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ORIGINAL PAPER

Short ($\mathbf{SL}_2 \times \mathbf{SL}_2$)-structures on Lie algebras

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Abstract

S-structures on Lie algebras, introduced by Vinberg, represent a broad generalization of the notion of gradings by abelian groups. Gradings by, not necessarily reduced, root systems provide many examples of natural **S**-structures. Here we deal with a situation not covered by these gradings: the short ($\mathbf{SL}_2 \times \mathbf{SL}_2$)-structures, where the reductive group is the simplest semisimple but not simple reductive group. The algebraic objects that coordinatize these structures are the J-ternary algebras of Allison, endowed with a nontrivial idempotent.

Keywords S-structures · J-ternary algebras · Structurable algebras

Mathematics Subject Classification Primary 17B70

1 Introduction

All the algebras considered will be defined over an arbitrary ground field $\mathbb{F}$ *of characteristic* $\neq 2, 3$. *Tensor products over* $\mathbb{F}$ *will simply be written as* $\otimes$, *instead of* $\otimes_{\mathbb{F}}$. *Algebraic groups over* $\mathbb{F}$ *will be understood in the sense of affine group schemes of finite type.*

A grading on a Lie algebra $\mathcal{L}$ by an abelian group G is determined by a homomorphism $\mathbf{D}(G) \to \mathbf{Aut}(\mathcal{L})$, where $\mathbf{D}(G)$ is the diagonalizable (and hence reductive) group scheme whose representing Hopf algebra is the group algebra $\mathbb{F}G$ (see, e.g., [15, Chapter 1]).

In memory of Georgia Benkart.

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Vinberg considered a large extension of this idea in his paper entitled *Non-abelian gradings of Lie algebras* [27], by substituting the diagonalizable groups above by arbitrary reductive groups.

Definition 1.1 [27, Definition 0.1] Let $\mathbf{S}$ be a reductive algebraic group and let $\mathcal{L}$ be a Lie algebra. An $\mathbf{S}$-*structure* on $\mathcal{L}$ is a homomorphism $\Phi : \mathbf{S} \to \mathbf{Aut}(\mathcal{L})$ from $\mathbf{S}$ into the algebraic group of automorphisms of $\mathcal{L}$.

Actually, this definition makes sense for nonassociative algebras, or more general algebraic systems, not just for Lie algebras.

Let $\mathbf{S}$ be a reductive algebraic group and let $\mathfrak{s}$ be its Lie algebra, the differential $\mathrm{d}\Phi$ of an $\mathbf{S}$-structure on the Lie algebra $\mathcal{L}$ is a Lie algebra homomorphism $\mathrm{d}\Phi : \mathfrak{s} \to \mathrm{Der}(\mathcal{L})$.

Definition 1.2 With the notations above, the $\mathbf{S}$-structure $\Phi : \mathbf{S} \to \mathbf{Aut}(\mathcal{L})$ on the Lie algebra $\mathcal{L}$ is said to be *inner* if there is a one-to-one Lie algebra homomorphism $\iota : \mathfrak{s} \hookrightarrow \mathcal{L}$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{s} & \overset{\iota}{\hookrightarrow} & \mathcal{L} \\ & {\scriptstyle \mathrm{d}\Phi}\searrow & \downarrow{\scriptstyle \mathrm{ad}} \\ & & \mathrm{Der}(\mathcal{L}) \end{array} \tag{1.1}$$

Note that if $\Phi : \mathbf{S} \to \mathbf{Aut}(\mathcal{L})$ is a non inner $\mathbf{S}$-structure on the Lie algebra $\mathcal{L}$, then we can take the split extensions $\tilde{\mathcal{L}} = \mathcal{L} \oplus \mathfrak{s}$ as in [22, p. 18], where $\mathfrak{s}$ acts on $\mathcal{L}$ through $\mathrm{d}\Phi$, and this is endowed with a natural $\mathbf{S}$-structure. Hence it is not harmful to restrict to inner $\mathbf{S}$-structures.

Definition 1.1 is too general, so some restrictions on $\mathcal{L}$ as a module for the reductive group $\mathbf{S}$ must be imposed. Vinberg himself considered in [27] two different situations:

- A nontrivial $\mathbf{SL}_2$-structure on a Lie algebra $\mathcal{L}$ is called *very short* if $\mathcal{L}$ decomposes, as a module for $\mathbf{SL}_2$, as a sum of copies of the adjoint module and of the trivial one-dimensional module. Collecting isomorphic submodules, a very short $\mathbf{SL}_2$-structure gives an isotypic decomposition of the form

  $$\mathcal{L} = (\mathfrak{sl}_2 \otimes \mathcal{J}) \oplus \mathcal{D}$$

  and it turns out that the Lie bracket on $\mathcal{L}$ induces a Jordan product on $\mathcal{J}$. The subalgebra $\mathcal{D}$ acts by derivations on $\mathcal{J}$. All this goes back to [26].
- A nontrivial $\mathbf{SL}_3$-structure on a Lie algebra $\mathcal{L}$ is called *short* if $\mathcal{L}$ decomposes as the direct sum of one copy of the adjoint representations, copies of its natural three-dimensional module and of its dual, and copies of the trivial representation, so that the isotypic decomposition is:

  $$\mathcal{L} = \mathfrak{sl}_3 \oplus (V \otimes \mathcal{J}) \oplus (V^* \otimes \mathcal{J}') \oplus \mathcal{D}.$$

  For simple $\mathcal{L}$, $\mathcal{J}$ and $\mathcal{J}'$ may be identified, and inherit a structure of a cubic Jordan algebra (see [27, Eq. (31)]). (A more general situation was considered in [7].)

Stasenko [25] has recently considered *short* $\mathbf{SL}_2$-*structures* on simple Lie algebras over the complex numbers (see Definition 2.1). Although not with this terminology, the short $\mathbf{SL}_2$-structures were considered in [18]. They are intimately related to the J-ternary algebras of Allison [1], a connection that will be reviewed in Sect. 2.

The reader should note that there is no general definition of short and very short structures. The definitions depend on the algebraic group $\mathbf{S}$ used (and on the inspiration of the different authors to find a suitable name).

Another important source of nice $\mathbf{S}$-structures is provided by the gradings by root systems, initially considered by Berman and Moody [10] (see [9] and the references therein).

In the simply-laced case, a Lie algebra graded by such a root system contains a finite-dimensional split simple Lie algebra $\mathfrak{s}$ with such a root system, and it decomposes, as a module for $\mathfrak{s}$, as a direct sum of copies of the adjoint module and the trivial module, so the corresponding isotypic decomposition has two components: $\mathcal{L} = (\mathfrak{s} \otimes \mathcal{A}) \oplus \mathcal{D}$. If $\mathbf{S}$ is the simply connected group with Lie algebra $\mathfrak{s}$, the action of $\mathfrak{s}$ integrates to an $\mathbf{S}$-structure on $\mathcal{L}$.

In the non simply-laced case, the isotypic decomposition also includes copies of the irreducible module for $\mathfrak{s}$ whose highest weight is the highest short root: $\mathcal{L} = (\mathfrak{s} \otimes \mathcal{A}) \oplus (W \otimes \mathcal{B}) \oplus \mathcal{D}$. Finally, the case of the nonreduced root systems BC_r gives isotypic decompositions with four components, with two exceptions of five components (see [6, 9]): BC_1-graded Lie algebras with grading subalgebra of type D_1 (which reduces to 5-gradings), and BC_2-graded Lie algebras with grading subalgebra of type $D_2 = A_1 \times A_1$.

Another nice class of $\mathbf{S}$-structures has been given in [12], where the exceptional simple Lie algebras of type E_r, $r = 7, 8$, are shown to be endowed with a $\mathbf{SL}_2^r$-structure, such that the irreducible modules that appear are single copies of the adjoint modules for the factors of $\mathbf{SL}_2^r$, tensor products of the two-dimensional natural modules for some of these factors, and trivial modules. This allows us to coordinatize these exceptional Lie algebras in terms of some algebras related to some well-known binary codes.

The goal of this paper is to explore a new kind of $\mathbf{S}$-structures not covered by the results mentioned above and where the reductive group is not simple: the short ($\mathbf{SL}_2 \times \mathbf{SL}_2$)-structures defined in Sect. 3. These structures give isotypic decompositions with six components, because as a module for $\mathbf{SL}_2 \times \mathbf{SL}_2$, the Lie algebra decomposes as a direct sum of copies of the adjoint modules for any of the two factors of $\mathbf{SL}_2 \times \mathbf{SL}_2$, copies of the two-dimensional natural modules for each of these factors, copies of the tensor product of these two natural modules, and copies of the trivial module. [See (3.1).]

In particular, if we fix two of the factors $\mathbf{SL}_2$ of the $\mathbf{SL}_2^r$-structures in [12], a short ($\mathbf{SL}_2 \times \mathbf{SL}_2$)-structure is obtained.

Trying to obtain directly the properties and multilinear operations among the six different components in (3.1) from the Lie bracket on the Lie algebra is quite cumbersome and not much illuminating. However, as shown in Sect. 3, the diagonal embedding $\mathbf{SL}_2 \to \mathbf{SL}_2 \times \mathbf{SL}_2$ induces a short $\mathbf{SL}_2$-structure from any short ($\mathbf{SL}_2 \times \mathbf{SL}_2$)-structure. This will be the clue to describe the short ($\mathbf{SL}_2 \times \mathbf{SL}_2$)-structures on a Lie algebra in terms of J-ternary algebras endowed with an idempotent (Theorem 3.3).

Section 4 will be devoted to show examples of J-ternary algebras with idempotents and the corresponding short ($\mathbf{SL}_2 \times \mathbf{SL}_2$)-structures on certain Lie algebras, and Sect. 5 will be devoted to using the results in [25] to describe, in terms of the examples in Sect. 4, all short ($\mathbf{SL}_2 \times \mathbf{SL}_2$)-structures on the finite-dimensional simple Lie algebras over an algebraically closed field of characteristic 0.

In concluding this introduction, let us mention that Lie algebras $\mathcal{L}$ endowed with a homomorphism $\Phi : \mathsf{G} \to \mathbf{Aut}(\mathcal{L})$, where G is a constant group scheme, have been considered too in the literature and shown to be "coordinatized" by some nonassociative algebras (see, e.g., [16–18]).

2 Short $\mathbf{SL}_2$-structures and J-ternary algebras

Short $\mathbf{SL}_2$-structures on simple Lie algebras over the complex numbers have been considered by Stasenko [25, Definition 2]. Hence we extend his definition over arbitrary fields.

Definition 2.1 An $\mathbf{SL}_2$-structure $\Phi : \mathbf{SL}_2 \to \mathbf{Aut}(\mathcal{L})$ on a Lie algebra $\mathcal{L}$ is said to be *short* if $\mathcal{L}$ decomposes, as a module for $\mathbf{SL}_2$ via Φ, into a direct sum of copies of the adjoint, natural, and trivial modules.

Therefore, the isotypic decomposition of $\mathcal{L}$ allows us to describe $\mathcal{L}$ as follows:

$$\mathcal{L} = \big(\mathfrak{sl}(V) \otimes \mathcal{J}\big) \oplus \big(V \otimes \mathcal{T}\big) \oplus \mathcal{D}, \tag{2.1}$$

for vector spaces $\mathcal{J}$, $\mathcal{T}$, and $\mathcal{D}$, where V is the natural two-dimensional representation of $\mathbf{SL}_2 \simeq \mathbf{SL}(V)$. The action of $\mathbf{SL}_2$ is given by the adjoint action of $\mathbf{SL}_2$ on $\mathfrak{sl}(V)$, its natural action on V, and the trivial action on $\mathcal{J}$, $\mathcal{T}$ and $\mathcal{D}$. The subspace $\mathcal{D}$, being the subspace of fixed elements by $\mathbf{SL}_2$, is a subalgebra of $\mathcal{L}$.

This section is devoted to reviewing the connection between short $\mathbf{SL}_2$-structures on a Lie algebra and the J-ternary algebras of Allison [1], developed in [18].

The following well known results will be needed. See, e.g., [18, Lemma 2.1].

Lemma 2.2 *Let V be a two-dimensional vector space.*

(i) *The space* $\mathrm{Hom}_{\mathfrak{sl}(V)}\big(\mathfrak{sl}(V) \otimes \mathfrak{sl}(V), \mathfrak{sl}(V)\big)$ *of* $\mathfrak{sl}(V)$*-invariant linear maps* $\mathfrak{sl}(V) \otimes \mathfrak{sl}(V) \to \mathfrak{sl}(V)$ *is spanned by the (skew-symmetric) Lie bracket:*

$$f \otimes g \mapsto [f, g] = fg - gf.$$

(ii) *The space* $\mathrm{Hom}_{\mathfrak{sl}(V)}\big(\mathfrak{sl}(V) \otimes \mathfrak{sl}(V), \mathbb{F}\big)$ *is spanned by the trace map:*

$$f \otimes g \mapsto \mathrm{tr}(fg).$$

(iii) *The space* $\mathrm{Hom}_{\mathfrak{sl}(V)}\big(\mathfrak{sl}(V) \otimes V, V\big)$ *is spanned by the natural action:*

$$f \otimes v \mapsto f(v).$$

(iv) *The space* $\mathrm{Hom}_{\mathfrak{sl}(V)}\big(V \otimes V, \mathbb{F}\big)$ *is one-dimensional. Its nonzero elements are of the form*

$$u \otimes v \mapsto (u \mid v),$$

for a nonzero skew-symmetric bilinear form $(. \mid .)$ *on* V.

(v) *The space* $\mathrm{Hom}_{\mathfrak{sl}(V)}\big(V \otimes V, \mathfrak{sl}(V)\big)$ *is one-dimensional. Once a nonzero skew-symmetric bilinear form* $(. \mid .)$ *is fixed on* V*, this subspace is spanned by the following symmetric map:*

$$u \otimes v \mapsto \gamma_{u,v}\big(: w \mapsto (u \mid w)v + (v \mid w)u\big).$$

(vi) *The spaces* $\mathrm{Hom}_{\mathfrak{sl}(V)}\big(\mathfrak{sl}(V) \otimes \mathfrak{sl}(V), V\big)$, $\mathrm{Hom}_{\mathfrak{sl}(V)}\big(\mathfrak{sl}(V) \otimes V, \mathfrak{sl}(V)\big)$, $\mathrm{Hom}_{\mathfrak{sl}(V)}\big(\mathfrak{sl}(V) \otimes V, \mathbb{F}\big)$*, and* $\mathrm{Hom}_{\mathfrak{sl}(V)}\big(V \otimes V, V\big)$ *are all trivial.*

Moreover, $\mathrm{Hom}_{\mathfrak{sl}(V)}$ *may be replaced by* $\mathrm{Hom}_{\mathbf{SL}(V)}$ *all over.*

Remark 2.3 Lemma 2.2 is even valid in characteristic 3. Actually, if the characteristic of our ground field $\mathbb{F}$ is 3, then [8, Theorem 1.11] gives the following isomorphisms of $\mathfrak{sl}(V)$-modules:

$$V \otimes V \simeq \mathfrak{sl}(V) \oplus \mathbb{F}, \quad \mathfrak{sl}(V) \otimes V \simeq Q(1), \quad \mathfrak{sl}(V) \otimes \mathfrak{sl}(V) \simeq \mathfrak{sl}(V) \oplus Q(0),$$

where $Q(1)$ is an indecomposable module with a unique maximal submodule W such that $Q(1)/W$ is isomorphic to V, and $Q(0)$ is an indecomposable module with a unique maximal submodule W such that $Q(0)/W$ is isomorphic to the trivial module $\mathbb{F}$. It follows at once that all the spaces in items **(i)–(v)** are one-dimensional, and those in item **(vi)** are trivial.

Let us return to our assumption that the characteristic of the ground field is $\neq 2, 3$. *This assumption will be kept throughout the paper with no further mention.*

From now on, we will fix a nonzero skew-symmetric bilinear form $(. \mid .)$ on our two-dimensional vector space V.

Let $\mathcal{L}$ be a Lie algebra with an inner $\mathbf{SL}_2$-structure and isotypic decomposition as in (2.1). The $\mathbf{SL}_2$-structure being inner forces $\mathcal{J}$ to contain a distinguished element 1, such that $\mathfrak{sl}(V) \otimes 1$ is the image of ι in (1.1).

The $\mathbf{SL}_2$-invariance or, equivalently, the $\mathfrak{sl}(V)$-invariance, of the Lie bracket in our Lie algebra $\mathcal{L}$ gives, for any $f, g \in \mathfrak{sl}(V)$, $u, v \in V$, $a, b \in \mathcal{J}$, $x, y \in \mathcal{T}$, and $D \in \mathcal{D}$, the following conditions:

$$\begin{aligned} [f \otimes a, g \otimes b] &= [f, g] \otimes a \cdot b + 2\mathrm{tr}(fg) D_{a,b}, \\ [f \otimes a, u \otimes x] &= f(u) \otimes a \bullet x, \\ [u \otimes x, v \otimes y] &= \gamma_{u,v} \otimes \langle x \mid y \rangle + \big(u \mid v\big) d_{x,y}, \\ [D, f \otimes a] &= f \otimes D(a), \\ [D, u \otimes x] &= u \otimes D(x), \end{aligned} \tag{2.2}$$

for suitable bilinear maps

$$\begin{aligned} &\mathcal{J} \times \mathcal{J} \to \mathcal{J}: \quad (a, b) \mapsto a \cdot b \quad \text{(symmetric)}, \\ &\mathcal{J} \times \mathcal{J} \to \mathcal{D}: \quad (a, b) \mapsto D_{a,b} \quad \text{(skew-symmetric)}, \\ &\mathcal{J} \times \mathcal{T} \to \mathcal{T}: \quad (a, x) \mapsto a \bullet x, \\ &\mathcal{T} \times \mathcal{T} \to \mathcal{J}: \quad (x, y) \mapsto \langle x \mid y \rangle \quad \text{(skew-symmetric)}, \\ &\mathcal{T} \times \mathcal{T} \to \mathcal{D}: \quad (x, y) \mapsto d_{x,y} \quad \text{(symmetric)}, \\ &\mathcal{D} \times \mathcal{J} \to \mathcal{J}: \quad (D, a) \mapsto D(a), \\ &\mathcal{D} \times \mathcal{T} \to \mathcal{T}: \quad (D, x) \mapsto D(x), \end{aligned} \tag{2.3}$$

such that

$$1 \cdot a = a, \quad D_{1,a} = 0, \quad \text{and} \quad 1 \bullet x = x. \tag{2.4}$$

The Jacobi identity on $\mathcal{L}$ also shows that all these maps are invariant under the action of the Lie subalgebra $\mathcal{D}$. The next result summarizes the properties of these maps:

Theorem 2.4 [18, Theorem 2.2] *A Lie algebra $\mathcal{L}$ is endowed with an inner short $\mathbf{SL}_2$-structure if and only if there is a two-dimensional vector space V such that $\mathcal{L}$ is, up to isomorphism, the Lie algebra in* (2.1)*, with Lie bracket given in* (2.2)*, for suitable bilinear maps given in* (2.3)*, satisfying the following conditions:*

- *$\mathcal{J}$ is a unital Jordan algebra with the multiplication $a \cdot b$.*

- *$\mathcal{T}$ is a special unital Jordan module for $\mathcal{J}$ with the action $a \bullet x$. That is, the map $\mathcal{J} \to \mathrm{End}(\mathcal{T})^{(+)}$, given by $a \mapsto (x \mapsto a \bullet x)$, is a homomorphism of unital Jordan algebras. (For an associative algebra $\mathcal{A}$, $\mathcal{A}^{(+)}$ denotes the special Jordan algebra with multiplication $a \cdot b = \frac{1}{2}(ab + ba)$.)*
  In other words, the following equation holds for $a, b \in \mathcal{J}$ and $x \in \mathcal{T}$:
$$(a \cdot b) \bullet x = \frac{1}{2}\big(a \bullet (b \bullet x) + b \bullet (a \bullet x)\big). \tag{2.5}$$
- *For any $a, b, c \in \mathcal{J}$ and $x, y, z \in \mathcal{T}$, the following identities hold:*
$$D_{a,b}(c) = a \cdot (b \cdot c) - b \cdot (a \cdot c), \tag{2.6}$$
$$D_{a\cdot b,c} + D_{b\cdot c,a} + D_{c\cdot a,b} = 0, \tag{2.7}$$
$$4D_{a,b}(x) = a \bullet (b \bullet x) - b \bullet (a \bullet x), \tag{2.8}$$
$$4D_{a,\langle x|y\rangle} = -d_{a\bullet x,y} + d_{x,a\bullet y}, \tag{2.9}$$
$$2a \cdot \langle x \mid y\rangle = \langle a \bullet x \mid y\rangle + \langle x \mid a \bullet y\rangle, \tag{2.10}$$
$$d_{x,y}(a) = \langle a \bullet x \mid y\rangle - \langle x \mid a \bullet y\rangle, \tag{2.11}$$
$$d_{x,y}(z) - d_{z,y}(x) = \langle x \mid y\rangle \bullet z - \langle z \mid y\rangle \bullet x + 2\langle x \mid z\rangle \bullet y. \tag{2.12}$$
- *For any $D \in \mathcal{D}$, the linear endomorphism of $\mathcal{J} \oplus \mathcal{T}$, given by $a + x \mapsto D(a) + D(x)$, for $a \in \mathcal{J}$ and $x \in \mathcal{T}$, is an even derivation of the $\mathbb{Z}/2$-graded algebra with even part $\mathcal{J}$, odd part $\mathcal{T}$, and multiplication given by the formula:*
$$(a + x) \diamond (b + y) = \big(a \cdot b + \langle x \mid y\rangle\big) + \big(a \bullet y + b \bullet x\big), \tag{2.13}$$
  for $a, b \in \mathcal{J}$ and $x, y \in \mathcal{T}$. □

As remarked in [18], all this is strongly related to the J-ternary algebras considered by Allison [1].

Definition 2.5 [6, (3.12)] Let $\mathcal{J}$ be a unital Jordan algebra with multiplication $a \cdot b$, for $a, b \in \mathcal{J}$. Let $\mathcal{T}$ be a unital special Jordan module for $\mathcal{J}$ with action $a \bullet x$ for $a \in \mathcal{J}$ and $x \in \mathcal{T}$. Assume $\langle . \mid .\rangle : \mathcal{T} \times \mathcal{T} \to \mathcal{J}$ is a skew-symmetric bilinear map and $(., ., .) : \mathcal{T} \times \mathcal{T} \times \mathcal{T} \to \mathcal{T}$ is a trilinear product on $\mathcal{T}$. Then the pair $(\mathcal{J}, \mathcal{T})$ is called a *J-ternary algebra* if the following axioms hold for any $a \in \mathcal{J}$ and $x, y, z, w, v \in \mathcal{T}$:

(JT1) $a \cdot \langle x \mid y\rangle = \frac{1}{2}\big(\langle a \bullet x \mid y\rangle + \langle x \mid a \bullet y\rangle\big)$,
(JT2) $a \bullet (x, y, z) = (a \bullet x, y, z) - (x, a \bullet y, z) + (x, y, a \bullet z)$,
(JT3) $(x, y, z) = (z, y, x) - \langle x \mid z\rangle \bullet y$,
(JT4) $(x, y, z) = (y, x, z) + \langle x \mid y\rangle \bullet z$,
(JT5) $\langle (x, y, z) \mid w\rangle + \langle z \mid (x, y, w)\rangle = \langle x \mid \langle z \mid w\rangle \bullet y\rangle$,
(JT6) $(x, y, (z, w, v)) = ((x, y, z), w, v) + (z, (y, x, w), v) + (z, w, (x, y, v))$.

Theorem 2.4 in [18] becomes now the following result:

Theorem 2.6 *Let $\mathcal{L}$ be a Lie algebra endowed with a short inner $\mathbf{SL}_2$-structure with isotypic decomposition in (2.1). Then the pair $(\mathcal{J}, \mathcal{T})$ is a J-ternary algebra with the triple product on $\mathcal{T}$ given by the next formula:*
$$(x, y, z) = \frac{1}{2}\Big(-d_{x,y}(z) + \langle x \mid y\rangle \bullet z\Big), \tag{2.14}$$
for all $x, y, z \in \mathcal{T}$, where $\langle\,|\,\rangle$, $\bullet$, and $d_{x,y}$ are defined as in (2.2) and (2.3).

Conversely, if $(\mathcal{J}, \mathcal{T})$ *is a* J*-ternary algebra with bilinear maps* $a \cdot b$*,* $a \bullet x$*,* $\langle x \mid y\rangle$*, and trilinear map* (x, y, z)*, as in Definition 2.5, then the vector space*

$$\mathcal{L}(\mathcal{J}, \mathcal{T}) := (\mathfrak{sl}(V) \otimes \mathcal{J}) \oplus (V \otimes \mathcal{T}) \oplus \mathcal{D}, \tag{2.15}$$

is a Lie algebra with an inner $\mathbf{SL}_2$*-structure with the bracket defined as in* (2.2)*, and where* $\mathcal{D}$ *is the subalgebra of the Lie algebra of even derivations of the* $\mathbb{Z}/2$*-graded algebra* $\mathcal{J} \oplus \mathcal{T}$ *defined in* (2.13) *spanned by the maps* $D_{a,b}$ *for* $a, b \in \mathcal{J}$*, and* $d_{x,y}$ *for* $x, y \in \mathcal{T}$*, defined as follows:*

$$\begin{aligned} D_{a,b}(c) &= a \cdot (b \cdot c) - b \cdot (a \cdot c), \\ D_{a,b}(x) &= \frac{1}{4}\big(a \bullet (b \bullet x) - b \bullet (a \bullet x)\big), \\ d_{x,y}(a) &= \langle a \bullet x \mid y\rangle - \langle x \mid a \bullet y\rangle, \\ d_{x,y}(z) &= \langle x \mid y\rangle \bullet z - 2(x, y, z), \end{aligned} \tag{2.16}$$

for $a, b, c \in \mathcal{J}$ *and* $x, y, z \in \mathcal{T}$*.*

Under the conditions of Theorem 2.6, the J-ternary algebra $(\mathcal{J}, \mathcal{T})$ is said to *coordinatize* the $\mathbf{SL}_2$-structure on $\mathcal{L}$.

A word of caution is needed here. If $\mathcal{L}$ is a Lie algebra endowed with an inner $\mathbf{SL}_2$-structure as in Theorem 2.6, and $(\mathcal{J}, \mathcal{T})$ is the associated J-ternary algebra that coordinatizes it, the Lie algebra $\mathcal{L}(\mathcal{J}, \mathcal{T})$ is not necessarily isomorphic to our original $\mathcal{L}$. It may even fail to be isomorphic to a subalgebra of $\mathcal{L}$. Actually, $\mathcal{L}(\mathcal{J}, \mathcal{T})$ is *centrally isogenous* to the subalgebra of the original $\mathcal{L}$ generated by the isotypic components $\mathfrak{sl}(V) \otimes \mathcal{J}$ and $V \otimes \mathcal{T}$. That is, their universal central extensions coincide (see [6, 10] and the references therein). We will not go into details.

3 Short ($\mathbf{SL}_2 \times \mathbf{SL}_2$)-structures

We start by introducing a new class of $\mathbf{S}$-structures: the short ($\mathbf{SL}_2 \times \mathbf{SL}_2$)-structures.

Definition 3.1 An ($\mathbf{SL}_2 \times \mathbf{SL}_2$)-structure $\Phi : \mathbf{SL}_2 \times \mathbf{SL}_2 \to \mathbf{Aut}(\mathcal{L})$ on a Lie algebra $\mathcal{L}$ is said to be *short* if $\mathcal{L}$ decomposes, as a module for $\mathbf{SL}_2 \times \mathbf{SL}_2$ via Φ, into a direct sum of copies of the following modules:

- the adjoint module for each of the two copies of $\mathbf{SL}_2$,
- the natural two-dimensional modules V_1 and V_0 for each of the two copies of $\mathbf{SL}_2$ [the weird numbering 1, 0 is justified by the Peirce decomposition in (3.5)],
- the tensor product $V_1 \otimes V_0$, and
- the trivial one-dimensional module.

Remark 3.2 Due to Lemma 2.2, where $\mathbf{SL}_2$-invariance may be substituted by $\mathfrak{sl}_2$-invariance, the reader who feels uncomfortable with affine group schemes may consider an alternate definition of *short* $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$*-structure* on a Lie algebra $\mathcal{L}$, as a homomorphism of Lie algebras $\Psi : \mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \to \mathrm{Der}(\mathcal{L})$ such that, as a module for $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ via Ψ, $\mathcal{L}$ decomposes into a direct sum of copies of the adjoint module for each of the two copies of $\mathfrak{sl}_2$, of the natural two-dimensional modules V_1 and V_0 for each of the two copies of $\mathfrak{sl}_2$, of the tensor product $V_1 \otimes V_0$, and of the trivial one-dimensional module.

Assume that $\Phi : \mathbf{SL}_2 \times \mathbf{SL}_2 \to \mathcal{L}$ is an inner short $(\mathbf{SL}_2 \times \mathbf{SL}_2)$-structure on the Lie algebra $\mathcal{L}$, then its isotypic decomposition allows us to describe $\mathcal{L}$ as follows:

$$\mathcal{L} = \big(\mathfrak{sl}(V_1) \otimes \mathcal{J}_1\big) \oplus \big(\mathfrak{sl}(V_0) \otimes \mathcal{J}_0\big) \oplus \big((V_1 \otimes V_0) \otimes \mathcal{J}_{\frac{1}{2}}\big) \oplus \big(V_1 \otimes \mathcal{T}_1\big) \oplus \big(V_0 \otimes \mathcal{T}_0\big) \oplus \mathcal{S}. \tag{3.1}$$

Note that the invariance of the bracket under the action of $\mathbf{SL}_2 \times \mathbf{SL}_2 \simeq \mathbf{SL}(V_1) \times \mathbf{SL}(V_0)$ forces that $\big(\mathfrak{sl}(V_i) \otimes \mathcal{J}_i\big) \oplus \big(V_i \otimes \mathcal{T}_i\big) \oplus \mathcal{S}$ is a subalgebra with a short inner $\mathbf{SL}_2$-structure, for $i = 0, 1$, and hence that the bracket in $\mathcal{L}$ induces a structure of J-ternary algebra on $(\mathcal{J}_i, \mathcal{T}_i)$, for $i = 1, 0$.

In order to get more information, it is not a good idea to expand blindly the Lie bracket on $\mathcal{L}$. Instead, consider the diagonal subgroup of $\mathbf{SL}_2 \times \mathbf{SL}_2$. The composition

$$\mathbf{SL}_2 \xrightarrow{\Delta} \mathbf{SL}_2 \times \mathbf{SL}_2 \xrightarrow{\Phi} \mathbf{Aut}(\mathcal{L}), \tag{3.2}$$

where Δ is the diagonal embedding $g \mapsto (g, g)$, gives an inner $\mathbf{SL}_2$-structure on $\mathcal{L}$. This structure is short because, as $\mathbf{SL}_2$-modules via $\Phi \circ \Delta$, $\mathfrak{sl}(V_i)$ is the adjoint module and V_i is the two-dimensional natural module, for $i = 1, 0$, and $V_1 \otimes V_0$ decomposes as the direct sum of an adjoint module and a trivial module. (See items (iv) and (v) of Lemma 2.2.)

The next theorem is our main result. It describes the short inner $(\mathbf{SL}_2 \times \mathbf{SL}_2)$-structures on a Lie algebra in terms of J-ternary algebras $(\mathcal{J}, \mathcal{T})$ where the unital Jordan algebra $\mathcal{J}$ contains a nontrivial distinguished idempotent.

Theorem 3.3 *Let $\mathcal{L}$ be a Lie algebra endowed with a short inner $(\mathbf{SL}_2 \times \mathbf{SL}_2)$-structure $\Phi : \mathbf{SL}_2 \times \mathbf{SL}_2 \to \mathbf{Aut}(\mathcal{L})$. Let $(\mathcal{J}, \mathcal{T})$ be the J-ternary algebra that coordinatizes the short $\mathbf{SL}_2$-structure $\Psi = \Phi \circ \Delta$ in (3.2), with isotypic decomposition given by (2.1):*

$$\mathcal{L} = \big(\mathfrak{sl}(V) \otimes \mathcal{J}\big) \oplus \big(V \otimes \mathcal{T}\big) \oplus \mathcal{D}, \tag{3.3}$$

where V is the two-dimensional natural module for $\mathbf{SL}_2$.

Then the unital Jordan algebra $\mathcal{J}$ contains an idempotent $e = e^{\cdot 2} \neq 0, 1$ such that the image of the Lie algebra homomorphism ι in (1.1) is

$$\big(\mathfrak{sl}(V) \otimes e\big) \oplus \big(\mathfrak{sl}(V) \otimes (1 - e)\big).$$

Conversely, if $\mathcal{L}$ is a Lie algebra endowed with an inner $\mathbf{SL}_2$-structure $\Psi : \mathbf{SL}_2 \to \mathbf{Aut}(\mathcal{L})$ with isotypic decomposition in (3.3), and such that the unital Jordan algebra $\mathcal{J}$ contains a nontrivial idempotent e, then $\mathcal{L}$ is endowed with a short inner $(\mathbf{SL}_2 \times \mathbf{SL}_2)$-structure whose isotypic decomposition is the following:

$$\begin{aligned}\mathcal{L} = &\big(\mathfrak{sl}(V) \otimes \mathcal{J}_1\big) \oplus \big(\mathfrak{sl}(V) \otimes \mathcal{J}_0\big) \oplus \Big(\big(\mathfrak{sl}(V) \otimes \mathcal{J}_{\frac{1}{2}}\big) \oplus D_{e,\mathcal{J}_{\frac{1}{2}}}\Big)\\ &\oplus \big(V \otimes \mathcal{T}_1\big) \oplus \big(V \otimes \mathcal{T}_0\big) \oplus \mathcal{S},\end{aligned} \tag{3.4}$$

where $\mathcal{J} = \mathcal{J}_1 \oplus \mathcal{J}_{\frac{1}{2}} \oplus \mathcal{J}_0$ is the Peirce decomposition of $\mathcal{J}$ relative to the idempotent e:

$$\mathcal{J}_1 = \{a \in \mathcal{J} \mid e \cdot a = a\}, \quad \mathcal{J}_{\frac{1}{2}} = \left\{a \in \mathcal{J} \mid e \cdot a = \frac{1}{2}a\right\}, \quad \mathcal{J}_0 = \{a \in \mathcal{J} \mid e \cdot a = 0\}, \tag{3.5}$$

$\mathcal{T} = \mathcal{T}_1 \oplus \mathcal{T}_0$ is the induced decomposition on $\mathcal{T}$:

$$\mathcal{T}_1 = \{x \in \mathcal{T} \mid e \bullet x = x\}, \quad \mathcal{T}_0 = \{x \in \mathcal{T} \mid e \bullet x = 0\}, \tag{3.6}$$

and where the following assertions hold:

- *for any $a \in \mathcal{J}_1$, $\mathfrak{sl}(V) \otimes a$ is a copy of the adjoint module for the first copy of* $\mathbf{SL}_2$*,*
- *for any $a \in \mathcal{J}_0$, $\mathfrak{sl}(V) \otimes a$ is a copy of the adjoint module for the second copy of* $\mathbf{SL}_2$*,*
- *for any $a \in \mathcal{J}_{\frac{1}{2}}$, $\big(\mathfrak{sl}(V) \otimes a\big) \oplus \mathbb{F}D_{e,a}$ is a copy of the tensor product of the natural module for the first copy of* $\mathbf{SL}_2$ *and the natural module for the second copy of* $\mathbf{SL}_2$*,*
- *for any $x \in \mathcal{T}_1$, $V \otimes x$ is a copy of the natural module for the first copy of* $\mathbf{SL}_2$*,*
- *for any $x \in \mathcal{T}_0$, $V \otimes x$ is a copy of the natural module for the second copy of* $\mathbf{SL}_2$*,*
- *$\mathcal{S}$ is the subspace of fixed elements by* $\mathbf{SL}_2 \times \mathbf{SL}_2$*. Moreover, the subspace of fixed elements by* $\mathbf{SL}_2$ *under $\Psi = \Phi \circ \Delta$ is the direct sum $\mathcal{D} = \mathcal{S} \oplus D_{e,\mathcal{J}_{\frac{1}{2}}}$.*

Proof For the first part, as the ($\mathbf{SL}_2 \times \mathbf{SL}_2$)-structure Φ is inner, $\mathcal{L}$ contains a subalgebra isomorphic to $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$, which in the decomposition (3.3) appears as

$$\big(\mathfrak{sl}(V) \otimes e_1\big) \oplus \big(\mathfrak{sl}(V) \otimes e_2\big),$$

for orthogonal idempotents $e_1, e_2 \in \mathcal{J}$; that is, $e_1^{\cdot 2} = e_1, e_2^{\cdot 2} = e_2, e_1 \cdot e_2 = 0$ and $1 = e_1 + e_2$. Now it is enough to take $e = e_1$.

Conversely, let $\mathcal{L}$ be a Lie algebra endowed with a short inner $\mathbf{SL}_2$-structure $\Psi : \mathbf{SL}_2 \to \mathbf{Aut}(\mathcal{L})$ with isotypic decomposition as in (3.3). Let $e \in \mathcal{J}$ be a nontrivial idempotent (i.e., $e = e^{\cdot 2} \neq 0, 1$), and let $\mathcal{J} = \mathcal{J}_1 \oplus \mathcal{J}_{\frac{1}{2}} \oplus \mathcal{J}_0$ be the corresponding Peirce decomposition as in (3.5). Equation (2.5) gives $e \bullet x = e \bullet (e \bullet x)$ for any $x \in \mathcal{T}$, so that $\mathcal{T}$ decomposes as in (3.6).

We need a short digression before continuing with the proof. The vector space $\mathfrak{gl}(V)$ of linear endomorphisms of our two-dimensional vector space V is a module for $\mathfrak{sl}(V) \oplus \mathfrak{sl}(V)$ by means of the action

$$(f_1, f_2) \cdot g := f_1 g - g f_2, \tag{3.7}$$

for $f_1, f_2 \in \mathfrak{sl}(V)$ and $g \in \mathfrak{gl}(V)$. Moreover, once we fix, as we have done before, a nonzero skew-symmetric bilinear form $(.\mid.)$ on V, $\mathfrak{gl}(V)$ is isomorphic to $V \otimes V$ by means of the linear map determined as follows:

$$u \otimes v \mapsto u(v \mid .)\big(: w \mapsto (v \mid w)u\big),$$

for $u, v, w \in V$. This linear isomorphism is an isomorphism of $\mathfrak{sl}(V) \oplus \mathfrak{sl}(V)$-modules, where the first (resp. second) copy of $\mathfrak{sl}(V)$ acts on the first (resp. second) copy of V in $V \otimes V$.

Also note that for any $f \in \mathfrak{sl}(V)$, the Cayley-Hamilton equation gives $f^2 = -\det(f)\mathrm{id} = \frac{1}{2}\mathrm{tr}(f^2)\mathrm{id}$, and hence, by linearization we get

$$fg + gf = \mathrm{tr}(fg)\mathrm{id}, \tag{3.8}$$

for any $f, g \in \mathfrak{sl}(V)$. On the other hand, $[f, g] = fg - gf$ which, together with (3.8) gives the formulas:

$$fg = \frac{1}{2}\big([f, g] + \mathrm{tr}(fg)\mathrm{id}\big), \quad gf = \frac{1}{2}\big(-[f, g] + \mathrm{tr}(fg)\mathrm{id}\big), \tag{3.9}$$

for $f, g \in \mathfrak{sl}(V)$, and hence we have

$$(f_1, f_2) \cdot g = f_1 g - g f_2 = \frac{1}{2}\big([f_1 + f_2, g] + \mathrm{tr}((f_1 - f_2)g)\mathrm{id}\big), \tag{3.10}$$

for any $f_1, f_2, g \in \mathfrak{sl}(V)$.

(Note that $\mathfrak{sl}(V) \oplus \mathfrak{sl}(V)$ can be substituted by $\mathbf{SL}(V) \times \mathbf{SL}(V)$ above, with (3.7) changed to $(f_1, f_2) \cdot g = f_1 g f_2^{-1}$, for rational points $f_1, f_2 \in \mathbf{SL}(V)$ and $g \in \mathfrak{gl}(V)$.)

Now, for $\mathcal{L}$ as above, endowed with a short inner $\mathbf{SL}_2$-structure Ψ, (2.2) gives, for any $f_1, f_2, g \in \mathfrak{sl}(V)$ and $a \in \mathcal{J}_{\frac{1}{2}}$, the following:

$$
\begin{aligned}
[f_1 \otimes e + f_2 \otimes (1-e), g \otimes a] &= \frac{1}{2}[f_1 + f_2, g] \otimes a + 2\mathrm{tr}((f_1 - f_2)g) D_{e,a},\\
[f_1 \otimes e + f_2 \otimes (1-e), D_{e,a}] &= \frac{1}{4}(f_1 - f_2) \otimes a,
\end{aligned}
$$

where we have used $D_{1,a} = 0$ and $D_{e,a}(e) = -\frac{1}{4}a$, which follow from (2.4) and (2.6).

Comparing with (3.10), it turns out that the linear map

$$
\begin{aligned}
\bigl(\mathfrak{sl}(V) \otimes a\bigr) \oplus \mathbb{F}D_{e,a} &\longrightarrow \mathfrak{gl}(V)\\
f \otimes a \quad &\mapsto \quad f,\\
D_{e,a} \quad &\mapsto \quad \frac{1}{4}\mathrm{id},
\end{aligned}
$$

is an isomorphism of $\mathfrak{sl}(V) \oplus \mathfrak{sl}(V) \simeq \bigl(\mathfrak{sl}(V) \otimes e\bigr) \oplus \bigl(\mathfrak{sl}(V) \otimes (1-e)\bigr)$-modules.

On the other hand, for any $a \in \mathcal{J}_{\frac{1}{2}}$, Eq. (2.6) gives $D_{e,a}(e) = \bigl(\frac{1}{4} - \frac{1}{2}\bigr)\, a = -\frac{1}{4}a$, so the linear map

$$
\mathcal{J}_{\frac{1}{2}} \to \mathcal{D}, \quad a \mapsto D_{e,a}
$$

is one-to-one, and $\mathcal{D}$ decomposes as $\mathcal{D} = D_{e,\mathcal{J}_{\frac{1}{2}}} \oplus \mathcal{S}$, with

$$
\mathcal{S} = \{D \in \mathcal{D} \mid D(e) = 0\}.
$$

It then follows that $\mathcal{S}$ is the centralizer of the subalgebra $\bigl(\mathfrak{sl}(V) \otimes e\bigr) \oplus \bigl(\mathfrak{sl}(V) \otimes (1-e)\bigr) \simeq \mathfrak{sl}(V) \oplus \mathfrak{sl}(V)$.

Therefore, the decomposition in (3.4) gives a decomposition of $\mathcal{L}$, as a module for $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \simeq \bigl(\mathfrak{sl}(V) \otimes e\bigr) \oplus \bigl(\mathfrak{sl}(V) \otimes (1-e)\bigr)$, into a direct sum of copies of the adjoint module for the first copy of $\mathfrak{sl}_2$, copies of the adjoint module for the second copy of $\mathfrak{sl}_2$, copies of the tensor product of the natural modules for the two copies of $\mathfrak{sl}_2$, copies of the natural module for the first copy of $\mathfrak{sl}_2$, copies for the natural module for the second copy of $\mathfrak{sl}_2$, and copies of the trivial one-dimensional module for $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$. This shows that $\mathcal{L}$ is endowed with a short inner $(\mathbf{SL}_2 \times \mathbf{SL}_2)$-structure. □

Remark 3.4 Let $\mathcal{L}$ be a Lie algebra endowed with a short inner $\mathbf{SL}_2$-structure $\Psi : \mathbf{SL}_2 \to \mathbf{Aut}(\mathcal{L})$ with isotypic decomposition as in (3.3), and such that the unital Jordan algebra $\mathcal{J}$ contains a nontrivial idempotent e. Then (2.7) with $c = e$, $a \in \mathcal{J}_1$, and $b \in \mathcal{J}_0$ gives

$$
D_{\mathcal{J}_1,\mathcal{J}_0} = 0,
$$

and now, with $b = c = e$ and $a \in \mathcal{J}_1$, gives $D_{e,\mathcal{J}_1} = 0$. (This should be familiar to experts in Jordan algebras, but note that here $D_{a,b}$ is an element of the subalgebra $\mathcal{D}$, which acts by derivations on $\mathcal{J}$, but it is not, in general, the Lie algebra of derivations of $\mathcal{J}$.)

Also, (2.10) with $a = e$ gives

$$
\langle \mathcal{T}_i \mid \mathcal{T}_i \rangle \subseteq \mathcal{J}_i \ (i = 1, 0), \qquad \langle \mathcal{T}_1 \mid \mathcal{T}_0 \rangle \subseteq \mathcal{J}_{\frac{1}{2}},
$$

and (2.9) with $a = e$ shows $d_{x.y} = 4D_{e,\langle x|y\rangle}$ for $x \in \mathcal{T}_0$ and $y \in \mathcal{T}_1$, which implies

$$
d_{\mathcal{T}_0,\mathcal{T}_1} \subseteq D_{e,\mathcal{J}_{\frac{1}{2}}}.
$$

All these conditions follow too from the fact that the bracket in $\mathcal{L}$ is invariant under the action of $\big(\mathfrak{sl}(V) \otimes e\big) \oplus \big(\mathfrak{sl}(V) \otimes (1-e)\big)$.

Remark 3.5 The Lie algebras graded by the nonreduced root system BC_2 and with grading subalgebra of type C_2 are also coordinatized by J-ternary algebras with a proper idempotent plus some extra restrictions [6, Theorem 6.66]. The connection with Theorem 3.3 is as follows. Take two two-dimensional vector spaces V_1, V_2, endowed with nonzero alternating forms. Identify the simple Lie algebra of type C_2 with $\mathfrak{s} = \mathfrak{sp}(V_1 \perp V_2)$. Then $\mathfrak{sl}(V_1) \oplus \mathfrak{sl}(V_2) \simeq \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ is naturally a subalgebra of $\mathfrak{s}$, and $\mathfrak{s}$ decomposes, as a module for $\mathfrak{sl}(V_1) \oplus \mathfrak{sl}(V_2)$, as $\mathfrak{sl}(V_1) \oplus \mathfrak{sl}(V_2) \oplus (V_1 \otimes V_2)$. The $\mathfrak{s}$-modules that appear in a BC_2-graded Lie algebra with grading subalgebra $\mathfrak{s}$ decompose, as modules for $\mathfrak{sl}(V_1) \oplus \mathfrak{sl}(V_2)$, into a direct sum of copies of the adjoint modules $\mathfrak{sl}(V_1)$ and $\mathfrak{sl}(V_2)$, natural modules V_1 and V_2, $V_1 \otimes V_2$, and the trivial module. Hence any BC_2-graded Lie algebra with a grading subalgebra of type C_2 is naturally endowed with a short ($\mathbf{SL}_2 \times \mathbf{SL}_2$)-structure.

4 Examples

In this section, several examples of J-ternary algebras will be reviewed. Any nontrivial idempotent in the corresponding Jordan algebras provides examples of short ($\mathbf{SL}_2 \times \mathbf{SL}_2$)-structures on the associated Lie algebra. The first class of examples of J-ternary algebras are defined in terms of unital associative algebras with involution and left modules for them, while the second class of examples is related to structurable algebras.

The classification of the reduced finite-dimensional J-ternary algebras was obtained by Hein [19, 20].

4.1 Prototypical example

Let $(\mathcal{A}, *)$ be a unital associative algebra with involution, and let $\mathcal{T}$ be a left $\mathcal{A}$-module endowed with a skew-hermitian form $\mathrm{h} : \mathcal{T} \times \mathcal{T} \to \mathcal{A}$. That is, h is $\mathbb{F}$-bilinear, and $\mathrm{h}(ax, y) = a\mathrm{h}(x, y)$ and $\mathrm{h}(x, y) = -\mathrm{h}(y, x)^*$, for any $a \in \mathcal{A}$ and $x, y \in \mathcal{T}$.

As above, let V be a two-dimensional vector space endowed with a nonzero skew-symmetric bilinear form $(.\mid .)$, and consider the left $\mathcal{A}$-module

$$\mathcal{W} = (V \otimes \mathcal{A}) \oplus \mathcal{T},$$

where $V \otimes \mathcal{A}$ is a left $\mathcal{A}$-module in the natural way: $a(u \otimes b) := u \otimes (ab)$, for any $a, b \in \mathcal{A}$ and $u \in V$. Extend h to a skew-hermitian form, also denoted by h, on $\mathcal{W}$ by imposing that $V \otimes \mathcal{A}$ and $\mathcal{T}$ are orthogonal relative to h, and defining its restriction to $V \otimes \mathcal{A}$ as follows:

$$\mathrm{h}(u \otimes a, v \otimes b) := 2(u \mid v)ab^*, \tag{4.1}$$

for any $u, v \in V$ and $a, b \in \mathcal{A}$.

Denote by τ the involution on $\mathrm{End}_{\mathcal{A}}(\mathcal{W})$ induced by h:

$$\mathrm{h}\big(f(x), y\big) = \mathrm{h}\big(x, f^{\tau}(y)\big)$$

for any $f \in \mathrm{End}_{\mathcal{A}}(\mathcal{W})$ and any $x, y \in \mathcal{W}$, and denote too by τ the restriction of τ to $\mathrm{End}_{\mathcal{A}}(\mathcal{T})$.

For any $x, y \in \mathcal{W}$, the $\mathcal{A}$-linear map defined as follows:

$$\varphi_{x,y} = \mathrm{h}(., x)y + \mathrm{h}(., y)x : z \mapsto \mathrm{h}(z, x)y + \mathrm{h}(z, y)x, \tag{4.2}$$

is skew-symmetric relative to τ. That is, $\varphi_{x,y}^{\tau} = -\varphi_{x,y}$, or $\varphi_{x,y}$ belongs to the Lie algebra $\mathrm{Skew}\big(\mathrm{End}_{\mathcal{A}}(\mathcal{W}), \tau\big)$. Also, for any $a \in \mathcal{A}$ and $x, y \in \mathcal{W}$, we have

$$\varphi_{ax,y} = \varphi_{x,a^*y}$$

because h is skew-hermitian. And for any $f \in \mathrm{Skew}\big(\mathrm{End}_{\mathcal{A}}(\mathcal{W}), \tau\big)$ and any $x, y \in \mathcal{W}$, the following equation follows at once:

$$[f, \varphi_{x,y}] = \varphi_{f(x),y} + \varphi_{x,f(y)}. \tag{4.3}$$

For any endomorphism φ of the left regular module $\mathcal{A}$, there is an element $a \in \mathcal{A}$ such that $\varphi(1) = a^*$. Hence, for any $b \in \mathcal{A}$, we get $\varphi(b) = \varphi(b1) = b\varphi(1) = ba^*$. As a consequence, the associative algebra $\mathrm{End}_{\mathcal{A}}(\mathcal{A})$ is isomorphic to $\mathcal{A}$ by means of the assignment:

$$a \mapsto \big(R_{a^*} : b \mapsto ba^*\big), \tag{4.4}$$

and hence we can identify the associative algebras

$$\mathrm{End}_{\mathcal{A}}(V \otimes \mathcal{A}) \simeq \mathrm{End}_{\mathbb{F}}(V) \otimes \mathrm{End}_{\mathcal{A}}(\mathcal{A}) \simeq \mathrm{End}_{\mathbb{F}}(V) \otimes \mathcal{A},$$

so that the element $f \otimes a \in \mathrm{End}_{\mathbb{F}}(V) \otimes \mathcal{A}$ is identified with the endomorphism $f \otimes R_{a^*} \in \mathrm{End}_{\mathcal{A}}(V \otimes \mathcal{A})$.

Also, for any $f \in \mathrm{End}_{\mathbb{F}}(V)$, $u, v \in V$, and $a, b, c \in \mathcal{A}$, we have:

$$\mathrm{h}(f(u) \otimes ba^*, v \otimes c) = 2\big(f(u) \mid v\big)ba^*c^* = 2\big(u \mid f^{\star}(v)\big)b(ca)^*,$$

where $f^{\star}$ is the adjoint of f relative to $(.\mid .)$: $\big(f(u) \mid v\big) = \big(u \mid f^{\star}(v)\big)$ for any $u, v \in V$, so that $f^{\star} = -f$ if and only if $f \in \mathfrak{sl}(V)$. Hence, with the identification above $\mathrm{End}_{\mathcal{A}}(V \otimes \mathcal{A}) \simeq \mathrm{End}_{\mathbb{F}}(V) \otimes \mathcal{A}$, the adjoint $(f \otimes a)^{\tau}$ equals $f^{\star} \otimes a^*$, and the Lie algebra $\mathrm{Skew}\big(\mathrm{End}_{\mathcal{A}}(V \otimes \mathcal{A}), \tau\big)$ appears as:

$$\mathrm{Skew}\big(\mathrm{End}_{\mathcal{A}}(V \otimes \mathcal{A}), \tau\big) \simeq \big(\mathfrak{sl}(V) \otimes \mathcal{H}(\mathcal{A}, *)\big) \oplus \big(\mathrm{id}_V \otimes \mathrm{Skew}(\mathcal{A}, *)\big),$$

where $\mathcal{H}(\mathcal{A}, *)$ denotes the subspace of symmetric elements of $\mathcal{A}$ relative to $*$.

Inside the Lie algebra $\mathrm{Skew}\big(\mathrm{End}_{\mathcal{A}}(\mathcal{W}), \tau\big)$ there is the subalgebra

$$\mathcal{L} = \mathrm{Skew}\big(\mathrm{End}_{\mathcal{A}}(V \otimes \mathcal{A}), \tau\big) \oplus \mathrm{Skew}\big(\mathrm{End}_{\mathcal{A}}(\mathcal{T}), \tau\big) \oplus \varphi_{V \otimes \mathcal{A}, \mathcal{T}},$$

which can be identified with

$$\big(\mathfrak{sl}(V) \otimes \mathcal{H}(\mathcal{A}, *)\big) \oplus \varphi_{V \otimes \mathcal{A}, \mathcal{T}} \oplus \Big(\big(\mathrm{id}_V \otimes \mathrm{Skew}(\mathcal{A}, *)\big) \oplus \mathrm{Skew}\big(\mathrm{End}_{\mathcal{A}}(\mathcal{T}), \tau\big)\Big).$$

Also, for any $u \in V$, $a \in \mathcal{A}$, and $x \in \mathcal{T}$, we have $\varphi_{u \otimes a, x} = \varphi_{u \otimes 1, a^* x}$, so that $\varphi_{V \otimes \mathcal{A}, \mathcal{T}} = \varphi_{V \otimes 1, \mathcal{T}}$, and this allows us to identify $\varphi_{V \otimes \mathcal{A}, \mathcal{T}}$ with $V \otimes \mathcal{T}$, where $u \otimes x$ corresponds to $\varphi_{u \otimes 1, x}$ for $u \in V$ and $x \in \mathcal{T}$.

With these identifications we may write

$$\mathcal{L} = \big(\mathfrak{sl}(V) \otimes \mathcal{H}(\mathcal{A}, *)\big) \oplus (V \otimes \mathcal{T}) \oplus \mathcal{D}, \tag{4.5}$$

with $\mathcal{D} = \big(\mathrm{id}_V \otimes \mathrm{Skew}(\mathcal{A}, *)\big) \oplus \mathrm{Skew}\big(\mathrm{End}_{\mathcal{A}}(\mathcal{T}), \tau\big)$. This shows that $\mathcal{L}$ is endowed with a short $\mathbf{SL}_2$-structure, with J-ternary algebra $\big(\mathcal{J} = \mathcal{H}(\mathcal{A}, *), \mathcal{T}\big)$.

Let us see what the operations in $(\mathcal{J}, \mathcal{T})$ look like, according to (2.2) and (2.3).

- For $f, g \in \mathfrak{sl}(V)$ and $a, b \in \mathcal{H}(\mathcal{A}, *)$, working inside $\operatorname{End}_{\mathbb{F}}(V) \otimes \mathcal{A}$ and using (3.9), we get

$$\begin{aligned}[f \otimes a, g \otimes b] &= fg \otimes ab - gf \otimes ba \\ &= [f, g] \otimes a \cdot b + \frac{1}{2}\operatorname{tr}(fg)\mathrm{id}_V \otimes [a, b],\end{aligned}$$

and this shows that the Jordan product on $\mathcal{J} = \mathcal{H}(\mathcal{A}, *)$ is the natural one: $a \cdot b = \frac{1}{2}(ab + ba)$, while $D_{a,b}$ is given by $D_{a,b} = \mathrm{id}_V \otimes 4[a, b] \in \mathrm{id}_V \otimes \operatorname{Skew}(\mathcal{A}, *)$.
- For $f \in \mathfrak{sl}(V)$, $a \in \mathcal{H}(\mathcal{A}, *)$, $v \in V$, and $x \in \mathcal{T}$, we get:

$$[f \otimes a, \varphi_{v\otimes 1,x}] = \varphi_{f(v)\otimes a^*,x} = \varphi_{f(v)\otimes 1,ax},$$

so the action of $\mathcal{J}$ on $\mathcal{T}$ is also the natural one: $a \bullet x = ax$.
Also, for any $s \in \operatorname{Skew}(\mathcal{A}, *)$, $a \in \mathcal{H}(\mathcal{A}, *)$ and $x \in \mathcal{T}$, we have

$$[\mathrm{id}_V \otimes s, \varphi_{v\otimes 1,x}] = \varphi_{v\otimes s^*,x} = \varphi_{v\otimes 1,sx}. \tag{4.6}$$

- Finally, for $u, v \in V$ and $x, y \in \mathcal{T}$, we get

$$\begin{aligned}[\varphi_{u\otimes 1,x}, \varphi_{v\otimes 1,y}] &= \varphi_{\varphi_{u\otimes 1,x}(v\otimes 1),y} + \varphi_{v\otimes 1,\varphi_{u\otimes 1,x}(y)} \\ &= 2(v \mid u)\varphi_{x,y} + \varphi_{v\otimes 1,u\otimes \mathrm{h}(y,x)}.\end{aligned} \tag{4.7}$$

But for $u, v, w \in V$ and $a \in \mathcal{H}(\mathcal{A}, *)$, the map $\varphi_{v\otimes 1,u\otimes a}$ sends $w \otimes 1$ to

$$2(w \mid v)u \otimes a + 2(w \mid u)v \otimes a^*.$$

On the other hand, for $a = c + s$ with $c = c^* \in \mathcal{H}(\mathcal{A}, *)$ and $s = -s^* \in \operatorname{Skew}(\mathcal{A}, *)$, using that $(u \mid v)w + (v \mid w)u + (w \mid u)v$ is 0, we get

$$\begin{aligned}&2(w \mid v)u \otimes a + 2(w \mid u)v \otimes a^* \\ &\quad= 2(w \mid v)u \otimes (c + s) + 2(w \mid u)v \otimes (c - s) \\ &\quad= -2\gamma_{u,v}(w) \otimes c - 2\big((v \mid w)u + (w \mid u)v\big) \otimes s \\ &\quad= \gamma_{u,v}(w) \otimes (-a - a^*) + (u \mid v)w \otimes (a - a^*) \\ &\quad= \Big(\gamma_{u,v} \otimes (-a - a^*) - (u \mid v)\mathrm{id}_V \otimes (a - a^*)\Big)(w \otimes 1)\end{aligned}$$

(recall that the element $f \otimes a \in \operatorname{End}_{\mathbb{F}}(V) \otimes \mathcal{A}$ is identified with the endomorphism $f \otimes R_{a^*} \in \operatorname{End}_{\mathcal{A}}(V \otimes \mathcal{A})$), so that we have

$$\varphi_{v\otimes 1,u\otimes a} = \gamma_{u,v} \otimes (-a - a^*) - (u \mid v)\mathrm{id}_V \otimes (a - a^*).$$

Then (4.7) gives the following identity:

$$\begin{aligned}&[\varphi_{u\otimes 1,x}, \varphi_{v\otimes 1,y}] \\ &\quad= \gamma_{u,v} \otimes \big(\mathrm{h}(x, y) - \mathrm{h}(y, x)\big) + (u \mid v)\Big(-2\varphi_{x,y} - \big(\mathrm{id}_V \otimes \big(\mathrm{h}(x, y) + \mathrm{h}(y, x)\big)\big)\Big),\end{aligned}$$

and comparing with (2.2) we obtain the following values:

$$\begin{aligned}\langle x \mid y\rangle &= \mathrm{h}(x, y) - \mathrm{h}(y, x) \ \big(= \mathrm{h}(x, y) + \mathrm{h}(x, y)^* \in \mathcal{H}(\mathcal{A}, *)\big), \\ d_{x,y} &= -2\varphi_{x,y} - \mathrm{id}_V \otimes \big(\mathrm{h}(x, y) + \mathrm{h}(y, x)\big) \\ &\qquad\qquad \in \operatorname{Skew}\big(\operatorname{End}_{\mathcal{A}}(\mathcal{T}), \tau\big) \oplus \big(\mathrm{id}_V \otimes \operatorname{Skew}(\mathcal{A}, *)\big).\end{aligned} \tag{4.8}$$

Now Equations (2.14), (4.2), and (4.8), provide the triple product:

$$
\begin{aligned}
(x, y, z) &= \varphi_{x,y}(z) + \frac{1}{2}\big(\mathrm{h}(x, y) + \mathrm{h}(y, x)\big)z + \frac{1}{2}\big(\mathrm{h}(x, y) - \mathrm{h}(y, x)\big)z \\
&= \mathrm{h}(z, x)y + \mathrm{h}(z, y)x + \frac{1}{2}\big(\mathrm{h}(x, y) + \mathrm{h}(y, x)\big)z + \frac{1}{2}\big(\mathrm{h}(x, y) - \mathrm{h}(y, x)\big)z \\
&= \mathrm{h}(x, y)z + \mathrm{h}(z, x)y + \mathrm{h}(z, y)x. \qquad (4.9)
\end{aligned}
$$

Therefore, our J-ternary algebra $(\mathcal{J}, \mathcal{T})$ is the *prototypical example* of J-ternary algebra in [6, Example 3.14]. We summarize it in the following result:

Proposition 4.1 *Let $(\mathcal{A}, *)$ be a unital associative algebra with involution and let $\mathcal{T}$ be a left $\mathcal{A}$-module endowed with a skew-hermitian form $\mathrm{h} : \mathcal{T} \times \mathcal{T} \to \mathcal{A}$. Then the pair $(\mathcal{J}, \mathcal{T})$, where $\mathcal{J} = \mathcal{H}(\mathcal{A}, *)$ is the Jordan algebra of symmetric elements of $\mathcal{A}$ relative to the involution $*$, is a J-ternary algebra with the following operations:*

- $a \cdot b = \frac{1}{2}(ab + ba)$ *for any* $a, b \in \mathcal{J} = \mathcal{H}(\mathcal{A}, *)$,
- $a \bullet x = ax$ *for any* $a \in \mathcal{J}$ *and* $x \in \mathcal{T}$,
- $\langle x \mid y\rangle = \mathrm{h}(x, y) - \mathrm{h}(y, x)$ *for* $x, y \in \mathcal{T}$, *and*
- $(x, y, z) = \mathrm{h}(x, y)z + \mathrm{h}(z, x)y + \mathrm{h}(z, y)x$ *for* $x, y, z \in \mathcal{T}$.

Let us have a look at some particular cases of this prototypical example.

To begin with, let W and Z be two vector spaces, let $\mathcal{B}$ be the associative algebra $\mathrm{End}_{\mathbb{F}}(W)$ of linear endomorphisms of W, and let $\mathcal{A} = \mathcal{B} \oplus \mathcal{B}^{op}$ be the direct sum of $\mathcal{B}$ and its opposite algebra (defined over the same vector space but with new multiplication $a.b = ba$). Consider the exchange involution on $\mathcal{A}$: $(a, b)^{ex} = (b, a)$. The Jordan algebra $\mathcal{J} = \mathcal{H}(\mathcal{A}, ex)$ is isomorphic to the Jordan algebra $\mathcal{B}^{(+)}$ obtained by the symmetrization of the multiplication in $\mathcal{B}$.

The vector space W is the natural left module for $\mathcal{B}$, and hence it is a left module for $\mathcal{A}$ annihilated by $\mathcal{B}^{op}$. The dual vector space W^* is a right $\mathcal{B}$-module, and hence a left $\mathcal{B}^{op}$-module, so it becomes a left $\mathcal{A}$-module annihilated by $\mathcal{B}$. Then $\mathcal{T} := (W \otimes Z^*) \oplus (W^* \otimes Z)$ becomes a left $\mathcal{A}$-module in a natural way (the action of $\mathcal{A}$ on Z and Z^* is trivial), and it is endowed with a skew-hermitian form h in which the subspaces $W \otimes Z^*$ and $W^* \otimes Z$ are totally isotropic, and where we have

$$\mathrm{h}(w \otimes \alpha, \omega \otimes z) = \alpha(z)w\omega(.) \in \mathrm{End}_{\mathbb{F}}(W) = \mathcal{B} \subseteq \mathcal{A},$$

for $w \in W$, $\omega \in W^*$, $z \in Z$, and $\alpha \in Z^*$.

Then $\big(\mathcal{J} = \mathcal{B}^{(+)}, \mathcal{T}\big)$ is a J-ternary algebra and if W and Z are finite-dimensional, the Lie algebra $\mathcal{L}$ in (4.5) is naturally isomorphic to the general linear Lie algebra $\mathfrak{gl}\big((V \otimes W) \oplus Z\big)$. Its derived algebra $\mathfrak{sl}\big((V \otimes W) \oplus Z\big)$ shares the same J-ternary algebra.

On the other hand, if W is a vector space endowed with a symmetric (respectively skew-symmetric) bilinear form $\mathrm{b}_W : W \times W \to \mathbb{F}$, and Z is another vector space endowed with a skew-symmetric (resp. symmetric) bilinear form $\mathrm{b}_Z : Z \times Z \to \mathbb{F}$, then $\mathcal{T} = W \otimes Z$ is naturally a left module for the algebra $\mathcal{A} = \mathrm{End}_{\mathbb{F}}(W)$, which is endowed with the involution attached to b_W, while $\mathcal{T}$ is endowed with the skew-hermitian form

$$
\begin{aligned}
\mathrm{h} : \mathcal{T} \times \mathcal{T} &\longrightarrow \mathcal{A} \\
(w_1 \otimes z_1, w_2 \otimes z_2) &\mapsto \mathrm{b}_Z(z_1, z_2)w_1\mathrm{b}_W(w_2, .).
\end{aligned}
$$

For finite-dimensional W and Z, the Lie algebra $\mathcal{L}$ in (4.5) is the symplectic Lie algebra $\mathfrak{sp}\big((V \otimes W) \perp Z\big)$ (resp. the orthogonal Lie algebra $\mathfrak{so}\big((V \otimes W) \perp Z\big)$), where the bilinear form on $(V \otimes W) \perp Z$ is the orthogonal sum of the form b_Z on Z and the tensor product of the skew-symmetric form on V and the form b_W on W.

4.2 Structurable algebras and *J*-ternary algebras

Let $\mathcal{L}$ be a Lie algebra with a short $\mathbf{SL}_2$-structure and isotypic decomposition as in (2.1). The Lie bracket is then given by (2.2), and the triple product on $\mathcal{T}$ by (2.14).

Fix a symplectic basis $\{p, q\}$ in V: $(p \mid q) = 1$, and identify $\mathfrak{sl}(V)$ with $\mathfrak{sl}_2$ using this basis. Then we have $E = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) = \frac{1}{2}\gamma_{p,p}$, $H = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) = -\gamma_{p,q}$, and $F = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) = -\frac{1}{2}\gamma_{q,q}$. The adjoint action of H gives a 5-grading:

$$\mathcal{L} = \mathcal{L}_{-2} \oplus \mathcal{L}_{-1} \oplus \mathcal{L}_0 \oplus \mathcal{L}_1 \oplus \mathcal{L}_2,$$

where $\mathcal{L}_2 = E \otimes \mathcal{J}$, $\mathcal{L}_{-2} = F \otimes \mathcal{J}$, $\mathcal{L}_1 = p \otimes \mathcal{T}$, $\mathcal{L}_{-1} = q \otimes \mathcal{T}$, and $\mathcal{L}_0 = (H \otimes \mathcal{J}) \oplus \mathcal{D}$.

Identify $\mathcal{J}$ with $\mathcal{L}_2$ by means of $a \leftrightarrow E \otimes a$, and $\mathcal{T}$ with $\mathcal{L}_1$ by means of $x \leftrightarrow p \otimes x$. Writing F for $F \otimes 1$ and using (2.2) gives, for $a, b \in \mathcal{J}$ and $x, y, z \in \mathcal{T}$, the following equations:

- $[[E \otimes a, F], E \otimes b] = [H \otimes a, E \otimes b] = 2E \otimes a \cdot b$, so that the Jordan product in $\mathcal{J}$ becomes the product in $\mathcal{L}_2$ given by

$$A \cdot B = \frac{1}{2}[[A, F], B], \tag{4.10}$$

  for $A, B \in \mathcal{L}_2$.
- $[[E \otimes a, F], p \otimes x] = [H \otimes a, p \otimes x] = p \otimes a \bullet x$, and hence $\mathcal{L}_1$ becomes a special module for the Jordan algebra $\mathcal{L}_2$ with the action given by

$$A \bullet X = [[A, F], X], \tag{4.11}$$

  for $A \in \mathcal{L}_2$ and $X \in \mathcal{L}_1$.
- $[p \otimes x, p \otimes y] = \gamma_{p,p} \otimes \langle x \mid y \rangle = 2E \otimes \langle x \mid y \rangle$, so that the product $\mathcal{L}_1 \times \mathcal{L}_1 \to \mathcal{L}_2$ in the J-ternary algebra $(\mathcal{L}_2, \mathcal{L}_1)$ is given by

$$\langle X \mid Y \rangle = \frac{1}{2}[X, Y], \tag{4.12}$$

  for $X, Y \in \mathcal{L}_1$.
- $[[p\otimes x, [p\otimes y, F]], p\otimes z] = -[[p\otimes x, q\otimes y], p\otimes z] = -[\gamma_{p,q}\otimes\langle x \mid y\rangle + d_{x,y}, p\otimes z] = p \otimes \big(\langle x \mid y \rangle \bullet z - d_{x,y}(z)\big) = 2p \otimes (x, y, z)$ [recall (2.14)], and hence the triple product in $\mathcal{L}_1$ is given by

$$(X, Y, Z) = \frac{1}{2}[[X, [Y, F]], Z], \tag{4.13}$$

  for $X, Y, Z \in \mathcal{L}_1$.

This is summarized in the next result.

Proposition 4.2 *Let $\mathcal{L}$ be a 5-graded Lie algebra, and assume that there are elements $E \in \mathcal{L}_2$ and $F \in \mathcal{L}_{-2}$, such that* span $\{E, F, H = [E, F]\}$ *is a subalgebra isomorphic to $\mathfrak{sl}_2$, with $\mathcal{L}_i = \{X \in \mathcal{L} \mid [H, X] = iX\}$, for $i = -2, -1, 0, 1, 2$. (In particular $[H, E] = 2E$ and $[H, F] = -2F$.) Then the pair $(\mathcal{L}_2, \mathcal{L}_1)$ is a J-ternary algebra with the operations in* (4.10), (4.11), (4.12), *and* (4.13).

In particular, consider a structurable algebra $(\mathcal{A}, \bar{\ })$, that is, a unital algebra with involution such that the operators $V_{x,y}$ defined by

$$V_{x,y}(z) = \{x, y, z\} := (x\bar{y})z + (z\bar{y})x - (z\bar{x})y$$

satisfy

$$[V_{x,y}, V_{z,w}] = V_{V_{x,y}(z),w} - V_{z,V_{y,x}(w)},$$

for all $x, y, z, w \in \mathcal{A}$ (see [2]). Let $\mathcal{H}$ and $\mathcal{S}$ denote, respectively, the subspace of symmetric ($\bar{h} = h$) and skew-symmetric ($\bar{s} = -s$) elements in $\mathcal{A}$. The inner structure Lie algebra Instrl($\mathcal{A}$, -) of $\mathcal{A}$ is the Lie subalgebra of $\mathfrak{gl}(\mathcal{A})$ spanned by the $V_{x,y}$'s.

The Kantor construction (see [3] or [6, 6.4]) gives the 5-graded Lie algebra

$$\mathcal{L} = K(\mathcal{A}, \text{-}) = \mathcal{S}^{\sim} \oplus \mathcal{A}^{\sim} \oplus \mathrm{Instrl}(\mathcal{A}, \text{-}) \oplus \mathcal{A} \oplus \mathcal{S},$$

where $\mathcal{L}_{-2} = \mathcal{S}^{\sim}$ is a copy of $\mathcal{S}$, $\mathcal{L}_{-1} = \mathcal{A}^{\sim}$ is a copy of $\mathcal{A}$ (the elements in these cases will be written as $s^{\sim}$ for $s \in \mathcal{S}$, or $x^{\sim}$ for $x \in \mathcal{A}$), $\mathcal{L}_0 = \mathrm{Instrl}(\mathcal{A}, \text{-})$, $\mathcal{L}_1 = \mathcal{A}$ and $\mathcal{L}_2 = \mathcal{S}$. The Lie bracket in $\mathcal{L}$ is given by the following equations:

$$\begin{aligned}
[T, x] &= T(x), & [T, x^{\sim}] &= T^{\epsilon}(x)^{\sim},\\
[T, s] &= T(s) + s\overline{T(1)}, & [T, s^{\sim}] &= \big(T^{\epsilon}(s) + s\overline{T^{\epsilon}(1)}\big)^{\sim},\\
[x, y] &= 2(x\bar{y} - y\bar{x}), & [x^{\sim}, y^{\sim}] &= 2(x\bar{y} - y\bar{x})^{\sim},\\
[x, y^{\sim}] &= 2V_{x,y}, & [s, t^{\sim}] &= L_s L_t \Big(= \frac{1}{2}(V_{st,1} - V_{s,t})\Big),\\
[x, s^{\sim}] &= -(sx)^{\sim}, & [x^{\sim}, s] &= -sx,
\end{aligned}$$

for any $x, y \in \mathcal{A}$, $s, t \in \mathcal{S}$, and $T \in \mathrm{Instrl}(\mathcal{A}, \text{-})$, and where L_s is the operator of left multiplication by s, and $V^{\epsilon}_{x,y} := -V_{y,x}$ for any $x, y \in \mathcal{A}$.

Proposition 4.3 *Let $(\mathcal{A}, \text{-})$ be a structurable algebra, and let $s \in \mathcal{S}$ be a skew-symmetric element such that the left multiplication L_s is bijective, then the pair $(\mathcal{S}, \mathcal{A})$ is a J-ternary algebra with the following operations:*

$$\begin{aligned}
a \cdot b &= \frac{1}{2}\big(a(sb) + b(sa)\big),\\
a \bullet x &= a(sx),\\
\langle x \mid y\rangle &= x\bar{y} - y\bar{x},\\
(x, y, z) &= -V_{x,sy}(z),
\end{aligned}$$

for $a, b \in \mathcal{S}$ and $x, y, z \in \mathcal{A}$.

Proof In a slightly different way, this appears without proof in [6, Remark 6.7]. We will prove it here as a consequence of Proposition 4.2.

Since L_s is invertible, there is an element $s' \in \mathcal{S}$ such that $L_s^{-1} = L_{s'}$ [5, Proposition 11.1] and then the elements $E = s'$ and $F = s^{\sim}$ satisfy the conditions of Proposition 4.2, with $H = [E, F] = V_{1,1} = \mathrm{id}$. Therefore, $(\mathcal{S}, \mathcal{A})$ is a J-ternary algebra with the operations defined as in (4.10), (4.11), (4.12), and (4.13):

$$\begin{aligned}
a \cdot b &= \frac{1}{2}[[a, s^{\sim}], b] = \frac{1}{2}[L_a L_s, b] = \frac{1}{2}\big(a(sb) + b(\overline{as})\big) = \frac{1}{2}\big(a(sb) + b(sa)\big),\\
a \bullet x &= [[a, s^{\sim}], x] = [L_a L_s, x] = a(sx),\\
\langle x \mid y\rangle &= \frac{1}{2}[x, y] = x\bar{y} - y\bar{x},\\
(x, y, z) &= \frac{1}{2}[[x, [y, s^{\sim}]], z] = \frac{1}{2}[[x, -(sy)^{\sim}], z] = -[V_{x,sy}, z] = -V_{x,sy}(z),
\end{aligned}$$

for $a, b \in \mathcal{S}$ and $x, y, z \in \mathcal{A}$, as required. □

As a particular case, consider the case of the structurable algebra obtained as the tensor product of a Cayley (or octonion) algebra and another unital composition algebra: $\mathcal{A} = \mathcal{C}_1 \otimes \mathcal{C}_2$, where the involution is the tensor product of the canonical involutions in the Cayley algebra $\mathcal{C}_1$ and in the composition algebra $\mathcal{C}_2$.

Denote by n_i the norm in $\mathcal{C}_i$, and by $\mathcal{S}_i$ the subspace of trace zero elements (i.e., $\bar{s} = -s$) in $\mathcal{C}_i$, for $i = 0, 1$. The skew-symmetric part of $\mathcal{A}$ is $\mathcal{S} = \mathcal{S}_1 \otimes 1 + 1 \otimes \mathcal{S}_2$, which we will identify with $\mathcal{S}_1 \oplus \mathcal{S}_2$. Fix an element $s \in \mathcal{S}_1$ with $\mathrm{n}_1(s) \neq 0$.

As in [4] consider the *Albert form* $Q : \mathcal{S} \to \mathbb{F}$, which is the nondegenerate quadratic form given by

$$Q(s_1 + s_2) := \frac{1}{\mathrm{n}_1(s)}\big(\mathrm{n}_1(s_1) - \mathrm{n}_2(s_2)\big),$$

for $s_1 \in \mathcal{S}_1$ and $s_2 \in \mathcal{S}_2$. Consider also the linear map $\sharp$ given by

$$(s_1 + s_2)^\sharp := \mathrm{n}_1(s)(s_1 - s_2).$$

Write $c = -\frac{1}{\mathrm{n}_1(s)}s$. The pair $(\mathcal{S}, \mathcal{A})$ is a J-ternary algebra (Proposition 4.3), and the Jordan product in $\mathcal{S}$ satisfies, for any $a \in \mathcal{S}$, the following:

$$a^{\cdot 2} = asa = \frac{1}{\mathrm{n}_1(s)}(as^\sharp a) = \frac{1}{\mathrm{n}_1(s)}\big(Q(a)s - Q(a, s)a\big) = Q(a, c)c - Q(a)c, \qquad (4.14)$$

where we have used [4, (3.7)]. Therefore $\mathcal{S}$ is the Jordan algebra denoted by $\mathcal{J}ord(Q, c)$ in [23, I.3.7] (a *quadratic factor* in the notation there), or the *Jordan algebra of the quadratic form* $-Q|_{(\mathbb{F}s)^\perp}$ in the notation of [21, I.4]. The unity is the element c. (Here $(\mathbb{F}s)^\perp$ denotes the subspace of $\mathcal{S}$ orthogonal to s.)

Our structurable algebra $\mathcal{A} = \mathcal{C}_1 \otimes \mathcal{C}_2$ is a special module for the Jordan algebra $(\mathcal{S}, \cdot)$ with $a \bullet x = a(sx)$. For any $a \in \mathcal{S}$ orthogonal to s (relative to Q), and any $x \in \mathcal{A}$ we get

$$a \bullet (a \bullet x) = a^{\cdot 2} \bullet x = -Q(a)c \bullet x = -Q(a)x,$$

and hence the action of $\mathcal{S}$ on $\mathcal{A}$ extends to a homomorphism of associative algebras

$$\theta : \mathfrak{Cl}\big((\mathbb{F}s)^\perp, -Q|_{(\mathbb{F}s)^\perp}\big) \to \mathrm{End}_{\mathbb{F}}(\mathcal{A}), \qquad (4.15)$$

where $\mathfrak{Cl}\big((\mathbb{F}s)^\perp, -Q|_{(\mathbb{F}s)^\perp}\big)$ is the associated Clifford algebra. This is the unital special universal envelope of the Jordan algebra $\mathcal{S}$ [21, II.11]. Actually, the Jordan algebra $(\mathcal{S}, \cdot)$ embeds in $\mathfrak{Cl}\big((\mathbb{F}s)^\perp, -Q|_{(\mathbb{F}s)^\perp}\big)^{(+)}$, with $\alpha c + u \in \mathcal{S}$ being sent to $\alpha 1 + u \in \mathfrak{Cl}\big((\mathbb{F}s)^\perp, -Q|_{(\mathbb{F}s)^\perp}\big)$, for any $\alpha \in \mathbb{F}$ and $u \in (\mathbb{F}s)^\perp$ (recall that c is the unity of the Jordan algebra $(\mathcal{S}, \cdot)$).

Remark 4.4 The Clifford algebra $\mathfrak{Cl}\big((\mathbb{F}s)^\perp, -Q|_{(\mathbb{F}s)^\perp}\big)$ is isomorphic to the even Clifford algebra $\mathfrak{Cl}^+\big(\mathcal{S}, Q\big)$. The natural isomorphism takes any $a \in (\mathbb{F}s)^\perp$ to the element $\frac{1}{\mathrm{n}_1(s)}a \diamond s$, where $\diamond$ denotes the (associative) product in $\mathfrak{Cl}\big(\mathcal{S}, Q\big)$.

If $\mathcal{C}_2$ is associative, and hence of dimension at most 4, then $\mathcal{A}$ is a free right $\mathcal{C}_2$-module of dimension 8 and the image of θ in (4.15) is contained in $\mathrm{End}_{\mathcal{C}_2}(\mathcal{A}) \simeq \mathrm{End}_{\mathbb{F}}(\mathcal{C}_1) \otimes \mathcal{C}_2$. Actually, [4, Theorem 4.5] shows that the image of θ is precisely $\mathrm{End}_{\mathcal{C}_2}(\mathcal{A})$. Moreover, if $\mathcal{C}_2$ is commutative (so its dimension is 1 or 2), by dimension count, θ gives an isomorphism $\mathfrak{Cl}\big((\mathbb{F}s)^\perp, -Q|_{(\mathbb{F}s)^\perp}\big) \simeq \mathrm{End}_{\mathcal{C}_2}(\mathcal{A})$. If $\mathcal{C}_2$ is a quaternion algebra, then $\mathfrak{Cl}\big((\mathbb{F}s)^\perp, -Q|_{(\mathbb{F}s)^\perp}\big)$ has dimension 2^9, while $\mathrm{End}_{\mathcal{C}_2}(\mathcal{A}) \simeq \mathrm{Mat}_8(\mathcal{C}_2)$ has dimension $4 \times 8^2 = 2^8$. In this case $\mathfrak{Cl}\big((\mathbb{F}s)^\perp, -Q|_{(\mathbb{F}s)^\perp}\big)$ is isomorphic to the tensor product of the even Clifford algebra, which is simple, and the two-dimensional center. The existence of θ shows that $\mathfrak{Cl}\big((\mathbb{F}s)^\perp, -Q|_{(\mathbb{F}s)^\perp}\big)$

is not simple, and hence its center is isomorphic to $\mathbb{F}\times\mathbb{F}$. It follows that $\mathfrak{Cl}\big((\mathbb{F}s)^\perp, -Q|_{(\mathbb{F}s)^\perp}\big)$ is isomorphic to $\mathrm{Mat}_8(\mathcal{C}_2) \times \mathrm{Mat}_8(\mathcal{C}_2)$.

If $\mathcal{C}_2$ is not associative, then it is an eight-dimensional Cayley algebra, and the dimension of $\mathfrak{Cl}\big((\mathbb{F}s)^\perp, -Q|_{(\mathbb{F}s)^\perp}\big)$ is 2^{13}, while the dimension of $\mathrm{End}_{\mathbb{F}}(\mathcal{A})$ is $64^2 = 2^{12}$. As in the previous case, we conclude that $\mathfrak{Cl}\big((\mathbb{F}s)^\perp, -Q|_{(\mathbb{F}s)^\perp}\big)$ is isomorphic to $\mathrm{Mat}_{64}(\mathbb{F}) \times \mathrm{Mat}_{64}(\mathbb{F})$.

Remark 4.5 In a different setting, the J-ternary algebras obtained from the structurable algebras $\mathcal{C}_1 \otimes \mathcal{C}_2$ have been considered too in [11].

5 Short $(\mathbf{SL}_2 \times \mathbf{SL}_2)$-structures on finite-dimensional simple Lie algebras

The aim of this section is to take advantage of the description of the short $\mathbf{SL}_2$-structures on the finite-dimensional simple Lie algebras in [25], obtained by a careful analysis of the 5-gradings on these algebras, together with Theorem 3.3, to describe the short $(\mathbf{SL}_2 \times \mathbf{SL}_2)$-structures on these algebras. All these structures are obtained from the examples considered in the previous section.

Throughout the section, the ground field $\mathbb{F}$ will be assumed to be algebraically closed and of characteristic 0.

5.1 Special linear Lie algebras

The results in [25, Sect. 4.2] show that for a finite-dimensional simple Lie algebra of type A (special linear), the only $\mathbf{SL}_2$-structures are given by viewing it as $\mathfrak{sl}\big((V\otimes W)\oplus Z\big)$, where V is our two-dimensional vector space endowed with a nonzero alternating bilinear form $(.\mid .)$, and W and Z are two other vector spaces. In this case, the associated J-ternary algebra $(\mathcal{J}, \mathcal{T})$ is given in Sect. 4.1 (prototypical example), with $\mathcal{J} = \mathrm{End}_{\mathbb{F}}(W)^{(+)}$ and $\mathcal{T} = (W\otimes Z^*)\oplus(W^*\otimes Z)$.

The proper idempotents $e \neq 0, 1$ of $\mathrm{End}_{\mathbb{F}}(W)^{(+)}$ exist only if $\dim_{\mathbb{F}} W > 1$, and they are just the projections relative to a splitting $W = W_1 \oplus W_0$: $e(w_1) = w_1$ and $e(w_0) = 0$, for $w_1 \in W_1$ and $w_0 \in W_0$. For any such idempotent there is an associated short $(\mathbf{SL}_2 \times \mathbf{SL}_2)$-structure, and the components $\mathcal{J}_1$, $\mathcal{J}_0$, $\mathcal{J}_{\frac12}$ in (3.1) are the Peirce components relative to e: $\mathcal{J}_1 = \mathrm{End}_{\mathbb{F}}(W_1)^{(+)}$, $\mathcal{J}_0 = \mathrm{End}_{\mathbb{F}}(W_0)^{(+)}$, and $\mathcal{J}_{\frac12} = \mathrm{Hom}_{\mathbb{F}}(W_1, W_0) \oplus \mathrm{Hom}_{\mathbb{F}}(W_0, W_1)$; while $\mathcal{T}_i = (W_i \otimes Z^*) \oplus (W_i^* \otimes Z)$, for $i = 1, 0$.

5.2 Orthogonal Lie algebras

Also, the results in [25, Sect. 4.2] show that for a finite-dimensional simple Lie algebra of type B or D (orthogonal), the only $\mathbf{SL}_2$-structures are given by viewing it as $\mathfrak{so}\big((V\otimes W)\perp Z\big)$, with V as above, W endowed with a nondegenerate skew-symmetric bilinear form b_W, and Z endowed with a nondegenerate symmetric bilinear form b_Z. The symmetric bilinear form on $(V\otimes W)\perp Z$ is the orthogonal sum of $(.\mid .)\otimes \mathrm{b}_W$ and b_Z. In this case, Sect. 4.1 shows that the associated J-ternary algebra $(\mathcal{J}, \mathcal{T})$ is given by $\mathcal{J} = \mathcal{H}\big(\mathrm{End}_{\mathbb{F}}(W), \tau\big)$, where τ is the symplectic involution relative to b_W, and $\mathcal{T} = W \otimes Z$.

The Jordan algebra $\mathcal{J} = \mathcal{H}\big(\mathrm{End}_{\mathbb{F}}(W)^{(+)}, \tau\big)$ contains proper idempotents if and only if the dimension of W is at least 4 (note that this dimension is always even). Any proper idempotent is the projection relative to an orthogonal decomposition $W = W_1 \perp W_0$

relative to b_W. For any such idempotent, the components $\mathcal{J}_1$, $\mathcal{J}_0$, $\mathcal{J}_{\frac{1}{2}}$ in (3.1) are the Peirce components relative to e: $\mathcal{J}_1 = \mathcal{H}\big(\mathrm{End}_{\mathbb{F}}(W_1)^{(+)}, \tau\big)$, $\mathcal{J}_0 = \mathcal{H}\big(\mathrm{End}_{\mathbb{F}}(W_0)^{(+)}, \tau\big)$, and $\mathcal{J}_{\frac{1}{2}} = \{f \in \mathcal{H}\big(\mathrm{End}_{\mathbb{F}}(W)^{(+)}, \tau\big) \mid f(W_1) \subseteq W_0,\ f(W_0) \subseteq W_1\}$; while $\mathcal{T}_i = W_i \otimes Z$, $i = 1, 0$.

5.3 Symplectic Lie algebras

Finally, [25, Sect. 4.2] shows that for a finite-dimensional simple Lie algebra of type C (symplectic), the only $\mathbf{SL}_2$-structures are given by viewing it as $\mathfrak{sp}\big((V \otimes W) \perp Z\big)$, with V as above, W endowed with a nondegenerate symmetric bilinear form b_W, and Z endowed with a nondegenerate skew-symmetric bilinear form b_Z. The skew-symmetric bilinear form on $(V \otimes W) \perp Z$ is the orthogonal sum of $(. \mid .) \otimes b_W$ and b_Z. In this case, Sect. 4.1 shows that again the associated J-ternary algebra $(\mathcal{J}, \mathcal{T})$ is given by $\mathcal{J} = \mathcal{H}\big(\mathrm{End}_{\mathbb{F}}(W), \tau\big)$, where τ is the orthogonal involution relative to b_W, and $\mathcal{T} = W \otimes Z$.

The Jordan algebra $\mathcal{J} = \mathcal{H}\big(\mathrm{End}_{\mathbb{F}}(W)^{(+)}, \tau\big)$ contains proper idempotents if and only if $\dim_{\mathbb{F}} W > 1$. Any proper idempotent is again the projection relative to an orthogonal decomposition $W = W_1 \perp W_0$ relative to b_W. For any such idempotent, the components $\mathcal{J}_1$, $\mathcal{J}_0$, $\mathcal{J}_{\frac{1}{2}}$ in (3.1) are the Peirce components relative to e: $\mathcal{J}_1 = \mathcal{H}\big(\mathrm{End}_{\mathbb{F}}(W_1)^{(+)}, \tau\big)$, $\mathcal{J}_0 = \mathcal{H}\big(\mathrm{End}_{\mathbb{F}}(W_0)^{(+)}, \tau\big)$, and $\mathcal{J}_{\frac{1}{2}} = \{f \in \mathcal{H}\big(\mathrm{End}_{\mathbb{F}}(W)^{(+)}, \tau\big) \mid f(W_1) \subseteq W_0,\ f(W_0) \subseteq W_1\}$; while $\mathcal{T}_i = W_i \otimes Z$, $i = 1, 0$.

5.4 Exceptional Lie algebras

The results in [25, Sect. 4.3] show that for any of the finite-dimensional exceptional simple Lie algebras there is a short $\mathbf{SL}_2$-structure whose associated Jordan algebra $\mathcal{J}$ is just the ground field $\mathbb{F}$. Then $\mathcal{T}$, endowed with the skew-symmetric bilinear form $\langle . \mid . \rangle$ and the triple product $[x, y, z] := d_{x,y}(z)$ is a *symplectic triple system* (see [28]). Alternatively (see [14, Theorem 4.7]), the triple product $\{x, y, z\} := d_{x,y}(z) - \langle x \mid z\rangle y - \langle y \mid z\rangle x$, is either trivial or endows $\mathcal{T}$ with the structure of a *Freudenthal triple system* (see [24]). The classification of the finite-dimensional simple symplectic triple systems may be consulted in [13, Theorem 2.21]. With one exception, they are obtained from the simple structurable algebras whose subspace of skew-symmetric elements has dimension one.

Apart from the short $\mathbf{SL}_2$-structures above, where the Jordan algebra $\mathcal{J}$ has dimension 1 and hence has no proper idempotent, [25, Sect. 4.3] shows that there is no other short $\mathbf{SL}_2$-structure for the simple Lie algebra of type G_2, and exactly one other short $\mathbf{SL}_2$-structure in the remaining cases: F_4, E_6, E_7, and E_8. Therefore, Sect. 4.2 shows that the associated J-ternary algebra $(\mathcal{J}, \mathcal{T})$ can be obtained from a structurable algebra $\mathcal{A} = \mathcal{C}_1 \otimes \mathcal{C}_2$, for a Cayley algebra $\mathcal{C}_1$ and a unital composition algebra $\mathcal{C}_2$ of dimension 1 for case F_4, dimension 2 for E_6, dimension 4 for E_7, and dimension 8 for E_8. In all these cases, the Jordan algebra $\mathcal{J}$ is the Jordan algebra of a quadratic form, and there is a unique orbit, under its automorphism group, of proper idempotents. As above, for any such idempotent, the components $\mathcal{J}_1$, $\mathcal{J}_0$, $\mathcal{J}_{\frac{1}{2}}$ in (3.1) are the Peirce components relative to e: $\mathcal{J}_1 = \mathbb{F}e$, $\mathcal{J}_0 = \mathbb{F}(1-e)$ [recall that the unity 1 is the element c in (4.14)], and $\mathcal{J}_{\frac{1}{2}} = \big(\mathbb{F}e + \mathbb{F}(1-e)\big)^{\perp}$; while $\mathcal{T}_i = \{x \in \mathcal{T} \mid e \bullet x = ix\}$, $i = 1, 0$.

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