty} \mathfrak{S}_n(a; \lambda, q)\frac{t^n}{n!}.$$

Therefore

$$\sum_{n=0}^{\infty} \mathfrak{S}_n(x, a; \lambda, q)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\sum_{j=0}^{n} \binom{n}{j} (x \ln a)^{n-j}\, \mathfrak{S}_n(a; \lambda, q)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the following theorem:

Theorem 2 *Let $n \in \mathbb{N}_0$. Then we have*

$$\mathfrak{S}_n(x, a; \lambda, q) = \sum_{j=0}^{n} \binom{n}{j} (x \ln a)^{n-j}\, \mathfrak{S}_n(a; \lambda, q). \tag{42}$$

Remark 2 Substituting $\lambda = 1$ into (40), we have

$$\mathfrak{s}_n(a, q) = \mathfrak{S}_n(a; \lambda, q)$$

(cf. [33]). Substituting $q \to 1$ and $a = e$ into (40), we have

$$\mathfrak{B}_n(\lambda) = \mathfrak{S}_n(e; \lambda, 1)$$

and

$$\mathfrak{B}_{n-1}(\lambda) = \left(\frac{\log \lambda}{n}\right) \mathcal{B}_n(\lambda) + \mathcal{B}_{n-1}(\lambda).$$

By using (39), we give p-adic integral representation of the special numbers $\mathfrak{S}_n(a; \lambda, q)$ by the following theorem:

Theorem 3 *Let $n \in \mathbb{N}_0$. Then we have*

$$\mathfrak{S}_n(a; \lambda, q) = \int_{\mathbb{Z}_p} \lambda^x \,(x \log a)^n \, d\mu_q(x). \tag{43}$$

Remark 3 Using (43), we have the following well-known the Witt's formula for the Bernoulli numbers, B_n:

$$\int\limits_{\mathbb{Z}_p} x^n d\mu_1(x) = B_n, \tag{44}$$

(cf. [40]). We also easily see that the Bernoulli polynomials are defined by

$$\int\limits_{\mathbb{Z}_p} (z+x)^n d\mu_1(x) = B_n(z) \tag{45}$$

(cf. [23, 24, 40]; see also the references cited in each of these earlier works).

Kim et al. [27] defined Witt-type identities for the Daehee numbers of the first kind by the following integral representation as follows:

$$D_n = \int\limits_{\mathbb{Z}_p} t_{(n)} d\mu_1(t). \tag{46}$$

and

$$\int\limits_{\mathbb{Z}_p} x_{(n)} d\mu_1(x) = \sum_{l=0}^{n} S_1(n,l) B_l. \tag{47}$$

Kim et al. [27] defined the Daehee numbers of the second kind as follows:

$$\widehat{D_n} = \int\limits_{\mathbb{Z}_p} t^{(n)} d\mu_1(t). \tag{48}$$

Kim et al. [27] defined the Daehee polynomials of the first and second kind, respectively, as follows:

$$D_n(x) = \int\limits_{\mathbb{Z}_p} (x+t)_{(n)} d\mu_1(t) \tag{49}$$

and

$$\widehat{D_n}(x) = \int\limits_{\mathbb{Z}_p} (x+t)^{(n)} d\mu_1(t). \tag{50}$$

2.3 Some properties of the fermionic *p*-adic integral

Here, we give some properties of the fermionic p-adic integral. By using integral equations of the fermionic p-adic integral, we give generating functions for special numbers and polynomials. Using these generating functions, some identities and formulas including these numbers and polynomials and also combinatorial sums are given. We also give interpolation function for these numbers.

Let $f \in C^1(\mathbb{Z}_p \to \mathbb{K})$. Kim [25] gave the following integral equation for the fermionic p -adic integral on $\mathbb{Z}_p$:

$$\int_{\mathbb{Z}_p} E^n\left[f\left(x\right)\right] d\mu_{-1}\left(x\right) + (-1)^{n+1} \int_{\mathbb{Z}_p} f\left(x\right) d\mu_{-1}\left(x\right) = 2\sum_{j=0}^{n-1} (-1)^{n-1-j} f(j), \quad (51)$$

where n is a positive integer.

Substituting $n = 1$ into (51), we have the following very useful integral equation, which is used to construct generating functions associated with Euler-type numbers and polynomials:

$$\int_{\mathbb{Z}_p} f\left(x+1\right) d\mu_{-1}\left(x\right) + \int_{\mathbb{Z}_p} f\left(x\right) d\mu_{-1}\left(x\right) = 2f(0) \quad (52)$$

(cf. [25]).

By using (30) and (52), the well-known Witt's formula for the Euler numbers and polynomials are given as follows, respectively:

$$E_n = \int_{\mathbb{Z}_p} x^n d\mu_{-1}\left(x\right) \quad (53)$$

and

$$E_n(z) = \int_{\mathbb{Z}_p} \left(z+x\right)^n d\mu_{-1}\left(x\right) \quad (54)$$

(cf. [24]; see also the references cited in each of these earlier works).

Theorem 4 *Let $n \in \mathbb{N}_0$. Then we have*

$$\int_{\mathbb{Z}_p} \binom{x}{n} d\mu_{-1}\left(x\right) = (-1)^n 2^{-n}. \quad (55)$$

Theorem 4 was proved by Kim et al. [31, Theorem 2.3].

Substituting $x_{(n)} = n!\binom{x}{n}$ into (55), we have the following well-known identity:

$$\int_{\mathbb{Z}_p} x_{(n)} d\mu_{-1}\left(x\right) = (-1)^n 2^{-n} n! \quad (56)$$

(cf. [31]).

Recently, by using the fermionic p-adic integral on $\mathbb{Z}_p$, Kim et al. [31] defined the following Changhee numbers of the first and second kind, respectively:

$$Ch_n = \int_{\mathbb{Z}_p} x_{(n)} d\mu_{-1}\left(x\right) \quad (57)$$

and

$$\widehat{Ch}_n = \int_{\mathbb{Z}_p} x^{(n)} d\mu_{-1}\left(x\right). \quad (58)$$

Kim et al. [31] defined the first and second kind Changhee polynomials, respectively, as follows:

$$Ch_n(x) = \int_{\mathbb{Z}_p} (x+t)_{(n)} \, d\mu_{-1}(t) \tag{59}$$

or

$$\widehat{Ch}_n(x) = \int_{\mathbb{Z}_p} (x+t)^{(n)} \, d\mu_{-1}(t) . \tag{60}$$

Therefore, by using Theorem 4, we have

$$\begin{aligned} \int_{\mathbb{Z}_p} \binom{x+n-1}{n} d\mu_{-1}(x) &= \sum_{m=0}^{n} \binom{n-1}{n-m} \int_{\mathbb{Z}_p} \binom{x}{m} d\mu_{-1}(x) \\ &= \sum_{m=1}^{n} (-1)^m \binom{n-1}{m-1} 2^{-m} \\ &= \sum_{m=0}^{n} (-1)^m \binom{n-1}{n-m} 2^{-m} \end{aligned} \tag{61}$$

(cf. [27, 31, 43]). By using (61), we have

$$\int_{\mathbb{Z}_p} (x+n-1)_{(n)} \, d\mu_{-1}(x) = n! \sum_{m=0}^{n} (-1)^m \binom{n-1}{n-m} 2^{-m}. \tag{62}$$

By using (51), Kim [26] modified (30). He gave the following integral equation:

$$q^d \int_{\mathbb{Z}_p} E^d f(x) \, d\mu_{-q}(x) + \int_{\mathbb{Z}_p} f(x) \, d\mu_{-q}(x) = [2] \sum_{j=0}^{d-1} (-1)^j q^j f(j), \tag{63}$$

where d is an positive odd integer.

2.4 Generating functions with help of integral equation of the fermionic *p*-adic *q*-integral on $\mathbb{Z}_p$

Here, we give generating functions with help of integral equation of the fermionic p-adic q-integral on $\mathbb{Z}_p$. By using these functions, we give new families of special numbers and polynomials including Euler-type numbers and polynomials.

Combining (38) with (63), we get the following integral equation.

$$\int_{\mathbb{Z}_p} \left(\lambda a^t\right)^x d\mu_{-q}(x) = \frac{[2]}{\left(\lambda q a^t\right)^d + 1} \sum_{j=0}^{d-1} (-1)^j \left(\lambda q a^t\right)^j . \tag{64}$$

By using (64), we define the following generating function for special numbers $\mathfrak{N}_n(a; q, \lambda, d)$:

$$F_1(t, a; q, \lambda, d) = \frac{[2]}{\left(\lambda q a^t\right)^d + 1} \sum_{j=0}^{d-1} (-1)^j \left(\lambda q a^t\right)^j = \sum_{n=0}^{\infty} \mathfrak{N}_n(a; q, \lambda, d) \frac{t^n}{n!}. \tag{65}$$

We define the following generating function for special polynomials $\mathfrak{N}_n(x, a; q, \lambda, d)$:

$$F_2(t, x, a; q, \lambda, d) = a^{tx} F_1(t, a; q, \lambda, d) = \sum_{n=0}^{\infty} \mathfrak{N}_n(x, a; q, \lambda, d)\frac{t^n}{n!}. \tag{66}$$

Observe that

$$\mathfrak{N}_n(a; q, \lambda, d) = \mathfrak{N}_n(0, a; q, \lambda, d).$$

Combining (66) with (65), we get

$$\sum_{n=0}^{\infty} \mathfrak{N}_n(x, a; q, \lambda, d)\frac{t^n}{n!} = \sum_{n=0}^{\infty} (x \ln a)^n \frac{t^n}{n!} \sum_{n=0}^{\infty} \mathfrak{N}_n(a; q, \lambda, d)\frac{t^n}{n!}.$$

Therefore

$$\sum_{n=0}^{\infty} \mathfrak{N}_n(x, a; q, \lambda, d)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\sum_{j=0}^{n} \binom{n}{j} (x \ln a)^{n-j}\, \mathfrak{N}_n(a; q, \lambda, d)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the following theorem:

Theorem 5 *Let $n \in \mathbb{N}_0$. Then we have*

$$\mathfrak{N}_n(x, a; q, \lambda, d) = \sum_{j=0}^{n} \binom{n}{j} (x \ln a)^{n-j}\, \mathfrak{N}_n(a; q, \lambda, d). \tag{67}$$

Substituting $q \to 1$ and $a = e$ into (65), we have

$$\frac{2}{\lambda^d e^{td} + 1} \sum_{j=0}^{d-1} (-1)^j \left(\lambda e^t\right)^j = \sum_{n=0}^{\infty} \mathfrak{N}_n(e; 1, \lambda, d)\frac{t^n}{n!}.$$

Remark 4 Combining (65) with (4), we have

$$\mathfrak{N}_n(e; 1, \lambda, d) = d^n \sum_{j=0}^{d-1} (-1)^j \lambda^j \mathcal{E}_n\left(\frac{j}{d}, \lambda^d\right).$$

3 Integral formulas for the Volkenborn integral

In this section, we give some integral formulas for the Volkenborn integral including the falling factorial and the rising factorial with their identities and relations, the combinatorial sums, the special numbers such as the Bernoulli numbers, the Stirling numbers and the Lah numbers.

Lemma 1 *Let $n \in \mathbb{N}_0$. Then we have*

$$\int_{\mathbb{Z}_p} x x_{(n)} d\mu_1(x) = (-1)^{n+1} \frac{n!}{n^2 + 3n + 2}. \tag{68}$$

Proof Since

$$xx_{(n)}=x_{(n+1)}+nx_{(n)}. \tag{69}$$

By applying the Volkenborn integral to the both sides of the above equation, and using (34), we arrive at the desired result. □

By combining (69) with (10), we have

$$xx_{(n)}=\sum_{k=0}^{n}\left(S_1(n+1,k)+nS_1(n,k)\right)x^k+x^{n+1}. \tag{70}$$

By combining the above equation with (9), and using $S_1(n,k)=0$ if $k<0$, we get

$$xx_{(n)}=\sum_{k=1}^{n}S_1(n,k-1)x^k+x^{n+1}.$$

By applying the Volkenborn integral to the both sides of the above equation, and using (44), we also arrive at the following theorem:

Theorem 6 *Let $n\in\mathbb{N}_0$. Then we have*

$$\int_{\mathbb{Z}_p} xx_{(n)}d\mu_1(x)=\sum_{k=1}^{n}S_1(n,k-1)B_k+B_{n+1}. \tag{71}$$

Theorem 7 *Let $n\in\mathbb{N}$. Then we have*

$$\int_{\mathbb{Z}_p} xx^{(n)}d\mu_1(x)=\sum_{k=1}^{n}(-1)^{k+1}\binom{n-1}{k-1}\frac{n!}{k^2+3k+2}. \tag{72}$$

Proof Using (1) and (24), we get

$$xx^{(n)}=\sum_{k=1}^{n}|L(n,k)|\,xx_{(k)}. \tag{73}$$

By applying the Volkenborn integral to the above equation and using (68), after some elementary calculations, we get the desired result. □

Theorem 8 *Let $n\in\mathbb{N}_0$. Then we have*

$$\int_{\mathbb{Z}_p} xx^{(n)}d\mu_1(x)=\sum_{k=1}^{n}C(n,k)B_{k+1}. \tag{74}$$

Proof By applying the Volkenborn integral to the following equation

$$xx^{(n)}=\sum_{k=1}^{n}C(n,k)x^{k+1}, \tag{75}$$

we get

$$\int_{\mathbb{Z}_p} xx^{(n)}d\mu_1(x)=\sum_{k=1}^{n}C(n,k)\int_{\mathbb{Z}_p}x^{k+1}d\mu_1(x).$$

By combining the above equation with (44), we get the desired result. □

Theorem 9 *Let $n \in \mathbb{N}_0$. Then we have*

$$\int_{\mathbb{Z}_p} \frac{x_{(n+1)}}{x} d\mu_1(x) = \sum_{k=0}^{n} (-1)^n n_{(n-k)} \frac{k!}{k+1}.$$

Proof In order to prove this theorem, we need the following identity:

$$x_{(n+1)} = x \sum_{k=0}^{n} (-1)^{n-k} n_{(n-k)} x_{(k)} \tag{76}$$

(cf. [39, p. 58]). By applying the Volkenborn integral to the both sides of the above equation, and using (46), we arrive at the desired result. □

By applying the Volkenborn integral to (76), we have

$$\int_{\mathbb{Z}_p} x_{(n+1)} d\mu_1(x) = \sum_{k=0}^{n} (-1)^{n-k} n_{(n-k)} \int_{\mathbb{Z}_p} x x_{(k)} d\mu_1(x).$$

By combining (35), (82) and (68) with the above equation, we arrive the at the following combinatorial sum:

Theorem 10 *Let $n \in \mathbb{N}_0$. Then we have*

$$\sum_{k=0}^{n} n_{(n-k)} \frac{k!}{k^2+3k+2} = \frac{(n+1)!}{n+2}.$$

Applying the Volkenborn integral to the following equation:

$$(x+1)_{(n+1)} = x x_{(n)} + x_{(n)}, \tag{77}$$

we obtain

$$\int_{\mathbb{Z}_p} (x+1)_{(n+1)} d\mu_1(x) = \int_{\mathbb{Z}_p} x x_{(n)} d\mu_1(x) + \int_{\mathbb{Z}_p} x_{(n)} d\mu_1(x).$$

Combining the above equation with (68) and (109), we get

$$\int_{\mathbb{Z}_p} (x+1)_{(n+1)} d\mu_1(x) = (-1)^{n+1} \frac{n!}{n^2+3n+2} + \frac{(-1)^n}{n+1} n!.$$

After some elementary calculations in the above equation, we arrive at the following result:

Corollary 1 *Let $n \in \mathbb{N}_0$. Then we have*

$$\int_{\mathbb{Z}_p} (x+1)_{(n+1)} d\mu_1(x) = \frac{(-1)^n}{n+2} n!.$$

Remark 5 By (112), we have

$$\sum_{k=0}^{n} \binom{x}{k} \binom{m}{n-k} = \binom{x+m}{n}.$$

By applying the Volkenborn integral to the above equation, we get the following formula:

$$\int_{\mathbb{Z}_p} \binom{x+m}{n} d\mu_1(x) = \sum_{m=0}^{n} (-1)^k \binom{m}{n-k} \frac{1}{k+1}.$$

By applying the Volkenborn integral with respect to x and y to (112), we have

$$\begin{aligned} &\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p} \sum_{k=0}^{n} \binom{x}{k}\binom{y}{n-k} d\mu_1(y)\, d\mu_1(y) \\ &= \int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p} \binom{x+y}{n} d\mu_1(y)\, d\mu_1(y). \end{aligned} \tag{78}$$

By combining the following identity with the above equation:

$$\binom{x+y}{n} = \frac{1}{n!}(x+y)_{(n)} = \frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k} x_{(k)} y_{(n-k)}, \tag{79}$$

and using (46) and (110), we also get the following lemma:

Lemma 2 *Let $n \in \mathbb{N}_0$. Then we have*

$$\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p} \binom{x+y}{n} d\mu_1(y)\, d\mu_1(y) = \sum_{k=0}^{n}(-1)^n \frac{1}{(k+1)(n-k+1)}. \tag{80}$$

By combining (78) with the following identity:

$$\binom{x+y}{n} = \frac{1}{n!}(x+y)_{(n)} = \frac{1}{n!}\sum_{k=0}^{n} S_1(n,k)(x+y)^k,$$

and using (46) and (110), we also get the following lemma:

Lemma 3 *Let $n \in \mathbb{N}_0$. Then we have*

$$\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p} \binom{x+y}{n} d\mu_1(y)\, d\mu_1(y) = \frac{1}{n!}\sum_{k=0}^{n}\sum_{j=0}^{k}\binom{k}{j} S_1(n,k) B_j B_{k-j}. \tag{81}$$

Lemma 4 *Let $n \in \mathbb{N}_0$. Then we have*

$$\int_{\mathbb{Z}_p} \binom{x+1}{n} d\mu_1(x) = \frac{(-1)^{n+1}}{n^2+n}. \tag{82}$$

Proof In [40], Schikhof gave the following integral formula:

$$\int_{\mathbb{Z}_p} f(x+n)\, d\mu_1(x) = \int_{\mathbb{Z}_p} f(x)\, d\mu_1(x) + \sum_{k=0}^{n-1} f'(k), \tag{83}$$

where

$$f'(x) = \frac{d}{dx}\{f(x)\}.$$

By substituting

$$f(x) = \binom{x}{n}$$

into (83), we get

$$\int_{\mathbb{Z}_p} \binom{x+1}{n} d\mu_1(x) = \int_{\mathbb{Z}_p} \binom{x}{n} d\mu_1(x) + \frac{d}{dx}\left\{\binom{x}{n}\right\}|_{x=0},$$

where

$$\begin{aligned}\frac{d}{dx}\left\{\binom{x}{n}\right\}|_{x=0} &= \left\{\frac{1}{n!}(x)_n \sum_{k=0}^{n-1} \frac{1}{x-k}\right\}|_{x=0} \\ &= (-1)^{n-1}\frac{1}{n}.\end{aligned}$$

Therefore

$$\int_{\mathbb{Z}_p} \binom{x+1}{n} d\mu_1(x) = \frac{(-1)^n}{n+1} + (-1)^{n-1}\frac{1}{n}.$$

After some elementary calculations, we get the desired result. □

Remark 6

$$\Delta\binom{x}{n} = \binom{x}{n-1}$$

and

$$\Delta\binom{x}{n} = \binom{x+1}{n} - \binom{x}{n}.$$

Therefore

$$\binom{x+1}{n} = \binom{x}{n} + \binom{x}{n-1} \tag{84}$$

(cf. [19, p. 69, Eq. (7)]). By applying the Volkenborn integral to the above well-known identities, we also get another proof of (83).

Theorem 11 *Let $n \in \mathbb{N}_0$. Then we have*

$$\int_{\mathbb{Z}_p} (x+1)_{(n)}\, d\mu_1(x) = (-1)^{n+1}\frac{n!}{n^2+n}.$$

Proof Since

$$\Delta x_{(n)} = (x+1)_{(n)} - x_{(n)}, \tag{85}$$

we have

$$(x+1)_{(n)} = x_{(n)} + n x_{(n-1)} \tag{86}$$

(cf. [39, p. 58]). By applying the Volkenborn integral to the above equation and combining with (109), we get the desired result. □

By applying the Volkenborn integral to the equation (85) and combining with (109), we arrive at the following corollary:

Corollary 2 *Let $n \in \mathbb{N}$. Then we have*

$$\int_{\mathbb{Z}_p} \Delta x_{(n)} d\mu_1(x) = (-1)^{n+1}(n-1)!.$$

By applying the Volkenborn integral to the equation (23), we obtain

$$\int_{\mathbb{Z}_p} (-x)_{(n)} d\mu_1(x) = \sum_{k=0}^{n} L(n,k) \int_{\mathbb{Z}_p} x_{(k)} d\mu_1(x),$$

where $n \in \mathbb{N}_0$. By using (109), we get

$$\int_{\mathbb{Z}_p} (-x)_{(n)} d\mu_1(x) = \sum_{k=0}^{n} (-1)^k \frac{k! L(n,k)}{k+1}.$$

Substituting (20) into the above equation, we arrive at the following theorem:

Theorem 12 *Let $n \in \mathbb{N}$. Then we have*

$$\int_{\mathbb{Z}_p} (-x)_{(n)} d\mu_1(x) = \sum_{k=1}^{n} (-1)^{k+n} \binom{n-1}{k-1} \frac{n!}{k+1}.$$

Corollary 3 *Let $n \in \mathbb{N}_0$. Then we have*

$$\int_{\mathbb{Z}_p} \binom{x+1}{n+1} d\mu_1(x) = \frac{(-1)^n}{n^2+3n+2}. \tag{87}$$

Proof By applying the Volkenborn integral to (77) and using (68) and (108), we get the desired result. □

Remark 7 Replacing n by $n+1$ in (82), we also get (87).

In [36], Osgood and Wu gave the following identity:

$$(xy)_{(k)} = \sum_{l,m=1}^{k} C_{l,m}^{(k)} (x)_l (x)_m \tag{88}$$

where

$$C_{l,m}^{(k)} = \sum_{j=1}^{k} (-1)^{k-j} S_1(k,j) S_2(j,l) S_2(j,m)$$

$C_{l,m}^{(k)} = C_{m,l}^{(k)}$, $C_{1,1}^{(1)} = 1$, $C_{1,1}^{(2)} = 0$, $C_{1,2}^{(3)} = 0 = C_{2,1}^{(3)}$. By applying the Volkenborn integral to the equation (88) with respect to x and y, we arrive at the following lemma:

Lemma 5 *Let $k \in \mathbb{N}_0$. Then we have*

$$\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}(xy)_{(k)}d\mu_1(x)\,d\mu_1(y)=\sum_{l,m=1}^{k}D_lD_mC_{l,m}^{(k)}$$

or

$$\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}(xy)_{(k)}d\mu_1(x)\,d\mu_1(y)=\sum_{l,m=1}^{k}(-1)^{l+m}\frac{l!m!}{(l+1)(m+1)}C_{l,m}^{(k)}. \tag{89}$$

Lemma 6 *Let $k \in \mathbb{N}_0$. Then we have*

$$\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}(xy)_{(k)}d\mu_1(x)\,d\mu_1(y)=\sum_{m=0}^{k}S_1(k,m)\,(B_m)^2\,. \tag{90}$$

Proof By using (17), we get

$$(xy)_{(k)}=\sum_{m=0}^{k}S_1(k,m)x^my^m \tag{91}$$

By applying the Volkenborn integral to (91) with respect to x and y, and using (44), we get the desired result. □

Theorem 13 *Let $n \in \mathbb{N}_0$. Then we have*

$$\int_{\mathbb{Z}_p}x\binom{x-2}{n-1}d\mu_1(x)=(-1)^n\sum_{k=1}^{n}\frac{k}{k+1}.$$

Proof Gould [18, Vol. 3, Eq. (4.20)] defined the following identity:

$$(-1)^nx\binom{x-2}{n-1}=\sum_{k=1}^{n}(-1)^k\binom{x}{k}k.$$

By applying the Volkenborn integral to the above equation, and using (34), we get the desired result. □

Theorem 14 *Let $n \in \mathbb{N}_0$. Then we have*

$$\int_{\mathbb{Z}_p}\binom{n-x}{n}d\mu_{-1}(x)=(-1)^nH_n,$$

where H_n denotes the harmonic numbers given by

$$H_n=\sum_{k=0}^{n}\frac{1}{k+1}.$$

Proof Gould [18, Vol. 3, Eq. (4.19)] defined the following identity:

$$(-1)^n\binom{n-x}{n}=\sum_{k=0}^{n}(-1)^k\binom{x}{k}.$$

By applying the Volkenborn integral to the above integral, and using (34), we get the desired result. □

Theorem 15 *Let $m \in \mathbb{N}$ and $n \in \mathbb{N}_0$. Then we have*

$$\int_{\mathbb{Z}_p} \binom{mx}{n} d\mu_1(x) = \sum_{k=0}^{n} \frac{(-1)^k}{k+1} \sum_{j=0}^{k} (-1)^j \binom{k}{j} \binom{mk-mj}{n}.$$

Proof Gould [17, Eq. (2.65)] gave the following identity:

$$\binom{mx}{n} = \sum_{k=0}^{n} \binom{x}{k} \sum_{j=0}^{k} (-1)^j \binom{k}{j} \binom{mk-mj}{n}. \tag{92}$$

By applying the Volkenborn integral to the above equation, and using (34), we arrive at the desired result. □

Theorem 16 *Let $n, r \in \mathbb{N}_0$. Then we have*

$$\int_{\mathbb{Z}_p} \binom{x}{n}^r d\mu_1(x) = \sum_{k=0}^{nr} \frac{(-1)^k}{k+1} \sum_{j=0}^{k} (-1)^j \binom{k}{j} \binom{k-j}{n}^r. \tag{93}$$

Proof Gould [17, Eq. (2.66)] gave the following identity:

$$\binom{x}{n}^r = \sum_{k=0}^{nr} \binom{x}{k} \sum_{j=0}^{k} (-1)^j \binom{k}{j} \binom{k-j}{n}^r. \tag{94}$$

By applying the Volkenborn integral to the above equation, and using (34), we arrive at the desired result. □

Remark 8 Substituting $r = 1$ into (93), since $\binom{k-j}{n} = 0$ if $k-j < n$, we arrive at the equation (34).

Theorem 17 *Let $n \in \mathbb{N}$ with $n > 1$. Then we have*

$$\int_{\mathbb{Z}_p} \left\{ x\binom{x-2}{n-1} + x(x-1)\binom{n-3}{n-2} \right\} d\mu_1(x) = (-1)^n \sum_{k=0}^{n} \frac{k^2}{k+1}.$$

Proof In [17, Eq. (2.15)], Gould gave the following identity for $n > 1$:

$$x\binom{x-2}{n-1} + x(x-1)\binom{n-3}{n-2} = \sum_{k=0}^{n} (-1)^k \binom{x}{k} k^2, \tag{95}$$

where $n \in \mathbb{N}$ with $n > 1$. By applying the Volkenborn integral to the above equation, and using (34), we arrive at the desired result. □

Theorem 18 *Let $n \in \mathbb{N}_0$. Then we have*

$$\int_{\mathbb{Z}_p} \binom{x+n}{n} d\mu_1(x) = \sum_{k=0}^{n} \frac{(-1)^k}{k+1} \sum_{j=0}^{k} (-1)^j \binom{k}{j} \binom{k-j+n}{n} \tag{96}$$

and

$$\int_{\mathbb{Z}_p} \binom{x+n}{n} d\mu_1(x) = \sum_{k=0}^{n} B_k \sum_{j=0}^{n} \binom{n}{j} \frac{S_1(j,k)}{j!}. \tag{97}$$

Proof In [17, Eq. (2.64) and Eq. (6.17)], Gould gave the following identities:

$$\binom{x+n}{n}=\sum_{k=0}^{n}\binom{x}{k}\sum_{j=0}^{k}(-1)^{j}\binom{k}{j}\binom{k-j+n}{n} \tag{98}$$

and

$$\binom{x+n}{n}=\sum_{k=0}^{n}x^{k}\sum_{j=0}^{n}\binom{n}{j}\frac{S_1(j,k)}{j!}. \tag{99}$$

By applying the Volkenborn integral to the above equations, and using (34) and (44), respectively, we arrive at the desired result. □

Theorem 19 *Let $n\in\mathbb{N}_0$. Then we have*

$$\int_{\mathbb{Z}_p}\binom{x+n+\frac{1}{2}}{n}d\mu_1(x)=\binom{2n}{n}\sum_{k=0}^{n}(-1)^{k}\binom{n}{k}\frac{2^{2k-2n}(2n+1)}{(k+1)(2k+1)\binom{2k}{k}}.$$

Proof Gould [18, Vol. 3, Eq. (6.26)] defined the following identity:

$$\binom{x+n+\frac{1}{2}}{n}=(2n+1)\binom{2n}{n}\sum_{k=0}^{n}\binom{n}{k}\binom{x}{k}\frac{2^{2k-2n}}{(2k+1)\binom{2k}{k}}.$$

By applying the Volkenborn integral to the above integral, and using (34), we get the desired result. □

Theorem 20 *Let $m,n\in\mathbb{N}_0$. Then we have*

$$\int_{\mathbb{Z}_p}x^{m}x_{(n)}d\mu_1(x)=\sum_{k=0}^{n}S_1(n,k)B_{k+m}.$$

Proof Multiplying both sides of the equation (10) by x^m, we get

$$x^{m}x_{(n)}=\sum_{k=0}^{n}S_1(n,k)x^{m+k}$$

By applying the Volkenborn integral to the above integral, and using (44), we get the desired result. □

In order to give a formula for the following integral

$$\int_{\mathbb{Z}_p}x_{(m)}x_{(n)}d\mu_1(x),$$

we need the following well-known identity

$$x_{(m)}x_{(n)}=\sum_{k=0}^{m}\binom{m}{k}\binom{n}{k}k!x_{(m+n-k)}, \tag{100}$$

where the coefficients of $x_{(n+n-k)}$ are called connection coefficients and they have a combinatorial interpretation as the number of ways to identify k elements each from a set of size m and a set of size n (cf. [49]).

By using (10), we have

$$x_{(m)}x_{(n)} = \sum_{j=0}^{n}\sum_{l=0}^{m} S_1(n,k)S_1(m,l)x^{j+l}.$$

By applying the Volkenborn integral to the above equation, and using (44), we get the following lemma:

Lemma 7 *Let $m, n \in \mathbb{N}_0$. Then we have*

$$\int_{\mathbb{Z}_p} x_{(n)}x_{(m)}d\mu_1(x) = \sum_{j=0}^{n}\sum_{l=0}^{m} S_1(n,k)S_1(m,l)B_{j+l}. \tag{101}$$

By combining (100) with (10), we get

$$x_{(m)}x_{(n)} = \sum_{k=0}^{m}\binom{m}{k}\binom{n}{k}k!\sum_{l=0}^{m+n-k} S_1(m+n-k,l)x^l.$$

By applying the Volkenborn integral to the above equation, we get the following lemma:

Lemma 8 *Let $m, n \in \mathbb{N}_0$. Then we have*

$$\int_{\mathbb{Z}_p} x_{(n)}x_{(m)}d\mu_1(x) = \sum_{k=0}^{m}\binom{m}{k}\binom{n}{k}k!\sum_{l=0}^{m+n-k} S_1(m+n-k,l)B_l. \tag{102}$$

4 Application of the Volkenborn integral to the falling factorial and rising factorial

In this section, we give some applications of the Volkenborn integral on $\mathbb{Z}_p$ to the falling factorial and rising factorial. With the aid of these applications, we derive some integral formulas including the Bernoulli numbers and polynomials, the Euler numbers and polynomials, the Stirling numbers, the Lah numbers and the combinatorial sums.

By using the same spirit of the Bernoulli polynomial of the second kind which are also called Cauchy numbers of the first kind, by applying the Volkenborn integral to the rising factorial and the falling factorial, respectively, we derive various formulas, identities, relations and combinatorial sums including the Bernoulli numbers, the Stirling numbers, the Lah numbers, the Daehee numbers and the Changhee numbers.

In [46], similar to the Cauchy numbers defined by aid of the Riemann integral, we studied the Bernoulli numbers sequences by using p -adic integral.

Let $Y_1(0:B) = B_0 = 1$ and $Y_1(1:B) = B_1 = -\frac{1}{2}$. Let $n \in \mathbb{N}$ with $n > 1$. Assuming that $x_j \in \mathbb{Z}$ and $j \in \{1, 2, \ldots, n-1\}$. We define the following sequence $(Y_1(n:B))$ associated with the Bernoulli numbers:

$$Y_1(n:B) = B_n + \sum_{j=1}^{n-1}(-1)^j x_j B_{n-j}. \tag{103}$$

Combining (103) with (46), we get a relation between the sequence of $(Y_1(n:B))$ and the Daehee numbers given as follows:

$$Y_1(n:B) = D_n. \tag{104}$$

Few values of (103) are computed by (46) as follows:

$$\begin{aligned} Y_1(2:B) &= B_2 - B_1, \\ Y_1(3:B) &= B_3 - 3B_2 + 2B_1, \\ Y_1(4:B) &= B_4 - 6B_3 + 11B_2 - 6B_1, \ldots. \end{aligned}$$

Therefore, we have

$$\begin{aligned} Y_1(2:B) &= B_2 - x_1B_1 = \frac{2}{3}, \\ Y_1(3:B) &= B_3 - x_1B_2 + x_2B_1 = -\frac{3}{2}, \\ Y_1(4:B) &= B_4 - x_1B_3 + x_2B_2 - x_3B_1 = \frac{24}{5}, \ldots. \end{aligned}$$

Due to the work of Kim et al. [27], we see that the coefficients x_j are computed by the Stirling numbers of the first kind.

Let $Y_2(0:B) = B_0$ and $Y_2(1:B) = B_1$. Let $n \in \mathbb{N}$ with $n > 1$. Assuming that $x_j \in \mathbb{Z}$ and $j \in \{1, 2, \ldots, n-1\}$. We define the following sequence $(Y_2(n:B))$ associated with the Bernoulli numbers as follows:

$$Y_2(n:B) = B_n + \sum_{j=1}^{n-1} x_j B_{n-j}, \tag{105}$$

Combining (105) with (48), a relation between the sequence of $(Y_2(n:B))$ and the Daehee numbers (of the second kind) given as follows:

$$Y_2(n:B) = \widehat{D_n}.$$

Few values of (105) are computed by (48) as follows:

$$\begin{aligned} Y_2(2:B) &= B_2 + B_1, \\ Y_2(3:B) &= B_3 + 3B_2 + 2B_1, \\ Y_2(4:B) &= B_4 + 6B_3 + 11B_2 + 6B_1, \ldots. \end{aligned}$$

Therefore, we have

$$\begin{aligned} Y_2(2:B) &= B_2 + x_1B_1 = -\frac{1}{3}, \\ Y_2(3:B) &= B_3 + x_1B_2 + x_2B_1 = -\frac{1}{2}, \\ Y_2(4:B) &= B_4 + x_1B_3 + x_2B_2 + x_3B_1 = -\frac{6}{5}, \ldots. \end{aligned}$$

Let $Y_1(x, 0:B(x)) = Y_2(x, 0:B(x)) = B_0(x) = 1$ and $Y_1(x, 1:B(x)) = Y_2(x, 1:B(x)) = B_1(x) = x - \frac{1}{2}$. Let $n \in \mathbb{N}$ with $n > 1$. Assuming that $x_j \in \mathbb{Z}$ and $j \in \{1, 2, \ldots, n-1\}$. We now define the following sequences associated $(Y_1(x, n:B(x)))$ and $(Y_2(x, n:B(x)))$ with the Bernoulli polynomials, respectively:

$$Y_1(x, n:B(x)) = B_n(x) + \sum_{j=1}^{n-1} (-1)^j x_j B_{n-j}(x) \tag{106}$$

and

$$Y_2(x, n : B(x)) = B_n(x) + \sum_{j=1}^{n-1} x_j B_{n-j}(x). \tag{107}$$

Few values of (106) are given as follows:

$$\begin{aligned} Y_1(x, 2 : B(x)) &= B_2(x) - x_1 B_1(x) = x^2 - 2x + \frac{2}{3}, \\ Y_1(x, 3 : B(x)) &= B_3(x) - x_1 B_2(x) + x_2 B_1(x) = x^3 - \frac{9}{2}x^2 + \frac{11}{2}x - \frac{3}{2}, \\ Y_1(x, 4 : B(x)) &= B_4(x) - x_1 B_3(x) + x_2 B_2(x) - x_3 B_1(x) = x^4 - 8x^3 - x^2 - 14x \\ &\quad + \frac{24}{5}, \ldots \end{aligned}$$

Few values of (107) are given as follows:

$$\begin{aligned} Y_2(x, 2 : B(x)) &= B_2(x) + x_1 B_1(x) = x^2 - \frac{1}{3}, \\ Y_2(x, 3 : B(x)) &= B_3(x) + x_1 B_2(x) + x_2 B_1(x) = x^3 + \frac{3}{2}x^2 - \frac{1}{2}x - \frac{1}{2}, \\ Y_2(x, 4 : B(x)) &= B_4(x) + x_1 B_3(x) + x_2 B_2(x) + x_3 B_1(x) = x^4 + 4x^3 + 3x^2 - 2x \\ &\quad - \frac{6}{5}, \ldots \end{aligned}$$

Observe that setting $x = 0$ into the above equations, we have

$$Y_1(n : B) = Y_1(0, n : B(0))$$

and

$$Y_2(n : B) = Y_2(0, n : B(0)).$$

5 Formulas for the sequence $Y_1(n : B)$

To use the Volkenborn integral and its integral equations, we give some formulas and identities for the sequence $(Y_1(n : B))$. We also gives some p-adic integral formulas including falling factorial.

An explicit formula for the sequence $(Y_1(n : B))$ is given by the following theorem, which was proved by different methods (cf. [15, 27], [38, p. 117], [45, 46]).

Theorem 21 *Let $n \in \mathbb{N}_0$. Then we have*

$$Y_1(n : B) = (-1)^n \frac{n!}{n+1}. \tag{108}$$

Proof By (104), we know that the sequence $(Y_1(n : B))$ is related to the numbers D_n. We now briefly give the proof. Since

$$x_{(n)} = n! \binom{x}{n},$$

by using (34), we get

$$\int_{\mathbb{Z}_p} x_{(n)} d\mu_1(x) = n! \int_{\mathbb{Z}_p} \binom{x}{n} d\mu_1(x) \\ = \frac{(-1)^n}{n+1} n! = Y_1(n:B). \tag{109}$$

Thus, we get the desired result. □

Remark 9 By combining (109), (46) and (47), we have

$$\sum_{l=0}^{n} S_1(n,l) B_l = \frac{(-1)^n}{n+1} n! \tag{110}$$

where $n \in \mathbb{N}_0$. Note that the proof of (110) is also given by Riordan [38, p. 117]. See also (cf. [15, 27, 45, 46]). That is, the equation (104) holds true.

Two kinds of recurrence relations for the numbers $Y_1(n:B)$ are given by the following theorem:

Theorem 22 *Let $n \in \mathbb{N}_0$. Then we have*

$$Y_1(n+1:B) + nY_1(n:B) = \sum_{k=1}^{n} S_1(n,k-1) B_k + B_{n+1}$$

and

$$Y_1(n+1:B) + nY_1(n:B) = (-1)^{n+1} \frac{n!}{n^2+3n+2}. \tag{111}$$

Proof By applying the Volkenborn integral to the both sides of equation (69) and using (46), (68) and (71), after some elementary calculations, we arrive at the desired result. □

Theorem 23 *Let $n \in \mathbb{N}_0$. Then we have*

$$\int_{\mathbb{Z}_p} Y_1(y,n:B(y)) d\mu_1(y) = (-1)^n \sum_{k=0}^{n} \frac{n!}{(k+1)(n-k+1)}.$$

Proof The well-known Chu-Vandermonde identity is defined as follows:

$$\sum_{k=0}^{n} \binom{x}{k} \binom{y}{n-k} = \binom{x+y}{n}. \tag{112}$$

By applying the Volkenborn integral with respect to x and y to the left hand side (LHS) of the equation (112), and using (34), we get

$$LHS = (-1)^n \sum_{k=0}^{n} \frac{1}{(k+1)(n-k+1)}. \tag{113}$$

By applying the Volkenborn integral with respect to x and y to the right hand side (RHS) of the equation (112), and using (49) and (50), we obtain

$$RHS = \frac{1}{n!} \int_{\mathbb{Z}_p} Y_1(y,n:B(y)) d\mu_1(y). \tag{114}$$

Combining (113) with (114), after some elementary calculations, we arrived at the desired result. □

By applying the Volkenborn integral to the equation (100), and using (46), (68) and (108), we get the following lemma:

Lemma 9 *Let $m, n \in \mathbb{N}_0$. Then we have*

$$\int_{\mathbb{Z}_p} x_{(m)} x_{(n)} d\mu_1(x) = \sum_{k=0}^{m} \binom{m}{k}\binom{n}{k} k! Y_1(m+n-k : B) \tag{115}$$

and

$$\int_{\mathbb{Z}_p} x_{(m)} x_{(n)} d\mu_1(x) = \sum_{k=0}^{m} (-1)^{m+n-k} \binom{m}{k}\binom{n}{k} \frac{k!(m+n-k)!}{m+n-k+1}. \tag{116}$$

Remark 10 Since $D_n = Y_1(n : B)$, we rewrite the equation (115) as follows:

$$\int_{\mathbb{Z}_p} x_{(m)} x_{(n)} d\mu_1(x) = \sum_{k=0}^{m} \binom{m}{k}\binom{n}{k} k! D_{m+n-k}.$$

6 Formulas for the sequence $Y_2(n : B)$

To use the Volkenborn integral and its integral equations, here we give some formulas and identities for the sequence $(Y_2(n : B))$. We give some p-adic integral formulas including raising factorial.

A computation formula for the sequence $Y_2(n : B)$ is given by the following theorems:

By using (10) and (22), we have the following well-known relation:

$$(x+n-1)_{(n)} = \sum_{m=0}^{n+1} (-1)^{m+n} S_1(n, m) x^m. \tag{117}$$

By applying the Volkenborn integral to the equation (117), and using (44), we get the following identities:

$$\int_{\mathbb{Z}_p} (x+n-1)_{(n)} d\mu_1(x) = \sum_{m=0}^{n+1} (-1)^{m+n} S_1(n, m) B_m \tag{118}$$

and

$$\int_{\mathbb{Z}_p} (-x)_{(n)} d\mu_1(x) = \sum_{m=0}^{n+1} (-1)^{m} S_1(n, m) B_m.$$

By using (36), we also have

$$\int_{\mathbb{Z}_p} \binom{x+n-1}{n} d\mu_1(x) = \sum_{m=0}^{n} (-1)^m \binom{n-1}{n-m} \frac{1}{m+1}$$

and

$$\int_{\mathbb{Z}_p} (x+n-1)_{(n)} d\mu_1(x) = \sum_{m=0}^{n} (-1)^m \binom{n-1}{n-m} \frac{n!}{m+1}. \tag{119}$$

Therefore

$$Y_2(n:B) = \int_{\mathbb{Z}_p} x^{(n)} d\mu_1(x)$$

which is associated with the Daehee numbers of the second kind. Thus we get the following theorem:

Theorem 24 *Let $n \in \mathbb{N}$. Then we have*

$$Y_2(n:B) = \frac{1}{n!} \sum_{m=1}^{n} (-1)^m \binom{n-1}{n-m} \frac{1}{m+1}.$$

Theorem 25 *Let $n \in \mathbb{N}_0$. Then we have*

$$Y_2(n:B) = \sum_{k=0}^{n} C(n,k) B_k. \tag{120}$$

Proof By applying the Volkenborn integral to equation (11) and using (44), we get the desired result. □

Theorem 26 *Let $n \in \mathbb{N}$. Then we have*

$$Y_2(n:B) = \sum_{k=1}^{n} (-1)^k \binom{n-1}{k-1} \frac{n!}{k+1}. \tag{121}$$

Proof By applying the Volkenborn integral to the equation (24), we get

$$Y_2(n:B) = \sum_{k=1}^{n} |L(n,k)| \int_{\mathbb{Z}_p} x_{(k)} d\mu_1(x)$$

By substituting (109) into the above equation, we get the desired result. □

By applying the Volkenborn integral to the equation (24), we get a formula for the sequence $Y_2(n:B)$ given by the following corollary:

Corollary 4 *Let $n \in \mathbb{N}$. Then we have*

$$Y_2(n:B) = \sum_{k=1}^{n} (-1)^k \frac{|L(n,k)|\, k!}{k+1}. \tag{122}$$

Substituting (10) into (24), we have

$$x^{(n)} = \sum_{k=1}^{n} |L(n,k)| \sum_{j=0}^{k} S_1(k,j) x^j. \tag{123}$$

By applying the Volkenborn integral to the above equation, we arrive a formula for the numbers $Y_2(n:B)$ given by the following theorem:

Theorem 27 *Let $n \in \mathbb{N}_0$. Then we have*

$$Y_2(n:B)=\sum_{k=1}^{n}\sum_{j=0}^{k}|L(n,k)|\,S_1(k,j)B_j. \tag{124}$$

By combining (110) and (47) with equation (124), we get the following corollary:

Corollary 5 *Let $n \in \mathbb{N}$. Then we have*

$$Y_2(n:B)=\sum_{k=1}^{n}|L(n,k)|\,D_k.$$

A recurrence relation of the sequence $Y_2(n:B)$ is given by the following theorem.

Theorem 28 *Let $n \in \mathbb{N}$. Then we have*

$$Y_2(n+1:B)-nY_2(n:B)=\sum_{k=1}^{n}(-1)^{k+1}\binom{n-1}{k-1}\frac{n!}{k^2+3k+2}. \tag{125}$$

Proof We set

$$(x+n)x^{(n)}=\sum_{k=1}^{n}|L(n,k)|\,xx_{(k)}+n\sum_{k=1}^{n}|L(n,k)|\,x_{(k)}.$$

From the above equation, we get

$$x^{(n+1)}=\sum_{k=1}^{n}|L(n,k)|\,xx_{(k)}+n\sum_{k=1}^{n}|L(n,k)|\,x_{(k)}$$

By applying the Volkenborn integral to the above equation, and using (68), we get

$$Y_2(n+1:B)=\sum_{k=1}^{n}(-1)^{k+1}\,|L(n,k)|\,\frac{k!}{k^2+3k+2}+nY_2(n:B). \tag{126}$$

Combining (126) with (21), we arrive at the desired result. □

A relationship between the numbers $Y_1(n:B)$ and $Y_2(n:B)$ is given by the following theorem:

Theorem 29 *Let $n \in \mathbb{N}$. Then we have*

$$Y_2(n:B)=\sum_{m=1}^{n}|L(n,k)|\,Y_1(m:B).$$

Proof By applying the Volkenborn integral to the equation (23), and using (108) and (121), we get the desired result. □

7 Integral formulas for the fermionic *p*-adic integral

In this section, we give some integral formulas for the fermionic p-adic integral including the falling factorial and the rising factorial with their identities and relations, the combinatorial sums, the special numbers such as the Euler numbers, the Stirling numbers and the Lah numbers.

Theorem 30 *Let $n \in \mathbb{N}$. Then we have*

$$\int_{\mathbb{Z}_p} x x_{(n)} d\mu_{-1}(x) = (-1)^n \frac{(n-1)}{2^{n+1}} n!. \tag{127}$$

Proof By applying the p-adic fermionic integral to the both sides of equation (69), we have

$$\int_{\mathbb{Z}_p} x x_{(n)} d\mu_{-1}(x) = \int_{\mathbb{Z}_p} x_{(n+1)} d\mu_{-1}(x) + n \int_{\mathbb{Z}_p} x_{(n)} d\mu_{-1}(x).$$

Combining the above equation with (56), we arrive at the desired result. □

Theorem 31 *Let $n \in \mathbb{N}_0$. Then we have*

$$\int_{\mathbb{Z}_p} x x^{(n)} d\mu_{-1}(x) = \sum_{k=1}^{n} (-1)^k \binom{n-1}{k-1} \frac{(k-1)}{2^{k+1}} n!. \tag{128}$$

Proof By applying the p-adic fermionic integral to the both sides of equation (73), we have

$$\int_{\mathbb{Z}_p} x x^{(n)} d\mu_{-1}(x) = \sum_{k=1}^{n} |L(n,k)| \int_{\mathbb{Z}_p} x x_{(k)} d\mu_{-1}(x).$$

Combining the above equation with (127), we obtain

$$\int_{\mathbb{Z}_p} x x^{(n)} d\mu_{-1}(x) = \sum_{k=1}^{n} (-1)^k \frac{(k-1)\,|L(n,k)|}{2^{k+1}} k!.$$

Combining the above equation with (21), we arrive at the desired result. □

Theorem 32 *Let $n \in \mathbb{N}_0$. Then we have*

$$\int_{\mathbb{Z}_p} (x+1)_{(n)} d\mu_{-1}(x) = (-1)^{n+1} \frac{1}{2^n} n!. \tag{129}$$

Proof By applying the p-adic fermionic integral to equation (86), we obtain

$$\int_{\mathbb{Z}_p} (x+1)_{(n)} d\mu_{-1}(x) = n \int_{\mathbb{Z}_p} (x)_{(n-1)} d\mu_{-1}(x) + \int_{\mathbb{Z}_p} (x)_{(n)} d\mu_{-1}(x).$$

Combining the above equation with (56), we get

$$\int_{\mathbb{Z}_p} (x+1)_{(n)} d\mu_{-1}(x) = (-1)^{n-1} n \frac{(n-1)!}{2^{n-1}} + (-1)^n \frac{n!}{2^n}.$$

After some elementary calculations, we arrive at the desired result. □

Theorem 33 *Let $n \in \mathbb{N}_0$. Then we have*

$$\int_{\mathbb{Z}_p} \frac{x_{(n+1)}}{x} d\mu_{-1}(x) = \sum_{k=0}^{n} (-1)^n n_{(n-k)} \frac{k!}{2^k}.$$

Proof By applying the p-adic fermionic integral to equation (76), and using (142), we arrive at the desired result. □

Remark 11 By applying the p-adic fermionic integral to equation (84) with (55), we get

$$\int_{\mathbb{Z}_p} \binom{x+1}{n} d\mu_{-1}(x) = (-1)^{n+1} \frac{1}{2^n}.$$

By using the above equation, we also get another proof of (129).

By applying the p-adic fermionic integral to the equation (88) with respect to x and y, we arrive at the following lemma:

Lemma 10 *Let $k \in \mathbb{N}_0$. Then we have*

$$\int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} (xy)_{(k)} d\mu_{-1}(x)\, d\mu_{-1}(y) = \sum_{l,m=1}^{k} (-1)^{l+m} 2^{-m-l} l! m! C_{l,m}^{(k)}. \tag{130}$$

Lemma 11 *Let $k \in \mathbb{N}_0$. Then we have*

$$\int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} (xy)_{(k)} d\mu_{-1}(x)\, d\mu_{-1}(y) = \sum_{m=0}^{k} S_1(k,m)\,(E_m)^2. \tag{131}$$

Proof By applying the p-adic fermionic integral to equation (91) with respect to x and y, and using (53), we get the desired result. □

Theorem 34 *Let $n \in N$ with $n > 1$. Then we have*

$$\int_{\mathbb{Z}_p} \left\{ x \binom{x-2}{n-1} + x(x-1) \binom{n-3}{n-2} \right\} d\mu_{-1}(x) = (-1)^n \sum_{k=0}^{n} \frac{k^2}{2^k}.$$

Proof By applying the p-adic fermionic integral to the equation (95) with (55), we get desired result. □

Theorem 35 *Let $n \in \mathbb{N}_0$. Then we have*

$$\int_{\mathbb{Z}_p} \binom{x+n}{n} d\mu_{-1}(x) = \sum_{k=0}^{n} \frac{(-1)^k}{2^k} \sum_{j=0}^{k} (-1)^j \binom{k}{j} \binom{k-j+n}{n} \tag{132}$$

and

$$\int_{\mathbb{Z}_p} \binom{x+n}{n} d\mu_{-1}(x) = \sum_{k=0}^{n} E_k \sum_{j=0}^{n} \binom{n}{j} \frac{S_1(j,k)}{j!}. \tag{133}$$

Proof By applying the p-adic fermionic integral to the equations (98) and (99) with (55) and (53), we get desired result. $\square$

Theorem 36 *Let $m \in \mathbb{N}$ and $n \in \mathbb{N}_0$. Then we have*

$$\int\limits_{\mathbb{Z}_p} \binom{mx}{n} d\mu_{-1}(x) = \sum_{k=0}^{n} \frac{(-1)^k}{2^k} \sum_{j=0}^{k} (-1)^j \binom{k}{j} \binom{mk-mj}{n}.$$

Proof By applying the p-adic fermionic integral to the equation (92) with (55), we get desired result. $\square$

Theorem 37 *Let $n, r \in \mathbb{N}_0$. Then we have*

$$\int\limits_{\mathbb{Z}_p} \binom{x}{n}^r d\mu_{-1}(x) = \sum_{k=0}^{nr} \frac{(-1)^k}{2^k} \sum_{j=0}^{k} (-1)^j \binom{k}{j} \binom{k-j}{n}^r. \tag{134}$$

Proof By applying the p-adic fermionic integral to the equation (94) with (55), we get desired result. $\square$

Remark 12 Substituting $r = 1$ into (134), since $\binom{k-j}{n} = 0$ if $k - j < n$, we arrive at the equation (55).

Remark 13 By (112), we have

$$\sum_{k=0}^{n} \binom{x}{k} \binom{m}{n-k} = \binom{x+m}{n}.$$

By applying the fermionic integral to the above equation, we get the following formula:

$$\int\limits_{\mathbb{Z}_p} \binom{x+m}{n} d\mu_{-1}(x) = \sum_{m=0}^{n} (-1)^k \binom{m}{n-k} 2^{-k}.$$

Theorem 38 *Let $n \in \mathbb{N}$. Then we have*

$$\int\limits_{\mathbb{Z}_p} x \binom{x-2}{n-1} d\mu_{-1}(x) = (-1)^n \sum_{k=1}^{n} k 2^{-k}.$$

Proof Gould [18, Vol. 3, Eq. (4.20)] defined the following identity:

$$(-1)^n x \binom{x-2}{n-1} = \sum_{k=1}^{n} (-1)^k \binom{x}{k} k.$$

By applying the p-adic fermionic integral to the above integral, and using (55), we get the desired result. $\square$

Theorem 39 *Let $n \in \mathbb{N}_0$. Then we have*

$$\int\limits_{\mathbb{Z}_p} \binom{n-x}{n} d\mu_{-1}(x) = (-1)^n \sum_{k=1}^{n} 2^{-k}.$$

Proof Gould [18, Vol. 3, Eq. (4.19)] gave the following identity:

$$(-1)^n \binom{n-x}{n} = \sum_{k=1}^{n} (-1)^k \binom{x}{k}.$$

By applying the p-adic fermionic integral to the above integral, and using (55), we get the desired result. □

Theorem 40 *Let $n \in \mathbb{N}_0$. Then we have*

$$\int_{\mathbb{Z}_p} \binom{x+n+\frac{1}{2}}{n} d\mu_{-1}(x) = (2n+1)\binom{2n}{n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{2^{k-2n}}{(2k+1)\binom{2k}{k}}.$$

Proof Gould [18, Vol. 3, Eq. (6.26)] defined the following identity:

$$\binom{x+n+\frac{1}{2}}{n} = (2n+1)\binom{2n}{n} \sum_{k=0}^{n} \binom{n}{k}\binom{x}{k} \frac{2^{2k-2n}}{(2k+1)\binom{2k}{k}}.$$

By applying the p-adic fermionic integral to the above integral, and using (55), we get the desired result. □

Remark 14 By applying the p-adic fermionic integral to equation (22) after that combining with (135), we have the following well-known identity:

$$\int_{\mathbb{Z}_p} (x+n-1)_{(n)}\, d\mu_{-1}(x) = n! \sum_{m=0}^{n} (-1)^m \binom{n-1}{n-m} 2^{-m} \tag{135}$$

and

$$\begin{aligned}\int_{\mathbb{Z}_p} x^{(n)} d\mu_{-1}(x) &= (-1)^n \int_{\mathbb{Z}_p} (-x)_{(n)}\, d\mu_{-1}(x) \\ &= n! \sum_{m=0}^{n} (-1)^m \binom{n-1}{n-m} 2^{-m}\end{aligned} \tag{136}$$

(cf. [27, 31, 43]).

8 Application of the *p*-adic fermionic integral to the falling factorial and rising factorial

In this section, we give some applications of the p-adic fermionic integral on $\mathbb{Z}_p$ to falling factorial and rising factorial. Here, we derive some integral formulas including the Euler numbers and polynomials, the Stirling numbers, the Lah numbers and combinatorial sums.

By using Euler numbers of the first kind, we define the following sequences, which are associated with p-adic fermionic integral. Let $x_j \in \mathbb{Z}$ and $j \in \{1, 2, \ldots, n-1\}$ with $n > 1$. Let $y_1(0 : E) = y_2(0 : E) = E_0 = 1$ and $y_1(1 : E) = y_2(1 : E) = E_1 = -\frac{1}{2}$. We define the sequences $(y_1(n : E))$ and $(y_2(n : E))$:

$$y_1(n : E) = E_n + \sum_{j=1}^{n-1} (-1)^j x_j E_{n-j}, \tag{137}$$

and

$$y_2(n:E) = E_n + \sum_{j=1}^{n-1} x_j E_{n-j}. \tag{138}$$

The sequences $(y_1(n:E))$ and $(y_2(n:E))$ can be computed by the first and the second kind Changhee numbers. Combining (137) with (57) and (103) with (58), we easily give the following relations for the general terms of the related sequences:

$$y_1(n:E) = Ch_n$$

and

$$y_2(n:E) = \widehat{Ch}_n. \tag{139}$$

Few values of (137) are computed by (57) as follows:

$$\begin{aligned}
y_1(2:E) &= E_2 - E_1, \\
y_1(3:E) &= E_3 - 3E_2 + 2E_1, \\
y_1(4:E) &= E_4 - 6E_3 + 11E_2 - 6E_1, \ldots
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
y_1(2:E) &= E_2 - x_1E_1 = \frac{1}{2}, \\
y_1(3:E) &= E_3 - x_1E_2 + x_2E_1 = -\frac{3}{4}, \\
y_1(4:E) &= E_4 - x_1E_3 + x_2E_2 - x_3E_1 = \frac{3}{2}, \ldots
\end{aligned}$$

Few values of (138) are computed by (58) as follows:

$$\begin{aligned}
y_2(2:E) &= E_2 + E_1, \\
y_2(3:E) &= E_3 + 3E_2 + 2E_1, \\
y_2(4:E) &= E_4 + 6E_3 + 11E_2 + 6E_1, \ldots
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
y_2(2:E) &= E_2 + x_1E_1 = -\frac{1}{2}, \\
y_2(3:E) &= E_3 + x_1E_2 + x_2E_1 = -\frac{3}{4}, \\
y_2(4:E) &= E_4 + x_1E_3 + x_2E_2 + x_3E_1 = -\frac{3}{2}, \ldots
\end{aligned}$$

Let $x_j \in \mathbb{Z}$ and $j \in \{1, 2, \ldots, n-1\}$ with $n > 1$. Let $y_1(x, 0 : E(x)) = y_2(x, 0 : E(x)) = E_0(x) = 1$ and $y_1(x, 1 : E(x)) = y_2(x, 1 : E(x)) = E_1(x) = x - \frac{1}{2}$. We define the sequences $(y_1(x, n : E(x)))$ and $(y_2(x, n : E(x)))$, including the Euler polynomials, related to the polynomials $Ch_n(x)$ and $\widehat{Ch_n}(x)$ as follows:

$$y_1(x, n : E(x)) = E_n(x) + \sum_{j=1}^{n-1} (-1)^j x_j E_{n-j}(x), \tag{140}$$

and

$$y_2(x, n : E(x)) = E_n(x) + \sum_{j=1}^{n-1} x_j E_{n-j}(x). \tag{141}$$

Few values of (140) are given as follows:

$$\begin{aligned} y_1(x, 2 : E) &= E_2(x) - x_1 E_1(x) = x^2 - 2x + \frac{1}{2}, \\ y_1(x, 3 : E) &= E_3(x) - x_1 E_2(x) + x_2 E_1(x) = x^3 - \frac{9}{2}x^2 + 5x, -\frac{3}{4}, \\ y_1(x, 4 : E) &= E_4(x) - x_1 E_3(x) + x_2 E_2(x) - x_3 E_1(x) = x^4 - 8x^3 + 20x^2 - 16x \\ &\quad + \frac{3}{2}, \ldots \end{aligned}$$

Few values of (141) are given as follows:

$$\begin{aligned} y_2(x, 2 : E) &= E_2(x) + x_1 E_1(x) = x^2 - \frac{1}{2}, \\ y_2(x, 3 : E) &= E_3(x) + x_1 E_2(x) + x_2 E_1(x) = x^3 + \frac{3}{2}x^2 - x - \frac{3}{4}, \\ y_2(x, 4 : E) &= E_4(x) + x_1 E_3(x) + x_2 E_2(x) + x_3 E_1(x) = x^4 + 4x^3 + 2x^2 - 4x \\ &\quad - \frac{3}{2}, \ldots \end{aligned}$$

Observe that when $x = 0$, the sequences $(y_1(x, n : E(x)))$ and $(y_2(x, n : E(x)))$ reduces to the following sequences, respectively:

$$y_1(n : E) = y_1(0, n : E(0))$$

and

$$y_2(n : E) = y_2(0, n : E(0)).$$

9 Formulas for the sequence $y_1(n : E)$

Using the p-adic fermionic integral and its integral equations, we give some formulas and identities for the sequence $(y_1(n : E))$. We also give some p-adic fermionic integral formulas including the falling factorial.

Explicit formula for the sequence $y_1(n : E)$ is given by the following theorem, which was proved by different method (see, for details, cf. [27, 31, 45, 46]).

Theorem 41 *Let $n \in \mathbb{N}_0$. Then we have*

$$y_1(n : E) = (-1)^n 2^{-n} n!.$$

Proof We know that the numbers forming the sequence $(y_1(n : E))$ are related to the numbers Ch_n. By using same computation of the numbers Ch_n, this theorem is also proved before. By using different method, we now briefly give proof of the theorem. Since

$$x_{(n)} = n! \binom{x}{n},$$

by using (55), we get

$$\int_{\mathbb{Z}_p} x_{(n)} d\mu_{-1}(x) = n! \int_{\mathbb{Z}_p} \binom{x}{n} d\mu_{-1}(x)$$
$$= \frac{(-1)^n}{2^n} n! = y_1(n : E). \tag{142}$$

Thus, we get the desired result. □

We give a recurrence relation the sequence $(y_1(n : E))$ by the following theorem:

Theorem 42 *Let $n \in \mathbb{N}_0$. Then we have*

$$y_1(n+1 : E) + n y_1(n : E) = (-1)^n \frac{n!(n-1)}{2^{n+1}}.$$

Proof By applying the p-adic fermionic integral to the both sides of the following well-known equation

$$x x_{(n)} = x_{(n+1)} + n x_{(n)},$$

and using (55) and (57), respectively, we get

$$\int_{\mathbb{Z}_p} x x_{(n)} d\mu_{-1}(x) = (-1)^n \frac{n!(n-1)}{2^{n+1}},$$

and

$$\int_{\mathbb{Z}_p} x x_{(n)} d\mu_{-1}(x) = y_1(n+1 : E) + n y_1(n : E).$$

Combining the above equation with (127), we arrive at the desired result. □

10 Formulas for the sequence $y_2(n : E)$

By using the p-adic fermionic integral and its integral equations, we derive some formulas and identities for the sequence $(y_2(n : E))$. We also gives some p-adic fermionic integral formulas including the raising factorial, combinatorial sums and special numbers.

Theorem 43 *Let $n \in \mathbb{N}$. Then we have*

$$y_2(n : E) = \sum_{k=1}^{n} (-1)^k \frac{|L(n,k)|}{2^k} k!. \tag{143}$$

Proof By applying the p-adic fermionic integral to the equation (24), we get

$$y_2(n : E) = \sum_{k=1}^{n} |L(n,k)| \int_{\mathbb{Z}_p} x_{(k)} d\mu_{-1}(x)$$

By substituting (56) into the above equation, we get the desired result. □

Combining (21) with (143), we arrive at the following result:

Corollary 6 *Let $n \in \mathbb{N}$. Then we have*

$$y_2(n:E) = n!\sum_{k=1}^{n}(-1)^k\binom{n-1}{k-1}2^{-k}.$$

By applying the p-adic fermionic integral to the equation (11) and using (53), we get the following theorem, which is modified equation (139):

Theorem 44 *Let $n \in \mathbb{N}_0$. Then we have*

$$y_2(n:E) = \sum_{k=1}^{n} C(n,k)E_k.$$

By applying the p-adic fermionic integral to the equation (123), and using (53), we arrive at another formula for the numbers $y_2(n:E)$ by the following theorem:

Theorem 45 *Let $n \in \mathbb{N}$. Then we have*

$$y_2(n:E) = \sum_{k=1}^{n}\sum_{j=0}^{k}|L(n,k)|\,S_1(k,j)E_j.$$

By applying the p-adic fermionic integral to the equation (117), and using (53), we get the following identity:

$$\int_{\mathbb{Z}_p}(x+n-1)_{(n)}d\mu_{-1}(x) = \sum_{m=0}^{n+1}(-1)^{m+n}S_1(n,m)E_m. \tag{144}$$

By combining the above equation with (135), we get the following formula for the sequence $y_2(n:E)$:

Theorem 46 *Let $n \in \mathbb{N}_0$. Then we have*

$$y_2(n:E) = \sum_{m=0}^{n+1}(-1)^{m+n}S_1(n,m)E_m.$$

We give a recurrence relation the sequence $(y_2(n:E))$ by the following theorem:

Theorem 47 *Let $n \in \mathbb{N}$. Then we have*

$$y_2(n+1:E) - ny_2(n:E) = \sum_{k=1}^{n}(-1)^k\binom{n-1}{k-1}\frac{(k-1)}{2^{k+1}}n!.$$

Proof Using (1), we have

$$\frac{x^{(n)}}{x^{(n+1)}} = \frac{1}{x+n}.$$

Therefore, we have

$$x^{(n+1)} = xx^{(n)} + nx^{(n)}. \tag{145}$$

By applying the p-adic fermionic integral to (145), we get

$$y_2(n+1:E) = \int_{\mathbb{Z}_p} xx^{(n)}d\mu_{-1}(x) + ny_2(n:E).$$

Combining the above equation wit (128), we arrive at the desired result. □

11 Identities for combinatorial sums including special numbers

In this section, by using integral formulas, we derive many novel combinatorial sums including the Bernoulli numbers, the Euler numbers, the Stirling numbers, the Eulerian numbers, the harmonic numbers and the Lah numbers.

Combining (116) and (101), we arrive at the following theorem:

Theorem 48 *Let $m, n \in \mathbb{N}_0$. Then we have*

$$\sum_{j=0}^{n}\sum_{l=0}^{m} S_1(n,k)S_1(m,l)B_{j+l} = \sum_{k=0}^{m}(-1)^{m+n-k}\binom{m}{k}\binom{n}{k}\frac{k!(m+n-k)!}{m+n-k+1}. \tag{146}$$

Combining (101) and (102), we arrive at the following theorem:

Theorem 49 *Let $m, n \in \mathbb{N}_0$. Then we have*

$$\sum_{j=0}^{n}\sum_{l=0}^{m} S_1(n,k)S_1(m,l)B_{j+l} = \sum_{k=0}^{m}\binom{m}{k}\binom{n}{k}k!\sum_{l=0}^{m+n-k} S_1(m+n-k,l)B_l. \tag{147}$$

By combining (146) and (147), we get the following combinatorial sum by the following the following corollary:

Corollary 7 *Let $m, n \in \mathbb{N}_0$. Then we have*

$$\sum_{k=0}^{m}\binom{m}{k}\binom{n}{k}k!\sum_{l=0}^{m+n-k} S_1(m+n-k,l)B_l = \sum_{k=0}^{m}(-1)^{m+n-k}\binom{m}{k}\binom{n}{k} \times\frac{k!(m+n-k)!}{m+n-k+1}.$$

By combining (80) and (81), we arrive at the following theorem:

Theorem 50 *Let $n \in \mathbb{N}_0$. Then we have*

$$\sum_{k=0}^{n}\sum_{j=0}^{k}\binom{k}{j}S_1(n,k)B_jB_{k-j} = \sum_{k=0}^{n}(-1)^n\frac{n!}{(k+1)(n-k+1)}.$$

Combining (68) with (71), we arrive at the following theorem:

Theorem 51 *Let $n \in \mathbb{N}_0$. Then we have*

$$\sum_{k=1}^{n} S_1(n,k-1)B_k = \frac{(-1)^{n+1}n!}{n^2+3n+2} - B_{n+1}.$$

Substituting (120) into (125), we get we get

$$\sum_{k=1}^{n+1} C(n+1,k)B_k - n\sum_{k=1}^{n} C(n,k)B_k = \sum_{k=1}^{n}(-1)^{k+1}\left|L(n,k)\right|\frac{k!}{k^2+3k+2}.$$

After some elementary calculation in the above equation, we arrive at the following theorem:

Theorem 52 *Let $n \in \mathbb{N}_0$. Then we have*

$$\sum_{k=1}^{n}\left(C(n+1,k)-nC(n,k)\right)B_k=\sum_{k=1}^{n}(-1)^{k+1}\left|L(n,k)\right|\frac{k!}{k^2+3k+2}-B_{n+1}.$$

Combining (118) with (119), we get the following theorem:

Theorem 53 *Let $n \in \mathbb{N}$. Then we have. Then we have*

$$\sum_{m=0}^{n}(-1)^m\binom{n-1}{n-m}\frac{n!}{m+1}=\sum_{m=0}^{n+1}(-1)^m S_1(n,m)B_m.$$

Combining (72)with (74) we arrive at the following theorem:

Theorem 54 *Let $n \in \mathbb{N}$. Then we have*

$$\sum_{k=1}^{n}C(n,k)B_{k+1}=\sum_{k=1}^{n}(-1)^{k+1}\binom{n-1}{k-1}\frac{n!}{k^2+3k+2}.$$

By combining (96) with (97), we get

$$\sum_{k=0}^{n}\sum_{j=0}^{n}\binom{n}{j}\frac{S_1(j,k)B_k}{j!}=\sum_{k=0}^{n}\frac{(-1)^k}{k+1}\sum_{j=0}^{k}(-1)^j\binom{k}{j}\binom{k-j+n}{n}.$$

By substituting (108), into the above equation, we arrive at the following theorem:

Theorem 55 *Let $n \in \mathbb{N}_0$. Then we have*

$$\sum_{k=0}^{n}\sum_{j=0}^{n}\binom{n}{j}\frac{S_1(j,k)B_k}{j!}=\sum_{k=0}^{n}\sum_{j=0}^{k}(-1)^j\binom{k}{j}\binom{k-j+n}{n}\frac{Y_1(k:B)}{k!}.$$

By combining (89) with (90), we arrive at the following theorem:

Theorem 56 *Let $k \in \mathbb{N}_0$. Then we have*

$$\sum_{l,m=1}^{k}C_{l,m}^{(k)}D_l D_m=\sum_{m=0}^{k}S_1(k,m)\left(B_m\right)^2$$

and

$$\sum_{m=0}^{k}S_1(k,m)\left(B_m\right)^2=\sum_{l,m=1}^{k}(-1)^{l+m}\frac{l!m!}{(l+1)(m+1)}C_{l,m}^{(k)}.$$

By combining (130) with (131), we arrive at the following theorem:

Theorem 57 *Let $k \in \mathbb{N}_0$. Then we have*

$$\sum_{l,m=1}^{k}(-1)^{l+m}l!m!2^{-m-l}C_{l,m}^{(k)}=\sum_{m=0}^{k}S_1(k,m)\left(E_m\right)^2.$$

By combining (144) with (135), we obtain the following theorem:

Theorem 58 *Let $n \in \mathbb{N}$. Then we have*

$$\sum_{m=0}^{n}(-1)^m \binom{n-1}{n-m}\frac{n!}{2^m} = \sum_{m=0}^{n+1}(-1)^{m+n} S_1(n,m)E_m.$$

By combining left-hand side of (96) with (97), we get the following theorem:

Theorem 59 *Let $n \in \mathbb{N}_0$. Then we have*

$$\sum_{k=0}^{n}\sum_{j=0}^{n}\binom{n}{j}\frac{S_1(j,k)B_k}{j!} = \sum_{k=0}^{n}\sum_{j=0}^{k}(-1)^{k+j}\binom{k}{j}\binom{k-j+n}{n}\frac{1}{k+1}.$$

Theorem 60 *Let $n \in \mathbb{N}$. Then we have*

$$B_n = \sum_{j=0}^{n}\frac{j!}{n!}\sum_{m=0}^{j}\sum_{k=0}^{j}(-1)^{j+k+m}\binom{j-1}{j-m}\binom{n+1}{j-k}\frac{k^n}{m+1}.$$

Proof By applying the Volkenborn integral to the following identity, which derived from the work of Golud [17, Eq. (4.1)]

$$x^n = \sum_{j=0}^{n}\binom{x+j-1}{n}\sum_{k=0}^{j}(-1)^{j+k}\binom{n+1}{j-k}k^n, \tag{148}$$

we get

$$\int_{\mathbb{Z}_p} x^n d\mu_1(x) = \sum_{j=0}^{n}\sum_{k=0}^{j}(-1)^{j+k}\binom{n+1}{j-k}k^n\int_{\mathbb{Z}_p}\binom{x+j-1}{n}d\mu_1(x).$$

Combining the above equation with (37) and (44), we get

$$B_n = \sum_{j=0}^{n}\sum_{k=0}^{j}(-1)^{j+k}\binom{n+1}{j-k}\frac{j!}{n!}k^n\sum_{m=0}^{j}(-1)^m\binom{j-1}{j-m}\frac{1}{m+1}.$$

Thus, proof of this theorem is completed. □

By combining the left-hand side of (132) and (133), we get the following theorem:

Theorem 61 *Let $n \in \mathbb{N}_0$. Then we have*

$$\sum_{k=0}^{n}\sum_{j=0}^{n}\binom{n}{j}\frac{S_1(j,k)E_k}{j!} = \sum_{k=0}^{n}\sum_{j=0}^{k}(-1)^{j+k}\binom{k}{j}\binom{k-j+n}{n}\frac{1}{2^k}.$$

Theorem 62 *Let $n \in \mathbb{N}_0$. Then we have*

$$E_n = \sum_{j=0}^{n}\frac{j!}{n!}\sum_{m=0}^{j}\sum_{k=0}^{j}(-1)^{j+k+m}\binom{j-1}{j-m}\binom{n+1}{j-k}\frac{k^n}{2^m}.$$

Proof By applying the p-adic fermionic integral to equation (148), we obtain

$$\int_{\mathbb{Z}_p} x^n d\mu_{-1}(x) = \sum_{j=0}^{n}\sum_{k=0}^{j}(-1)^{j+k}\binom{n+1}{j-k}k^n \int_{\mathbb{Z}_p}\binom{x+j-1}{n}d\mu_{-1}(x).$$

Combining the above equation with (62) and (53), we get

$$E_n = \sum_{j=0}^{n}\sum_{k=0}^{j}(-1)^{j+k}\binom{n+1}{j-k}\frac{j!}{n!}k^n\sum_{m=0}^{j}(-1)^m\binom{j-1}{j-m}\frac{1}{2^m}.$$

Thus, proof of this theorem is completed. □

Theorem 63 *Let $n \in \mathbb{N}_0$. Then we have*

$$\sum_{m=0}^{n} S_2(n,m)L(n,k) = \sum_{m=0}^{n}\binom{n}{m}S_2(n-m,k)w_g^{(k)}(m).$$

Proof Substituting $t = e^t - 1$ into (19), and combining with equation (16) and (7), we get the following functional equation:

$$F_L(e^t-1,k) = F_S(t,k;1)F_{Fu}(t,k). \tag{149}$$

By using (149), we get

$$\sum_{n=0}^{\infty} L(n,k)\frac{(e^t-1)^n}{n!} = \sum_{n=k}^{\infty} S_2(n,k)\frac{t^n}{n!}\sum_{n=0}^{\infty} w_g^{(k)}(n)\frac{t^n}{n!}.$$

By using the Cauchy product rule from the above equation, we obtain

$$\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} S_2(n,m)L(n,k)\right)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}S_2(n-m,k)w_g^{(k)}(m)\right)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we get the desired result. □

Remark 15 Substituting $k = 1$ into equation (7), we arrive at equation (6).

Theorem 64 *Let $n \in N_0$. Then we have*

$$\sum_{k=0}^{n} C(n,k)B_k = \sum_{k=1}^{n}(-1)^k\frac{n!}{k+1}\binom{n-1}{k-1}.$$

Proof Combining (121) with (120), we derived the desired result. □

Theorem 65 *Let $n \in \mathbb{N}_0$. Then we have*

$$\sum_{j=0}^{n}\sum_{k=0}^{\left[\frac{n-j}{2}\right]}\binom{n}{j}S_{12}(n-j,k)B_{k+j} = (-1)^n\frac{n!}{n+1},$$

$$\sum_{j=0}^{n}\sum_{k=0}^{\left[\frac{n-j}{2}\right]}\binom{n}{j}S_{12}(n-j,k)B_{k+j} = \sum_{j=0}^{n}s_1(n,j)B_j,$$

and

$$\sum_{j=0}^{n}\sum_{k=0}^{\left[\frac{n-j}{2}\right]}\binom{n}{j}S_{12}(n-j,k)E_{k+j}=(-1)^{n}\frac{n!}{2^{n}}.$$

Proof In order to prove the assertions of the theorem, we apply the Volkenborn integral and the p-adic fermionic integral to the following identity (cf. [12, p. 123]):

$$t_{(n)}=\sum_{j=0}^{n}\sum_{k=0}^{\left[\frac{n-j}{2}\right]}\binom{n}{j}S_{12}(n-j,k)t^{j+k}, \tag{150}$$

after some elementary evaluations, we get the desired result. □

Integrating both sides of (150) from 0 to 1 (with the Riemann integral sense) and using the definition of the Bernoulli numbers of the second kind, we arrive at the following corollary:

Corollary 8 *Let $n \in \mathbb{N}_0$. Then we have*

$$b_n(0)=\sum_{j=0}^{n}\sum_{k=0}^{\left[\frac{n-j}{2}\right]}\binom{n}{j}\frac{S_{12}(n-j,k)}{j+k+1}.$$

Theorem 66 *Let $n \in \mathbb{N}_0$. Then we have*

$$\sum_{j=0}^{n}S_1(n,k)B_k=\frac{(-1)^n n!}{n+1},$$

and

$$\sum_{j=0}^{n}S_1(n,k)E_k=\frac{(-1)^n n!}{2^n}.$$

Proof In [17, Vol. 7, Eq. (5.59)], Gould gave the following identity:

$$\binom{x}{n}=\sum_{j=0}^{n}\frac{S_1(n,k)}{n!}x^k. \tag{151}$$

Applying the Volkenborn integral and the p-adic fermionic integral to the above identity with the Riemann integral (from 0 to 1), respectively, we get the assertions of the theorem. □

Remark 16 Different proofs of the Theorem 66 were given by Kim et al. [31] and [27].

Applying the Riemann integral to the equation (151) from 0 to 1, and using (14), we also arrive at the following well-known identity:

$$b_n(0)=\sum_{j=0}^{n}\frac{1}{k+1}S_1(n,k). \tag{152}$$

Remark 17 Equation (152) has been proved by means of the different methods. For example, see [39].

By applying the Volkenborn integral to the falling factorial, we have the following well-known theorem:

Theorem 67 *Let* $n \in \mathbb{N}$. *Then we have*

$$B_n = \sum_{k=0}^{n-1} (-1)^k \frac{k!}{k+1} S_2(n,k). \tag{153}$$

Proof We modify (17) as follows

$$x^n = \sum_{k=0}^{n} S_2(n,k) \left(x x_{(k-1)} - (k-1) x_{(k-1)} \right).$$

By applying the Volkenborn integral to the above equation, and using (44), we get

$$B_n = \sum_{k=0}^{n} S_2(n,k) \left(\int_{\mathbb{Z}_p} x x_{(k-1)} d\mu_1(x) - (k-1) \int_{\mathbb{Z}_p} x_{(k-1)} d\mu_1(x) \right). \tag{154}$$

By substituting (109) and (68) into the above equation, after some elementary calculations, we arrive at the desired result. □

Remark 18 Equation (153) has been proved by various different methods, see, for detail (cf. [6], [12, p. 117], [38, 47]).

By substituting (109) into (153), we get a relation between the Bernoulli numbers and the numbers $Y_1(n : B)$ and the Daehee numbers by the following corollary:

Corollary 9 *Let* $n \in \mathbb{N}$. *Then we have*

$$B_n = \sum_{k=0}^{n-1} Y_1(k : B) S_2(n,k)$$

and

$$B_n = \sum_{k=0}^{n-1} D_k S_2(n,k).$$

Theorem 68 *Let* $n \in \mathbb{N}_0$. *Then we have*

$$B_n = \sum_{k=0}^{n} S_2(n,k) \sum_{j=0}^{k-1} S_1(k,j) B_j + \sum_{k=0}^{n} S_2(n,k) B_k.$$

Proof By combining (9) and (70) with (154), we get

$$B_n = \sum_{k=0}^{n} S_2(n,k) \left(\sum_{j=0}^{k-1} S_1(k-1,k-1) B_j + B_k - (k-1) \int_{\mathbb{Z}_p} x_{(k-1)} d\mu_1(x) \right).$$

By substituting (46) into the above equation, we obtain

$$
\begin{aligned}
B_n &= \sum_{k=0}^{n} S_2(n,k)\left(\sum_{j=0}^{k-1} S_1(k-1,k-1)B_j + B_k - (k-1)\sum_{j=0}^{k-1} S_1(k-1,j)B_j\right) \\
&= \sum_{k=0}^{n} S_2(n,k)\sum_{j=0}^{k-1}\left(S_1(k-1,j-1)-(k-1)S_1(k-1,j)\right)B_j + \sum_{k=0}^{n} S_2(n,k)B_k.
\end{aligned}
$$

By combining the above equation with (9), after some elementary calculations, we arrive at the desired result. □

12 Conclusion

The methods of the present paper are two-folds. The first is to use the methods of generating functions for special numbers and polynomials and their relationships between each others. By using the generating function method, various properties of these numbers and polynomials are investigated. Due to presence of novel applications of the generating functions in mathematics, mathematical physics, and the other areas, some results of present paper can be used by not only mathematicians, but also physicists and engineers. The second is to use the p-adic q-integrals and their integral equations. By applying these integrals equations to some special functions and polynomials, we derive various useful and elegant formulas including combinatorial sums and the p-adic q-integrals. With the help of those formulas and necessary comparisons, we find new and novel relations, identities, combinatorial sums and formulas covering many important numbers and polynomials such as the Bernoulli numbers and polynomials, the Euler numbers and polynomials, the Frobenius–Euler numbers and polynomials, the Stirling numbers, the Harmonic numbers, the Lah numbers and the others. It is possible to say that due to presence of applications of the p-adic q-integrals and their integral equations in ultrametric calculus, combinatorial physics, quantum physics, physical models and the other areas, some of our results may contribute to not only mathematics and applied physics, but also theoretical physics, and also other related areas. We also give applications related to the Lah numbers and the Laguerre polynomials. We give relations between the Laguerre polynomials, the Lah numbers, the Bernoulli numbers and the Euler numbers. It is well-known that these the Laguerre polynomials and the Laguerre differential equations have been many applications both in mathematics and in physics. Additionally, special functions, special numbers and special polynomials, studied in this paper, form the basis of the studies of mathematicians, physicists and other scientists because special numbers and special polynomials are used in modeling, making algorithms, coding theory, combinatorics problems, partition theory, approximation theory and estimation etc.

Details of the conclusion about this paper and its applications may be summarized as follows:

By applying the p-adic q-integrals to differentiable functions, novel p -adic integral formulas, significant relations and results have been obtained.

By applying the Volkenborn integral to falling and rising factorial polynomials, new number sequences containing the Bernoulli numbers and polynomials have been constructed. The recurrence relations of these number sequences and their relation to the special numbers are given. In addition, a great number of (p-adic) Volkenborn integral formulas including falling and rising factorial polynomials have been obtained thanks to this technique. With the help of these integral formulas, a large number of identities, combinatorial sums and relations were obtained.

Similarly, another new number sequences containing Euler numbers and polynomials were defined with the aid of p-adic fermionic integral. By applying similar methods to Euler numbers and polynomials, the recurrence relation of these sequences and the relations with the special numbers were given. By applying fermionic integral to falling and rising factorial polynomials, many fermionic integral formulas including the Euler numbers, the Lah numbers, the Bernoulli numbers and the Stirling numbers have been obtained.

By taking the different combinations of these formulas, not only new identities, but also elegant combinatorial sums containing the Lah numbers, the Stirling numbers, the Bernoulli numbers, the Euler numbers, the harmonic numbers and the other special numbers have been obtained.

It is worth to note that these integral formulas will contribute to primarily ultrametric calculus, and then many areas of physics such as especially combinational physics, q-quantum mechanic, q-quantum model. This paper's formulas, identities, relations and combinatorial sums will shed light on the fields of researchers working in both theoretical and applied sciences.

Appendix A Some Some tables containing *p*-adic integral formulas

Here, we finalize this paper along with an appendix of some tables containing p-adic integral formulas obtained in this paper.

Table 1 Some Volkenborn integral formulas containing falling factorials

$$\int_{\mathbb{Z}_p} (x+n-1)_{(n)} d\mu_1(x) = \sum_{m=0}^{n} (-1)^m \binom{n-1}{n-m} \frac{n!}{m+1}.$$

$$\int_{\mathbb{Z}_p} (x+1)_{(n)} \, d\mu_1(x) = (-1)^{n+1} \frac{n!}{n^2+n}.$$

$$\int_{\mathbb{Z}_p} (-x)_{(n)} \, d\mu_1(x) = \sum_{k=1}^{n} (-1)^{k+n} \binom{n-1}{k-1} \frac{n!}{k+1}.$$

$$\int_{\mathbb{Z}_p} (mx)_{(n)} \, d\mu_1(x) = \sum_{k=0}^{n} \frac{(-1)^k}{k+1} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (mk-mj)_{(n)}.$$

$$\int_{\mathbb{Z}_p} x_{(m)} x_{(n)} d\mu_1(x) = \sum_{k=0}^{m} (-1)^{m+n-k} \binom{m}{k} \binom{n}{k} \frac{k!(m+n-k)!}{m+n-k+1}.$$

$$\int_{\mathbb{Z}_p} \left(x_{(n)}\right)^r \, d\mu_1(x) = \sum_{k=0}^{nr} \frac{(-1)^k}{k+1} \sum_{j=0}^{k} (-1)^j \binom{k}{j} \left((k-j)_{(n)}\right)^r.$$

Table 2 Some fermionic p-adic integral formulas containing falling factorials

$$\int_{\mathbb{Z}_p} (x+n-1)_{(n)}\, d\mu_{-1}(x) = n! \sum_{m=0}^{n} (-1)^m \binom{n-1}{n-m} 2^{-m}.$$

$$\int_{\mathbb{Z}_p} (mx)_{(n)}\, d\mu_{-1}(x) = \sum_{k=0}^{n} \frac{(-1)^k}{2^k} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (mk-mj)_{(n)}.$$

$$\int_{\mathbb{Z}_p} \left(x_{(n)}\right)^r d\mu_{-1}(x) = \sum_{k=0}^{nr} \frac{(-1)^k}{2^k} \sum_{j=0}^{k} (-1)^j \binom{k}{j} \left((k-j)_{(n)}\right)^r.$$

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