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ORIGINAL PAPER

Embedding the free topological group $F(X^n)$ into $F(X)$

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Abstract
In 1976, Nickolas showed that for each natural n, the free topological group $F(X^n)$ is topologically isomorphic to a subgroup of $F(X)$ provided X is a compact space or, more generally, a k_ω-space. We complement the Nickolas' embedding theorem by showing that it remains true for every topological space X such that all finite powers of X are pseudocompact. For example, all pseudocompact k-spaces enjoy this property. Also, we extend the embedding theorem to the class of NC_ω-spaces that includes, in particular, the k_ω-spaces and the well-ordered spaces of ordinals $[0, \alpha)$, for every ordinal α. Our results are quite sharp because we present a first example of a Tychonoff space Z such that $F(Z)$ does not contain an isomorphic copy of the group $F(Z^2)$. In addition, our space Z is countably compact, separable, and its square Z^2 is not pseudocompact.

Keywords Free topological group · C-embedding · Pseudocompact space · Countably compact space

Mathematics Subject Classification 22A05 (primary) · 54C45 · 54D20 (secondary)

1 Introduction

All topological spaces under consideration are assumed to be Tychonoff.

Free topological groups were introduced in 1941 by Markov in the short note [15]. The complete construction appeared 4 years later (see [16]), where certain basic properties of free topological groups $F(X)$ and $A(X)$ over a Tychonoff space X were established. Specifically, Markov responded negatively to Kolmogorov's question regarding whether every Hausdorff topological group is a normal space. An explicit description of the topology of the free topological group on a compact space was given in the late 1940s by Graev (see [6]). A construction of the free locally convex space $L(X)$ on a space X is presented in [23].

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Definition 1.1 [15] Let X be a Tychonoff space. A topological group $F(X)$ is called the *(Markov) free topological group* over X if $F(X)$ satisfies the following conditions:

(i) there is a continuous mapping $\gamma : X \to F(X)$ such that $\gamma(X)$ algebraically generates $F(X)$;
(ii) if $f : X \to G$ is a continuous mapping to a topological group G, then there exists a continuous homomorphism $\bar{f} : F(X) \to G$ such that $f = \bar{f} \circ \gamma$.

Replacing in Definition 1.1 'topological group' with 'topological abelian group', one obtains the *free abelian topological group* $A(X)$ on the space X. In both cases, the mapping γ is a topological embedding [16].

In a similar way, one can define free locally convex spaces:

Definition 1.2 (See [15, 23]) Let X be a Tychonoff space. The *free locally convex space* $L(X)$ on X is a pair consisting of a locally convex space $L(X)$ and a continuous mapping $\gamma : X \to L(X)$ such that $\gamma(X)$ forms a Hamel basis for $L(X)$, and every continuous mapping f from X to a locally convex space E extends to a unique continuous linear operator $\bar{f} : L(X) \to E$ with $f = \bar{f} \circ \gamma$.

The free topological groups $F(X)$, $A(X)$ as well as the free locally convex space $L(X)$ always exist and are unique up to isomorphism. In what follows we identify X with its homeomorphic copy $\gamma(X)$.

A topological space X is called a *k-space* whenever $A \subset X$ is closed if and only if the intersection $A \cap K$ is closed in K for every compact set $K \subset X$. A topological space X is called a *k_ω-space* if X is the union of an increasing sequence $\{X_n : n \in \omega\}$ of compact subspaces, with the property that a subset $A \subset X$ is closed if and only if the intersection $A \cap X_n$ is closed in X_n for each $n \in \omega$. In this case, the representation $X = \bigcup_{n\in\omega} X_n$ is called a *k_ω-decomposition* for X. Every locally compact σ-compact space is a k_ω-space. In particular, all open and all closed subsets of a Euclidean space are k_ω-spaces.

Given a subset Y of a space X and an integer $n \geq 0$, we denote by $F_n(Y, X)$ and $A_n(Y, X)$ the subsets of $F(X)$ and $A(X)$, respectively, consisting of words of length at most n with letters from the set Y. If Y is closed in X, then $F_n(Y, X)$ and $A_n(Y, X)$ are closed in the respective groups $F(X)$ and $A(X)$, for each integer $n \geq 0$ (one can combine Theorems 7.1.13 and 7.4.5 in [1]).

Topological groups $F(X)$ and $A(X)$ are never compact, but if X is a k_ω-space, so are the groups $F(X)$ and $A(X)$ (see [1, Theorem 7.4.1]):

Theorem 1.3 *Let X be a k_ω-space with a k_ω-decomposition $X = \bigcup_{n\in\omega} X_n$. Then both $F(X)$ and $A(X)$ are k_ω-spaces with the corresponding k_ω-decompositions $F(X) = \bigcup_{n\in\omega} F_n(X_n)$ and $A(X) = \bigcup_{n\in\omega} A_n(X_n)$.*

Denote by $\mathbb{I}$ the closed unit interval $[0, 1]$. The following results obtained in [12] are relevant for our paper.

Theorem 1.4 (See [12]) *For a Tychonoff space X, the following are equivalent:*

(i) *$A(X)$ is topologically isomorphic to a subgroup of $A(\mathbb{I})$;*
(ii) *$F(X)$ is topologically isomorphic to a subgroup of $F(\mathbb{I})$;*
(iii) *X is a k_ω-space such that every compact subspace of X is finite-dimensional and metrizable.*

Theorem 1.5 (See [12]) *Let X be a finite-dimensional compact metrizable space. Then $L(X)$ isomorphically embeds into $L(\mathbb{I})$.*

So, as an easy corollary, we have the following assertion: for every finite-dimensional compact metrizable space X containing a homeomorphic copy of the segment $\mathbb{I}$, and every integer $n \geq 1$, the groups $A(X^n)$ and $F(X^n)$ are topologically isomorphic to subgroups of $A(X)$ and $F(X)$, respectively; and, similarly, $L(X^n)$ is topologically isomorphic to a linear subspace of $L(X)$.

Topological monomorphisms in Theorems 1.4 and 1.5 cannot be replaced by topological isomorphisms since the existence of a topological isomorphism between $F(X)$ and $F(Y)$, for compact metrizable spaces X and Y, implies the equality $\dim X = \dim Y$, by a result due to Graev [6]. This fact was generalized to arbitrary Tychonoff spaces X and Y by Pestov [21].

Motivated by these results, one can ask the following question:

Question 1.6 *Let G be any of the topological functors F, A, L. Is it true that for every Tychonoff space X and every integer $n \geq 1$, there exists a topological monomorphism of $G(X^n)$ into $G(X)$? What if X is a compact metrizable space?*

For the functors $G = A$ and $G = L$, Question 1.6 has been answered negatively for compact metrizable spaces X in [11]. Denote the topological sum of two copies of a space X by $X \oplus X$. It is known that if X is a Cook continuum, then the free abelian topological group $A(X \oplus X)$ does not embed into $A(X)$ as a topological subgroup. Similarly, for this X, the free locally convex space $L(X \oplus X)$ does not embed into $L(X)$ as a topological linear subspace (for details, see [11, 13]). Since $G(X^2)$ contains a topological and isomorphic copy of $G(X \oplus X)$ if X is a compact space with $|X| \geq 2$ (see Corollary 3.3), we conclude that $G(X^2)$ does not admit a topological monomorphism into $G(X)$ if X is a Cook continuum and $G \in \{A, L\}$.

The case of the (non-abelian) functor F is very different. For this functor, the positive answer to the second part of Question 1.6 has long been known.

Theorem 1.7 (Nickolas, see [18]) *Let X be a k_ω-space. Then $F(X^n)$ is topologically isomorphic to a subgroup of $F(X)$, for each integer $n \geq 1$. For the special case $n = 2$, a topological monomorphism φ of $F(X^2)$ to $F(X)$ restricted to the subspace X^2 of $F(X^2)$ is given by $\varphi(x, y) = xyx$ for all $x, y \in X$.*

In fact, the topological embedding in Theorem 1.7 is *closed*, so $F(X^n)$ is topologically isomorphic to a closed subgroup of $F(X)$. This follows from the combination of Theorem 1.3 and Graev's result stating that a Hausdorff topological group is Weil complete (hence, Raĭkov complete) provided it is a k_ω-space [7, Corollary 3]. One can also recall that the class of k_ω-spaces is finitely productive.

Notably, the article [18] by P. Nickolas does not contain any comments on whether Theorem 1.7 remains valid for all Tychonoff spaces.

Our goal in this paper is twofold: to extend the embedding Theorem 1.7 to wider classes of spaces and to provide a negative answer to the first part of Question 1.6 for the functor F.

Adapting the arguments in the original proof of Theorem 1.7, we show in Theorem 3.5 that the conclusion of the theorem remains true for any space X such that every finite power X^n is pseudocompact. For example, all pseudocompact k-spaces enjoy this property. Afterwards, in Theorem 4.12, we extend Theorem 1.7 to the class of NC_ω-spaces which contains as a proper subclass all k_ω-spaces. Examples of NC_ω-spaces include the ordinal spaces $[0, \alpha)$ for every ordinal α. We show that the class of NC_ω-spaces is invariant under basic topological operations. The topological sum of countably many copies of the space $[0, \omega_1)$ provides an example of a NC_ω-space that is not a k_ω-space and is not pseudocompact.

We give a negative answer to the first part of Question 1.6 for the functor $G = F$ by presenting a first example of a Tychonoff space Z such that $F(Z^2)$ is not topologically isomorphic to a subgroup of $F(Z)$ (Theorem 5.4). Since our space Z is countably compact, this shows that the embedding Theorem 3.5 is nearly optimal.

Concluding Sect. 6 contains some additional remarks and open questions.

2 Preliminaries

For a given Tychonoff space X, there are at least three distinct ways to "complete" it, by taking the *Stone–Čech compactification* βX of X, the *Hewitt–Nachbin realcompactification* υX of X, and the *Dieudonné completion* μX of X, respectively. These completions are related by the inclusions

$$X \subseteq \mu X \subseteq \upsilon X \subseteq \beta X.$$

We recall that μX is the completion of X with respect to the finest compatible uniformity on X. A space X is *Dieudonné complete* if $X = \mu X$; equivalently, X is homeomorphic to a closed subspace of a product of metrizable spaces [5, 8.5.13].

A subspace X of a space Y is *C-embedded* (*C^*-embedded*) in Y if every continuous (continuous bounded) function $f\colon X \to \mathbb{R}$ extends to a continuous function $f\colon Y \to \mathbb{R}$. The space υX is the biggest subspace Y of βX containing X such that X is C-embedded in Y [5, Theorem 3.11.10]. A space X is *Hewitt–Nachbin complete* or *realcompact* if $X = \upsilon X$; equivalently, X is homeomorphic to a closed subspace of a power of the real line $\mathbb{R}$. If there are no uncountable measurable cardinals, then the equality $\mu X = \upsilon X$ holds for every space X. A more exact version of this result states that if every discrete family of open sets in X has a non-measurable cardinality, then $\mu X = \upsilon X$ [5, 8.5.13(h)].

The notion of pseudocompactness is central to this article. A Tychonoff space X is called *pseudocompact* if every continuous real-valued function on X is bounded. One of the equivalent internal characterizations of pseudocompactness is as follows: a Tychonoff space X is pseudocompact if and only if every locally finite family of open sets in X is finite. A classical result states that every pseudocompact Dieudonné complete space is compact, so for a pseudocompact space X, $\mu X = \upsilon X = \beta X$ holds. The reader may consult the monograph [10] for a systematic study of pseudocompact spaces.

Given a subset X of a space B, we denote by $F(X, B)$ the subgroup of $F(B)$ algebraically generated by X. It is known that if X is closed in B, then $F(X, B)$ is a closed subgroup of $F(B)$ [1, Theorem 7.4.5]. However, the canonical continuous monomorphism of $F(X)$ to $F(B)$ that maps $F(X)$ onto $F(X, B)$ is not necessarily a topological embedding [29]. We manage to overcome this obstacle.

For our purposes in this paper the space X will frequently be viewed as a dense subspace of one of the aforementioned completions βX, υX or μX. As one of the main technical tools, we use repeatedly the following results which are due to Nummela and Pestov (see [20, 22] or [1, Theorem 7.7.3, Corollary 7.7.5]).

Theorem 2.1 (E. Nummela, V. Pestov) *Let X be a dense subspace of a space B. Then the subgroup $F(X, B)$ of $F(B)$ is topologically isomorphic to the free topological group $F(X)$ if and only if $X \subseteq B \subset \mu X$, that is, every continuous pseudometric on X admits a continuous extension over B.*

The next corollary of Theorem 2.1 explains why pseudocompactness plays a crucial role throughout our paper.

Corollary 2.2 (E. Nummela, V. Pestov) *The subgroup $F(X, \beta X)$ of $F(\beta X)$ is topologically isomorphic to the free topological group $F(X)$ if and only if X is pseudocompact.*

3 Extending the embedding theorem

It is known that the free topological group $F(X)$ is Raĭkov complete for every k_ω-space X [1, Theorem 7.4.11], so Theorem 1.7 provides a topological isomorphism between $F(X^n)$ and a *closed* subgroup of $F(X)$, for each $n \in \mathbb{N}$. If the space X is Dieudonné complete, a similar argument guarantees that any topological monomorphism of $F(X^n)$ to $F(X)$, if it exists, is closed (apply [25, Theorem 1]).

We show below that Theorem 1.7 remains true for any space X such that all finite powers of X are pseudocompact, maintaining the embedding to be closed. This is an important matter since we usually deal with spaces that are not Dieudonné complete.

Theorem 3.1 *Let X and Y be spaces such that the product $X \times Y$ is pseudocompact. Then the free topological group $F(X \oplus Y)$ contains a closed subgroup topologically isomorphic to $F(X \times Y)$.*

Proof Denote the topological sum $X \oplus Y$ by Z. Then the space Z is pseudocompact and $\beta Z = \beta X \oplus \beta Y$. Hence, by Corollary 2.2, $F(Z)$ can be identified with the subgroup $F(Z, \beta Z)$ of $F(\beta Z)$ algebraically generated by the subset Z of βZ. Let $K = \beta X \times \beta Y$. Again, since $X \times Y$ is pseudocompact, the group $F(X \times Y)$ is topologically isomorphic to the subgroup $F(X \times Y, K)$ of $F(K)$.

Similarly to [18, Proposition 1], we consider the continuous mapping $\varphi\colon K \to F(\beta Z)$ defined by the rule

$$\varphi(x, y) = xyx, \quad \text{for all } x \in \beta X \text{ and } y \in \beta Y.$$

Then φ is a topological embedding—a simple verification shows that the arguments from [18] remain valid in this more general situation. Furthermore, the subgroup G of $F(\beta Z)$ generated by $\varphi(K)$ is algebraically the free group on the compact set $\varphi(K)$. Again, this follows from [18, Lemma 1].

The mapping φ extends to a continuous homomorphism $\hat{\varphi}\colon F(K) \to F(\beta Z)$. Since $\varphi(K)$ is a free set of generators for the group G, we see that $\hat{\varphi}$ is a monomorphism. Applying [18, Lemma 1] once again we obtain the inclusion $G \cap F_n(\beta Z) \subseteq \langle\varphi(K)\rangle_n$ for each $n \in \omega$, where $\langle\varphi(K)\rangle_n$ is the set of elements of G that have length at most n with respect to the algebraic basis $\varphi(K)$. According to [14, Corollary 2], the latter implies that G is the free topological group on K, so $\hat{\varphi}$ is a topological isomorphism of $F(K)$ onto G (the result formulated in Corollary 2 of [14] refers to the Graev free topological groups, but its proof works without changes for Markov free topological groups as well). Hence, the restriction of $\hat{\varphi}$ to the subgroup $F(X \times Y, K)$ of $F(K)$ is a topological monomorphism of $F(X \times Y, K)$ to $F(\beta Z)$. It is also clear from our choice of φ and $\hat{\varphi}$ that $\hat{\varphi}\big(F(X \times Y, K)\big) \subset F(Z, \beta Z)$.

To see that $\hat{\varphi}(F(X \times Y, K)) \cong F(X \times Y)$ is closed in $F(Z, \beta Z) \cong F(Z) = F(X \oplus Y)$ it suffices to note that the groups $F(K)$ and $\hat{\varphi}(F(K))$ are Raĭkov complete and that the equality

$$\hat{\varphi}\big(F(X \times Y, K)\big) = \hat{\varphi}(F(K)) \cap F(Z, \beta Z)$$

is valid. □

If K is a nonempty compact subset of a pseudocompact space X, then the product $X \times K$ is also pseudocompact. By applying Theorem 3.1 we can conclude that the group $F(X \times K)$

is topologically isomorphic to a closed subgroup of $F(X \oplus K)$. In the following proposition we refine this result and show that, actually, $F(X \times K)$ is topologically isomorphic to a closed subgroup of $F(X)$ in this case.

Proposition 3.2 *Let $K_1, \ldots, K_m$ be nonempty compact subsets of a space X and $K = K_1 \times \cdots \times K_m$, where $m \in \mathbb{N}$. If X is either pseudocompact or a k_ω-space, then the group $F(X \times K)$ is topologically isomorphic to a closed subgroup of $F(X)$.*

Proof The product of a pseudocompact (or k_ω-) space with a compact space is again a pseudocompact (or k_ω-) space. Therefore, it suffices to consider the case $m = 1$ and then apply induction on m.

We assume first that X is a k_ω-space. Then $X \times X$ is also a k_ω-space and $X \times K$ is its closed subset. By [14, Theorem 3], the group $F(X \times K)$ is topologically isomorphic to the closed subgroup $F(X \times K, X \times X)$ of $F(X \times X)$. According to Theorem 1.7, the latter group is topologically isomorphic to a closed subgroup of $F(X)$. Hence, $F(X \times K)$ is topologically isomorphic to a closed subgroup of $F(X)$.

Next we consider the case where X is pseudocompact. We cannot apply Theorem 3.1 directly because the product X^2 is not necessarily pseudocompact. Therefore, we adapt the arguments from the proof of the theorem.

Let βX be the Stone-Čech compactification of X. Since the spaces X and $Y = X \times K$ are pseudocompact, it follows from [4, Theorem 3.1] that βY can be naturally identified with the product $P = \beta X \times K$. Also, by Corollary 2.2, $F(Y)$ is topologically isomorphic to the subgroup $F(Y, P)$ of $F(P)$ algebraically generated by the dense C-embedded subset Y of P. We denote by j_Y the topological monomorphism of $F(Y)$ to $F(P)$ extending the identity embedding of Y to P. Also, since P is a closed subset of the compact space $\beta X \times \beta X$, the identity embedding $i_P\colon P \to \beta X \times \beta X$ extends to a topological monomorphism $j_P\colon F(P) \to F(\beta X \times \beta X)$. Hence, the composition $j_P \circ j_Y$ is a topological monomorphism of $F(Y)$ to $F(\beta X \times \beta X)$. Notice that the restriction of $j_P \circ j_Y$ to Y is the identity embedding of Y to $\beta X \times \beta X$, so $j_P \circ j_Y$ is a topological isomorphism of $F(Y)$ onto the subgroup $F(Y, \beta X \times \beta X)$ of the group $F(\beta X \times \beta X)$.

Let $\varphi\colon F(\beta X \times \beta X) \to F(\beta X)$ be the closed topological monomorphism described in Theorem 1.7 for $n = 2$. Then we have the inclusions

$$\varphi\big(F(Y, \beta X \times \beta X)\big) \subset \varphi\big(F(X \times X, \beta X \times \beta X)\big) \subset F(X, \beta X) \cong F(X).$$

Since the above inclusions are topological monomorphisms, we see that $F(Y) \cong F(Y, \beta X \times \beta X)$ is topologically isomorphic to a subgroup of $F(X)$. It remains to verify that $\varphi \circ j_P \circ j_Y\colon F(Y) \to F(X)$ is a closed embedding. This follows from the equality

$$\varphi\big(F(Y, \beta X \times \beta X\big) = \varphi\big(F(P, \beta X \times \beta X)\big) \cap F(X, \beta X)$$

and the facts that $F(P, \beta X \times \beta X)$ is a closed subgroup of $F(\beta X \times \beta X)$ and that φ is a closed embedding of $F(\beta X \times \beta X)$ to $F(\beta X)$. □

Corollary 3.3 *Let X be a space with $|X| \geq 2$. If X is either pseudocompact or a k_ω-space, then the group $F(X \oplus X)$ is topologically isomorphic to a closed subgroup of $F(X)$.*

Proof It suffices to consider the case of a pseudocompact space X. Take distinct points $a, b \in X$ and identify $X \oplus X$ with the closed subspace $Y = X \times K$ of the product $X \times X$, where $K = \{a, b\}$. Since K is evidently a compact subset of X, we can apply Proposition 3.2 to conclude that $F(X \times K)$ is topologically isomorphic to a closed subgroup of $F(X)$. □

Theorem 3.4 *Let X be a space such that $X \times X$ is pseudocompact. Then the group $F(X \times X)$ is topologically isomorphic to a closed subgroup of $F(X)$.*

Proof Firstly, since the product $X \times X$ is pseudocompact, taking $Y = X$ in Theorem 3.1 we obtain that $F(X \times X)$ is topologically isomorphic to a closed subgroup of $F(X \oplus X)$. Secondly, by Corollary 3.3, $F(X \oplus X)$ admits a topological isomorphism onto a closed subgroup of $F(X)$. Therefore, $F(X)$ contains a closed isomorphic copy of $F(X \times X)$. □

Finally, we extend the above theorem to $F(X^n)$ for every natural n.

Theorem 3.5 *Let X be a space such that all finite powers of X are pseudocompact. Then for every integer $n \geq 1$, $F(X)$ contains a closed subgroup topologically isomorphic to $F(X^n)$.*

Proof According to Theorem 3.4, $F(X^2)$ is topologically isomorphic to a closed subgroup of $F(X)$. This result applied to X^2 in place of X shows that $F(X^4)$ is topologically isomorphic to a closed subgroup of $F(X^2)$. Thus, $F(X)$ contains a closed copy of $F(X^4)$. Continuing the argument, we see that $F(X)$ contains a closed isomorphic copy of $F(X^{2^k})$, for each natural k. Finally, for arbitrary natural numbers n and m with $1 \leq n < m$, there exists a continuous open retraction of X^m onto X^n, so [1, Exercise 7.7.a] implies that $F(X^n)$ is topologically isomorphic to a closed subgroup of $F(X^m)$. For every $n \geq 1$, choose $k \in \mathbb{N}$ such that $n < 2^k$, and the required conclusion follows. □

Given a Tychonoff space X, we call it an *FP-space* if all finite powers of X are pseudocompact. Not every pseudocompact (nor even countably compact) space is an FP-space [19, 27]. Clearly, all compact spaces are FP-spaces. However, the class of FP-spaces is considerably wider than the class of compact spaces. It is known that the product of any family of pseudocompact k-spaces is pseudocompact [10, Theorem 1.4.9]. Therefore, any product of pseudocompact k-spaces is an FP-space.

Corollary 3.6 *Let X be a pseudocompact k-space (for instance, a pseudocompact locally compact space or a pseudocompact sequential space). Then for each integer $n \geq 1$, $F(X)$ contains a closed subgroup topologically isomorphic to $F(X^n)$.*

The following two examples give more information on the class of FP-spaces and its permanence properties.

Example 3.7 (a) Every *Isbell–Mrówka Ψ-space* is locally compact and pseudocompact (see [10, Chapter 8]), hence it is an FP-space.

(b) Let X be a closed subset of a Σ-product of any family of compact spaces [5, 2.7.14]. Then X is an *ω-bounded* space, i.e. the closure of every countable subset in X is compact, and therefore X is an FP-space.

Example 3.8 Later, in Proposition 5.1 we present two FP-spaces X and Y such that their topological sum $X \oplus Y$ is not an FP-space.

4 Free topological groups on NC_ω-spaces

In order to extend Theorem 1.7 to a class of spaces strictly wider than the class of k_ω-spaces, we follow [1, Section 7.8].

Let us say that X is an *NC-space* if X^n is normal and countably compact for each natural n. Given a space Y and a family $\mathcal{F} = \{Y_n : n \in \omega\}$ of subspaces of Y with $Y = \bigcup_{n\in\omega} Y_n$, we also say that Y is the *direct limit* of the family $\mathcal{F}$ if a subset C of Y is closed in Y if and only if $C \cap Y_n$ is closed in Y_n for each $n \in \omega$.

The following result is proved in [1, Theorem 7.8.8].

Theorem 4.1 *If X is an NC-space, then the groups $F(X)$ and $A(X)$ are direct limits of the respective families $\{F_n(X) : n \in \omega\}$ and $\{A_n(X) : n \in \omega\}$.*

If the group $G(X)$, where $G \in \{F, A\}$, satisfies the conclusion of Theorem 4.1, we say that $G(X)$ has the *direct limit property.*

Definition 4.2 A space X is called an NC_ω-*space* if there exists an increasing sequence $\{X_n : n \in \omega\}$ of closed NC-subspaces of X such that $X = \bigcup_{n\in\omega} X_n$ and X is the direct limit of this sequence.

It is clear that every k_ω-decomposition of a space is an NC_ω-decomposition, whence it follows that every k_ω-space is an NC_ω-space. The converse is false because the space $[0, \omega_1)$ of countable ordinals with the order topology is an NC-space (hence, NC_ω-space) that is not σ-compact and thus is not a k_ω-space. It is also clear that a closed subspace of an NC_ω-space is an NC_ω-space. The following lemma provides additional information in this direction.

Lemma 4.3 *Every ordinal space $X = [0, \alpha)$ with the topology generated by the natural well-ordering is an NC_ω-space.*

Proof If α is a successor ordinal, i.e., $\alpha = \beta + 1$, then the space $X = [0, \beta]$ is compact and there is nothing to prove. If $cf(\alpha) > \omega$, then X is an NC-space (see [1, Lemma 7.8.14]). Finally, if $cf(\alpha) = \omega$, take a strictly increasing sequence $\{\alpha_n : n \in \omega\}$ of ordinals such that $\alpha = \sup_{n\in\omega} \alpha_n$. Then $X = \bigcup_{n\in\omega}[0, \alpha_n]$ is a k_ω-decomposition for X and, hence, a NC_ω-decomposition for X. □

Corollary 4.4 *The product $[0, \omega_1)\times\omega$, where ω carries the discrete topology, is an NC_ω-space which is neither a k_ω-space nor pseudocompact.*

To analyze the free topological groups on NC_ω-spaces, it is necessary to establish a number of crucial topological properties of this class of spaces.

Lemma 4.5 *Every NC_ω-space X is normal.*

Proof Let A and B be nonempty closed disjoint subsets of X. Let also $X = \bigcup_{n\in\omega} X_n$ be an NC_ω-decomposition for X.

Take $n_0 \in \omega$ such that both sets $A_{n_0} = A \cap X_{n_0}$ and $B_{n_0} = B \cap X_{n_0}$ are nonempty. Since X_{n_0} is normal, there exist disjoint open sets U_{n_0} and V_{n_0} in X_{n_0} such that $A_{n_0} \subseteq U_{n_0}$, $B_{n_0} \subseteq V_{n_0}$ and $\overline{U}_{n_0} \cap \overline{V}_{n_0} = \emptyset$. Assume that for some $n \geq n_0$ we have defined open sets U_n and V_n in X_n such that $A_n = A \cap X_n \subseteq U_n$, $B_n = B \cap X_n \subseteq V_n$ and $\overline{U}_n \cap \overline{V}_n = \emptyset$. Then $A^*_{n+1} = (A \cap X_{n+1}) \cup \overline{U}_n$ and $B^*_{n+1} = (B \cap X_{n+1}) \cup \overline{V}_n$ are closed disjoint subsets of X_{n+1}. Hence, there exist open sets U_{n+1} and V_{n+1} in X_{n+1} such that $A^*_{n+1} \subseteq U_{n+1}$, $B^*_{n+1} \subseteq V_{n+1}$ and $\overline{U}_{n+1} \cap \overline{V}_{n+1} = \emptyset$. This completes our construction of the sequences $\{U_n : n \geq n_0\}$ and $\{V_n : n \geq n_0\}$.

Let $U = \bigcup_{n\geq n_0} U_n$ and $V = \bigcup_{n\geq n_0} V_n$. Then the sets $U \cap X_n = \bigcup_{k\geq n}(U_k \cap X_n)$ and $V \cap X_n = \bigcup_{k\geq n}(V_k \cap X_n)$ are open in X_n for each $n \in \omega$. Since X is the direct limit of the subspaces X_n, we conclude that U and V are open in X. It is clear from the choice of the sets U_n and V_n that $A \subseteq U$, $B \subseteq V$ and $U \cap V = \emptyset$. Hence, X is normal. □

Corollary 4.6 *Every open F_σ-set in an NC_ω-space is also an NC_ω-space.*

Proof Let O be a nonempty open F_σ-set in an NC_ω-space X. Denote $O = \bigcup_{n\in\omega} F_n$, where each F_n is closed in X. We can assume that $F_n \subseteq F_{n+1}$ for each $n \in \omega$. Let also $X = \bigcup_{n\in\omega} X_n$ be an NC_ω-decomposition for X. For every $n \in \omega$, we put $K_n = F_n \cap X_n$. Clearly, each K_n is an NC-space. Using the normality of X (see Lemma 4.5), one can verify that $O = \bigcup_{n\in\omega} K_n$ is an NC_ω-decomposition for the space O. □

Lemma 4.7 *Let $X = \bigcup_{n\in\omega} X_n$ be an NC_ω-decomposition for a space X. Then for each integer $k \geq 1$, X^k is an NC_ω-space with NC_ω-decomposition $X^k = \bigcup_{n\in\omega} X_n^k$. Hence, every finite power of an NC_ω-space is again an NC_ω-space.*

Proof Since the class of NC_ω-spaces is closed hereditary, it suffices to verify that X^2 is an NC_ω-space with NC_ω-decomposition $X^2 = \bigcup_{n\in\omega} X_n^2$. Notice that each summand X_n^2 is a closed NC-subspace of X^2.

Assume that F is a nonempty subset of X^2 such that $F \cap X_n^2$ is closed in X_n^2 for each $n \in \omega$. We claim that $X^2 \setminus F$ is open in X^2, so F is closed. Take an arbitrary point $x_0 \in X^2 \setminus F$ and choose an integer $n_0 \geq 0$ such that $x_0 \in X_{n_0}^2$ and $F_{n_0} = F \cap X_{n_0}^2 \neq \emptyset$. Since F_{n_0} is closed in $X_{n_0}^2$ and the space $X_{n_0}^2$ is normal, there exist open sets U_{n_0} and V_{n_0} in X_{n_0} such that $x_0 \in U_{n_0} \times V_{n_0}$ and $(\overline{U}_{n_0} \times \overline{V}_{n_0}) \cap F_{n_0} = \emptyset$.

Assume that we have defined sets $U_{n_0}, \dots, U_n$ and $V_{n_0}, \dots, V_n$ for some $n \geq n_0$ such that U_k and V_k are open in X_k and $(\overline{U}_k \times \overline{V}_k) \cap (F \cap X_k^2) = \emptyset$ if $n_0 \leq k \leq n$ and $\overline{U}_k \subseteq U_{k+1}$, $\overline{V}_k \subseteq V_{k+1}$ if $k < n$. Let $F_{n+1} = F \cap X_{n+1}^2$. Since the space X_{n+1}^2 is normal and the closed subsets $\overline{U}_n \times \overline{V}_n$ and F_{n+1} of X_{n+1}^2 are disjoint, we can find open sets U_{n+1} and V_{n+1} in X_{n+1} such that $\overline{U}_n \subseteq U_{n+1}$, $\overline{V}_n \subseteq V_{n+1}$ and $(\overline{U}_{n+1} \times \overline{V}_{n+1}) \cap F_{n+1} = \emptyset$. This completes our construction of the sequences $\{U_n : n_0 \leq n < \omega\}$ and $\{V_n : n_0 \leq n < \omega\}$.

Let $U = \bigcup_{n\geq n_0} U_n$ and $V = \bigcup_{n\geq n_0} V_n$. Then the sets U and V are open in X. Clearly, $x_0 \in U_0 \times V_0 \subseteq U \times V$. It also follows from our construction that $(U_n \times V_n) \cap F_n = \emptyset$ for each $n \geq n_0$. Therefore, $(U \times V) \cap F = \emptyset$. This proves that the set $X^2 \setminus F$ is open in X^2, as claimed. Hence, $X^2 = \bigcup_{n\in\omega} X_n^2$ is an NC_ω-decomposition for X^2. □

Combining Lemmas 4.5 and 4.7 we deduce the following result.

Corollary 4.8 *Every finite power of an NC_ω-space is normal.*

We have to warn the reader that the product of two NC_ω-spaces or even NC-spaces can fail to be normal. To see this, it suffices to take $X = \omega_1$ and $Y = \omega_1 + 1$, where both spaces carry their usual order topology. Then X and Y are NC-spaces, but the product $X \times Y$ is not normal, hence, not a NC_ω-space (see also Remark 6.2).

Proposition 4.9 *Let $f : X \to Y$ be a continuous closed mapping of an NC_ω-space X onto a Hausdorff space Y. Then Y is also an NC_ω-space.*

Proof Let $X = \bigcup_{n\in\omega} X_n$ be an NC_ω-decomposition for X. For every $n \in \omega$, let $Y_n = f(X_n)$. Then Y_n is closed in Y and $Y_n \subseteq Y_{n+1}$, for each $n \in \omega$.

It is known that normality is invariant under continuous closed mappings [5, Theorem 1.5.20]. So each Y_n is an NC-space. Therefore, it suffices to verify that Y is the direct limit of the sequence $\{Y_n : n \in \omega\}$.

Let K be a subset of Y such that $K_n = K \cap Y_n$ is closed in Y_n for each $n \in \omega$. We claim that the intersection $F \cap X_n$ is closed in X for each $n \in \omega$, where $F = f^{-1}(K)$. Indeed, we have the equalities

$$f^{-1}(K_n) \cap X_n = f^{-1}(K) \cap f^{-1}(Y_n) \cap X_n = F \cap X_n.$$

Since K_n is closed in Y, we see that $F \cap X_n$ is closed in X. So F is closed in X because $\bigcup_{n\in\omega} X_n$ is an NC_ω-decomposition for X. This implies that $K = f(F)$ is closed in Y. Therefore, $\bigcup_{n\in\omega} Y_n$ is an NC_ω-decomposition for Y. □

The following key result clarifies the relationship between NC_ω-spaces and k_ω-spaces.

Lemma 4.10 *If X is an NC_ω-space with NC_ω-decomposition $X = \bigcup_{n\in\omega} X_n$, then the Hewitt–Nachbin completion υX is a k_ω-space with k_ω-decomposition $\upsilon X = \bigcup_{n\in\omega} K_n$, where $K_n = cl_{\upsilon X} X_n$ for each $n \in \omega$.*

Proof Every closed subspace of a Hewitt–Nachbin complete space is Hewitt–Nachbin complete [5, Theorem 3.11.4]. Since the closure of a pseudocompact subspace of a space is pseudocompact and a pseudocompact Hewitt–Nachbin complete space is compact [5, Theorem 3.11.1], we conclude that the closure of every pseudocompact subspace of a Hewitt–Nachbin complete space is compact. Therefore, each K_n is a compact subset of υX.

Let τ be the topology of the space υX, that is, the topology υX inherits from βX. Denote by τ^* the finer topology on υX such that F is closed in $X^* = (X, \tau^*)$ iff $F \cap K_n$ is closed in K_n for each $n \in \omega$. In other words, $\bigcup_{n\in\omega} K_n$ is a k_ω-decomposition for X^*. Then X^* is a normal space, by Lemma 4.5 (every k_ω-space is an NC_ω-space). Clearly, X is a dense subspace of X^*.

We claim that X is C-embedded in X^*. Indeed, consider a continuous real-valued function f on X. The space X is normal, by Lemma 4.5. Since X_n is a closed subset of X and the function $f_n = f{\restriction}X_n$ is bounded on the countably compact space X_n, we see that X_n is C^*-embedded in K_n and $K_n \cong \beta X_n$. Hence, f_n extends to a continuous real-valued function g_n on K_n. Since the spaces K_n are Hausdorff, it follows that $g_n{\restriction}K_k = g_k$ whenever $0 \le k < n$. Denote by g the function on the set υX that extends each g_n. Since g_n is continuous for each $n \in \omega$, the function g is continuous on the space X^*. Hence, X is C-embedded in X^*, as claimed.

Because X^* is normal, hence Tychonoff, and X is a C-embedded subspace of X^*, we conclude that X^* is homeomorphic to the subspace υX of βX and so υX is a k_ω-space. This completes the proof. □

We also need a version of Lemma 4.10 for finite powers of X.

Lemma 4.11 *Let X be an NC_ω-space. Then for each integer $k \ge 1$, the identity mapping of X^k onto itself extends to a homeomorphism of $\upsilon(X^k)$ onto $(\upsilon X)^k$.*

Proof For $k = 1$, the required conclusion follows from Lemma 4.10. So we assume that $k > 1$. Let $i_k\colon X^k \to (\beta X)^k$ and $j_k\colon X^k \to \beta(X^k)$ be natural topological embeddings. The identity mapping of X^k onto itself extends to a continuous mapping $f_k\colon \beta(X^k) \to (\beta X)^k$. Then $i_k = f_k \circ j_k$. Let also $X = \bigcup_{n\in\omega} X_n$ be an NC_ω-decomposition for X. In what follows we identify υX and $\upsilon(X^k)$ with the corresponding subspaces of βX and $\beta(X^k)$, respectively. Lemma 4.10 implies that $\upsilon X = \bigcup_{n\in\omega} cl_{\beta X} X_n$ and $\upsilon(X^k) = \bigcup_{n\in\omega} cl_{\beta(X^k)} X_n^k$. In the latter two equalities, the closures in υX and $\upsilon(X^k)$ can be used interchangeably with the closures in βX and $\beta(X^k)$, respectively.

For every $n \in \omega$, the closed subspace X_n^k of the normal space X^k is C^*-embedded, whence it follows that $cl_{\beta(X^k)} X_n^k \cong \beta(X_n^k)$. Since X_n^k is countably compact (hence, pseudocompact), Glicksberg's theorem [5, 3.12.21(c)] implies that $\beta(X_n^k) \cong (\beta X_n)^k$. We conclude that the closure of X_n^k in $\beta(X^k)$ can be identified with the subspace $(\beta X_n)^k$ of $(\beta X)^k$. In fact, one can easily verify that the restriction of f_k to $cl_{\beta(X^k)} X_n^k$ is a homeomorphism onto the subspace $(\beta X_n)^k$ of $(\beta X)^k$—this is just a more precise form of the aforementioned Glicksberg's theorem. Hence, f_k maps $\upsilon(X^k)$ onto the subspace $(\upsilon X)^k$ of $(\beta X)^k$ in a one-to-one fashion.

Take a closed subset F of $\upsilon(X^k)$. Then $F_n = F \cap cl_{\beta(X^k)} X_n^k$ is closed in the compact space $cl_{\beta(X^k)} X_n^k$, so F_n is compact. Hence $f_k(F) \cap (cl_{(\beta X)^k} X_n^k) = f_k(F_n)$ is closed in $cl_{(\beta X)^k} X_n^k = (cl_{\beta X} X_n)^k$, for each $n \in \omega$. Since, by Lemmas 4.7 and 4.10, $(\upsilon X)^k = \bigcup_{n\in\omega} (cl_{\beta X} X_n)^k$ is the k_ω-decomposition for $(\upsilon X)^k$, we see that the image $f_k(F)$ is closed in $(\upsilon X)^k$. We have thus proved that the restriction of f_k to $\upsilon(X^k)$ is a closed continuous bijection of $\upsilon(X^k)$ onto $(\upsilon X)^k$. Therefore, this restriction is a homeomorphism and the spaces $\upsilon(X^k)$ and $(\upsilon X)^k$ are homeomorphic. □

Now we can present one of the main results of this section, which extends Theorem 1.7 to the broader class of NC_ω-spaces and complements its conclusion by making an embedding of $F(X^n)$ to $F(X)$ *closed*. It is worth noting in this respect that neither NC-spaces nor NC_ω-spaces need to be Dieudonné complete, so the free topological group $F(X)$ on an NC_ω-space X is not necessarily Raĭkov complete. In fact, the group $F(X)$ on an NC_ω-space X is Raĭkov complete if and only if X is a k_ω-space. This follows from the facts that the Raĭkov completeness of $F(X)$ implies the Dieudonné completeness of X (see [29, p. 659]) and that every Dieudonné complete NC_ω-space is a k_ω-space (apply Lemma 4.10 and the equality $\upsilon X = \mu X$, where μX is the Dieudonné completion of an NC_ω-space X).

Theorem 4.12 *Let X be an NC_ω-space. Then for every integer $n \geq 1$, $F(X)$ contains a closed subgroup topologically isomorphic to $F(X^n)$.*

Proof Making use of Lemma 4.10, consider X as a dense subspace of the k_ω-space υX. Note that in a pseudocompact space, every discrete family of open subsets is finite. Hence, every discrete family of open sets in X is countable. Also, X is C-embedded in υX. Hence, all the requirements of the Uspenskij's criterion in [29, Theorem 2] are met, and we deduce that $F(X)$ is topologically isomorphic to the subgroup $F(X, \upsilon X)$ of $F(\upsilon X)$ generated by X.

One applies the same argument along with Lemma 4.11 to conclude that for each integer $k \geq 1$, $F(X^k)$ is topologically isomorphic to the subgroup $F\big(X^k, (\upsilon X)^k\big)$ of $F((\upsilon X)^k)$. In particular, $F(X^2)$ is topologically isomorphic to the subgroup $F\big(X^2, (\upsilon X)^2\big)$ of $F((\upsilon X)^2)$.

As in the proof of Theorem 3.5, it suffices to verify that $F(X^2)$ is topologically isomorphic to a closed subgroup of $F(X)$. Let $\varphi\colon F((\upsilon X)^2) \to F(\upsilon X)$ be a continuous homomorphism extending the mapping $f\colon (\upsilon X)^2 \to F(\upsilon X)$, $f(x, y) = xyx$. Since υX and $(\upsilon X)^2$ are k_ω-spaces, it follows from Theorem 1.7 that φ is a topological monomorphism. It also follows from the choice of f and φ that $\varphi\big(F(X^2, (\upsilon X)^2)\big) \subset F(X, \upsilon X)$. Therefore, $\varphi\big(F(X^2, (\upsilon X)^2)\big)$ is a topological and isomorphic copy of the group $F(X^2)$ in $F(X, \upsilon X)$. Finally, since the group $F((\upsilon X)^2)$ is Raĭkov complete, the equality

$$\varphi\big(F(X^2, (\upsilon X)^2)\big) = \varphi\big(F((\upsilon X)^2)\big) \cap F(X, \upsilon X)$$

implies that this copy is closed in $F(X, \upsilon X) \cong F(X)$. □

5 The embedding theorem fails for a countably compact space

The main result of this section is Theorem 5.4 stating that the free topological group $F(Z)$ does not contain an isomorphic topological copy of $F(Z^2)$ for some countably compact space

Z. Its proof makes use of the spaces S and T from the following simple proposition that has its roots in [3, Theorem 1] and [9, Example 8].

Proposition 5.1 *There exist countably compact separable Tychonoff spaces S and T without isolated points such that the product $S \times T$ is not pseudocompact.*

Proof Let X and Y be subspaces of $\beta\omega$ such that $|X| = |Y| = \mathfrak{c}$, $X \cap Y = \omega$, and the spaces X^ω, Y^ω are countably compact—one can take X and Y as in [9, Lemma 5]. It follows from $X \cap Y = \omega$ that both X and Y are separable and that the product $X \times Y$ is not pseudocompact since the set $\{(n, n) : n \in \omega\}$ does not have accumulation points in $X \times Y$. The spaces X and Y are zero-dimensional as subspaces of $\beta\omega$.

Let $S = X^\omega$ and $T = Y^\omega$. It is clear that the spaces S and T are countably compact, separable, and do not have isolated points. Since X is a continuous image of S and Y is a continuous image of T, we see that $X \times Y$ is a continuous image of $S \times T$. Hence, the space $S \times T$ cannot be pseudocompact. □

Remark 5.2 One can easily verify that the spaces S in T in Proposition 5.1 have a stronger property, namely, $K \times L$ is not pseudocompact whenever K and L are regular closed subsets of S and T, respectively. Applying the construction of W. Comfort and J. van Mill in [3, Theorem 1.1], one can obtain a weaker version of the latter fact, with *pseudocompact* spaces S and T such that $K \times L$ is not pseudocompact for any nonempty regular closed subsets $K \subset S$ and $L \subset T$.

It is also worth mentioning that answering a question in [3], E. Reznichenko constructed in [24] *homogeneous* spaces S and T such that S^ω and T^ω are countably compact and the product $S \times T$ is not pseudocompact.

We say that a subset B of a space X is *bounded* in X if every continuous real-valued function defined on X is bounded on B. Hence, X is bounded in itself iff it is pseudocompact. A subset Y of X is *σ-bounded* in X if Y is the union of countably many bounded subsets of X (see [1, p. 400]).

As usual, we use βX and μX to denote the Stone–Čech and Dieudonné completion of a space X, respectively.

Theorem 5.3 *Let X and Y be spaces such that $F(Y)$ is topologically isomorphic to a subgroup of $F(X)$. If X is pseudocompact, then the space Y is σ-bounded and there exists a topological monomorphism of $F(\mu Y)$ to $F(\beta X)$.*

Proof Let $\varphi\colon F(Y) \to F(X)$ be a topological monomorphism. Let also $P = \varphi(Y)$ and $P_n = P \cap F_n(X)$, where $n \in \omega$. Then $P = \bigcup_{n\in\omega} P_n$. We claim that each P_n is precompact in $F(X)$.

Every pseudocompact subspace of a topological group is precompact in the group (this follows from [26, page 154, Statement A]). In particular, X is a precompact subset of $F(X)$. Hence the subset $X^* = X \cup \{e\} \cup X^{-1}$ of $F(X)$ is also precompact (we apply Corollary 3.7.11 of [1] here). The product of two (equivalently, finitely many) precompact sets in a topological group is again precompact [1, Corollary 3.7.11]. Since

$$F_k(X) = \underbrace{X^* \cdots X^*}_{k \text{ times}}$$

(the product on the right side of the equality is taken in the free group $F(X)$), we conclude that $F_k(X)$ and its subset P_k are precompact in $F(X)$. This proves our claim.

Denote by G the image $\varphi(F(Y))$ considered as a subgroup of $F(X)$. Since $P_k \subset P \subset G$ and $G \subset F(X)$, it follows that P_k is a precompact subset of the group G (the property of being a precompact subset is independent of whether the subset is related to the ambient group $F(X)$ or its subgroup G). Hence, the subset $\varphi^{-1}(P_k)$ of Y is precompact in $F(Y)$. According to [1, Lemma 7.5.2] the latter implies that $\varphi^{-1}(P_k)$ is bounded in Y. The equality $Y = \bigcup_{k\in\omega} \varphi^{-1}(P_k)$ enables us to conclude that the space Y is σ-bounded.

Using Corollary 2.2, we identify $F(X)$ with the subgroup $F(X, \beta X)$ of $F(\beta X)$ algebraically generated by the subset X of βX.

Every continuous pseudometric on Y extends to a continuous pseudometric on μY. Hence, by Theorem 2.1, $F(Y)$ is a dense topological subgroup of the group $F(\mu Y)$.

Since the group $F(\beta X)$ is Raĭkov complete, the topological monomorphism $\varphi\colon F(Y) \to F(X) \subset F(\beta X)$ admits an extension to a continuous homomorphism $\psi\colon F(\mu Y) \to F(\beta X)$. Then ψ is also a topological monomorphism, according to [1, Corollary 3.6.18]. This completes the proof of the theorem. □

Below we apply the necessary condition for the embeddability of $F(Y)$ in $F(X)$, presented in Theorem 5.3 for a pseudocompact space X. The next theorem, which is one of the main results of our article, demonstrates that Theorem 3.5 is quite precise.

Theorem 5.4 *There exists a countably compact separable space Z such that $F(Z^2)$ does not embed as a topological subgroup into $F(Z)$.*

Proof Let S and T be as in Proposition 5.1 and $Z = S \oplus T$ be the topological sum of S and T. We recall that $S = X^\omega$ and $T = Y^\omega$, where X and Y are (countably compact) subspaces of $\beta\omega$ satisfying $X \cap Y = \omega$. Note that all the spaces X, Y, S and T are countably compact and separable, and so is Z. By Theorem 5.3, it suffices to verify that the space Z^2 is not σ-bounded.

Suppose for a contradiction that Z^2 is σ-bounded. Then $S \times T$ is also σ-bounded as a clopen subset of Z^2. Let $S \times T = \bigcup_{n\in\omega} B_n$, where each B_n is a bounded subset of Z^2 and, hence, of $S \times T$.

For every $k \in \omega$, denote by p_k the projection of $(\beta\omega)^\omega$ to the kth factor $\beta\omega_{(k)}$. Since $p_k(S) \cap p_k(T) = X \cap Y = \omega$ for each $k \in \omega$ and the set $\Delta = \{(n, n) : n \in \omega\}$ is discrete and clopen in $X \times Y$, the intersection $\Delta \cap (p_k \times p_k)(B_n)$ is finite for each $n \in \omega$. Otherwise, this intersection would be unbounded in $X \times Y$, thus contradicting the fact that a continuous image of a bounded set is bounded in the codomain, $X \times Y$ in our case.

Applying a diagonal argument we choose, for every $k \in \omega$, an integer $n_k \in \omega$ such that $(n_k, n_k) \notin (p_k \times p_k)(B_k)$. Then the point $x = (n_k)_{k\in\omega} \in \omega^\omega$ is in $(S \times T) \setminus \bigcup_{k\in\omega} B_k$, which is a contradiction. Therefore, neither $S \times T$ nor Z^2 is σ-bounded. This implies the conclusion of the theorem, as was previously stated. □

6 Remarks and problems

In Sect. 3, the validity of Nickolas' result (Theorem 1.7) has been extended in different ways by Theorems 3.5 and 4.12. It is only natural to seek a common generalization for the latter two theorems. We suggest a candidate for this type of generalization.

Problem 6.1 *Let $X = \bigcup_{k\in\omega} X_k$ be a Tychonoff space, where $X_k \subset X_{k+1}$ and each X_k is a closed C^*-embedded subspace of X. Assume also that X_k is an FP-space for each $k \in \omega$. Does $F(X)$ contain a (closed) copy of the group $F(X^n)$, for each integer $n \geq 1$?*

Remark 6.2 According to [28, Theorem 3.13], the free topological group $F(X)$ of a pseudocompact space X is the inductive limit of its closed subspaces $F_n(X)$, with $n \in \omega$, if and only if X is an NC-space. Let $X = \omega_1$ and $Y = \omega_1 + 1$ be the spaces of ordinals that carry the usual order topology. Clearly, both X and Y are NC-spaces. The topological sum $L = X \oplus Y$ of X and Y is normal, countably compact and locally compact, but it fails to be an NC-space. Indeed, it is well-known that $\beta X \cong Y$. Also, the square $L \times L$ contains the closed subspace $X \times Y$. The latter space is not normal by the Tamano theorem [5, Theorem 5.1.38] because X is not paracompact. Therefore, the square $L \times L$ is not normal either. We conclude that L is not an NC-space and that the group $F(L)$ is not the the inductive limit of its closed subspaces $F_n(L)$. We see, in particular, that the class of NC-spaces is not finitely productive (take the product $X \times Y$). A similar conclusion was obtained in [28, Theorem 4.1] by means of a considerably longer argument.

Notice that L is an FP-space because it is pseudocompact and locally compact. Hence Corollary 3.6 applies to the space L.

Remark 6.3 Let X be an ordinal space $[0, \alpha)$, where α is an infinite ordinal. Lemma 4.3 together with Theorem 4.12 imply that for each integer $n \geq 1$, $F(X)$ contains a closed subgroup topologically isomorphic to $F(X^n)$. However, this statement is meaningful only for the uncountable ordinals α. In fact, it follows from [2, Theorem 3.1] that $F(X)$ and $F(X^n)$ are topologically isomorphic for each integer $n \geq 1$ and every *countable* ordinal α. If, for instance, $\alpha = \omega_1$ or $\alpha = \omega_1 + 1$, then $F(X)$ and $F(X^n)$ are not topologically isomorphic, by the results of [8], so the embedding of $F(X^n)$ into $F(X)$ indeed makes sense.

It has been noticed by many authors that the free topological groups in the sense of Markov or Graev share many topological and algebraic properties. Denote the free topological group in the sense of Graev by $FG(X)$. It is known that for every X, the free topological group in the sense of Markov $F(X)$ is isomorphic to the free topological group in the sense of Graev $FG(Y)$, where Y is obtained from X by adding the isolated point e, the identity element of $FG(Y)$ (see [1, Exercise 7.1(b)]).

It is also known that for every Tychonoff space X, the Graev free topological group $FG(X)$ is a topological subgroup of the Markov free topological group $F(X)$, see [17]. We do not know whether the converse holds true:

Problem 6.4 *Is $F(X)$ topologically isomorphic to a subgroup of $FG(X)$?*

Remark 6.5 All the main results of the article are valid for embeddings of the free topological groups in the sense of Graev, regardless of the solution to Problem 6.4. It is known or can be easily verified that *all* theorems on Markov free topological groups involved in the proofs of Theorems 3.5, 4.12 and 5.4 are valid for Graev free topological groups.

Problem 6.6 *Let X be an arbitrary Tychonoff space. Let $\varphi\colon X \to F(X)$ be a mapping defined by $\varphi(x) = x^n$ for each $x \in X$, where $n \geq 2$ is a natural number. Is it true that the extension of φ to a continuous homomorphism $\widetilde{\varphi}\colon F(X) \to F(X)$ is a topological isomorphism of $F(X)$ onto a closed subgroup of $F(X)$?*

A very well-known fact says that the free algebraic group on two generators $\mathbb{F}_2$ contains an isomorphic copy of the free algebraic group $\mathbb{F}_\infty$ on a countably infinite set of generators. Consequently, the two problems that follow can be viewed as topological versions of this purely algebraic result. In the first of them, we propose to generalize Corollary 3.3.

Problem 6.7 *It is true that for every Tychonoff space X with $|X| \geq 2$, the group $F(X)$ contains a (closed) subgroup topologically isomorphic to $F(X \oplus X)$ (or $F(X \times \mathbb{N})$, where $\mathbb{N}$ carries the discrete topology)?*

Problem 6.8 *Let X be a space containing a nontrivial convergent sequence $\mathbb{S}$. Does $F(X)$ contains a (closed) subgroup topologically isomorphic either to $F(X \oplus \mathbb{S})$ or $F(X \times \mathbb{S})$? What happens if one replaces $\mathbb{S}$ with the closed unit interval $[0, 1]$?*

Proposition 3.2 implies an affirmative answer to Problem 6.8 in the special case of a pseudocompact or k_ω-space X.

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Declarations

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