Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat. (2023) 117:119
https://doi.org/10.1007/s13398-023-01446-2

SURVEY

Derivations and loops of some evolution algebras

Yolanda Cabrera Casado[1] · **Paula Cadavid**[2] · **Tiago Reis**[3,4]

Received: 25 March 2022 / Accepted: 26 April 2023 / Published online: 12 May 2023

Abstract
In this work we study the space of derivations of non-degenerate evolution algebras. We improve some results obtained recently in the literature and, as a consequence, we advance in the description of the derivations for n-dimensional Volterra evolution algebras. In addition, we introduce the notion of loop of an evolution algebra and we analyze under which conditions the set of loops is invariant under change of basis.

Keywords Genetic algebra · Volterra evolution algebra · Derivation · Graph

Mathematics Subject Classification 17A36 · 05C25 · 17D92 · 17D99

1 Introduction

1.1 Evolution algebras and its derivations

The evolution algebras are non-associative algebras introduced by Tian and Vojtechovsky [16], who established the theoretical foundations of these structures. In addition

Yolanda Cabrera Casado is supported by the Spanish Ministerio de Ciencia e Innovación through Project PID2019-104236GB-I00/AEI/10.13039/501100011033 and by the Junta de Andalucía through Projects FQM-336 and UMA18-FEDERJA-119, all of them with FEDER funds.

✉ Yolanda Cabrera Casado
yolandacc@uma.es

Paula Cadavid
paula.cadavid@ufabc.edu.br; pacadavid@gmail.com

Tiago Reis
treis@utfpr.edu.br

1 Departamento de Matemática Aplicada, E.T.S. Ingeniería Informática, Universidad de Málaga, Campus de Teatinos s/n., 29071 Málaga, Spain

2 Centro de Matemática, Computação e Cognição, Universidade Federal do ABC, Avenida dos Estados, 5001-Bangu, Santo André, SP, Brazil

3 Universidade Tegnológica Federal do Paraná, Av. Alberto Carazzai, 1640, Campus Cornélio Procópio, PR, Brazil

4 Universidade Federal do ABC, Avenida dos Estados, 5001 - Bangu, Santo André, SP, Brazil

to emerging to model non-Mendelian genetic, in Tian [15] identified a whole series of connections with other areas such as graph theory, group theory and discrete-time Markov chains, among others. For a recent review of the advances in this type of algebras, we refer the reader to Ceballos et al. [10].

In this work, we are interested in studying the space of derivations of some evolution algebras. We point out that although there are many works describing partially such a space, by using different approaches, a complete characterization is still an unfinished task. For any evolution algebra, Tian [15] described the derivations in terms of a system of equations which becomes the starting point for the characterization of the derivations for different families of evolution algebras. The study of the derivations of evolution algebras with non-singular structure matrices was done in [8] for complex algebras and extended in [12] for algebras over a field with any characteristic. While [9] gave a complete characterization for the space of derivations of two-dimensional evolution algebras, Alsarayreh et al. [1] studied the derivations of certain three-dimensional evolution algebras (solvable and nilpotent). Later, Qaralleh and Mukhamedov [13] provided a description of the derivations of three-dimensional Volterra evolution algebras. In [7] the authors provided a characterization for the case of evolution algebras associated to graphs over a field of zero characteristic, which was generalized later for fields of any characteristic in [14]. The novelty in the approach developed in [7, 14] rely on the connection between the set of equation mentioned above and the structural properties of the considered graph. Such an approach was explored in [3], where the authors studied the space of derivations of some non-degenerate irreducible evolution algebras depending on the twin partition of an associated directed graph.

One of the contributions of our work is to provide a characterization, in a sense to be defined later, of the space of derivations of non-degenerate evolution algebras. We improve some results obtained recently in the literature and, as a consequence, we advance in the description of the derivations for n-dimensional Volterra evolution algebras.

1.2 Loops of an evolution algebra

Our approach to dealing with derivations is inspired by a combination of arguments developed by [3, 4]. While the former explores the structure of a directed graph associated to an evolution algebra, the last relies on a partition of the considered basis. One of the peculiarities of evolution algebras is that they are not defined by identities, so their study usually follows a different strategy than the one associated to other non-associative algebras like Jordan, Lie or power-associative algebras. A usual approach to deal with an evolution algebra is to fix its natural basis. However, many properties are not invariant through the chosen basis. Some examples of this are the connectedness of the associated direct graph (see [11, Example 2.5]) or the skew-symmetry of the structure matrix (see Example 2.1). Therefore an interesting task is to know which properties are invariant under the chosen basis. Motivated by this question we study the phenomenon showed in Example 2.1, where we change the basis of a Volterra evolution algebra and as a consequence, we verify that each element of the diagonal of the structure matrix remains equal to zero. We prove that this is true in general for Volterra evolution algebras. Moreover, it was trying to answer this question that we solve a more general problem; namely, when the number of zeros in the diagonal of a structure matrix of an evolution algebra is invariant under the change of natural basis. We point out that this problem has been addressed previously in [4, Proposition 2.13] for the case where the algebra is perfect.

The nonzero elements belonging to the diagonal of the structure matrix are what we call the loops of an evolution algebra. Although our results related to this part of the paper are of independent interest, we observe that knowing the loops of an evolution algebra was useful to the study of its derivations in [3].

1.3 Organization of the paper

Now, we will show how this paper is organized. In Sect. 2 we introduce the preliminary definitions and notations. Taking into account [2, Theorem 2.11] we define a natural decomposition of a natural basis of an evolution algebra which will be an important tool for later results.

We begin Sect. 3 by establishing a characterization of space of derivations for a non-degenerate evolution algebra in Proposition 3.1. As a consequence, the Corollary 3.4 proves that the derivation is a block matrix, up to reordering. The Proposition 3.5 gives a connection between the set of derivations and the fact of having a unique natural basis (in the sense that whatever other natural basis can obtain by permutations or product by scalars). This fact generalizes the results of [8, Theorem 2.1], [12, Theorem 4.1 item (1)] and [3, Theorem 1]. In the particular case of $\dim(\mathcal{A}^2) = 1$ and the product of any two square elements of basis is different from zero, we provide necessary and sufficient conditions for a linear operator to be a derivation (Proposition 3.8). In fact, we show in Corollary 3.9 that, up to reordering, the derivation is a skew-symmetric matrix. One of the requirements is related to the fact that the matrix of derivation is a diagonal matrix. For this reason, we ask, on the one hand when the derivation will have, under suitable conditions, some of the entries of the main diagonal equals and, on the other hand, the main diagonal null. Proposition 3.12 and Theorem 3.13 answer these questions, respectively.

The Sect. 4 is devoted to studying of derivations in the case of Volterra evolution algebras. In the same way as before, we show a characterization of the space of derivations for this specific case. Fixed a Volterra evolution algebra and verifying that the product of any two square of elements of basis is different from zero, the main result of this section (Theorem 4.4) provides a way of finding another Volterra evolution algebra with structure matrix diagonal and space of derivations the same as the Volterra evolution algebra original. As the structure matrix is diagonal, calculating the set of derivations is equivalent to calculating the set of derivations over certain evolution ideals of the algebra (Corollary 3.2). In fact, what we need is to find conditions that ensure that the main diagonal of derivations is null. In Proposition 4.6 we claim that the requirement imposed on the elements of the natural basis in the Theorem 4.4 can be replaced by other properties related to the structure constants. Particularly, Propositions 4.9 and 4.11 give conditions under which there exists a derivation of a non-degenerate Volterra evolution algebra is not a diagonal matrix.

In Sect. 5 we start by defining the loops of an evolution algebra and study when this set is invariant under change of natural basis. First, we will consider the set of no loops and we prove in Theorem 5.3 that if an element of a natural decomposition is contained within the set of no-loops then its corresponding element in another natural decomposition is also contained within it. Next, we will focus on the set of loops and we have that if an evolution algebra has no loops relative to a natural basis then it has no loops relative to any natural basis (Corollary 5.5), as we have said before. Moreover, the Theorem 5.7 and Proposition 5.9 provide convenient criteria in terms of the elements of the natural decomposition for the number of loops to be invariant. By contrast, we also give in Theorem 5.8 some conditions to find a new natural basis such that the number of loops does not stay constant. A summarizing of conditions for invariability of the number of loops can be seen in Corollary 5.12.

2 Preliminaries

In what follows $\mathbb{K}$ will denote, unless we state otherwise, a field such that $\text{char}(\mathbb{K}) = 0$. In order to state the first definitions let $\Lambda := \{1, \ldots, n\}$. An n-dimensional $\mathbb{K}$-algebra $\mathcal{A}$ is called *evolution algebra* if it admits a basis $B = \{e_i\}_{i\in\Lambda}$ such that $e_i e_j = 0$ whenever $i \neq j$. A basis with this property is known as *natural basis*. The scalars $\omega_{ij} \in \mathbb{K}$ such that

$$e_i^2 = \sum_{k\in\Lambda} \omega_{ik} e_k$$

are called the *structure constants* of $\mathcal{A}$ relative to B and the matrix $M_B = (\omega_{ik})$ is called the *structure matrix* of $\mathcal{A}$ relative to B. When $\mathcal{A} = \mathcal{A}^2$ or equivalently when M_B is invertible, it is said that $\mathcal{A}$ is *perfect*.

If $u = \sum_{i\in\Lambda} \alpha_i e_i$ is an element of $\mathcal{A}$ then the *support of u relative to B* is defined as $\text{supp}_B(u) := \{i \in \Lambda : \alpha_i \neq 0\}$. In general, if $X \subseteq \mathcal{A}$, we have that $\text{supp}_B(X) = \cup_{x\in X} \text{supp}_B(x)$. For $u = e_i^2$ support of e_i^2 is called the *first-generation descendents* of i relative to the natural basis B, i.e., $D^1(i) = \{k \in \Lambda, : \ \omega_{ik} \neq 0\}$. By analogy, given a subset $U \subseteq \Lambda$, we let $D^1(U) := \text{supp}_B(W)$ where $W = \{e_i \in B : \ i \in U\}$. Similarly, we say that j is a *second-generation* descendent of i whenever $j \in D^1(D^1(i))$. Therefore

$$D^2(i) = \bigcup_{k\in D^1(i)} D^1(k).$$

By recurrence, we define the set of *mth-generation descendents* of i as

$$D^m(i) = \bigcup_{k\in D^{m-1}(i)} D^1(k).$$

Finally, the set of descendents of i is defined as

$$D(i) = \bigcup_{m\in\mathbb{N}} D^m(i).$$

An evolution algebra $\mathcal{A}$ is *non-degenerate* if there is a natural basis B such that $e_i^2 \neq 0$ for all $e_i \in B$. We remark that $\mathcal{A}$ is a non-degenerate evolution algebra if and only if $D^1(i) \neq \emptyset$ for all $i \in \Lambda$. By [11, Lemma 2.7] this definition does not depend on the chosen natural basis since $\text{ann}(\mathcal{A}) = \text{span}(\{e_i : e_i^2 = 0\})$ where $\text{ann}(\mathcal{A}) := \{x \in \mathcal{A} : x\mathcal{A} = 0\}$. Therefore $\mathcal{A}$ is non-degenerate if and only if $\text{ann}(\mathcal{A}) = 0$.

On the other hand, an evolution algebra $\mathcal{A}$ is *reducible* if there exist two nonzero ideals I and J of $\mathcal{A}$ such that $\mathcal{A} = I \oplus J$. In another case, it is called *irreducible*.

We follow Definitions 2.1 and 2.8 in [2]. An evolution algebra $\mathcal{A}$ has an *unique* natural basis if the subgroup of $\text{Aut}_{\mathbb{K}}(\mathcal{A})$ such that map natural basis into natural basis is precisely $S_n \rtimes (\mathbb{K}^\times)^n$. This group was depicted in [6] for $n = 3$. On the other hand, it is said that $\mathcal{A}$ has *Property* (2LI) if for any two different vectors e_i, e_j of a natural basis, $\{e_i^2, e_j^2\}$ is linearly independent. Note that any perfect evolution algebra has the Property (2LI) but the reciprocal is not true (see [2, Example 2.9]).

Let V be a $\mathbb{K}$-vector space and S be a subset of V. We denote by span(S) the vector subspace generated by S and rk(S) the *rank of* S, that is, the dimension of span(S) as vector space.

We recall that $\mathcal{A}$ is a *Volterra evolution algebra* if there exists a natural basis B of $\mathcal{A}$ such that M_B is a skew-symmetric matrix. In this case, we say that $\mathcal{A}$ is a *Volterra evolution algebra relative to* B. Since we are considering algebras over a field of zero characteristic, the matrix M_B has null diagonal. This family of algebras was introduced in [13] where the authors give a connection between this kind of algebras with the ergodicities of Volterra quadratic stochastic operators and, among other things, they show that these algebras are not nilpotent and they calculate its derivations for some cases.

Note that if $\mathcal{A}$ is a Volterra evolution algebra then is not true that for any natural basis B the structure matrix M_B is skew-symmetric.

Example 2.1 Let $\mathcal{A}$ be an evolution algebra, and let $B = \{e_1, e_2, e_3\}$ be a natural basis such that

$$M_B = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

Therefore $\mathcal{A}$ is a Volterra evolution algebra. On the other hand, if $B' = \{f_1, f_2, f_3\}$ is such that $f_1 = 2e_1 + e_3$, $f_2 = \frac{1}{2}e_1 + e_3$ and $f_3 = e_2$, then B' is a natural basis of $\mathcal{A}$ such that

$$M_{B'} = \begin{pmatrix} 0 & 0 & 3 \\ 0 & 0 & -\frac{3}{4} \\ -1 & 2 & 0 \end{pmatrix}$$

is non skew-symmetric.

Now, we recall some basic definitions and notations for directed graphs. A *directed graph* is a 4-tuple $E = (E^0, E^1, s_E, r_E)$ where E^0, E^1 are sets and $s_E, r_E \colon E^1 \to E^0$ are maps. The elements of E^0 are called the *vertices* of E and the elements of E^1 are the *arrows* or *directed edges* of E. For $f \in E^1$ the vertices $r(f)$ and $s(f)$ are called the *range* and the *source* of f, respectively. If E^0 and E^1 are both finite we say that E is *finite*. A vertex $v \in E^0$ is called *sink* if it verifies that $s(f) \neq v$, for every $f \in E^1$. A *path* or a *path from* $s(f_1)$ *to* $r(f_m)$ in E, μ, is a finite sequence of arrows $\mu = f_1 \dots f_m$ such that $r(f_i) = s(f_{i+1})$ for $i \in \{1, \dots, (m-1)\}$. In this case, we say that m is the *length* of the path μ and denote it by $|\mu| = m$. Let $\mu = f_1 \dots f_m$ be a path in E with $|\mu| = m \geq 1$. If $v = s(f_1) = r(f_n)$, then μ is called a *closed path based at* v. If $\mu = f_1 \dots f_m$ is a closed path based at v and $s(f_i) \neq s(f_j)$ for every $i \neq j$, then μ is called a *cycle based at* v or simply a *cycle*. A cycle of length 1 will be said to be a *loop*. Given a finite graph E, its *adjacency matrix* is the matrix $A_E = (a_{ij})$ where a_{ij} is the number of arrows from i to j. A graph E is said to satisfy Condition (Sing) if among two vertices of E^0 there is at most one arrow. If exist a path from i to j in E, then we define the *distance* from i to j as $\delta(i, j) = \min\{|\mu|, \mu \text{ is a path from } i \text{ to } j\}$.

There are several ways to associate a graph to an evolution algebra (see [5, 11]). We consider the directed graph described in [5] as follows. Given a natural basis $B = \{e_i\}_{i \in \Lambda}$ of an evolution algebra $\mathcal{A}$ and its structure matrix $M_B = (\omega_{ij}) \in M_\Lambda(\mathbb{K})$, consider the matrix $P = (a_{ij}) \in M_\Lambda(\mathbb{K})$ such that $a_{ij} = 0$ if $\omega_{ij} = 0$ and $a_{ij} = 1$ if $\omega_{ij} \neq 0$. The *graph associated to the evolution algebra* $\mathcal{A}$ (relative to the basis B), denoted by $E^B_{\mathcal{A}}$ (or simply by E if the algebra $\mathcal{A}$ and the basis B are understood) is the directed graph whose adjacency matrix is given by $P = (a_{ij})$. In this way, we only consider graphs satisfying Condition Sing.

By analogy with graph theory, we define the following notions. Let $\mathcal{A}$ be an evolution algebra with natural basis B and let $i, j \in \Lambda$. We say that i and j are *twins relative to* B if $D^1(i) = D^1(j)$. We notice that by defining the relation $\sim_{t_B}$ on the set of indices Λ by $i \sim_{t_B} j$ whether i and j are twins relative to B, then $\sim_{t_B}$ is an equivalence relation. An equivalence class of the twin relation $\sim_{t_B}$ is referred to as a *twin class relative to* B. In other words, the twin class of an index i, that we will denote by $\mathcal{T}(i)$, is the set $\mathcal{T}(i) := \{j \in \Lambda : i \sim_{t_B} j\}$. The set of all twin classes relative to B of Λ is denoted by $\Pi_B(\Lambda)$ and it is referred to as the *twin partition relative to* B of Λ. If $\mathcal{A}$ has no twins relative to B, that is, if for all $i, j \in \Lambda$, $i \neq j$, $D^1(i) \neq D^1(j)$, then we say that $\mathcal{A}$ is *twin-free relative to* B. These definitions depend on the chosen natural basis (see [3, Example 2]).

One of our main purposes is to study the derivations of Volterra evolution algebras. Given an (evolution) $\mathbb{K}$-algebra $\mathcal{A}$, a *derivation* of $\mathcal{A}$ is a linear map $d : \mathcal{A} \to \mathcal{A}$ such that

$$d(u \cdot v) = d(u) \cdot v + u \cdot d(v),$$

for all $u, v \in \mathcal{A}$. The space of all derivations of $\mathcal{A}$ is denoted by $\mathrm{Der}(\mathcal{A})$. In [15, Section 3.2.6], it was proved that, if $\mathcal{A}$ is an evolution $\mathbb{K}$-algebra with a natural basis $B = \{e_i\}_{i \in \Lambda}$ then a linear map d such that $d(e_i) = \sum_{k \in \Lambda} d_{ki} e_k$ is a derivation of the evolution algebra $\mathcal{A}$ if, and only if, it satisfies the following conditions:

$$\omega_{jk} d_{ij} + \omega_{ik} d_{ji} = 0, \quad \text{for} \quad i, j, k \in \Lambda \text{ such that } i \neq j, \tag{1}$$

$$\sum_{k \in \Lambda} \omega_{ik} d_{kj} = 2\omega_{ij} d_{ii}, \qquad \text{for} \quad i, j \in \Lambda, \tag{2}$$

From now on, we identify the linear map d with the matrix (d_{ij}) relative to the basis B.

Remark 2.2 According to [2, Theorem 2.11] if $\mathcal{A}$ is an evolution algebra with a natural basis $B = \{e_i\}_{i \in \Lambda}$ then we can write B as a disjoint union of subsets as follows:

$$B = B_0 \cup B_1 \cup \cdots \cup B_r, \tag{3}$$

where $\mathrm{ann}(\mathcal{A}) = \mathrm{span}(B_0)$, $\mathrm{rk}(\{e_i^2 : e_i \in B_t\}) = 1$ for $1 \leq t \leq r$ and $\mathrm{rk}(\{u^2, v^2\}) = 2$ if $u \in B_t$ and $v \in B_s$ with $t \neq s$. Therefore, if we define $\Lambda_t := \{k \in \Lambda : e_k \in B_t\}$, (3) implies that Λ can be also expressed as disjoint union of subsets:

$$\Lambda = \Lambda_0 \cup \Lambda_1 \cup \cdots \cup \Lambda_r, \tag{4}$$

where $\mathrm{rk}(\{e_i^2, e_j^2\}) = 1$ if $i \in \Lambda_t$, $j \in \Lambda_s$ and $t \neq s$ and $\mathrm{rk}(\{e_i^2, e_j^2\}) = 2$, if $i, j \in \Lambda_t$, for some $t \in \{1, \ldots, r\}$. So, in this last case,

$$e_i^2 = \alpha_{ji} e_j^2 \tag{5}$$

for some $\alpha_{ji} \in \mathbb{K}^\times$.

This observation leads us to the following definition.

Definition 2.3 In the same conditions of Remark 2.2, the partitions $B = B_0 \cup B_1 \cup \cdots \cup B_r$ and $\Lambda = \Lambda_0 \cup \Lambda_1 \cup \cdots \cup \Lambda_r$ are called *a natural decomposition* of B and *a natural decomposition of* Λ *relative to* B, respectively.

Definition 2.4 Let $\mathcal{A}$ an evolution algebra. We define

$$\Lambda(j) = \{k \in \Lambda \backslash \Lambda_0;\ e_k^2 \text{ and } e_j^2 \text{ are linearly dependent}\}.$$

Moreover, we can write $e_k^2 = \alpha_{jk} e_j^2$, for some $\alpha_{jk} \in \mathbb{K}^\times$ and $j, k \in \Lambda(j)$.

Remark 2.5 Under the conditions of Remark 2.2, if B' is another natural basis of $\mathcal{A}$ with a natural decomposition $B' = B'_0 \cup B'_1 \cup \cdots \cup B'_s$ then by [2, Remark 2.14] we know that $r = s$ and that it is possible to reorder B' in such a way that $\operatorname{span}(B_0) = \operatorname{span}(B'_0)$ and $B'_t \subseteq \operatorname{span}(B_0 \cup B_t)$. In addition, it is easy to check that $|B_t| = |B'_t|$ for every $t \in \{1, \dots, r\}$.

From now on, when we have natural decompositions of two natural bases B and B' of an evolution algebra $\mathcal{A}$ we suppose that both decompositions are written taking into account this reordering.

3 Derivations of a non-degenerate evolution algebra

In this section, we will investigate when a linear operator of a non-degenerate evolution algebra is a derivation. In particular, we will study the derivations of an evolution algebra with $\dim(\mathcal{A}^2) = 1$.

The following proposition improves [3, Proposition 1] in the sense that it provides a condition necessary and sufficient under which a linear operator $d\colon \mathcal{A} \to \mathcal{A}$ is a derivation of a non-degenerate evolution algebra $\mathcal{A}$.

Proposition 3.1 *Let $\mathcal{A}$ be a non-degenerate evolution algebra with a natural basis $B = \{e_i\}_{i\in\Lambda}$, structure matrix $M_B = (\omega_{ij})$ and let $d\colon \mathcal{A} \to \mathcal{A}$ be a linear map, $d = (d_{ij})$. Then $d \in \operatorname{Der}(\mathcal{A})$ if and only if d satisfies the following conditions:*

(i) *If $i, j \in \Lambda$, $i \neq j$, and $i \sim_{t_B} j$ then $d_{ji} = -\frac{\omega_{jk}}{\omega_{ik}} d_{ij}$, for all $k \in D^1(i)$.*
(ii) *If $i, j \in \Lambda$ and $i \nsim_{t_B} j$ then $d_{ji} = d_{ij} = 0$.*
(iii) *For any $i \in \Lambda$*

$$\sum_{k\in D^1(i)} \omega_{ik} d_{kj} = \begin{cases} 0, & \text{if } j \notin D^1(i), \\ 2\omega_{ij} d_{ii}, & \text{if } j \in D^1(i). \end{cases}$$

Proof If $d \in \operatorname{Der}(\mathcal{A})$ then d satisfies conditions (i) to (iii) by [3, Proposition 1]. Conversely, let $d\colon \mathcal{A} \to \mathcal{A}$ be a linear map satisfying conditions (i) to (iii). In order to prove that $d \in \operatorname{Der}(\mathcal{A})$, it will be necessary to check that d verifies (1) and (2). Let $i, j, k \in \Lambda$, $i \neq j$. If $i \sim_{t_B} j$, by (i), we have

$$\omega_{jk} d_{ij} + \omega_{ik} d_{ji} = \omega_{jk} d_{ij} + \omega_{ik}\Big(-\frac{\omega_{jk}}{\omega_{ik}} d_{ij}\Big) = 0, \quad \text{for all } k \in D^1(i).$$

Furthermore, if $k \notin D^1(i)$ then $w_{jk} = w_{ik} = 0$, which implies that $\omega_{jk} d_{ij} + \omega_{ik} d_{ji} = 0$. Otherwise if $i \nsim_{t_B} j$, by (ii), we have that $d_{ij} = d_{ji} = 0$. Therefore d satisfies (1).

Now note that for all $i, j \in \Lambda$ we have

$$\sum_{k=1}^{n} \omega_{ik} d_{kj} = \sum_{k\in D^1(i)} \omega_{ik} d_{kj} = \begin{cases} 0, & \text{if } j \notin D^1(i), \\ 2\omega_{ij} d_{ii}, & \text{if } j \in D^1(i). \end{cases} = 2\omega_{ij} d_{ii}.$$

This proves that d satisfies (2). □

Corollary 3.2 *Let $\mathcal{A}$ be a non-degenerate reducible evolution algebra with $\mathcal{A} = I_1 \oplus \ldots \oplus I_t$ where I_t is an ideal of $\mathcal{A}$ for every $t \in \{1, \dots, m\}$. Then $d = (d_{ij}) \in \operatorname{Der}\mathcal{A}$ is a block matrix. Moreover, d restricted to subspace I_t (up to reordering) is a derivation over I_t and its matrix is one of the blocks of $d = (d_{ij})$.*

Proof Let B be a natural basis of $\mathcal{A}$. By [5, Theorem 5.6] we know that the ideals $I_1, \ldots, I_m$ can be taken as evolution ideals. Concretely, by [5, Theorem 5.6] we get a partition of $B = B^1 \cup \cdots \cup B^m$ such that $I_t = \operatorname{span}(\{e_i : \ e_i \in B^t\})$ or equivalently the structure matrix relative to B is a block diagonal matrix. Since $i \nsim_{t_B} j$ for $e_i \in B^k$ and $e_j \in B^\ell$ with $k \neq \ell$ then by Proposition 3.1 (ii) we have that $d_{ij} = d_{ji} = 0$. It is easy to check that for every $t \in \{1, \ldots, m\}$, d restricted to I_t is a derivation and moreover, if d^t is the matrix of $d|_{I_t}$ relative to the natural basis B^t then

$$d = \operatorname{diag}(d^1, \ldots, d^m).$$

□

Corollary 3.3 *Let $\mathcal{A}$ be a non-degenerate evolution algebra with a natural basis $B = \{e_i\}_{i \in \Lambda}$ and structure matrix $M_B = (\omega_{ij})$. If $d = (d_{ij}) \in \operatorname{Der}(\mathcal{A})$ then*

(i) *If $i, j \in \Lambda$, $i \neq j$ and $d_{ij} \neq 0$ then $e_j^2 = \alpha_{ij} e_i^2$, for some $\alpha_{ij} \in \mathbb{K}^\times$.*
(ii) *If $i, j \in \Lambda$ and $j \in D^1(i)$ then*

$$\sum_{k \in \mathcal{T}(j)} \omega_{ik} d_{kj} = 2\omega_{ij} d_{ii}.$$

Proof In order to prove (i) consider $i, j \in \Lambda$ such that $i \neq j$ and $d_{ij} \neq 0$. By [3, Lemma 4] we have that $i \sim_{t_B} j$. If $|D^1(i)| = 1$ the proof is straightforward. In other case, by Proposition 3.1 (i), we have that

$$d_{ji} = -\frac{\omega_{jk}}{\omega_{ik}} d_{ij} = -\frac{\omega_{j\ell}}{\omega_{i\ell}} d_{ij}, \text{ for all } k, \ell \in D^1(i).$$

Therefore $\frac{\omega_{jk}}{\omega_{ik}} = \frac{\omega_{j\ell}}{\omega_{i\ell}}$. Fix $\ell \in D^1(i)$. Then

$$e_j^2 = \sum_{k \in D^1(i)} \omega_{jk} e_k = \sum_{k \in D^1(i)} \frac{\omega_{j\ell}}{\omega_{i\ell}} \omega_{ik} e_k = \frac{\omega_{j\ell}}{\omega_{i\ell}} \sum_{k \in D^1(i)} \omega_{ik} e_k = \frac{\omega_{j\ell}}{\omega_{i\ell}} e_i^2.$$

Taking $\alpha_{ij} := \frac{\omega_{j\ell}}{\omega_{i\ell}}$, we have that $e_j^2 = \alpha_{ij} e_i^2$, as required. For item (ii), we have that $d_{kj} = 0$ for all $k \notin \mathcal{T}(j)$ by Proposition 3.1(ii). Then, using Proposition 3.1 (iii), we get

$$2\omega_{ij} d_{ii} = \sum_{k \in D^1(i)} \omega_{ik} d_{kj} = \sum_{k \in \mathcal{T}(j)} \omega_{ik} d_{kj}.$$

□

Corollary 3.4 *Let $\mathcal{A}$ be a non-degenerate evolution algebra. Then $d = (d_{ij}) \in \operatorname{Der}(\mathcal{A})$ can be written as a block matrix.*

Proof We consider a natural decomposition $B = B_0 \cup \cdots \cup B_r$. We know by Corollary 3.3 (i) that $d_{ij} = d_{ji} = 0$ for every $e_i \in B_t$ and $e_j \in B_s$ with $t \neq s$. □

Observe that $\operatorname{span}(B_t)$ for $t \neq 0$ is not ideal in general, therefore d restricted to subspace $\operatorname{span}(B_t)$ is not necessarily a derivation.

Proposition 3.5 *Let $\mathcal{A}$ be a non-degenerate evolution algebra with $\operatorname{Der}(\mathcal{A}) \neq 0$. Then $\mathcal{A}$ does not have a unique natural basis.*

Proof Let B be a natural basis of $\mathcal{A}$ and let $d \in \mathrm{Der}(\mathcal{A})$, with $d \neq 0$. By [3, Lemma 1] there are $i, j \in \Lambda$ such that $i \neq j$ and $d_{ij} \neq 0$. Therefore by Corollary 3.3 (i) we have that $e_j^2 = \alpha_{ij} e_i^2$, for some $\alpha_{ij} \in \mathbb{K}^{\times}$. Thus $\mathcal{A}$ has not the Property (2LI) and by [2, Corollary 2.7] we get that $\mathcal{A}$ has not a unique natural basis. □

Remark 3.6 Note that the Proposition 3.5 is equivalent to say that if $\mathcal{A}$ has Property (2LI) then $\mathrm{Der}(\mathcal{A}) = 0$. Since all perfect evolution algebras and evolution algebras which are twin-free both have the Property (2LI), the Proposition 3.5 provides a generalization of the [8, Theorem 2.1], [12, Theorem 4.1 item (1)] and [3, Theorem 1].

However, the converse of Proposition 3.5 is not true as shown in the following example.

Example 3.7 Let $\mathcal{A}$ a non-degenerate two-dimensional evolution algebra with product $e_1^2 = e_2^2 = e_1 + e_2$. As $\mathcal{A}$ does not have Property (2LI), then does not have a unique natural basis. However, it is easy to check that $\mathrm{Der}(\mathcal{A}) = 0$.

Proposition 3.8 *Let $\mathbb{K}$ be an arbitrary field and let $\mathcal{A}$ be a non-degenerate evolution $\mathbb{K}$-algebra with $\dim(\mathcal{A}^2) = 1$. Consider $B = \{e_i\}_{i\in\Lambda}$ a natural basis and $M_B = (\omega_{ij})$ the structure matrix. For $i \in \Lambda$, let α_{1i} be no null scalars such that $e_i^2 = \alpha_{1i} e_1^2$. Suppose that $e_1^2 e_1^2 \neq 0$. Then $d \in \mathrm{Der}(\mathcal{A})$ if and only if it verifies the following conditions:*

(i) $d_{ii} = 0$ *for any* $i \in \Lambda$.
(ii) $d_{ij} = -\dfrac{\alpha_{1i}}{\alpha_{1j}} d_{ji}$ *for any* $i, j \in \Lambda$, $i \neq j$.
(iii) $\sum_{j\in\Lambda} \omega_{1j} d_{jk} = 0$ *for* $k \in \Lambda$.

Proof Firstly, observe that $\mathcal{A}^2 = \mathbb{K}e_1^2$. We write $e_1^2 e_1^2 = \gamma e_1^2$ for some $\gamma \in \mathbb{K}^{\times}$. Let $d \in \mathrm{Der}(\mathcal{A})$. We get that $d(e_i^2) = 2e_i d(e_i) = 2d_{ii} e_i^2$ for any $i \in \Lambda$. On the other hand, we have that $d(e_1^2 e_1^2) = 2e_1^2 d(e_1^2)$, so $\gamma d(e_1^2) = 2e_1^2 d(e_1^2)$. Therefore, if $\mathrm{char}(\mathbb{K}) = 2$, then $d(e_1^2) = 0$. If $\mathrm{char}(\mathbb{K}) \neq 2$, we get that $\gamma d_{11} e_1^2 = 2d_{11} e_1^2 e_1^2$. This implies that $\gamma d_{11} e_1^2 = 2d_{11}\gamma e_1^2$. So, as $\gamma \neq 0$ and $\mathcal{A}$ is non-degenerate, then $d_{11} = 0$. We conclude that $d(e_1^2) = 0$ in any case and therefore $d(e_i^2) = 0$ for any $i \in \Lambda$. Then $d_{ii} = 0$ because $d(e_i^2) = 2d_{ii}\alpha_{1i} e_1^2$. Let $i \neq j$ with $i, j \in \Lambda$, then $d(e_i e_j) = 0$, which implies that $d_{ij} e_j^2 + d_{ji} e_i^2 = 0$. So $e_1^2(d_{ij}\alpha_{1j} + d_{ji}\alpha_{1i}) = 0$ and by non-degeneracy of $\mathcal{A}$ $d_{ij} = -\frac{\alpha_{1i}}{\alpha_{1j}} d_{ji}$. Finally, as $d(e_1^2) = 2d(e_1)e_1$ by (2) we have that $\sum_j \omega_{1j} d_{jk} = 2d_{11}\omega_{1k}$ for every $k \in \Lambda$. The converse is straightforward. □

Corollary 3.9 *Let $\mathcal{A}$ be a non-degenerate evolution algebra such that $\dim(\mathcal{A}^2) = 1$ and natural basis $B = \{e_i\}_{i\in\Lambda}$. Suppose that $e_i^2 = e_1^2$ for any $i \in \Lambda$ and $e_1^2 e_1^2 \neq 0$. Then if $d \in \mathrm{Der}(\mathcal{A})$, the matrix of d relative to B is skew-symmetric, up to reordering.*

Remark 3.10 The converse of Corollary 3.9 is not true in general. Indeed, if we consider the 3-dimensional evolution algebra $\mathcal{A}$ with basis $\{e_i\}$ and product $e_i^2 = e_1$ for any $i = \{1, 2, 3\}$ then $d \in \mathrm{Der}(\mathcal{A})$ if and only if $d_{ii} = 0$ for any i, $d_{12} = d_{21} = d_{13} = d_{31} = 0$ and $d_{32} = -d_{23}$.

The condition $e_1^2 e_1^2 \neq 0$ can not be eliminated of the Proposition 3.8 as the following remark shows.

Remark 3.11 Let $\mathcal{A}$ be a non-degenerate evolution algebra with $\dim(\mathcal{A}^2) = 1$, natural basis $B = \{e_i\}_{i\in\Lambda}$ and structure matrix $M_B = (\omega_{ij})$. We can write $e_i^2 = \alpha_{1i} e_1^2$ for every $i \in \Lambda$. Suppose that $e_1^2 e_1^2 = 0$. Since $e_1^2 \neq 0$ there exists k such that $\omega_{1k} \neq 0$. Now, we will find a derivation such that $d_{ii} \neq 0$ for every $i \in \Lambda$. Indeed, it is enough to consider the derivation $d \in \mathrm{Der}(\mathcal{A})$ defined by

$$d_{ij} = \begin{cases} 1, & if \quad i = j, \\ -\dfrac{\alpha_{1i}\omega_{1i}}{\alpha_{1k}\omega_{1k}}, & if \quad j = k \text{ and } i \in \Lambda \backslash \{k\}, \\ 0, & if \quad i \neq j \text{ and } i, j \neq k, \\ \dfrac{\omega_{1j}}{\omega_{1k}}, & if \quad i \neq j \text{ and } i = k. \end{cases}$$

Proposition 3.12 *Let $\mathcal{A}$ be an evolution algebra with $\{e_i\}_{i\in\Lambda}$ natural basis and $d = (d_{ij}) \in \mathrm{Der}(\mathcal{A})$. Let $\{e_i^2\}_{i=1}^{\ell}$ be a basis of $\mathcal{A}^2$ and $e_j^2 = \sum_{k=1}^{\ell} \beta_{kj} e_k^2$. If $\beta_{kj} \neq 0$ for certain $k \in \Lambda$ then $d_{jj} = d_{kk}$ for any $j \in \Lambda$.*

Proof Let $d \in \mathrm{Der}(\mathcal{A})$. If $j \in \{1, \ldots, \ell\}$ then the statement is trivially true. We study now if $j \notin \{1, \ldots, \ell\}$. Applying d in the equality $e_j^2 = \sum_{k=1}^{\ell} \beta_{kj} e_k^2$ we get that $e_j^2 d_{jj} = \sum_{k=1}^{\ell} \beta_{kj} e_k^2 d_{kk}$. So $d_{jj} \sum_{k=1}^{\ell} \beta_{kj} e_k^2 = \sum_{k=1}^{\ell} \beta_{kj} e_k^2 d_{kk}$. Then $\sum_{k=1}^{\ell} \beta_{kj}(d_{jj} - d_{kk}) e_k^2 = 0$. Since $\{e_i^2\}_{i=1}^{\ell}$ is a basis of $\mathcal{A}^2$ then $\beta_{kj}(d_{jj} - d_{kk}) = 0$ for every $k \in \{1, \ldots, \ell\}$. If there exists $k \in \Lambda$ such that $\beta_{kj} \neq 0$ then $d_{jj} = d_{kk}$. □

Theorem 3.13 *Let $\mathcal{A}$ be a non-degenerate evolution algebra with $\{e_i\}_{i\in\Lambda}$ natural basis and $d = (d_{ij}) \in \mathrm{Der}(\mathcal{A})$. If $\{e_i^2\}_{i=1}^{\ell}$ is a basis of $\mathcal{A}^2$ with $e_i^2 e_i^2 \neq 0$ for any $i \in \Gamma_1 := \{1, \ldots, \ell\}$ then $d_{jj} = 0$ for any $j \in \Lambda$.*

Proof Let $d \in \mathrm{Der}(\mathcal{A})$. First, we can write $e_i^2 e_i^2 = \sum_{k=1}^{\ell} \lambda_{ik} e_k^2$ for any $i \in \Lambda$. Let $j \in \Lambda$ then if we apply the derivation d in both members of $e_j^2 e_j^2 = \sum_{k=1}^{\ell} \lambda_{jk} e_k^2$ we get $2e_j^2 d(e_j^2) = 2\sum_{k=1}^{\ell} \lambda_{ji} e_k d(e_k)$, so $2e_j^2 e_j^2 d_{jj} = \sum_{k=1}^{\ell} \lambda_{jk} e_k^2 d_{kk}$. Therefore $2\sum_{k=1}^{\ell} \lambda_{jk} e_k^2 d_{jj} = \sum_{k=1}^{\ell} \lambda_{jk} e_k^2 d_{kk}$, which implies $\sum_{k=1}^{\ell} \lambda_{jk}(2d_{jj} - d_{kk}) e_k^2 = 0$. Since $\{e_k^2\}_{k=1}^{\ell}$ is a basis of $\mathcal{A}^2$ then $\lambda_{jk}(2d_{jj} - d_{kk}) = 0$ for any $k \in \Gamma_1$. As $e_j^2 e_j^2 \neq 0$ and $\mathcal{A}$ is non-degenerate there exists some $j_1 \in \Gamma_1$ such that $\lambda_{jj_1} \neq 0$ and so $2d_{jj} - d_{j_1 j_1} = 0$. Firstly, we consider the set $R = \{j \in \Gamma_1 : \ 2d_{jj} - d_{jj} = 0\}$, then $d_{jj} = 0$ for any $j \in R$. Secondly, let $j_0 \in \Gamma_1$ such that $j_0 \notin R$. We can write the following chain of equalities:

$$\begin{aligned} 2d_{j_0 j_0} &= d_{j_1 j_1}, \\ 2d_{j_1 j_1} &= d_{j_2 j_2}, \\ &\vdots \\ 2d_{j_{s-1} j_{s-1}} &= d_{j_s j_s}, \end{aligned}$$

with $j_1, \ldots, j_s \in \Gamma_1$. Moreover either $j_s \in R$ or $j_s \notin R$ but as ℓ is finite, $j_s = j_m$ for certain $j_m \in \{j_0, j_1, \ldots, j_{s-2}\}$. In the first case we get that $d_{j_0 j_0} = d_{j_1 j_1} = \ldots = d_{j_s j_s} = 0$. In the second case, if we write $s = m + t$ for $t > 1$ then it is easy to check that $2^t d_{j_m j_m} = d_{j_{m+t} j_{m+t}}$ i.e., $2^t d_{j_m j_m} = d_{j_m j_m}$. Then $d_{j_m j_m} = d_{j_0 j_0} = \cdots = d_{j_{s-1} j_{s-1}} = 0$. Therefore if $j \in \Gamma_1$ we have proved that $d_{jj} = 0$. Let $j \notin \Gamma_1$. Now, we know that $2d_{jj} - d_{j_1 j_1} = 0$ for certain $j_1 \in \Gamma_1$ so $d_{jj} = 0$. □

Remark 3.14 In terms of matrices, for every $i \in \Lambda$ we can compute the product $e_i^2 e_i^2$ as $(M_B \circ M_B) \cdot M_B \cdot (e_1 \ldots e_n)^t$ where $\circ$ is the Hadamard product (element-wise multiplication).

The converse of Theorem 3.13 is not true in general as shown the following example.

Example 3.15 Let $\mathcal{A}$ an evolution algebra with $B = \{e_i\}_{i\in\Lambda}$ ($\Lambda = \{1, \dots, 5\}$) natural basis and multiplication table $e_1^2 = e_1+e_2+e_4+e_5$, $e_2^2 = e_1+e_2$, $e_3^2 = e_4+e_5$, $e_4^2 = -e_5^2 = e_3$ and $d = (d_{ij}) \in \mathrm{Der}(\mathcal{A})$. Since $\{e_1^2, e_2^2, e_4^2\}$ is a basis of $\mathcal{A}^2$ and $e_i^2e_i^2 \neq 0$ for every $i \in \{1, 2, 4\}$ then $d_{ii} = 0$ for every $i \in \Lambda$. But if we consider $\{e_1^2, e_3^2, e_4^2\}$ a basis of $\mathcal{A}^2$ then $e_3^2e_3^2 = 0$ and clearly $d_{ii} = 0$ for every $i \in \Lambda$.

4 Derivations of Volterra Evolution Algebras

Lemma 4.1 *Let $\mathcal{A}$ be a non-degenerate Volterra evolution algebra with natural basis $B = (\omega_{ij})$ and let $i, j \in \Lambda$ such that $\mathcal{T}(i) = \{i, j\}$ and there exists $\ell \in D^1(i)$ with $\omega_{\ell i}^3 \neq \omega_{j\ell}^3$. Suppose that $\ell \nsim_{t_B} k$ for every $k \in D^1(i)\backslash\{\ell\}$. Then $d_{ij} = d_{ji} = d_{ii} = d_{jj} = d_{\ell\ell} = 0$ for any $d = (d_{ij}) \in \mathrm{Der}(\mathcal{A})$.*

Proof Let $d = (d_{ij}) \in \mathrm{Der}(\mathcal{A})$. Since $\sum_{k\in D^1(i)} \omega_{ik}d_{k\ell} = 2\omega_{i\ell}d_{ii}$ and $\ell \nsim_{t_B} k$ we get that $d_{\ell\ell} = 2d_{ii}$ by Proposition 3.1 (ii). Likewise, $d_{\ell\ell} = 2d_{jj}$. On the other hand by Proposition 3.1 (i), $d_{ji} = -\frac{\omega_{j\ell}}{\omega_{i\ell}}d_{ij}$. Moreover, since $\sum_{k\in D^1(\ell)} \omega_{\ell k}d_{ki} = 2\omega_{\ell i}d_\ell$ and $\mathcal{T}(i) = \{i, j\}$ then $\omega_{\ell i}d_{ii} + \omega_{\ell j}d_{ji} = 2\omega_{\ell i}d_{\ell\ell}$. Similarly we have $\omega_{\ell i}d_{ij} + \omega_{\ell j}d_{jj} = 2\omega_{\ell j}d_{\ell\ell}$. So, we get the following homogeneous system of linear equations:

$$-3\omega_{\ell i}\omega_{i\ell}d_{ii} - \omega_{\ell j}\omega_{j\ell}d_{ij} = 0,$$
$$-3\omega_{\ell j}d_{ii} + \omega_{\ell i}d_{ij} = 0.$$

This system will have the trivial solution if and only if $\omega_{\ell i}^3 \neq \omega_{j\ell}^3$. □

Proposition 4.2 *Let $\mathcal{A}$ a non-degenerate Volterra evolution algebra with a natural basis $B = \{e_i\}_{i\in\Lambda}$ and structure matrix $M_B = (\omega_{ij})$. Consider a natural decomposition $\Lambda = \Lambda_1 \cup \cdots \cup \Lambda_r$ relative to B and $\alpha_{ij} \in \mathbb{K}^\times$ such that $e_j^2 = \alpha_{ij}e_i^2$ for $i, j \in \Lambda(i)$. Then $d = (d_{ij}) \in \mathrm{Der}(\mathcal{A})$ if and only if d satisfies the following conditions:*

(i) *If $i, j \in \Lambda$, $i \neq j$ and $\{i, j\} \nsubseteq \Lambda_t$ for any $t \in \{1, \dots, r\}$ then $d_{ij} = d_{ji} = 0$.*
(ii) *If $i, j \in \Lambda$, $i \neq j$ and $\{i, j\} \subseteq \Lambda_t$ for some $t \in \{1, \dots, r\}$ then $d_{ij} = -\alpha_{ji}d_{ji}$.*
(iii) *If $i, j \in \Lambda$ and $i \in D^1(j)$ then $2d_{ii} = \sum_{k\in\Lambda(j)} \alpha_{jk}d_{kj}$.*

Proof If $i \neq j$ and $\{i, j\} \nsubseteq \Lambda_t$ for any $t \in \{1, \dots, r\}$, then e_i^2 and e_j^2 are linearly independent. By Corollary 3.3 (i) we have $d_{ij} = d_{ji} = 0$, which proves item (i). Now, note that if $i \neq j$ and $\{i, j\} \subseteq \Lambda_t$ for some $t \in \{1, \dots, r\}$, then $e_i^2 = \alpha_{ji}e_j^2$ and $w_{ik} = \alpha_{ji}w_{jk}$ for all $k \in D^1(j)$. By Proposition 3.1 (i) we have

$$d_{ij} = -\frac{w_{ik}}{w_{jk}}d_{ji} = -\alpha_{ji}d_{ji},$$

which proves item (ii). Now, let $i, j \in \Lambda$. By item (i), if $k \notin \Lambda(j)$ then $d_{kj} = 0$. We have

$$\begin{aligned}\sum_{k\in\Lambda} w_{ik}d_{kj} &= \sum_{k\in\Lambda(j)} w_{ik}d_{kj} = \sum_{k\in\Lambda(j)} -w_{ki}d_{kj} = \sum_{k\in\Lambda(j)} -\alpha_{jk}w_{ji}d_{kj} \\ &= -w_{ji}\sum_{k\in\Lambda(j)} \alpha_{jk}d_{kj} = w_{ij}\sum_{k\in\Lambda(j)} \alpha_{jk}d_{kj} \end{aligned} \tag{6}$$

for any $i, j \in \Lambda$. On the other hand, using Eq. (2) we get $2w_{ij}d_{ii} = \sum_{k=1}^n w_{ik}d_{kj}$ then $2w_{ij}d_{ii} = w_{ij}\sum_{k\in\Lambda(j)} \alpha_{jk}d_{kj}$. Now, if $i \in D^1(j)$ then $2d_{ii} = \sum_{k\in\Lambda(j)} \alpha_{jk}d_{kj}$, which proves item (iii). Conversely, let $d=(d_{ij})$ satisfying conditions (i)–(iii). We will prove that d satisfies conditions (i)–(iii) of Proposition 3.1.

Let $i, j \in \Lambda, i \neq j$ and $i \sim_t j$.

Case 1 If $\{i, j\} \nsubseteq \Lambda_t$ for any $t \in \{1, \ldots, r\}$. Then by item (i) $d_{ij}=d_{ji}=0$.

Case 2 If $\{i, j\} \subseteq \Lambda_t$ for some $t \in \{1, \ldots, r\}$. Then by item (ii) $d_{ij}=-\alpha_{ji} d_{ji}$. Note that $\alpha_{ji}=\frac{w_{ik}}{w_{jk}}$ for all $k \in D^1(i)$.

Therefore, for both cases, we have that $d_{ji}=-\frac{w_{jk}}{w_{ik}} d_{ij}$ for all $k \in D^1(i)$.

Let $i, j \in \Lambda, i \neq j$ and $i \nsim_t j$. Then e_i^2 and e_j^2 are linearly independent and by Corollary 3.3 (i) we get that $d_{ij}=d_{ji}=0$.

Let $i \in \Lambda$, by Eq. (6) and item (iii) we obtain that

$$\sum_{k \in \Lambda} \omega_{ik} d_{kj}=w_{ij} \sum_{k \in \Lambda(j)} \alpha_{jk} d_{kj}= \begin{cases}0 & \text { if } j \notin D^1(i) \\ 2 \omega_{ij} d_{ii} & \text { if } j \in D^1(i) .\end{cases}$$

□

Corollary 4.3 *Let $\mathcal{A}$ be a non-degenerate Volterra evolution algebra and $d=(d_{ij}) \in \operatorname{Der}(\mathcal{A})$. If $i, \ell \in D^1(j)$ for some $j \in \Lambda$ then $d_{ii}=d_{\ell\ell}$.*

Proof The result is a direct consequence of Proposition 4.2 (iii). □

Theorem 4.4 *Let $\mathcal{A}$ be a non-degenerate Volterra evolution algebra with a natural basis $B=\{e_i\}_{i \in \Lambda}$ and structure matrix $M_B=(\omega_{ij})$. Let $\Lambda=\Lambda_1 \cup \cdots \cup \Lambda_r$ be a natural decomposition and $\alpha_{ij} \in \mathbb{K}^{\times}$ such that $e_j^2=\alpha_{ij} e_i^2$ for $i, j \in \Lambda(j)$. Define $\nu_t:=|\Lambda_t|$ for all $t \in\{1, \ldots, r\}$. If $e_i^2 e_i^2 \neq 0$ for all $i \in \Lambda$ then there exists a Volterra evolution algebra $\mathcal{A}'$ with a natural basis B' such that $\operatorname{Der}(\mathcal{A})=\operatorname{Der}(\mathcal{A}')$. Moreover, the structure matrix $M_{B'}$ is the following block diagonal matrix*

$$\operatorname{diag}(H_1, \ldots, H_c),$$

where $r=2c+q$, $q \in \{0,1\}$ and H_t is given by:

(i) *If $q=0$ then for $t \in \{1, \ldots, c\}$ we have that*

$$H_t=\begin{pmatrix}0 & F_t \\ -F_t^T & 0\end{pmatrix}, \tag{7}$$

where $F_t \in M_{\nu_t, \nu_{t+1}}(\mathbb{K})$ is a matrix without null entries.

(ii) *If $q=1$ then for $t \in \{1, \ldots, c-1\}$ the matrix H_t is of the form (7) and*

$$H_c=\begin{pmatrix}0 & F_c & F_{c+1} \\ -F_c^T & 0 & 0 \\ -F_{c+1}^T & 0 & 0\end{pmatrix} \tag{8}$$

where $F_c \in M_{\nu_{r-2}, \nu_{r-1}}(\mathbb{K})$ and $F_{c+1} \in M_{\nu_{r-2}, \nu_r}(\mathbb{K})$ are matrices without null entries.

Proof We define $s_0=0$ and $s_h=\sum_{k=1}^h \nu_k$ with $h \in \{1, \ldots, r\}$. Note that we can reorder Λ such that $\Lambda_h=\{s_{h-1}+1, \ldots, s_h\}$ for $h \in \{1, \ldots, r\}$. First, we assume that r is even. We are going to construct an evolution algebra $\mathcal{A}'$ with natural basis $B'=\{f_j\}_{j \in \Lambda}$. To describe the product in $\mathcal{A}'$, we consider $t \in \{1, \ldots, c\}$ and denote by $p=s_{2t-2}+1$ and $q=s_{2t-1}+1$. Now, with this notation we define

$$f_p^2:=\sum_{k \in \Lambda_{2t}} \alpha_{qk} f_k, \tag{9}$$

$$f_j^2 := \alpha_{pj} f_p^2, \text{ for all } j \in \Lambda_{2t-1} \backslash \{p\}, \tag{10}$$

$$f_q^2 := - \sum_{k \in \Lambda_{2t-1}} \alpha_{pk} f_k, \tag{11}$$

$$f_j^2 := \alpha_{qj} f_q^2, \text{ for all j} \in \Lambda_{2t} \backslash \{q\}. \tag{12}$$

Note that $D^1(\Lambda_{2t-1}) = \Lambda_{2t}$ and $D^1(\Lambda_{2t}) = \Lambda_{2t-1}$ for $t \in \{1, \dots, c\}$. So, we get that the structure matrix of $\mathcal{A}'$ relative to that basis is given by the diagonal matrix $\text{diag}(H_1, \dots, H_c)$ with

$$H_t = \begin{pmatrix} 0 & F_t \\ -F_t^T & 0 \end{pmatrix}$$

and

$$F_t := \begin{pmatrix} 1 & \alpha_{qq+1} & \dots & \alpha_{qs_{2t}} \\ \alpha_{pp+1} & \alpha_{pp+1}\alpha_{qq+1} & \dots & \alpha_{pp+1}\alpha_{qs_{2t}} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{ps_{2t-1}} & \alpha_{ps_{2t-1}}\alpha_{qq+1} & \dots & \alpha_{ps_{2t-1}}\alpha_{qs_{2t}} \end{pmatrix} \tag{13}$$

for every $t \in \{1, \dots, c\}$. Consequently $\mathcal{A}'$ is a Volterra evolution algebra. Observe that for all $i, j \in \Lambda(j)$ we have $e_i^2 = \alpha_{ji} e_j^2$ and $f_i^2 = \alpha_{ji} f_j^2$. As $e_i^2 e_i^2 \neq 0$ for all $i \in \Lambda$, then by Theorem 3.13 we obtain that $d_{ii} = 0$ for all $i \in \Lambda$. Thus, by Proposition 4.2 follows that $\text{Der}(\mathcal{A}) = \text{Der}(\mathcal{A}')$, as required.

Now, we assume that r is odd. In this case we consider an evolution algebra $\mathcal{A}'$ with natural basis $B' = \{f_j\}_{j \in \Lambda}$. To define the product in $\mathcal{A}'$ we consider $t \in \{1, \dots, c-1\}$ and we define f_j as in Eqs. (9–12) for $j \in \Lambda_{2t-1} \cup \Lambda_{2t}$. We denote by $p = s_{r-3} + 1$, $q = s_{r-2} + 1$ and $\ell = s_{r-1} + 1$ and we define

$$f_p^2 := \sum_{k \in \Lambda_{r-1}} \alpha_{qk} f_k + \sum_{k \in \Lambda_r} \alpha_{\ell k} f_k,$$

$$f_j^2 := \alpha_{pj} f_p^2, \text{ for all } j \in \Lambda_{r-2} \backslash \{p\},$$

$$f_q^2 := - \sum_{k \in \Lambda_{r-2}} \alpha_{pk} f_k,$$

$$f_j^2 := \alpha_{qj} f_q^2, \text{ for all } j \in \Lambda_{r-1} \backslash \{q\},$$

$$f_\ell^2 := - \sum_{k \in \Lambda_{r-2}} \alpha_{pk} f_k;$$

$$f_j^2 := \alpha_{\ell j} f_\ell^2, \text{ for all } j \in \Lambda_r \backslash \{\ell\}.$$

We have that $D^1(\Lambda_{2t-1}) = \Lambda_{2t}$ and $D^1(\Lambda_{2t}) = \Lambda_{2t-1}$ for $t \in \{1, \dots, c-1\}$, $D^1(\Lambda_{r-2}) = \Lambda_{r-1} \cup \Lambda_r$ and $D^1(\Lambda_{r-1}) = D^1(\Lambda_r) = \Lambda_{r-2}$. Furthermore, as in the previous case, we can reorder B' such that the structure matrix of $\mathcal{A}'$ relative to that basis is a block diagonal matrix $\text{diag}(H_1, \dots, H_c)$ where H_t is a matrix of the form (7), F_t of the form (13) for $t \in \{1, \dots, c-1\}$, H_c is of the form (8) and

$$F_c = \begin{pmatrix} 1 & \alpha_{qq+1} & \dots & \alpha_{qs_{r-1}} \\ \alpha_{pp+1} & \alpha_{pp+1}\alpha_{qq+1} & \dots & \alpha_{pp+1}\alpha_{qs_{r-1}} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{ps_{r-2}} & \alpha_{ps_{r-2}}\alpha_{qq+1} & \dots & \alpha_{ps_{r-2}}\alpha_{qs_{r-1}} \end{pmatrix},$$

$$F_{c+1} = \begin{pmatrix} 1 & \alpha_{\ell\ell+1} & \dots & \alpha_{\ell s_r} \\ \alpha_{pp+1} & \alpha_{pp+1}\alpha_{\ell\ell+1} & \dots & \alpha_{pp+1}\alpha_{\ell s_r} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{ps_{r-2}} & \alpha_{ps_{r-2}}\alpha_{\ell\ell+1} & \dots & \alpha_{ps_{r-2}}\alpha_{\ell s_r} \end{pmatrix}.$$

Analogously to other case, if $i, j \in \Lambda(j)$, we have $e_i^2 = \alpha_{ji} e_j^2$ and $f_i^2 = \alpha_{ji} f_j^2$ and $d_{ii} = 0$ for every $i \in \Lambda$. Consequently, $\mathrm{Der}(\mathcal{A}) = \mathrm{Der}(\mathcal{A}')$. □

Remark 4.5 Thanks to the Corollary 3.2 since $\mathcal{A}'$ is a non-degenerate reducible evolution algebra, to compute the set of derivations of $\mathcal{A}$ is equivalent to compute the set of derivations over the evolution ideals $I_i = \mathrm{span}(\{e_j \colon j \in \Lambda_i \cup \Lambda_{i+s}\})$ for any $i \in \{1, \dots, s-1\}$, $I_s = \mathrm{span}(\{e_j \colon j \in \Lambda_s \cup \Lambda_{2s}\})$ if $r = 2s$ and $I_s = \mathrm{span}(\{e_j \colon j \in \Lambda_s \cup \Lambda_{2s} \cup \Lambda_{2s+1}\})$ if $r = 2s+1$. So, we can reduce the dimension of evolution algebras whose space of derivations will be studied.

Proposition 4.6 *Let $\mathcal{A}$ be a non-degenerate Volterra evolution algebra with a natural basis $B = \{e_i\}_{i\in\Lambda}$ and $\Lambda = \Lambda_1 \cup \dots \cup \Lambda_r$ a natural decomposition of Λ relative to B. Let $\alpha_{ij} \in \mathbb{K}^\times$ such that $e_j^2 = \alpha_{ij} e_i^2$ for $i, j \in \Lambda(j)$. Suppose that there exists $i \in \Lambda$ such that for any $j \in D^1(i)$ it is verifies that $\sum_{k\in\Lambda(j)} \alpha_{jk}^3 = 0$ then $e_i^2 e_i^2 = 0$.*

Proof Let $M_B = (\omega_{ij})$ be the structure matrix of $\mathcal{A}$ relative to B and $i \in \Lambda$ such that for any $j \in D^1(i)$ we have that $\sum_{k\in\Lambda(j)} \alpha_{jk}^3 = 0$. Note that there exists $\{\ell_1, \dots, \ell_t\} \subseteq D^1(i)$ such that $D^1(i) = \Lambda(\ell_1) \cup \dots \cup \Lambda(\ell_t)$ with $\Lambda(\ell_h) \cap \Lambda(\ell_g) = \emptyset$ for $h \neq g$. Then

$$\begin{aligned} e_i^2 e_i^2 &= \sum_{k\in\Lambda(\ell_1)} w_{ik}^2 e_k^2 + \dots + \sum_{k\in\Lambda(\ell_t)} w_{ik}^2 e_k^2 \\ &= \sum_{k\in\Lambda(\ell_1)} w_{ki}^2 \alpha_{\ell_1 k} e_{\ell_1}^2 + \dots + \sum_{k\in\Lambda(\ell_t)} w_{ki}^2 \alpha_{\ell_t k} e_{\ell_t}^2 \\ &= w_{\ell_1 i}^2 e_{\ell_1}^2 \sum_{k\in\Lambda(\ell_1)} \alpha_{\ell_1 k}^3 + \dots + w_{\ell_t i}^2 e_{\ell_t}^2 \sum_{k\in\Lambda(\ell_t)} \alpha_{\ell_t k}^3. \end{aligned}$$

Since $\ell_h \in D^1(i)$ for $h \in \{1, \dots, t\}$ then by hypothesis $\sum_{k\in\Lambda(\ell_1)} \alpha_{\ell_1 k}^3 = \dots = \sum_{k\in\Lambda(\ell_t)} \alpha_{\ell_t k}^3 = 0$. Therefore $e_i^2 e_i^2 = 0$. □

Remark 4.7 Theorem 3.13 presents conditions for a non-degenerate evolution algebra to have only derivations with zero diagonal. If $\mathcal{A}$ is degenerate, then there always exists $d \in \mathrm{Der}(\mathcal{A})$ with non-zero diagonal entries. Indeed, let $\mathcal{A}$ be a degenerate evolution algebra with a natural basis $B = \{e_i\}_{i\in\Lambda}$ and $\ell \in \Lambda$ such that $e_\ell^2 = 0$. Consider the linear operator $d = (d_{ij})$ defined by

$$d_{ij} = \begin{cases} 1, & \text{if } i = j = \ell, \\ 0, & \text{if } i \neq \ell \text{ or } j \neq \ell. \end{cases}$$

Then the Eqs. (1) and (2) shows that $d \in \mathrm{Der}(\mathcal{A})$. The next two results show conditions for a non-degenerate Volterra evolution algebra to have derivations with non-zero diagonal entries

Remark 4.8 Let $\mathcal{A}$ be a Volterra evolution algebra with a natural basis $B = \{e_i\}_{i\in\Lambda}$ and let $\Lambda = \Lambda_1 \cup \dots \cup \Lambda_r$ be a natural decomposition of Λ relative to B. Note that if there exists a path from i to j then $\delta(i, \ell) = \delta(i, j)$ for all $\ell \in \Lambda(j)$. Therefore, $\delta(i, \Lambda(j)) = \delta(i, j)$. Analogously, we have that $\delta(\Lambda(i), \Lambda(j)) = \delta(i, j)$.

Proposition 4.9 *Let $\mathcal{A}$ be a non-degenerate Volterra evolution algebra with a natural basis $B = \{e_i\}_{i\in\Lambda}$ and $\Lambda = \Lambda_1 \cup \cdots \cup \Lambda_r$ a natural decomposition of Λ relative to B. Assume that the associated graph $E_{\mathcal{A}}^B$ does not have odd-length cycles. Moreover, suppose that there is $i \in \Lambda$ such that $\sum_{k\in\Lambda(j)} \alpha_{jk}^3 = 0$ for all $j \in \Lambda$ with $\delta(i, j)$ even. Then there exists $d = (d_{ij}) \in \mathrm{Der}(\mathcal{A})$ verifying $d_{kk} \neq 0$ for any $k \in D(i)$.*

Proof Firstly, we observe that if $i \in \Lambda$ then either $\Lambda_h \subseteq D(i)$ or $\Lambda_h \cap D(i) = \emptyset$ for every $h \in \{1, \dots, r\}$. Now, we can suppose without loss of generality that there exists a reordering of the natural decomposition in such a way that $\Lambda = \Lambda_1 \cup \cdots \cup \Lambda_t \cup \Lambda_{t+1} \cup \cdots \cup \Lambda_r$ where $i \in \Lambda_1$, $\Lambda_h \subseteq D(i)$ for all $h \in \{1, \dots, t\}$ and $\Lambda_h \cap D(i) = \emptyset$ for all $h \in \{t+1, \dots, r\}$. Let d be the linear map with diagonal matrix $d = \mathrm{diag}(C_1, \dots, C_r)$ where $C_k \in M_{|\Lambda_k|}(\mathbb{K})$ is defined as follows:

- If $k \in \{t+1, \dots, r\}$ then $C_k := 0$.
- If $k \in \{1, \dots, t\}$ and $\delta(i, \Lambda_k)$ is odd then $C_k =: 2I_{|\Lambda_k|}$.
- If $k \in \{1, \dots, t\}$, $\delta(i, \Lambda_k)$ is even and $\Lambda_k = \{k_1, \dots, k_s\}$ then

$$C_k := \begin{pmatrix} 1 & 0 & \dots & 0 & -3\alpha_{k_s k_1}^2 \\ 0 & 1 & \dots & 0 & -3\alpha_{k_s k_2}^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -3\alpha_{k_s k_{s-1}}^2 \\ 3\alpha_{k_s k_1} & 3\alpha_{k_s k_2} & \dots & 3\alpha_{k_s k_{s-1}} & 1 \end{pmatrix}.$$

To prove that $d = (d_{ij}) \in \mathrm{Der}(\mathcal{A})$ it is sufficient to verify that d satisfies the conditions of the Proposition 4.2. It is clear that the condition (i) of Proposition 4.2 holds.

Now we are going to verify that d satisfies the condition (ii) of Proposition 4.2. Let $\ell, j \in \Lambda$, $\ell \neq j$ and $\{\ell, j\} \subseteq \Lambda_k$. If $k \in \{t+1, \dots, r\}$ or $k \in \{1, \dots, t\}$ and $\delta(i, \Lambda_k)$ is odd, then $d_{\ell j} = d_{j\ell} = 0$ by the definition of d. Now, observe that if $\{\ell, j\} \subseteq \Lambda_k \setminus \{k_s\}$ where $\Lambda_k = \{k_1, \dots, k_s\}$ and $\delta(i, \Lambda_k)$ is even then $d_{\ell j} = d_{j\ell} = 0$. Finally, if $\{\ell, j\} \cap \{k_s\} \neq \emptyset$ we can assume without loss of generality that $\ell = k_s$. Then,

$$d_{\ell j} = d_{k_s j} = 3\alpha_{k_s j} = 3\alpha_{k_s j}\alpha_{k_s j}\alpha_{jk_s} = \alpha_{jk_s} 3\alpha_{k_s j}^2 = -\alpha_{j\ell} d_{j\ell}.$$

Next, we will show that d verifies the condition (iii) of Proposition 4.2. First, note that if $k \in \{1, \dots, t\}$ and $\delta(i, \Lambda_k)$ is odd, then $\delta(i, \Lambda_h)$ is even, for all Λ_h satisfying $\delta(\Lambda_h, \Lambda_k) = 1$. Indeed, if we suppose that $\delta(i, \Lambda_k)$ and $\delta(i, \Lambda_h)$ are odd for some Λ_h with $\delta(\Lambda_h, \Lambda_k) = 1$ then there are paths μ from i to j_1 and σ from i to j_2, where $j_1 \in \Lambda_k$ and $j_2 \in \Lambda_h$ such that $|\mu|$ and $|\sigma|$ are odd. As $\delta(j_1, j_2) = 1$ then we have a cycle of length $|\mu| + |\sigma| + 1$ which is odd, a contradiction. Analogously, if $k \in \{1, \dots, t\}$ and $\delta(i, \Lambda_k)$ is even, then $\delta(i, \Lambda_h)$ is odd for all Λ_h verifying $\delta(\Lambda_h, \Lambda_k) = 1$. Now, let $\ell, j \in \Lambda$ with $\ell \in D^1(j)$. Observe that $\Lambda(\ell) \cap \Lambda(j) = \emptyset$ because $\mathcal{A}$ is a Volterra evolution algebra and so $\omega_{jj} = 0$. We will distinguish several cases:

Case 1 If $\ell \in \Lambda_1 \cup \ldots \cup \Lambda_t$ and $j \in \Lambda_{t+1} \cup \cdots \cup \Lambda_r$ then $\ell \notin D^1(j)$, which is a contradiction.

Case 2 If $\ell, j \in \Lambda_{t+1} \cup \cdots \cup \Lambda_r$ then $d_{\ell\ell} = d_{\ell j} = d_{j\ell} = 0$ by definition and we get what we wanted.

Case 3 In this case we suppose that $\delta(i, \Lambda(\ell))$ is even. Necessarily $\delta(i, \Lambda(j))$ is odd since $\delta(\Lambda(j), \Lambda(\ell)) = 1$. Thus $d_{\ell\ell} = 1$, $d_{jj} = 2$ and $d_{kj} = 0$ for all $k \in \Lambda \setminus \{j\}$. Therefore

$$2d_{\ell\ell} = 2 = \alpha_{jj} d_{jj} = \sum_{k\in\Lambda(j)} \alpha_{jk} d_{kj}.$$

Case 4 Assume that $\delta(i, \Lambda(\ell))$ odd. As in the previous case, observe that $\delta(i, \Lambda(j))$ is even. Thus, we have that $d_{\ell\ell} = 2$. Let $\Lambda(j) = \{k_1, \ldots, k_s\}$. We are going to consider two cases:
Case 4.1 If $j \neq k_s$ then $d_{jj} = 1$, $d_{k_s j} = 3\alpha_{k_s j}$ and $d_{kj} = 0$ for $k \in \Lambda \backslash \{j, k_s\}$. Then

$$\sum_{k\in\Lambda(j)} \alpha_{jk} d_{kj} = \alpha_{jj} d_{jj} + \alpha_{jk_s} d_{k_s j} = 1 + \alpha_{jk_s} 3\alpha_{k_s j} = 4 = 2d_{\ell\ell}.$$

Case 4.2 If $j = k_s$ then $d_{k_s k_s} = 1$ and $d_{kk_s} = -3\alpha_{k_s k}^2$ for $k \in \{k_1, \ldots, k_{s-1}\}$. Then

$$\sum_{k\in\Lambda(j)} \alpha_{k_s k} d_{kk_s} = \alpha_{k_s k_s} d_{k_s k_s} - \sum_{\substack{k\in\Lambda(j)\\ k\neq k_s}} \alpha_{k_s k} 3\alpha_{k_s k}^2 = 1 - 3 \sum_{\substack{k\in\Lambda(j)\\ k\neq k_s}} \alpha_{k_s k}^3.$$

Since $\sum_{k\in\Lambda(j)} \alpha_{k_s k}^3 = 0$ we get that $\sum_{\substack{k\in\Lambda(j)\\ k\neq k_s}} \alpha_{k_s k}^3 = -1$. Therefore,

$$\sum_{k\in\Lambda(j)} \alpha_{k_s k} d_{kk_s} = 1 - 3 \sum_{\substack{k\in\Lambda(j)\\ k\neq k_s}} \alpha_{k_s k}^3 = 4 = 2d_{\ell\ell}.$$

□

The condition of having odd-length cycles can not be eliminated as shown the following example.

Example 4.10 Let $\mathcal{A}$ an evolution algebra with $B = \{e_i\}_{i\in\Lambda}$ ($\Lambda = \{1, \ldots, 7\}$) natural basis and multiplication table $e_1^2 = e_2 - e_3 + e_4 - e_5 + e_6 - e_7$, $e_2^2 = -e_3^2 = -e_1 + e_4 - e_5$, $e_4^2 = -e_5^2 = -e_1 - e_2 + e_3$ and $e_6^2 = -e_7^2 = -e_1$. Since that M_B is skew-symmetric, then $\mathcal{A}$ is a Volterra evolution algebra. Consider a natural decomposition of $\Lambda = \Lambda_1 \cup \Lambda_2 \cup \Lambda_3 \cup \Lambda_4$ relative to B, where $\Lambda_1 = \{1\}$, $\Lambda_2 = \{2, 3\}$, $\Lambda_3 = \{4, 5\}$ and $\Lambda_4 = \{6, 7\}$. Note that $7 \in \Lambda$ is such that $\sum_{k\in\Lambda_g} \alpha_{jk}^3 = 0$ for all $j \in \Lambda (j \in \Lambda_g)$ with $\delta(7, j)$ even. Therefore all hypotheses of Theorem 4.9 are verified, except that $E_{\mathcal{A}}^B$ has odd length cycles. Furthermore, an easy computation shows that $\mathrm{Der}(\mathcal{A}) = \{0\}$.

Proposition 4.11 *Let $\mathcal{A}$ be a non-degenerate Volterra evolution algebra with a natural basis $B = \{e_i\}_{i\in\Lambda}$, $\Lambda = \Lambda_1 \cup \cdots \cup \Lambda_r$ a natural decomposition of Λ relative to B and structure matrix $M_B = (\omega_{ij})$. If there is $i \in \Lambda$ such that $\sum_{k\in\Lambda_g} \alpha_{jk}^3 = 0$ for all $j \in D(i)$ where $i \in \Lambda_g$, then there exists $d = (d_{ij}) \in \mathrm{Der}(\mathcal{A})$ such that $d_{kk} \neq 0$ for all $k \in D(i)$.*

Proof First, we can reorder the natural decomposition in such a way that $\Lambda = \Lambda_1 \cup \cdots \cup \Lambda_t \cup \Lambda_{t+1} \cup \cdots \cup \Lambda_r$, with $D(i) = \Lambda_1 \cup \cdots \cup \Lambda_t$. Define $d = (d_{ij})$ as the diagonal matrix $\mathrm{diag}(C_1, \ldots, C_r)$ where $C_k \in M_{|\Lambda_k|}(\mathbb{K})$ is defined as below:

- If $k \in \{t+1, \ldots, r\}$ then $C_k := 0$.
- If $k \in \{1, \ldots, t\}$ and $\Lambda_k = \{k_1, \ldots, k_s\}$ then

$$C_k := \begin{pmatrix} 1 & 0 & \ldots & 0 & -\alpha_{k_s k_1}^2 \\ 0 & 1 & \ldots & 0 & -\alpha_{k_s k_2}^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & -\alpha_{k_s k_{s-1}}^2 \\ \alpha_{k_s k_1} & \alpha_{k_s k_2} & \ldots & \alpha_{k_s k_{s-1}} & 1 \end{pmatrix}.$$

In order to see that d is a derivation, we will prove that it satisfies the conditions (i)–(iii) of Proposition 4.2. Note that d verifies Proposition 4.2(i) by definition.

Let $p, q \in \Lambda$, $p \neq q$ and $\{p, q\} \subseteq \Lambda_h$ for $h \in \{1, \ldots, r\}$. If $h > t$ then $d_{pq} = d_{qp} = 0$. If $h \leq t$, we consider two cases.

Case 1 If $\{p, q\} \subseteq \Lambda_h \backslash \{k_s\}$ then $d_{pq} = d_{qp} = 0$.

Case 2 If $\{p, q\} \cap \{k_s\} \neq \emptyset$, assume, without loss of generality, that $p = k_s$. Then

$$d_{pq} = d_{k_s q} = \alpha_{k_s q} = \alpha_{k_s q}(\alpha_{k_s q}\alpha_{q k_s}) = \alpha_{q k_s}(\alpha_{k_s q}^2) = \alpha_{q k_s}(-d_{q k_s}) = -\alpha_{qp} d_{qp}.$$

Now, let $p, q \in \Lambda$, $q \in \Lambda(q) = \Lambda_h$ for some $h \in \{1, \ldots, r\}$ and $p \in D^1(q)$.

If $h > t$, then $d_{pp} = d_{kq} = 0$ for all $k \in \Lambda_h$ since $p \notin D(i)$ otherwise $q \in D(i)$ contradiction. So the condition (ii) of Proposition 4.2 is verified.

If $h \leq t$ therefore $d_{qq} = 1$. Moreover, $p \in D(i)$ then $d_{pp} = 1$. Writing $\Lambda_h = \{k_1, k_2, \ldots, k_s\}$, we will distinguish two cases.

Case 1 If $q \neq k_s$ then

$$\sum_{k \in \Lambda_h} \alpha_{qk} d_{kq} = d_{qq} + \alpha_{q k_s} d_{k_s q} = d_{qq} + \alpha_{q k_s}\alpha_{k_s q} = 2.$$

Case 2 If $q = k_s$ then

$$\sum_{k \in \Lambda_h} \alpha_{k_s k} d_{k k_s} = 1 + \sum_{\substack{k \in \Lambda_h \\ k \neq k_s}} \alpha_{k_s k}(-\alpha_{k_s k}^2) = 1 - \sum_{\substack{k \in \Lambda_h \\ k \neq k_s}} \alpha_{k_s k}^3 = 2.$$

In both cases, we have that $2d_{pp} = \sum_{k \in \Lambda_h} \alpha_{qk} d_{kq}$. □

5 The loops of an evolution algebra

By analogy with graph theory, we define when an element of a natural basis is called loop and then we start by studying what properties of this set is invariant under change of natural basis.

Definition 5.1 Let $\mathcal{A}$ be an evolution algebra with a natural basis $B = \{e_i\}_{i \in \Lambda}$ and structure matrix $M_B = (\omega_{ij})$. We say that e_i is a *loop relative to the basis* B if $\omega_{ii} \neq 0$. Otherwise, we say that e_i is a *no-loop*. We denote by $\mathrm{L}(\mathcal{A}, B)$ the set of loops and by $\mathrm{NL}(\mathcal{A}, B) := B \backslash \mathrm{L}(\mathcal{A}, B)$ the set of no-loops.

Remark 5.2 If $\mathcal{A}$ is an evolution algebra with a natural basis $B = \{e_i\}_{i \in \Lambda}$ then the following conditions are equivalent:

- $\mathrm{L}(\mathcal{A}, B) = \emptyset$.
- $i \notin \mathrm{supp}(e_i^2)$ for all $i \in \Lambda$.
- $e_i e_i^2 = 0$ for all $i \in \Lambda$.

Theorem 5.3 *Let $\mathcal{A}$ be an evolution algebra with natural basis B and B'. Let $B = B_0 \cup \cdots \cup B_r$ and $B' = B'_0 \cup \cdots \cup B'_r$ be natural decomposition of B and B' where B' is reordered in such a way that $B'_0 = B_0$ and $B'_t \subseteq \mathrm{span}(B_0 \cup B_t)$. Then $B_t \subseteq \mathrm{NL}(\mathcal{A}, B)$ if and only if $B'_t \subseteq \mathrm{NL}(\mathcal{A}, B')$.*

Proof Let $B = \{e_i\}_{i \in \Lambda}$ and $B' = \{f_i\}_{i \in \Lambda}$ and $M_B = (\omega_{ij})$ and $M_{B'} = (\omega'_{ij})$ the corresponding structure matrices. For B_0 the affirmation is trivial. Let B_t with $t \neq 0$ such that

$B_t \subseteq \mathrm{NL}(\mathcal{A}, B)$. Suppose, contrary to our claim, that there exists $f_j \in B'_t \cap \mathrm{L}(\mathcal{A}, B')$. Then according to Remark 5.2 we have that $f_j^2 f_j = \omega'_{jj} f_j^2 \neq 0$. Let $e_\ell \in B_t$ and $\alpha_{\ell k} \in \mathbb{K}^\times$ for $k \in \Lambda_t$ such that $e_k^2 = \alpha_{\ell k} e_\ell^2$. Then $D^1(k) = D^1(\ell)$, for $k \in \Lambda_t$. On the other hand, using Remarks 2.2 and 2.5 we can write $f_j = \sum_{k\in\Lambda_t} x_k e_k + \sum_{k\in\Lambda_0} x_k e_k$. Then

$$f_j^2 = \sum_{k\in\Lambda_t} x_k^2 e_k^2 = \sum_{k\in\Lambda_t} x_k^2 \alpha_{\ell k} e_\ell^2 = \beta \sum_{p\in D^1(\ell)} \omega_{\ell p} e_p,$$

where $\beta = \sum_{k\in\Lambda_t} x_k^2 \alpha_{\ell k}$. Therefore

$$\begin{aligned} 0 \neq f_j^2 f_j = f_j^2 \left(\sum_{k\in\Lambda_t} x_k e_k + \sum_{k\in\Lambda_0} x_k e_k \right) = f_j^2 \sum_{k\in\Lambda_t} x_k e_k \\ = \beta \left(\sum_{p\in D^1(\ell)} \omega_{\ell p} e_p \right) \left(\sum_{k\in\Lambda_t} x_k e_k \right). \end{aligned}$$

Thus there exists $e_q \in B_t$ such that $q \in D^1(\ell) = D^1(q)$, that is, $e_q \in \mathrm{L}(\mathcal{A}, B)$, contrary to our assumption. The proof of the reciprocal statement is analogous. □

Corollary 5.4 *Let $\mathcal{A}$ be an evolution algebra. Assume that there exists a natural basis B of $\mathcal{A}$ satisfying some of the following conditions:*

(i) *$\mathcal{A}$ has Property* (2LI).
(ii) $\mathrm{L}(\mathcal{A}, B) = \emptyset$.
(iii) $\mathcal{A}^2 = \mathcal{A}$.
(iv) *$\mathcal{A}$ is twin-free relative to B.*
(v) *$\mathcal{A}$ is a Volterra evolution algebra relative to B.*

Then $|\mathrm{L}(\mathcal{A}, B)| = |\mathrm{L}(\mathcal{A}, B')|$ for all natural basis B'.

Corollary 5.5 *Let $\mathcal{A}$ be an evolution algebra with natural basis B and B'. Then $\mathrm{L}(\mathcal{A}, B) = \emptyset$ if and only if $\mathrm{L}(\mathcal{A}, B') = \emptyset$.*

However, the number of loops is not invariant under the natural base change as the following example shows.

Example 5.6 Consider the evolution algebra $\mathcal{A}$ with natural basis $\{e_i\}_{i=1}^3$ such that $e_1^2 = e_2^2 = e_2 + e_3$ and $e_3^2 = e_1 - e_2 + e_3$. Observe that $\mathrm{L}(\mathcal{A}, B) = \{e_2, e_3\}$. If we take another natural basis $B' = \{f_i\}_{i=1}^3$ with $f_1 = e_1 + e_2$, $f_2 = e_3$ and $f_3 = e_1 - e_2$ then $\mathrm{L}(\mathcal{A}, B') = B'$.

Theorem 5.7 *Let $\mathcal{A}$ be an evolution algebra with a natural basis $B = \{e_i\}_{i\in\Lambda}$ and $\Lambda = \Lambda_0 \cup \Lambda_1 \cup \ldots \cup \Lambda_r$ be a natural decomposition of Λ relative to B. If for all $t \in \{1, \ldots, r\}$ such that $|\Lambda_t| > 1$ we have that $B_t \subseteq \mathrm{NL}(\mathcal{A}, B)$ then $|\mathrm{L}(\mathcal{A}, B)| = |\mathrm{L}(\mathcal{A}, B')|$ for any natural basis B' of $\mathcal{A}$.*

Proof Let $B' = \{f_i\}_{i\in\Lambda}$ be a natural basis of $\mathcal{A}$. We define $\Lambda^1 := \bigcup_{|\Lambda_t|=1} \Lambda_t$. By our assumption, $\mathrm{L}(\mathcal{A}, B) \subseteq \{e_i : i \in \Lambda^1\}$. Note that if $i \in \Lambda^1$ then, by Theorem 5.3, up to reordering, $e_i \in \mathrm{L}(\mathcal{A}, B)$ if and only if $f_i \in \mathrm{L}(\mathcal{A}, B')$. Therefore $|\mathrm{L}(\mathcal{A}, B)| = |\mathrm{L}(\mathcal{A}, B')|$. □

Theorem 5.8 *Let $\mathcal{A}$ be an evolution algebra with a natural basis $B = \{e_i\}_{i\in\Lambda}$. Consider a natural decomposition $B = B_0 \cup B_1 \cup \cdots \cup B_r$ and suppose that there exists $t \in \{1, \ldots, r\}$ such that $|B_t| > 1$ and satisfies some of the following conditions*

(i) $B_t \cap \mathrm{L}(\mathcal{A}, B) \neq \emptyset$ *and* $B_t \cap \mathrm{NL}(\mathcal{A}, B) \neq \emptyset$.

(ii) $B_t \subseteq \mathrm{L}(\mathcal{A}, B)$ *and there exist* $e_q, e_p \in B_t$ *such that* $\alpha_{qp} \neq -\left(\frac{\omega_{qq}}{\omega_{qp}}\right)^2$ *where* $e_p^2 = \alpha_{qp} e_q^2$.

(iii) $B_t \subseteq \mathrm{L}(\mathcal{A}, B)$ *and* $|B_t| > 2$.

Then there is a natural basis B' *such that* $|\mathrm{L}(\mathcal{A}, B)| \neq |\mathrm{L}(\mathcal{A}, B')|$.

Proof First, assume that B_t satisfies (i). Let $e_i \in B_t \cap \mathrm{L}(\mathcal{A}, B)$ and $e_j \in B_t \cap \mathrm{NL}(\mathcal{A}, B)$. Let $\alpha_{ij} \in \mathbb{K}^\times$ such that $e_j^2 = \alpha_{ij} e_i^2$ and $\gamma \in \mathbb{K}\backslash\{0, 1\}$ such that $\gamma^2 \neq \frac{-1}{\alpha_{ij}}$. Then $\omega_{ij} = \frac{1}{\alpha_{ij}} \omega_{jj} = 0$. Consider the basis $B' = \{f_k\}_{k \in \Lambda}$ where

$$f_k = \begin{cases} e_k, & \text{if } k \neq i, j, \\ e_i + \gamma e_j, & \text{if } k = i, \\ -\gamma \alpha_{ij} e_i + e_j, & \text{if } k = j. \end{cases}$$

Note that $f_i f_j = -\gamma \alpha_{ij} e_i^2 + \gamma e_j^2 = -\gamma \alpha_{ij} e_i^2 + \gamma \alpha_{ij} e_i^2 = 0$. So, B' is a natural basis for $\mathcal{A}$. Observe now that $f_k f_k^2 = 0$ if and only if $e_k e_k^2 = 0$ for all $k \in \Lambda\backslash\{i, j\}$. Now we will prove that $f_i f_i^2 \neq 0$ and $f_j f_j^2 \neq 0$. Indeed,

$$e_i e_i^2 = \omega_{ii} e_i^2,\ e_j e_j^2 = e_j e_i^2 = 0,\ f_i^2 = (1 + \alpha_{ij}\gamma^2) e_i^2 \text{ and } f_j^2 = (\alpha_{ij} + \alpha_{ij}^2 \gamma^2) e_i^2.$$

Then

$$\begin{aligned} f_i f_i^2 &= (e_i + \gamma e_j)(1 + \alpha_{ij}\gamma^2) e_i^2 = (1 + \alpha_{ij}\gamma^2)\omega_{ii} e_i^2 \neq 0 \\ f_j f_j^2 &= (-\gamma \alpha_{ij} e_i + e_j)(\alpha_{ij} + \alpha_{ij}^2 \gamma^2) e_i^2 = -\gamma \alpha_{ij}(\alpha_{ij} + \alpha_{ij}^2 \gamma^2)\omega_{ii} e_i^2 \neq 0 \end{aligned}$$

Therefore $|\mathrm{L}(\mathcal{A}, B)| = |\mathrm{L}(\mathcal{A}, B')| + 1$.

Now, assume that B_t satisfies (ii). Define $\gamma := -\alpha_{qp} \frac{\omega_{qp}}{\omega_{qq}}$ and $\beta := \frac{\omega_{qp}}{\omega_{qq}}$. Consider the set $B' = \{f_k\}_{k \in \Lambda}$ where

$$f_k = \begin{cases} e_k, & \text{if } k \neq q, p, \\ \gamma e_q + e_p, & \text{if } k = q, \\ e_q + \beta e_p, & \text{if } k = p. \end{cases}$$

By hypotheses $\gamma\beta \neq 1$ then B' is a basis for $\mathcal{A}$. Similarly to item (i), $f_k f_k^2 = 0$ if and only if $e_k e_k^2 = 0$ for all $k \in \Lambda\backslash\{q, p\}$. Therefore

$$\begin{aligned} f_q f_q^2 &= (\gamma e_q + e_p)(\gamma^2 + \alpha_{qp}) e_q^2 = (\gamma^2 + \alpha_{qp})(\gamma \omega_{qq} e_q^2 + \omega_{qp} e_p^2) \\ &= (\gamma^2 + \alpha_{qp})(\gamma \omega_{qq} + \omega_{qp}\alpha_{qp}) e_q^2 = 0. \end{aligned}$$

So $|\mathrm{L}(\mathcal{A}, B')| < |\mathrm{L}(\mathcal{A}, B)|$.

Finally, assume that B_t satisfies (iii). Suppose that $\alpha_{ji} = -\left(\frac{\omega_{jj}}{\omega_{ji}}\right)^2$ for all $j, i \in \Lambda_t$. Then we will get a contradiction. Indeed, let $q, p, \ell \in \Lambda_t$. Since $\alpha_{qp} = -\frac{\omega_{qq}^2}{\omega_{qp}^2}$ then $\omega_{pq} = -\frac{\omega_{qq}^3}{\omega_{qp}^2}$ and analogously $\omega_{\ell q} = -\frac{\omega_{qq}^3}{\omega_{qt}^2}$. Consequently $\omega_{pp} = -\frac{\omega_{qq}^2}{\omega_{qp}}$ and $\omega_{p\ell} = -\omega_{q\ell}\frac{\omega_{qq}^2}{\omega_{qp}^2}$. Moreover, using that $\alpha_{p\ell} = -\frac{\omega_{pp}^2}{\omega_{p\ell}^2}$ we get that $\omega_{tq} = \alpha_{pt}\omega_{pq} = \frac{\omega_{qq}^3}{\omega_{q\ell}^2}$, which is a contradiction. Therefore there exist $q, p \in \Lambda_t$ such that $\alpha_{qp} \neq -\frac{\omega_{qq}^2}{\omega_{qp}^2}$ and the affirmation follows from item (ii). $\square$

Proposition 5.9 *Let $\mathcal{A}$ be a non-degenerate evolution algebra and $B = \{e_i\}_{i\in\Lambda}$ a natural basis of $\mathcal{A}$. Consider a natural decomposition $B = B_1 \cup \cdots \cup B_r$ and suppose that $|B_s| \leq 2$ for all $s \in \{1, \ldots, r\}$ such that $B_s \cap \mathrm{L}(\mathcal{A}, B) \neq \emptyset$. If for every B_t, with $|B_t| = 2$, some of the following conditions is satisfied:*

(i) $B_t \subseteq \mathrm{NL}(\mathcal{A}, B)$,

(ii) $B_t \subseteq \mathrm{L}(\mathcal{A}, B)$ *and* $\alpha_{jk} = -\left(\frac{\omega_{jj}}{\omega_{jk}}\right)^2$ *for all* $j, k \in \Lambda_t$ *where* $e_k^2 = \alpha_{jk}e_j^2$,

then $|\mathrm{L}(\mathcal{A}, B)| = |\mathrm{L}(\mathcal{A}, B')|$ *for all natural basis* B' *of* $\mathcal{A}$.

Proof Let $\Lambda = \Lambda_1 \cup \ldots \cup \Lambda_r$ be a natural decomposition of Λ relative to B. Let us define

$$\Lambda^1 := \bigcup_{|\Lambda_t|=1} \Lambda_t,\ \Lambda^2 := \bigcup_{\substack{|\Lambda_t|=2\\ B_t\subseteq \mathrm{L}(\mathcal{A},B)}} \Lambda_t \text{ and } \Lambda^3 := \bigcup_{\substack{|\Lambda_t|\geq 2\\ B_t\subseteq \mathrm{NL}(\mathcal{A},B)}} \Lambda_t.$$

Suppose, contrary to our claim, that there is a natural basis $B' = \{f_k\}_{k\in\Lambda}$ of $\mathcal{A}$ such that $|\mathrm{L}(\mathcal{A}, B')| \neq |\mathrm{L}(\mathcal{A}, B)|$. Let $B' = B'_1 \cup \cdots \cup B'_r$ be a natural decomposition, ordered in such a way that $B'_t \subseteq \mathrm{span}(B_t)$ for any t and let $M_{B'} = (\omega'_{kj})$ be the structure matrix of $\mathcal{A}$ relative to B'. By Theorem 5.3 we have that $\left|\{k \in \Lambda^1 : \omega'_{kk} \neq 0\}\right| = \left|\{k \in \Lambda^1 : \omega_{kk} \neq 0\}\right|$ and $\omega'_{jj} = 0$ for all $j \in \Lambda^3$. Therefore there exists $h \in \{1, \ldots, r\}$ such that $B_h = \{e_i, e_\ell\} \subseteq \mathrm{L}(\mathcal{A}, B)$ and $B'_h = \{f_k, f_j\} \nsubseteq \mathrm{L}(\mathcal{A}, B')$. Without loss of generality, we assume that $f_k f_k^2 = 0$. Then we have that

$$f_k = x_{11}e_i + x_{12}e_\ell \text{ and } f_j = x_{21}e_i + x_{22}e_\ell,$$

where $x_{ij} \in \mathbb{K}$. Consequently, using that $e_\ell^2 = \alpha_{i\ell}e_i^2$, $e_i e_i^2 = \omega_{ii}e_i^2$ and $e_i^2 e_\ell = \omega_{i\ell}e_\ell^2$ it follows that

$$\begin{aligned} f_k f_k^2 &= (x_{11}e_i + x_{12}e_\ell)(x_{11}^2 + x_{12}^2\alpha_{i\ell})e_i^2 \\ &= (x_{11}^2 + x_{12}^2\alpha_{i\ell})(x_{11}\omega_{ii}e_i^2 + x_{12}\omega_{i\ell}e_\ell^2) \\ &= (x_{11}^2 + x_{12}^2\alpha_{i\ell})(x_{11}\omega_{ii} + x_{12}\omega_{i\ell}\alpha_{i\ell})e_i^2 = 0. \end{aligned} \tag{14}$$

As $f_k^2 \neq 0$ then $x_{11}^2 + x_{12}^2\alpha_{i\ell} \neq 0$. Thus

$$x_{11}\omega_{ii} - x_{12}\omega_{i\ell}\frac{\omega_{ii}^2}{\omega_{i\ell}^2} = \omega_{ii}\left(x_{11} - x_{12}\frac{\omega_{ii}}{\omega_{i\ell}}\right) = 0,$$

where we use that, by hypotheses, $\alpha_{i\ell} = -\frac{\omega_{ii}^2}{\omega_{i\ell}^2}$. The fact that $\omega_{ii} \neq 0$ implies $x_{11} = \frac{\omega_{ii}}{\omega_{i\ell}}x_{12}$. Therefore $x_{11}x_{12} \neq 0$ because $f_k \in B'$. On the other hand, since B' is a natural basis

$$f_k f_j = x_{11}x_{21}e_i^2 + x_{12}x_{22}e_\ell^2 = (x_{11}x_{21} + x_{12}x_{22}\alpha_{i\ell})e_i^2 = 0. \tag{15}$$

Now, by Eq. (15) we obtain that

$$x_{21} = \frac{x_{12}x_{22}}{x_{11}}\frac{\omega_{ii}^2}{\omega_{i\ell}^2} = x_{12}x_{22}\frac{\omega_{ii}^2}{\omega_{i\ell}^2}\frac{\omega_{i\ell}}{x_{12}\omega_{ii}} = \frac{\omega_{ii}}{\omega_{i\ell}}x_{22}.$$

Finally, we conclude that

$$x_{11}x_{22} - x_{12}x_{21} = x_{12}\frac{\omega_{ii}}{\omega_{i\ell}}x_{22} - x_{12}\frac{\omega_{ii}}{\omega_{i\ell}}x_{22} = 0.$$

Therefore f_k and f_j are linearly dependent, which is a contradiction. □

Proposition 5.10 *Let $\mathcal{A}$ be an evolution algebra with a natural basis $B = \{e_i\}_{i\in\Lambda}$ such that the structure matrix $M_B = (\omega_{ij})$ satisfies $\omega_{ij} = 0$ if and only if $\omega_{ji} = 0$, for all $i, j \in \Lambda$, $i \neq j$. Consider the natural decomposition $\Lambda = \Lambda_0 \cup \cdots \cup \Lambda_r$ relative to B. Suppose that $e_i \in \mathrm{NL}(\mathcal{A}, B)$. Then $e_j \in \mathrm{NL}(\mathcal{A}, B)$ for every $j \in \Lambda(i)$.*

Proof Since $\omega_{ii} = 0$ and $\mathrm{rk}(\{e_i^2, e_j^2\}) = 1$ we get that $\omega_{ji} = 0$. By symmetry $\omega_{ij} = 0$. Again, as $\mathrm{rk}(\{e_i^2, e_j^2\}) = 1$ then $\omega_{jj} = 0$. □

Remark 5.11 If A is an evolution algebra and B is a natural basis satisfying the assumptions of the previous proposition and $B = B_0 \cup \cdots \cup B_r$ is a natural decomposition then either $B_t \subseteq \mathrm{L}(\mathcal{A}, B)$ or $B_t \subseteq \mathrm{NL}(\mathcal{A}, B)$ for every $t \in \{0, 1, \ldots, r\}$.

Corollary 5.12 *Let $\mathcal{A}$ be a non-degenerate evolution algebra with a natural basis $B = \{e_i\}_{i\in\Lambda}$. Consider a natural decomposition $B = B_1 \cup \cdots \cup B_r$. Then the number of loops in $\mathcal{A}$ is invariant by change of natural basis if $|B_t| \leq 2$ for all $t \in \{1, \ldots, r\}$ such that $B_t \cap \mathrm{L}(\mathcal{A}, B) \neq \emptyset$ and for every s such that $|B_s| = 2$, some of the following conditions is satisfied:*

(i) $B_s \subseteq \mathrm{NL}(\mathcal{A}, B)$,
(ii) $B_s \subseteq \mathrm{L}(\mathcal{A}, B)$ *and* $\alpha_{jk} = -\left(\omega_{jj}/\omega_{jk}\right)^2$ *for all* $j, k \in \Lambda_s$ *where* $e_k^2 = \alpha_{jk} e_j^2$.

Otherwise the number of loops of $\mathcal{A}$ depend of the natural basis.

Proof The statement that the number of loops is invariant by the change of base follows from Proposition 5.9. On the other hand, by Theorem 5.8, in any other case the number of loops in $\mathcal{A}$ depends on the natural basis. □

Funding Funding for open access publishing: Universidad Málaga/CBUA

Data Availability The authors confirm that the data supporting the findings of this study are available within the article.

References

1. Alsarayreh, A., Qaralleh, I., Ahmmad, M.Z.: Derivation of three dimensional evolution algebras. JP J. Algebra Number Theory Appl. **39**(4), 425–444 (2017)
2. Boudi, N., Cabrera Casado, Y., Siles Molina, M.: Natural families in evolution algebras. Publ. Mate. **66**(1), 159–181 (2022)
3. Cabrera Casado, Y., Cadavid, P., Rodiño Montoya, M.L., Rodríguez, P.M.: On the characterization of the space of derivations in evolution algebras. Ann. Mate. Pura ed Appl. (1923–) **200**, 737–755 (2021)
4. Cabrera Casado, Y., Müge, K., Siles Molina, M.: Basic ideals in evolution algebras. Linear Algebra Appl. **570**, 148–180 (2019)
5. Cabrera Casado, Y., Siles Molina, M., Velasco, M.V.: Evolution algebras of arbitrary dimension and their decomposition. Linear Algebra Appl. **496**, 122–162 (2016)

6. Cabrera Casado, Y., Siles Molina, M., Velasco, M.V. : Classification of three-dimensional evolution algebras. Linear Algebra Appl. **524**, 68–108 (2017)
7. Cadavid, P., Rodiño Montoya, M.L., Rodriguez, P.M.: Characterization theorems for the space of derivations of evolution algebras associated to graphs. Linear Multilinear Algebra **68**(7), 1340–1354 (2020)
8. Camacho, L.M., Gómez, J.R., Omirov, B.A., Turdibaev, R.M.: The derivations of some evolution algebras. Linear Multilinear Algebra **61**, 309–322 (2013)
9. Cardoso, M.I., Gonçalves, D., Martín, D., Martín, C., Siles Molina, M.: Squares and associative representations of two dimensional evolution algebras. J. Algebra Appl. **20**(06), 2150090 (2021)
10. Ceballos, M., Falcón, R., Núñez-Valdés, J., Tenorio, A.: A historical perspective of Tian's evolution algebras. Expos. Math. **40**(3), 819–843 (2022)
11. Elduque, A., Labra, A.: Evolution algebras and graphs. J. Algebra Appl. **14**, 1550103 (2015)
12. Elduque, A., Labra, A.: Evolution algebras, automorphisms, and graphs. Linear Multilinear Algebra **69**(2), 331–342 (2021)
13. Qaralleh, I., Mukhamedov, F.: Volterra evolution algebras and their graphs. Linear Multilinear Algebra **69**(12), 2228–2244 (2021)
14. Reis, T., Cadavid, P.: Derivations of evolution algebras associated to graphs over a field of any characteristic. Linear Multilinear Algebra **70**(15), 2884–2897 (2022)
15. Tian, J.P.: Evolution Algebras and Their Applications. Springer, Berlin (2008)
16. Tian, J.P., Vojtechovsky, P.: Mathematical concepts of evolution algebras. In: Non-Mendelian Genetics Quasigroups Related Systems, vol. 14, pp. 111–122 (2006)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.