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ORIGINAL PAPER

# One dimensional equisymmetric strata in moduli space with genus 1 quotient surfaces

S. Allen Broughton[1] · Antonio F. Costa[2] · Milagros Izquierdo[3]

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**Abstract**

The complex orbifold structure of the moduli space of Riemann surfaces of genus $g$ ($g \geq 2$) produces a stratification into complex subvarieties named equisymmetric strata. Each equisymmetric stratum is formed by the surfaces where the group of automorphisms acts in a topologically equivalent way. The Riemann surfaces in the equisymmetric strata of dimension one are of two structurally different types. Type 1 equisymmetric strata correspond to Riemann surfaces where the group of automorphisms produces a quotient surface of genus zero, while those of Type 2 appear when such a quotient is a surface of genus one. Type 1 equisymmetric strata have been extensively studied by the authors of the present work in a previous recent paper, we now focus on Type 2 strata. We first establish the existence of such strata and their frequency of occurrence in moduli spaces. As a main result we obtain a complete description of Type 2 strata as coverings of the sphere branched over three points (Belyi curves) and where certain isolated points (punctures) have to be eliminated. Finally, we study in detail the doubly infinite family of Type 2 strata whose automorphism groups have order the product of two primes.

**Keywords** Riemann surface · Moduli space · Automorphism group · Belyi curve

**Mathematics Subject Classification** 14H15 · 30F60 · 32G15

✉ Antonio F. Costa
acosta@mat.uned.es

S. Allen Broughton
brought@rose-hulman.edu

Milagros Izquierdo
milagros.izquierdo@liu.se

1 Rose-Hulman Institute of Technology, Terre Haute, Indiana, USA

2 Universidad Nacional de Educación a Distancia, Madrid, Spain

3 Linköping University, Linköping, Sweden

## 1 Introduction

Let $g$ be an integer $\geq 2$, the moduli space $\mathcal{M}_g$ is the space of analytic structures on a closed topological surface of genus $g$, with suitable analytic structure. The Teichmüller space $\mathbb{T}_g$ is the space of hyperbolic geometric structures modulo isotopy and it is analytically equivalent to an open subset of $\mathbb{C}^{3g-3}$ (L. Bers embedding). The mapping class group $M_g$ acts upon $\mathbb{T}_g$ as a properly discontinuous group of bi-holomorphic transformations, more detail is given in Sect. 2. The quotient map $\mathbb{T}_g \to \mathbb{T}_g/M_g$ yields a regular, branched covering $\pi : \mathbb{T}_g \to \mathcal{M}_g$ providing a structure of a complex orbifold on $\mathcal{M}_g$ of complex dimension $3g - 3$, and whose singularity set is called the *branch locus*. (Specifically, the branch locus is the image in $\mathcal{M}_g$ of those points in $\mathbb{T}_g$ with non-trivial stabilizers.) The singularity structure of a complex orbifold produces a stratification of the moduli space into a finite disjoint union of *equisymmetric strata*. Each equisymmetric stratum corresponds to a collection of surfaces whose automorphism groups act in a *topologically equivalent* way (see Refs. [2, 3] and [12]).

The zero dimensional strata correspond to the well-studied *quasi-platonic surfaces*. At the other extreme are the open, dense stratum of surfaces with *no automorphisms*, of complex dimension $3g - 3$, and the *hyperelliptic locus* of dimension $2g - 1$.

In this paper we explore the topology of the complex 1-dimensional strata, which are punctured Riemann surfaces. Our objective is to describe these strata explicitly, as punctured Riemann surfaces, in terms of the data of the action of the automorphism group.

Let $\mathfrak{S}$ be a stratum. For every $S \in \mathfrak{S}$, $S/\mathrm{Aut}(S)$ is a surface of genus $h$ and $S \to S/\mathrm{Aut}(S)$ is branched over $r$ points. Hence the stratum $\mathfrak{S}$ is a "covering" of the moduli space $\mathcal{M}_{h,r}$ of complex 2-orbifolds of genus $h$ and $r$ conic points. (The "covering" may be branched and may not be surjective.) The moduli space $\mathcal{M}_{h,r}$ is a complex orbifold of dimension $3h-3+r$ (see [13]) and this number gives the dimension of the stratum $\mathfrak{S}$. If we assume $\dim_{\mathbb{C}} \mathfrak{S} = 1$, either $h = 0, r = 4$ or $h = 1, r = 1$. So, there are two types of strata of dimension 1:

**Type 1:** If $S$ is a surface in the stratum, $S/\mathrm{Aut}(S)$ is the sphere and $S \to S/\mathrm{Aut}(S)$ is branched over four points.

**Type 2:** If $S$ is a surface in the stratum, $S/\mathrm{Aut}(S)$ is a torus and $S \to S/\mathrm{Aut}(S)$ is branched over one point.

The first type has been studied in detail in [5]. In this paper we address strata of Type 2.

There are strata formed by surfaces that are regular branched coverings of a torus that are apparently of Type 2 but are actually of the first type. This phenomenon occurs because of the algebraic structure of the automorphism group of the covering (Sect. 3.1) and because the action of the mapping class group is not effective on the Teichmüller space for some types of orbifolds (see [13]). In Examples 3.2, 3.9 we present a case of an automorphism group and corresponding stratum that is of Type 2, so showing the existence of such strata. Immediately following, in Sect. 3.2, we present a summary enumeration of surfaces with small automorphism groups that are branched regular coverings of a torus and how many are Type 1 or Type 2.

In Sect. 3.3 we obtain a description of the strata of Type 2 (similar to that obtained in reference [5] for the strata of Type 1) in terms of the action of the automorphism group of the surfaces: see Theorem 3.5. As a consequence we have that all the equisymmetric strata of dimension one are Belyi curves with punctures: see Corollary 3.7.

Finally, in Sect. 4, we give a detailed description of a bi-infinite family of Type 2 strata, generalizing the case presented in Examples 3.2, 3.9.

## 2 Preliminaries

### 2.1 Riemann surfaces and coverings

A Riemann surface is a connected surface endowed with a complex analytical structure. Let $S$ be a compact Riemann surface of genus $g \geq 2$ and assume that $G$ is a group of automorphisms of $S$, i.e, $G \leq \mathrm{Aut}(S)$. Hence $S/G$ is an orbifold, and there is a Fuchsian group $\Gamma \leq PSL(2, \mathbb{R})$, such that:

$$\mathcal{H} \to S = \mathcal{H}/\pi_1(S) \to S/G = \mathcal{H}/\Gamma.$$

The group $\Gamma$ is the lifting of $G$ to the universal covering $\mathcal{H}$. Notice that $\Gamma$ is isomorphic to the orbifold fundamental group of $S/G$.

The Fuchsian group $\Gamma$ is isomorphic to an abstract group $\mathcal{G}_s$ with presentation

$$\left\langle \alpha_1, \beta_1, \ldots, \alpha_h, \beta_h, \gamma_1, \ldots, \gamma_r : \prod_{i=1}^{h}[\alpha_i, \beta_i] \prod_{i=1}^{r} \gamma_i = 1, \gamma_1^{m_1} = \cdots = \gamma_r^{m_r} = 1 \right\rangle \tag{2.1}$$

and we say that $\Gamma$ has *signature*

$$s = (h; m_1, \ldots, m_r). \tag{2.2}$$

There is a surface Fuchsian group $\Delta \trianglelefteq \Gamma$ (that is, a Fuchsian group with signature $(g; -)$), isomorphic to $\pi_1(S)$, and there is an epimorphism $\xi : \Gamma \to G$ such that $\ker \xi = \Delta$ (the monodromy of the covering, see [16]).

Given two Riemann surfaces $S_1$ and $S_2$ and two automorphism groups $G_1 \leq \mathrm{Aut}(S_1)$ and $G_2 \leq \mathrm{Aut}(S_2)$, we will say that the action of $G_1$ is topologically equivalent to the action of $G_2$ if there is a homeomorphism $h : S_1 \to S_2$ such that $G_2 = hG_1h^{-1}$. In terms of monodromies: if $\xi_1 : \Gamma_1 \to \Gamma_1/\Delta_1 \cong G_1$ and $\xi_2 : \Gamma_2 \to \Gamma_2/\Delta_2 \cong G_2$, the actions of $G_1$ and $G_2$ are topologically equivalent if there are isomorphisms $\sigma : \Gamma_1 \to \Gamma_2$ and $\tau : G_1 \to G_2$ such that $\xi_2 = \tau \circ \xi_1 \circ \sigma^{-1}$.

Assume now that the quotient orbifold $S/G = T_n$ is a torus with a conic point of order $n$ ($n > 1$) with a branched, holomorphic covering:

$$\pi_G : S \to S/G = T_n,$$

such that $\pi_G$ is branched over a point $P$ with branch index $n$. The branched covering $\pi_G : S \to T_n$ is determined by a monodromy $\xi : \pi_1 O(T_n) \to G$, where $\pi_1 O(T_n)$ is the orbifold fundamental group of $T_n$.

Hence, given a finite group $G$ and a fixed canonical presentation of $\pi_1 O(T_n)$:

$$\left\langle \alpha, \beta, \gamma : [\alpha, \beta]\gamma = 1; \gamma^n = 1 \right\rangle,$$

each monodromy $\xi : \pi_1 O(T_n) \to G$ is given by a vector

$$(\xi(\alpha), \xi(\beta), \xi(\gamma)) = (a, b, c)$$

with $[a, b]c = 1$, $c^n = 1$, and $(a, b)$ is a set of generators of $G$.

Let $g^G$ denote the conjugacy class of an element $g$ of order $n$ in $G$. For later use, we shall consider the following subsets of $G \times G$ and set of classes:

$$K_G(1:g^G) = \{(a,b) : [a,b]^{-1} \in g^G\}$$
$$K_G^{\circ}(1:g^G) = vectors\ in\ K_G(1:g^G) generating\ G$$
$$\widetilde{K_G^{\circ}}(1:g^G) = \{\mathrm{Aut}(G)\ -\ \text{classes of } K_G^{\circ}(1:g^G)\},$$

(see [5]).

### 2.2 Teichmüller and moduli spaces

The group $\mathcal{G}_s$, given in (2.1) is an abstract group isomorphic to all the Fuchsian groups of signature $s = (h; m_1, ..., m_r)$. The Teichmüller space of Fuchsian groups of signature $s$ is:

$$\mathbb{T}_s = \{\rho : \mathcal{G}_s \to PSL(2,\mathbb{R}) : s(\rho(\mathcal{G}_s)) = s\}/\text{conjugation in } PSL(2,\mathbb{R})$$

(see [13]). The Teichmüller space $\mathbb{T}_s$ has complex dimension $3g-3+r$ and is homeomorphic to an open ball.

Let $S$ be a compact Riemann surface of genus $g \geq 2$. If we have a normal inclusion $\pi_1(S) \lhd \mathcal{G}_s$, then there is a topological action of the group $G = \mathcal{G}_s/\pi_1(S)$ on surfaces of genus $g$. The inclusion $i : \pi_1(S) \to \mathcal{G}_s$ produces $i_*(\mathbb{T}_s) \subset \mathbb{T}_g$ in the following way: if $[\rho] \in \mathbb{T}_s$, $\rho : \mathcal{G}_s \to PSL(2,\mathbb{R})$, we define $i_*[\rho] = [\rho \circ i]$ where

$$i : \pi_1(S) \to \mathcal{G}_s\ ,\ \rho \circ i : \pi_1(S) \to PSL(2,\mathbb{R})$$

(see [12] and [13]).

The modular group, $M_s$, for the signature $s$ of a Fuchsian group $\Gamma$, is the mapping class group of $\mathcal{H}/\Gamma$, which in turn equals the group of orientation-preserving outer automorphisms of $\mathcal{G}_s$:

$$M_s = \mathrm{Aut}^+(\mathcal{G}_s)/\mathrm{Inn}(\mathcal{G}_s),$$

where $\mathrm{Inn}(\mathcal{G}_s)$ is the group of inner automorphisms of $\mathcal{G}_s$ (see [13]). The modular group $M_s$ acts upon $\mathbb{T}_s$ as $[\rho] \to \mu_*([\rho]) = [\rho \circ \mu^{-1}]$ where $\mu \in M_s$. The moduli space of signature $s$ is the quotient space

$$\mathcal{M}_s = \mathbb{T}_s/M_s.$$

The image of $i_*(\mathbb{T}_s)$ by $\mathbb{T}_g \to \mathcal{M}_g$ is $\overline{\mathcal{M}}^{G,a}$, where $\overline{\mathcal{M}}^{G,a}$ is the set of Riemann surfaces with automorphism group containing a subgroup acting in a topologically equivalent way to the action of $G$ on $S$ given by the inclusion $i$.

Now, given $\mu \in M_s$ and an inclusion $i : \pi_1(S) \to \mathcal{G}_s$ we have a map $i^*(\mu_*) : i_*(\mathbb{T}_s) \subset \mathbb{T}_g \to i_*(\mathbb{T}_s) \subset \mathbb{T}_g$ given by $i^*(\mu_*)[\rho \circ i] = [\rho \circ \mu^{-1} \circ i]$.

## 3 Type 2: orbit space of genus 1

First of all, we note that there are coverings $S \to S/G$, branched over a point and where $S/G$ is a torus, that actually define a stratum of the Type 1. These strata do not produce anything new from what has already been studied in [5]. Our first objective will be to characterise the situations in which a monodromy given by a class in $\widetilde{K_G^{\circ}}(1:g^G)$ actually defines a stratum of Type 1, and to obtain an example of a stratum of Type 2 monodromy that cannot be extended to Type 1.

### 3.1 Strata of Type 1 coming from apparent Type 2 strata

Let $S$ be a surface with a group of automorphisms $G$ such that $S/G$ is a torus with a single conic point. Consider the monodromy of $\pi_G : S \to S/G = T_n$:

$$\xi : \pi_1 O(T_n) = \left\langle \alpha, \beta, \gamma : \alpha\beta\alpha^{-1}\beta^{-1}\gamma = 1, \gamma^n = 1 \right\rangle \to G. \tag{3.1}$$

We note that the surface $S/G$ has an elliptic involution $\iota : T_n \to T_n$ fixing the conic point $P$ and three other points. Choosing the base point $P_0$ to be an appropriately selected fixed point of $\iota$, we have $\iota_* : \pi_1 O(T_n) \to \pi_1 O(T_n)$:

$$\iota_*(\alpha) = \alpha^{-1}, \iota_*(\beta) = \beta^{-1}, \iota_*(\gamma) = (\alpha\beta)^{-1}\gamma\alpha\beta \tag{3.2}$$

The quotient $T_n/\langle\iota\rangle$ is the sphere and $T_n \to T_n/\langle\iota\rangle$ has four branch points. The orbifold $T_n/\langle\iota\rangle$ has signature $(0; 2, 2, 2, 2n)$.

If the group $G$ has an automorphism $G \to G$ given by:

$$\xi(\alpha) \longmapsto \xi(\alpha)^{-1}; \xi(\beta) \longmapsto \xi(\beta)^{-1},$$

then the group $G$ is not the full group of automorphisms of the surfaces in the stratum (See also [15]). There is a group $H \supset G$, such that $H$ acts on $S$ and $S$ is in a stratum of Type 1, with orbit space of genus 0, i.e. $S/H$ is a sphere, and $S \to S/H$ is branched over 4 points with branch indices $(0; 2, 2, 2, 2n)$ (see [6] and [7]).

Reciprocally, if the action of the group $G$ can be extended to a group of automorphisms $H \supset G$ such that $S/H$ is a sphere and $S \to S/H$ is branched over 4 points with branch indices $(0; 2, 2, 2, 2n)$, then there in an automorphism $G \to G$ given by:

$$\xi(\alpha) \longmapsto \xi(\alpha)^{-1},\ \xi(\beta) \longmapsto \xi(\beta)^{-1} \tag{3.3}$$

***Example 3.1*** Consider the one dimensional stratum in $\mathcal{M}_4$ given by the monodromy:

$$\begin{aligned}
&\xi : \pi_1 O(T_5) = \left\langle \alpha, \beta, \gamma : \alpha\beta\alpha^{-1}\beta^{-1}\gamma = 1, \gamma^5 = 1 \right\rangle \\
&\to D_5 = \left\langle s, r : s^2 = r^5 = (sr)^2 = 1 \right\rangle \\
&\xi(\alpha) = s;\ \xi(\beta) = r; \xi(\gamma) = r^2
\end{aligned}$$

Since the group $D_5$ admits the automorphism $s \to s^{-1} = s$ and $r \to r^{-1}$ the action can be extended to an action of $D_{10}$ with quotient orbifold with signature $(0; 2, 2, 2, 10)$ (see [1]).

The following example shows that there are finite groups that do not admit such an automorphism $\xi(\alpha) \longmapsto \xi(\alpha)^{-1}, \xi(\beta) \longmapsto \xi(\beta)^{-1}$.

***Example 3.2*** Let us consider

$$G = C_q \rtimes C_3 = \left\langle s, t : s^q = t^3 = 1, t^2st = s^u \right\rangle$$

where $q \equiv 1 \bmod 3$, $q$ prime and with $u$ satisfying $u^2 \equiv -u-1 \bmod q$ (example: $q = 7, u = 2$).

By a straightforward computation and the above presentation of $C_q \rtimes C_3$, there is no automorphism $\alpha : C_q \rtimes C_3 \to C_q \rtimes C_3$ such that:

$$\alpha(ts) = (ts)^{-1}, \alpha(s) = s^{-1}$$

The stratum $\mathfrak{S}$ composed of surfaces of genus $\frac{3q-1}{2}$ that are coverings of tori with monodromy:

$$\xi : \pi_1 O(T_q) = \langle \alpha, \beta, \gamma : \alpha\beta\alpha^{-1}\beta^{-1}\gamma = 1, \gamma^q = 1 \rangle \to C_q \rtimes C_3$$
$$\xi(\alpha) = ts, \xi(\beta) = s, \xi(\gamma) = s^{u+2}$$

is of Type 2 but not of Type 1, since the action cannot be extended.

### 3.2 How many Type 1 and Type 2 actions?

In the two preceding examples we have shown that actions with signature $(1; n)$ can result in Type 1 and Type 2 actions and strata. We may use computer calculation to find a generous set of examples of $(1; n)$ actions of various groups $G$ that lift to $(0; 2, 2, 2, 2n)$ actions (extensible) and those that do not (inextensible). We used the Magma [14] small group data base to systematically search the 6064 groups with order in the range $2 \leq |G| \leq 200$ for such actions. The main purpose of our calculation was to get some sense on the proportion of Type 1 vs Type 2. The results are in Tables 1 and 2. The Magma code for the calculations are on this website [4].

Before explaining the tables, let us preview a construction of a stratum model $\mathcal{B}$ that we will use in Theorem 3.5, so that we can see the general nature of the model and its use in studying group actions on surfaces. Let $s$ be any signature and $\mathcal{G}_s$, $\mathbb{T}_s$, $M_s$, $\mathcal{M}_s$ be the abstract group and spaces introduced in Sect. 2.2. We define the sets $K_G^\circ(s)$ and $\widetilde{K_G^\circ}(s)$ similarly to the sets defined in Sect. 2. These $K$-sets are in 1-1 correspondence with monodromies $\xi : \mathcal{G}_s \to G$ and their $\mathrm{Aut}(G)$ classes. The action of $M_s$ on $\widetilde{K_G^\circ}(s)$ is defined through the action on monodromies. We define:

$$\mathcal{B}_{G,s} = \left(\mathbb{T}_s \times \widetilde{K_G^\circ}(s)\right)/M_s,$$

**Table 1** Action classes

| $\|G\|$ | # Groups | Type1 | Type2 | Total | % Type1 | % Type2 |
|---|---|---|---|---|---|---|
| 2–50 | 264 | 858 | 120 | 978 | 87.7 | 12.3 |
| 51–100 | 791 | 3207 | 740 | 4037 | 81.7 | 18.3 |
| 101–150 | 2834 | 6250 | 1624 | 7874 | 79.4 | 20.6 |
| 151–200 | 2183 | 10235 | 3320 | 13555 | 75.5 | 24.5 |
| Total | 6064 | 20640 | 5804 | 26444 | 78.1 | 21.9 |

**Table 2** Number of strata

| $\|G\|$ | # Groups | Type1 | Type2 | Total | % Type1 | % Type2 |
|---|---|---|---|---|---|---|
| 2–50 | 256 | 112 | 8 | 120 | 93.3 | 6.7 |
| 51–100 | 791 | 276 | 37 | 313 | 88.2 | 11.8 |
| 101–150 | 2834 | 408 | 55 | 463 | 88.1 | 11.9 |
| 151–200 | 2183 | 532 | 105 | 637 | 83.5 | 16.5 |
| Total | 6064 | 1328 | 205 | 1533 | 86.6 | 13.4 |

with covering map

$$\eta_{G,s} : \mathcal{B}_{G,s} \to \mathcal{M}_s, ([\rho], [v]) \to [\rho]M_s,$$

where $[\rho]M_s$ is the $M_s$ equivalence class of $[\rho]$, and $[v]$ is an $\mathrm{Aut}(G)$ class of generating vectors corresponding to an $\mathrm{Aut}(G)$ class of monodromies $\xi$. Each element of $\mathcal{B}_{G,s}$ corresponds uniquely to a conformal equivalence class of quotient maps $\mathcal{H}/\rho(\ker \xi) \to \mathcal{H}/\rho(\mathcal{G}_s)$, and every such map is captured by $\mathcal{B}_{G,s}$. In other words, $\mathcal{B}_{G,s}$ classifies, uniquely, conformal equivalence classes of $G$ actions with signature $s$, up to $\mathrm{Aut}(G)$ equivalence. The orbits of $\mathrm{M}_s$ on $\widetilde{K_G^\circ}(s)$ correspond to the components of $\mathcal{B}_{G,s}$. The behaviour of $\eta_{G,s}$ over the singular set classifies those surfaces $S$ for which there is a **normalizing** overgroup $H \rhd G$ of automorphisms of $S$. Non-normal overgroups need to be dealt with in other ways.

*Modular Companions*

Any two surfaces $S_1$ and $S_2$ lying over the same quotient $\mathcal{H}/\rho(\mathcal{G}_s)$, but not conformally equivalent, correspond to distinct points $([\rho], [v])$ and $\mu_*([\rho], [v])$ for some $\mu \in M_s$. The quotient orbifolds are the same since $\rho(\mathcal{G}_s)$ and $\rho(\mu(\mathcal{G}_s))$ are conjugate subgroups. But the surfaces are not conformally equivalent since the kernels $\ker \xi_{([\rho],[v])}$, $\ker \xi_{\mu_*([\rho],[v])}$ are different. We call the surfaces $S_1$ and $S_2$ *modular companions* since they have the same quotient orbifolds and their monodromies are modularly equivalent. They both lie in a single fibre of $\eta_{G,s}$ but in the same connected component of $\mathcal{B}_{G,s}$. It can be shown that for such surfaces there is a closed loop in $\mathcal{M}_s$ such that the surface $S_2$ is obtained from $S_1$ by "analytic continuation" along the path.

*Enumeration of group action classes*

Our first enumeration summary is the number of $\mathrm{Aut}(G)$ classes of $(1; n)$ generating vectors of $G$, with $n$ varying over all possibilities. There may be several classes for the same group, they are all counted separately. For a given group $G$, representatives of each $\mathrm{Aut}(G)$ class of $(1; n)$ generating vectors are computed, and then tested to see if there is a lift of the elliptic involution. For each group, a summary file was produced listing representatives, the automorphism that enabled the lift of the elliptic involution (if it existed), and the genus of the surface produced. For each genus so calculated the number of classes of vectors were recorded and enumerated. The genus counts are incomplete since

$$|G| = \frac{2g-2}{1-\frac{1}{n}},$$

and for a given genus $g$, $|G|$ could be out of range. Many groups tested were rejected out of hand without calculating: abelian, requiring more than 2 elements to generate $G$, or an automorphism group that is too large. Since $G = \langle a, b\rangle$ for a generating vector $(a, b, c)$, an automorphism class of generating vectors is of size at most $|G|^2$ and so $|\mathrm{Aut}(G)| \le |G|^2$ is a restriction.

*Enumeration of "strata"*

Our next calculation was the number of strata, or more precisely the number of components of $\mathcal{B}_{G,s}$. For those actions that have a lift of the elliptic involution, the components of $\mathcal{B}_{G,s}$ are not technically strata though each element of such a component defines a unique action on $S$ of a unique overgroup $H$ with $G \lhd H$, $|H/G| = 2$, and signature $(2, 2, 2, 2n)$. We did not determine the exact relation between the components of $\mathcal{B}_{G,s}$ and the strata of the $(2, 2, 2, 2n)$ action of $H$, though they are closely related.

Note the increasing percentage of Type 2 strata shown in the last columns in Tables 1 and 2.

***Remark 3.3*** It is a curiosity that only one group, namely SmallGroup (200,44) of order 200, had both Type 1 and Type 2 actions.

***Remark 3.4*** Any finite group $G$ that is the full group of automorphisms of surfaces in a Type 2 stratum is also the full group of automorphisms of surfaces of a Type 1 stratum. Assume $G$ has an action on genus $g$ surfaces producing a stratum of Type 2 in $\mathcal{M}_g$ and given by the monodromy:

$$\xi : \pi_1 O(T_n) = \left\langle \alpha, \beta, \gamma : \alpha\beta\alpha^{-1}\beta^{-1}\gamma = 1, \gamma^n = 1 \right\rangle \rightarrow G.$$

We can construct another monodromy

$$\xi' : \pi_1(\widehat{\mathbb{C}} - \{z_1, z_2, z_3, z_4\}) \rightarrow G$$

defined by:

$$\xi'(\gamma_1) = \xi(\alpha), \xi'(\gamma_2) = \xi(\beta), \xi'(\gamma_3) = \xi(\gamma)$$

Note that $\xi'(\gamma_4)$ cannot be 1. The monodromy $\xi'$ defines a stratum of Type 1. By the Riemann-Hurwitz formula the surfaces uniformized by $\ker \xi'$ have genus $g'$ satisfying $\frac{1}{3}g + \frac{2}{3} \leq g' < 3g - 2$.

*Note:* We are using monodromy in two different senses. One as a epimorphism from an orbifold fundamental group to $G$, the second as an epimorphism from the fundamental group of a punctured sphere to $G$.

### 3.3 Strata of Type 2

Let $S$ be a surface with a group of automorphisms $G$ such that $S/G$ is a torus and $\pi_G : S \rightarrow S/G$ is a covering branched over one point with branch index $n$. We denote $S/G = T_n$ the orbifold induced by $\pi_G$. We consider the monodromy of $\pi_G : S \rightarrow T_n$:

$$\xi : \pi_1 O(T_n) = \left\langle \alpha, \beta, \gamma : \alpha\beta\alpha^{-1}\beta^{-1}\gamma = 1, \gamma^n = 1 \right\rangle \rightarrow G.$$

*Note:* In the remainder of this section we assume that the finite group $G$ does not admit an automorphism $G \rightarrow G$ satisfying:

$$\xi(\alpha) \longmapsto \xi(\alpha)^{-1}; \xi(\beta) \longmapsto \xi(\beta)^{-1}.$$

The structures of $T_n$ are given by the moduli space $\mathcal{M}_{(1;n)}$. Note that $\mathcal{M}_{(1;n)} = \mathcal{M}_1$, i.e. the position of the branch point in $T_n$ does not play any role, all the points are conformally equivalent. Remember the well-known equality between modular groups: $M_{(1;n)} = M_1$ (pag. 54-55 of [10] or [2, 12]).

Given $a \in \mathrm{Aut}(\pi_1 O(T_n))$ we shall denote by $A$ the corresponding element in $\mathrm{Aut}(\pi_1 O(T_n))/\mathrm{Inn}(\pi_1 O(T_n))$. Let us consider the elements $x, y \in \mathrm{Aut}(\pi_1 O(T_n))$ defined in the following table: two dots and full stop, then the table and after the sentence: The

| $\theta$ | $\alpha$ | $\beta$ | $\gamma$ |
|---|---|---|---|
| $x$ | $\theta(x, \alpha) = \beta$ | $\theta(x, \beta) = \alpha^{-1}$ | $\theta(x, \gamma) = \alpha^{-1}\gamma\alpha$ |
| $y$ | $\beta$ | $\alpha^{-1}\beta$ | $\alpha^{-1}\gamma\alpha$ |
| $xy$ | $\alpha^{-1}$ | $\beta^{-1}\alpha^{-1}$ | $\beta^{-1}\alpha^{-1}\gamma\alpha\beta$ |

elements $X, Y$ given by $x, y$ provide the following presentation:

$$M_{(1;n)} = \langle X, Y : X^4 = Y^6 = 1, X^2 = Y^3 \rangle \simeq SL(2, \mathbb{Z}).$$

In $M_{(1;n)}$ we have $X^4 = Y^6 = 1$ and $X^2 = Y^3$ $(= -\,\mathrm{Id}_{SL(2,\mathbb{Z})}$, in the center of $SL(2, \mathbb{Z})$, and remark that this relation is not satisfied by $x, y$). Note that the action of $M_{(1;n)}$ is not effective in $\mathbb{T}_{(1;n)}$ (see [13], the element $X^2 = Y^3$ acts as the identity on $\mathbb{T}_{(1;n)}$).

The action $\theta$ of $M_{(1;n)}$ on $\pi_1 O(T_n)$ induces an action $\Theta$ on $\widetilde{K^\circ_G}(1 : g^G)$ via the monodromies as follows. First we define $\xi_{(a,b)} : \pi_1 O(T_n) \to G$, by $\xi_{(a,b)}(\alpha) = a$ and $\xi_{(a,b)}(\beta) = b$. If $\mu \in M_{(1;n)}$ and $[(a, b)] \in \widetilde{K^\circ_G}(1 : g^G)$, the action of $M_{(1;n)}$ on $\widetilde{K^\circ_G}(1 : g^G)$ is:

$$\Theta : (\mu, [(a, b)]) \to [(\xi_{(a,b)}(\mu^{-1}(\alpha)), \xi_{(a,b)}(\mu^{-1}(\beta)))],$$
$$([.] \text{ means } \mathrm{Aut}(G) - \text{class in } \widetilde{K^\circ_G}(1 : g^G)).$$

Note that

$$\xi_{(\xi_{(a,b)}(\mu^{-1}(\alpha)), \xi_{(a,b)}(\mu^{-1}(\beta)))} = \xi_{(a,b)} \circ \mu^{-1}$$

and $\ker \xi_{(\xi_{(a,b)}(\mu^{-1}(\alpha)), \xi_{(a,b)}(\mu^{-1}(\beta)))} = \mu(\ker \xi_{(a,b)})$.

Remark that the element $X^2 = Y^3$ does not act as the identity on $\widetilde{K^\circ_G}(1 : g^G)$ if

$$\xi(\alpha) \longmapsto \xi(\alpha)^{-1};\ \xi(\beta) \longmapsto \xi(\beta)^{-1}$$

defines an automorphism of $G$.

The moduli space $\mathcal{M}_{(1;n)}$ is a non-compact orbifold of complex dimension 1 with two conic points $P_X$ and $P_Y$ of order 2 and 3, that are the projections by $\mathbb{T}_{(1;n)} \to \mathcal{M}_{(1;n)}$ of the fixed points of $X$ and $Y$ in $\mathbb{T}_{(1;n)}$, and a puncture, denoted by $\infty$. Then $\mathcal{M}_{(1;n)} - \{P_X, P_Y\}$ is isomorphic to $\mathbb{C} - \{z_X, z_Y\}$ and

$$M_{(1;n)} / \langle X^2 = Y^3 \rangle = \pi_1 O(\mathcal{M}_{(1;n)}) \cong \pi_1(\mathbb{C} - \{z_X, z_Y\}, z_0) / \langle \gamma_X^2, \gamma_Y^3 \rangle)$$

where $\gamma_X$ and $\gamma_Y$ are represented by loops around the points $z_X, z_Y$ respectively and based at $z_0 \in \mathbb{C} - \{z_X, z_Y\}$.

### 3.4 Main theorem and consequences

**Theorem 3.5** *Let $G$ be a finite group and $[(a_0, b_0)]$ be an element of $\widetilde{K^\circ_G}(1 : g^G)$. Let $S$ be a Riemann surface, uniformized by a surface Fuchsian group $\Delta$ ($S = \mathcal{H}/\Delta$) such that $S/\mathrm{Aut}(S) = \mathcal{H}/\Gamma$ where $\Gamma$ has signature $(1; n)$. We assume that $\xi_\Gamma : \Gamma \to \Gamma/\Delta \cong G$ is a monodromy and that there is a canonical presentation of $\Gamma$*

$$\langle \alpha, \beta, \gamma : \alpha\beta\alpha^{-1}\beta^{-1}\gamma = 1, \gamma^n = 1 \rangle$$

*such that $(\xi_\Gamma(\alpha), \xi_\Gamma(\beta)) \in [(a_0, b_0)] \in \widetilde{K^\circ_G}(1 : g^G)$.*

*Let $\mathfrak{S}$ be the 1-dimensional stratum of Type 2 containing the surface $S$. Then, there is a Riemann surface $\mathcal{B}$ such that $\mathfrak{S}$ is isomorphic to $\mathcal{B} - \mathcal{I}$ where $\mathcal{I}$ is a set of isolated points in $\mathcal{B}$.*

*The surface $\mathcal{B}$ is a finite covering $\eta : \mathcal{B} \to \mathcal{M}_{(1;n)}$ branched over $\{P_X, P_Y\}$ such that:*

*(1) Let $\mathcal{O} = \{o_1 = [(a_0, b_0)], ..., o_l\}$ be the orbit of the action $\Theta$ on $\widetilde{K^\circ_G}(1 : g^G)$ containing $[(a_0, b_0)]$. The degree of the covering $\eta$ is the size of the orbit $\mathcal{O}$, i.e., $l$.*

*(2) The monodromy of the covering* $\eta : \mathcal{B} \to \mathcal{M}_{(1,n)}$ *is*

$$\omega_{\Theta} : \langle X, Y \rangle = \pi_1(\mathcal{M}_{(1;n)} - \{P_X, P_Y\}) \to M_{(1;n)} \xrightarrow{\Theta} \Sigma_{|\mathcal{O}|}$$

*where* $\Theta$ *has been defined above and* $\mathcal{B}$ *is the completion to a branched covering of* $\mathcal{M}_{(1;n)}$ *of the unbranched covering defined by* $\omega_{\Theta}^{-1}(Stab(o_1))$.

***Proof*** Let $\mathcal{G}_{(1;n)}$ be an abstract group with presentation given by the signature $(1; n)$:

$$\left\langle \alpha, \beta, \gamma : \alpha\beta\alpha^{-1}\beta^{-1}\gamma = 1, \gamma^n = 1 \right\rangle$$

Let

$$\begin{aligned} \mathcal{R} = \{\rho : \mathcal{G}_{(1;n)} \to \rho(\mathcal{G}_{(1;n)}) \leq PSL(2, \mathbb{R}) :\\ \rho(\mathcal{G}_{(1;n)}) \text{ is a Fuchsian group of signature } (1; n)\}, \end{aligned}$$

so that $\mathbb{T}_{(1;n)} = \mathcal{R}/$conjugation on $PSL(2, \mathbb{R})$. Let $\rho_\Gamma : \mathcal{G}_{(1;n)} \to \rho_\Gamma(\mathcal{G}_{(1;n)}) = \Gamma \leq PSL(2, \mathbb{R})$, where $S/\mathrm{Aut}(S) = \mathcal{H}/\Gamma$ and $\xi_\Gamma : \Gamma \to G$ where $S = \mathcal{H}/\ker \xi_\Gamma$, $(\xi_\Gamma(\alpha), \xi_\Gamma(\beta)) = (a_0, b_0)$. Hence $\xi_{(a_0,b_0)} = \xi_\Gamma \circ \rho_\Gamma : \mathcal{G}_{(1;n)} \to G$.

The actions of $M_{(1;n)}$ on $\mathbb{T}_{(1;n)}$ and $\widetilde{K_G^\circ}(1 : g^G)$ produce a natural product action on $\mathbb{T}_{(1;n)} \times \mathcal{O} = \{([\rho], [(a, b)])\}$. If $\mu \in M_{(1;n)}$,

$$\begin{aligned} \mu_*([\rho], [a, b]) &= ([\rho \circ \mu^{-1}], [(\xi_{(a,b)}(\mu^{-1}(\alpha)), \xi_{(a,b)}(\mu^{-1}(\beta)))]) \\ &= ([\rho \circ \mu^{-1}], \Theta(\mu, [(a, b)])) \\ &\in \mathbb{T}_{(1;n)} \times \widetilde{K_G^\circ}(1 : g^G). \end{aligned}$$

The quotient $\mathbb{T}_{(1;n)} \times \widetilde{K_G^\circ}(1 : g^G)$ by the action of $\Theta^{-1}(Stab(o_1))$ yields the Riemann surface $\mathcal{B}$ and the covering $\mathcal{B} \to \mathcal{M}_{(1;n)}$ is:

$$\eta : \mathcal{B} = \mathbb{T}_{(1;n)} \times \widetilde{K_G^\circ}(1 : g^G)/\Theta^{-1}(Stab(o_1)) \to \mathbb{T}_{(1;n)}/M_{(1;n)} = \mathcal{M}_{(1;n)}$$

The map $\xi_* : \mathbb{T}_{(1;n)} \times \widetilde{K_G^\circ}(1 : g^G)/M_{(1;n)} \to \mathcal{M}_g$, is given by $\xi_*([\rho], [(a, b)]) = [\mathcal{H}/\rho(\ker \xi_{(a,b)})] \in \mathcal{M}_g$.

If

$$\begin{aligned} ([\rho_\Gamma], [(a_0, b_0)]) &= \mu_*([\rho_\Gamma], [(a_0, b_0)]) \\ &= ([\rho_\Gamma \circ \mu^{-1}], [(\xi_{(a_0,b_0)}(\mu^{-1}(\alpha)), \xi_{(a_0,b_0)}(\mu^{-1}(\beta)))]) \end{aligned}$$

(i.e. $\mu \in \Theta^{-1}(Stab(o_1))$) then:

1. $[\rho_\Gamma] = [\rho_\Gamma \circ \mu^{-1}]$ and the Fuchsian groups $\Gamma$ and $\rho_\Gamma \circ \mu^{-1}(\mathcal{G}_{(1;n)})$ are conjugate in $PSL(2, \mathbb{R})$, so the Riemann orbifolds $\mathcal{H}/\Gamma$ and $\mathcal{H}/(\rho_\Gamma \circ \mu^{-1}(\mathcal{G}_{(1;n)}))$ are conformally equivalent.

2. $[(a_0, b_0)] = [\xi_{(a_0,b_0)}(\mu^{-1}(\alpha)), \xi_{(a_0,b_0)}(\mu^{-1}(\beta))]$ implies there is $\delta \in \mathrm{Aut}(G)$ such that $\xi_{(a_0,b_0)} = \delta \circ \xi_{(a_0,b_0)} \circ \mu^{-1}$, then the coverings

$$\mathcal{H}/\rho_\Gamma(\ker \xi_{(a_0,b_0)}) = \mathcal{H}/\ker \xi_\Gamma \to \mathcal{H}/\Gamma$$

and

$$\mathcal{H}/\rho_\Gamma \circ \mu^{-1}(\ker \xi_{(a_0,b_0)} \circ \mu^{-1}) \to \mathcal{H}/\rho_\Gamma \circ \mu^{-1}(\mathcal{G}_{(1;n)})$$

are conformally equivalent.

3. Finally,

$$\begin{aligned}\rho_\Gamma \circ \mu^{-1}(\ker \xi_{(a_0,b_0)} \circ \mu^{-1}) &= \rho_\Gamma \circ \mu^{-1}(\mu(\ker \xi_{(a_0,b_0)})) \\ &= \rho_\Gamma(\ker \xi_{(a_0,b_0)})\end{aligned}$$

so $\mathcal{H}/\rho_\Gamma \circ \mu^{-1}(\ker \xi_{(a_0,b_0)} \circ \mu^{-1})$ and $\mathcal{H}/\rho_\Gamma(\ker \xi_{(a_0,b_0)})$ correspond to the same point of $\mathfrak{S} \subset \mathcal{M}_g$.

Now we study when the map $\xi_* : \mathbb{T}_{(1;n)} \times \widetilde{K^\circ_G}(1 : g^G)/M_{(1;n)} \to \mathcal{M}_g$, is not injective. Assume $\xi_*([\rho], [(a, b)]) = \xi_*([\rho'], [(a', b')])$, if $([\rho], [(a, b)]) \neq ([\rho'], [(a', b')])$ then there are two different actions on $\mathcal{H}/\rho(\ker \xi) = \mathcal{H}/\rho'(\ker \xi)$ of two groups of automorphisms isomorphic to $G$. Hence the map $\xi_*$ is injective up to the preimage of surfaces $W \in \xi_*(\mathbb{T}_{(1;n)} \times \widetilde{K^\circ_G}(1 : g^G)/M_{(1;n)}) \subset \mathcal{M}_g$ with $\mathrm{Aut}(W) \gneqq G$, and these surfaces are not in $\mathfrak{S}$.

Let $\mathcal{Y}$ be the set of points in $\mathcal{M}_g$ corresponding to surfaces $R$ such that $\mathrm{Aut}(R) \gneqq H$ where the action of $H$ on $R$ is topologically equivalent to the action of $G$ on $S$. Note that these points are surfaces such that $R/\mathrm{Aut}(R)$ is a sphere and the covering $R \to R/\mathrm{Aut}(R)$ has three branch points, so that the set $\mathcal{Y}$ is a finite set of points.

We then have an isomorphism:

$$\mathbb{T}_{(1;n)} \times \widetilde{K^\circ_G}(1 : g^G)/M_{(1;n)} - \xi_*^{-1}(\mathcal{Y}) = \mathcal{B}\text{–}\mathcal{I} \to \mathfrak{S} \subset \mathcal{M}_g.$$

□

### 3.4.1 Genus of the stratum

As in [5] we can use the monodromy operators to compute the genus of the stratum, i.e., the genus of the projective completion of $\mathfrak{S}$. According to Theorem 3.5, the stratum $\mathfrak{S}$ is a Zariski open set of the Riemann surface $\mathcal{B}$ for which there is an unbranched cover from an open subset $\eta : \mathcal{B}^\circ \to \mathbb{C} - \{P_X, P_Y\}$, a thrice punctured sphere. The degree of the cover is $l$, and the monodromy of the cover is a transitive subgroup of $\Sigma_l$, defined by the action of $M_{(1,n)}$ on $\mathcal{O}$. The lifted monodromies of appropriately oriented small circles surrounding the punctures $P_X, P_Y, \infty$ are conjugates of the action of $X, Y$, and $(XY)^{-1}$ on the orbit $\mathcal{O}$. Let $l_X$, $l_Y$ and $l_\infty$ denote the number of orbits corresponding to the $X$, $Y$ and $(XY)^{-1}$ monodromies. Then the Riemann Hurwitz equation for the genus of the stratum, $g_\mathfrak{S} = g_\mathcal{B}$, is:

$$g_\mathfrak{S} = 1 + \frac{l - l_X - l_Y - l_\infty}{2}. \tag{3.4}$$

In Example 3.9, later in the paper, we get as claimed:

$$g_\mathfrak{S} = 1 + \frac{8 - 2 - 2 - 2}{2} = 2.$$

***Remark 3.6*** (Belyi cover) Note that $\mathcal{B}$ is a finite covering of $\mathcal{M}_{(1,n)}$ and that $\mathcal{B}$ can be completed to $\overline{\mathcal{B}}$ which is a covering of the sphere branched on three points ($\overline{\mathcal{B}}$ is a Belyi curve.) By [5] for Type 1 strata and Theorem 3.5 for Type 2 we have the following corollary:

**Corollary 3.7** *All one dimensional strata in the moduli space of Riemann surfaces are Belyi curves with punctures.*

***Remark 3.8*** The elliptic involution $\iota$ on $T_n$ acts upon monodromies as in equation (3.3). It follows that for every $\mathrm{Aut}(G)$ monodromy class $[\xi]$ is paired with another class $[\xi \circ \iota_*]$. It can be shown that $[\xi]$ and $[\xi \circ \iota_*]$ both belong to the same orbit of $M_{(1;n)}$, acting upon classes of monodromies. Therefore, the map, $[\xi] \to [\xi \circ \iota_*]$ is an involution fixing the components $\mathcal{B}$ and is also deck transformation of $\eta : \mathcal{B} \to \mathcal{M}_{(1;n)}$. The involution $[\xi] \to [\xi \circ \iota_*]$ has no fixed points because of the automorphism restriction on Type 2 actions - at least over the non-branch points of $\eta : \mathcal{B} \to \mathcal{M}_{(1;n)}$. It would be interesting to understand how the involution $[\xi] \to [\xi \circ \iota_*]$ interacts with the "dessin d'enfant" (see [11]) defining the Belyi curve $\overline{\mathcal{B}}$.

**Open question** Which Belyi curves appear as one dimensional equisymmetric strata? (We remark that all Belyi curves appear as Hurwitz spaces [8].)

We now apply Theorem 3.5 to construct the stratum corresponding to the Type 2 action defined in Example 3.2:

***Example 3.9*** We consider again the groups

$$G = C_q \rtimes C_3 = \langle s, t : s^q = t^3 = 1, t^2 st = s^u \rangle$$

where $q \equiv 1 \bmod 3$, $q$ prime and with $u$ satisfying $u^2 \equiv -u - 1 \bmod q$.

The points in $\mathfrak{S}$ correspond to surfaces of genus $\frac{3q-1}{2}$. Any generating vector for a monodromy giving surfaces in the stratum is conjugate to $(t, ts)$. There are eight $\mathrm{Aut}(G)-$classes of generating vectors in $\widetilde{K_G^{\circ}}(1 : g^G)$ with representatives:

| Class number in $\mathcal{O}$ | Representative generating vector: |
|---|---|
| 1 | $(t, ts)$ |
| 2 | $(t, t^2 s)$ |
| 3 | $(t^2, ts)$ |
| 4 | $(t^2, t^2 s)$ |
| 5 | $(s, t)$ |
| 6 | $(s, t^2)$ |
| 7 | $(t, s)$ |
| 8 | $(t^2, s)$ |

Now, the transitive action of the modular group $M_{(1;q)}$ on the monodromies is as follows:

(1) $\Theta(X) = (1, 2, 4, 3)(7, 6, 8, 5)$,
(2) $\Theta(Y) = (1, 7, 6, 4, 8, 5)(3, 2)$, and
(3) $\Theta(XY) = (1, 3, 7, 4, 2, 8)(5, 6)$.

So, $\mathfrak{S}$ is contained in a covering of degree 8 of $\mathcal{M}_{(1;q)}$ that is a Belyi curve of genus 2. The monodromy group has order 24 and is isomorphic to $\mathrm{SL}(2, 3)$.

## 4 Type 2 actions of non-abelian *pq* groups

We give a detailed analysis of Type 2 actions of non-abelian groups of order $pq$. We consider these examples since they are uncomplicated examples of non-abelian groups that yield Type 2 actions. There is a single stratum and we can get nice formulas for the number of epimorphisms, the genus of the stratum, and the monodromy of the cover $\mathcal{B} \to \mathcal{M}_{(1;q)}$.

Example 3.2 is a special, and exceptional, case with $p = 3$. The groups were extensively studied by Wolfart and Streit in [17] in the context of dessins d'enfant.

We perform this analysis in the following steps: introduce non-abelian $pq$ groups, identify and enumerate the generating vectors, identify $\widetilde{K}^{\circ}(1; y^G)$, determine the modular action, and then summarize all results in Proposition 4.2.

*Definition and properties of $pq$ groups*

Let $p < q$ be two primes such that $p$ divides $q - 1$. It is well known that there is exactly one isomorphism class of non-abelian groups of order $pq$, all isomorphic to $C_q \rtimes C_p$. We shall call such groups *(non-abelian) pq groups*. The groups have the following properties most of which we leave to the reader to prove or may be found in [9].

(1) A non-abelian $pq$ group $G$ has a presentation

$$G = \left\langle x, y : x^p = y^q = 1, y^x = y^r \right\rangle, \tag{4.1}$$

where $1 < r < q$ and $r^p = 1 \mod q$.

(2) The non-trivial elements comprise $(p-1)q$ elements of order $p$ and $q - 1$ elements of order $q$.

(3) The order of the automorphism group $\mathrm{Aut}(G)$ is $q(q-1)$, consisting of the products $U_u V_v$, $0 \le u < q$, $1 \le v < q$, where $U_u : x \to x^{y^u} = xy^{u-ru}$, $y \to y$ and $V_v : x \to x$, $y \to y^v$.

(4) The last item follows easily from the fact that $\mathrm{Aut}(G)$ acts trivially on the abelianization $\overline{G} = G/[G, G] = G/\langle y \rangle$.

*Identifying generating vectors*

In order to find Type 2 actions of $G$, we need to be able to identify generating vectors. If $(a, b, c)$ is a Type 2 generating vector for $G$ then $c^{-1} = [a, b] \in \langle y \rangle$ and so the signature of the action is $(1; q)$, Hence the genus of the constructed surface is:

$$g = \frac{p(q-1)+2}{2}.$$

Let us calculate the sizes of $K(1; y^G)$, $K^{\circ}(1; y^G)$, and $\widetilde{K}^{\circ}(1; y^G)$. To this end, let $a = x^u y^v$ and $b = x^w y^z$. Then

$$\begin{aligned}
c = [a, b]^{-1} &= bab^{-1}a^{-1} \\
&= x^w y^z x^u y^v y^{-z} x^{-w} y^{-v} x^{-u} \\
&= x^w x^u (y^{r^u z} y^v y^{-z}) x^{-w} y^{-v} x^{-u} \\
&= x^w x^u x^{-w} (y^{r^u z} y^v y^{-z})^{r^{-w}} y^{-v} x^{-u} \\
&= x^w x^u x^{-w} x^{-u} ((y^{r^u z} y^v y^{-z})^{r^{-w}} y^{-v})^{r^{-u}} \\
&= ((y^{r^u z} y^v y^{-z})^{r^{-w}} y^{-v})^{r^{-u}}
\end{aligned}$$

The exponent on $y$ is

$$\begin{aligned}
&(r^{-w-u}) \left(r^u z + v - z\right) - r^{-u} v \\
&\quad = r^{-w-u}(v - z) + r^{-w} z - r^{-u} v \\
&\quad = (r^{-w-u} - r^{-u}) v - (r^{-w-u} - r^{-w}) z \\
&\quad = r^{-u}(r^{-w} - 1) v - r^{-w}(r^{-u} - 1) z
\end{aligned} \tag{4.2}$$

Assume for the moment that $u \neq 0 \bmod p$. Then, $r^{-u} \neq 1 \bmod q$ and equating the term of (4.2) to zero mod $q$ yields:

$$z = \frac{r^{-w-u} - r^{-u}}{r^{-w-u} - r^{-w}} v = \frac{r^{-u}(r^{-w} - 1)}{r^{-w}(r^{-u} - 1)} v \tag{4.3}$$

So, $c = 1$ if and only if (4.3) holds. There is a similar equation if $w \neq 0 \bmod p$. So, for the $p^2q^2$ potential choices for $(a, b)$ we must remove $q^2$ choices for $u = w = 0 \bmod p$ and $q$ choices if at least one of $u \neq 0$, $w \neq 0$ holds. The total remaining is:

$$\left|K(1; y^G)\right| = p^2q^2 - q^2 - (p^2 - 1)q = (p^2 - 1)q(q - 1)$$

All of these are generating $(1; q)$ vectors since $\langle c \rangle = \langle y \rangle$ and one of $a, b$ has a nontrivial $x$ factor. Finally we determine

$$\left|K(1; y^G)\right| = \left|K^\circ(1; y^G)\right| = (p^2 - 1)q(q - 1) \tag{4.4}$$

and

$$\widetilde{K}^\circ(1; y^G) = \frac{(p^2 - 1)q(q - 1)}{\mathrm{Aut}(G)} = \frac{(p^2 - 1)q(q - 1)}{q(q - 1)} = p^2 - 1. \tag{4.5}$$

*Classes of generating vectors and the modular action*
Before proceeding we make a remark on the identification $M_{(1,q)}$ and $M_{1,1}$ with $M_1$.

***Remark 4.1*** Let $T$ be the smooth underlying space of $T_q$. The map $T_q \to T$ induces a map $\pi_1 O(T_q) \to \pi_1(T)$ which is simply the abelianization map $\alpha, \beta, \gamma \to \overline{\alpha}, \overline{\beta}, 0$. Any monodromy $\xi : \pi_1 O(T_q) \to G$, produces an abelianized monodromy $\overline{\xi}(\overline{\alpha}) = \overline{\xi(\alpha)}$ and $\overline{\xi}(\overline{\beta}) = \overline{\xi(\beta)}$. It follows that the $M_{1,q}$ action on $\pi_1 O(T_q)$ induces the following action of $M_1 = SL(2, Z)$ on generating vectors in $\overline{G} \times \overline{G}$ by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot (\overline{u}, \overline{v}) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \overline{u} \\ \overline{v} \end{bmatrix}.$$

Next we need to determine the $\mathrm{Aut}(G)$ classes of generating vectors. Suppose $(a, b) = (x^u y^v, x^w y^z)$ a generating $(1; q)$ vector. Letting $g \to \overline{g}$ be the abelianization map, we may identify $(\overline{a}, \overline{b})$ with $(u, w) \bmod p$. By previous discussion $(u, w) \neq (0, 0) \bmod p$. According to the list of properties of $pq$ groups, $\mathrm{Aut}(G)$ acts trivially on $\overline{G}$ so the totality of $(a, b)^{\mathrm{Aut}(G)}$ maps to $(u, w)$ under abelianization. From (4.3) we see that $q^2 - q$ of the remaining vectors are generating vectors. By cardinality arguments all of these vectors form a single $\mathrm{Aut}(G)$ class.

According to the discussion in the last paragraph, the orbits of the modular action on $\mathrm{Aut}(G)$ classes are determined entirely by the orbits of the action of $SL_2(\mathbb{Z})$ on $C_p \times C_p$. Since $SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z}_p)$ is surjective and the action factors through $SL_2(\mathbb{Z}_p)$ we need only determine the $SL_2(\mathbb{Z}_p)$ orbits.

One of the matrices $\begin{bmatrix} u & 1 \\ v & 0 \end{bmatrix}, \begin{bmatrix} u & 0 \\ v & 1 \end{bmatrix}$ is invertible so that $(u, v)$ is in the $GL_2(C_p)$ orbit of $(1, 0)$. Thus the nonzero vectors of $\mathbb{Z}_p \times \mathbb{Z}_p$ form a single $GL_2(\mathbb{Z}_p)$ orbit. Setting $V_0$

$= (1, 0)$ we easily compute

$$\begin{aligned} \left|\mathrm{Stab}(GL_2(\mathbb{Z}_p), V_0)\right| &= p(p-1), \\ \left|\mathrm{Stab}(SL_2(\mathbb{Z}_p), V_0)\right| &= p, \\ \frac{\left|GL_2(\mathbb{Z}_p)\right|}{\left|SL_2(\mathbb{Z}_p)\right|} &= p-1. \end{aligned}$$

It follows that $SL_2(\mathbb{Z}_p) \cdot V_0 = GL_2(\mathbb{Z}_p) \cdot V_0 = \mathbb{Z}_p \times \mathbb{Z}_p - (0, 0)$. Thus, *all* $\mathrm{Aut}(G)$ classes of generating vectors are equivalent under the modular action on $\pi_1 O(T_n)$.

By inspection, matrices representing $X$, $Y$ and $XY$ acting upon $\mathbb{Z}_p \times \mathbb{Z}_p$ are:

$$X = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, Y = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}, XY = \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix}.$$

The number of orbits for operators are: TABLE The formula for the genus gives us $g_{\mathfrak{S}} = 2$

| Operator | $p = 3$ | $p > 3$ |
|---|---|---|
| $X$ | 2 orbits of size 4 | $\frac{p^2-1}{4}$ orbits of size 4 |
| $Y$ | 1 orbit of size 6 | $\frac{p^2-1}{6}$ orbits of size 6 |
| | 1 orbit of size 2 | |
| $XY$ | 1 orbit of size 6 | $\frac{p-1}{2}$ orbits of size $2p$ |
| | 1 orbit of size 2 | $\frac{p-1}{2}$ orbits of size 2 |

for $p = 3$ as previously computed, and for primes $> 3$:

$$\begin{aligned} g_{\mathfrak{S}} &= 1 + \frac{p^2 - 1 - \frac{p^2-1}{4} - \frac{p^2-1}{6} - (p-1)}{2} \\ &= 1 + \frac{7p^2 - 12p + 5}{24} \\ &= 1 + \frac{(7p-5)(p-1)}{24} \end{aligned} \tag{4.6}$$

It is easily shown that for odd integers not divisible by 3, the expression is an integer.

Let us verify orbit sizes in the table. In the first two cases, if an orbit size is less than an element order, then a power of the operator fixes a non-zero vector. It follows that 1 is a zero mod $p$ of the characteristic polynomial of the operator power. There are five cases to check and the only non-trivial powers with fixed vectors are $Y^2$, $Y^4$ and $p = 3$. For $XY$ the orbits of size $2p$ have the form

$$\{(u + kv, v) : 0 \le k < p\} \cup \{(-u - kv, -v) : 0 \le k < p\}$$

unless $v = 0$. There are $\frac{p-1}{2}$ of these orbits, one for each pair $\{v, -v\}$. If $v = 0$ we get $\frac{p-1}{2}$ orbits of the form $\{(u, 0), (-u, 0)\}$.

We summarize the foregoing discussion in the following proposition.

**Proposition 4.2** *Let $G$ be a non-abelian $pq$-group formed from odd primes satisfying $p < q$ and $p|(q-1)$. Then, there is a one dimensional stratum, $\mathfrak{S}$, of surfaces of genus* $1 + \frac{(p-1)q}{2}$

*with a* $(1; q)$ *action of* $G$. *The stratum has genus* $2$ *if* $p = 3$, *and otherwise has genus* $1 + \frac{(7p-5)(p-1)}{24}$. *The map* $\overline{\mathfrak{S}} \to P^1(\mathbb{C})$ *is a non-regular branched covering of the sphere, of degree* $p^2 - 1$, *branched over three points. The full monodromy group of the covering is* $SL(2, p)$.



## References

1. Bartolini, G., Costa, A. F., Izquierdo, M.: On the orbifold structure of the moduli space of Riemann surfaces of genera four and five. Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Mat. RACSAM **108** (2014), no. 2, 769–793
2. Birman, J. S.: *Braids, links, and mapping class groups*. Annals of Mathematics Studies, No. 82. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, (1974)
3. Broughton, A.: The equisymmetric stratification of the moduli space and the Krull dimension of mapping class groups. Topology Appl. **37**, 101–113 (1990)
4. Broughton, S.A.: *Tilings, Geometry, and Automorphisms of Surfaces*. https://tilings.org/autosurf. Accessed 28 Oct 2023
5. Broughton, S. A., Costa, A. F., Izquierdo, M.: *One dimensional equisymmetric strata in moduli space*. Automorphisms of Riemann surfaces, subgroups of mapping class groups and related topics, 177–215, Contemp. Math., 776, Amer. Math. Soc., [Providence], RI, [2022], 2022
6. Bujalance, E., Cirre, F.J., Conder, M.: On extendability of group actions on compact Riemann surfaces. Trans. Amer. Math. Soc. **355**, 1537–1557 (2002)
7. Costa, A.F., Parlier, H.: Applications of a theorem of Singerman about Fuchsian groups. Arch. Math. (Basel) **91**(6), 536–543 (2008)
8. Diaz, S., Donagi, R., Harbater, D.: Every curve is a Hurwitz space. Duke Math. J. **59**(3), 737–746 (1989)
9. Dummit, D.S., Foote, R.M.: Abstract Algebra, 3rd edn. J. Wiley and Sons, Hoboken NJ (2003)
10. Farb, B., Margalit, D.: A Primer on Mapping Class Groups. Princeton University Press, Princeton, N. J. (2012)
11. Girondo, E., González-Díez, G.: Introduction to Compact Riemann Surfaces and Dessins d'Enfants. Cambridge University Press, Cambridge (2012)
12. Harvey, W.: On branch loci in Teichmüller space. Trans. Amer. Math. Soc. **153**, 387–399 (1971)
13. Macbeath, A. M., Singerman, D.: Spaces of subgroups and Teichmüller space. Proc. London Math. Soc. (3) **31** (1975), 2, 211–256
14. Magma computational algebra system, Computational Algebra Group, University of Sydney
15. Singerman, D.: Finitely maximal Fuchsian groups. J. Lond. Math. Soc. **6**(2), 29–38 (1972)
16. Singerman, D.: Subgroups of Fuschian groups and finite permutation groups. Bull. Lond. Math. Soc. **2**, 319–323 (1970)
17. Wolfart, J., Streit, M.: Characters and Galois invariants of regular dessins, Revista Matematica Complutense, vol. XIII, num. **1**, 49–81 (2000)