Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat. (2023) 117:91
https://doi.org/10.1007/s13398-023-01427-5

ORIGINAL PAPER

The second Hankel determinant of the logarithmic coefficients of strongly starlike and strongly convex functions

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Received: 21 August 2021 / Accepted: 20 March 2023 / Published online: 29 March 2023

Abstract
Sharp bounds are given for the second Hankel determinant of the logarithmic coefficients of strongly starlike and strongly convex functions.

Keywords Strongly starlike · Strongly convex · Carathéodory function · Hankel determinant · Logarithmic coefficient

Mathematics Subject Classification 30C45 · 30C50

1 Introduction

Denote by $\mathcal{H}$ the class of analytic functions in $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ with Taylor expansion

$$f(z) = \sum_{n=1}^{\infty} a_n z^n, \quad z \in \mathbb{D}, \tag{1}$$

and let $\mathcal{A}$ be the subclass of f normalized by $f'(0) = 1$. Let $\mathcal{S}$ denote the subclass of univalent functions in $\mathcal{A}$.

For $f \in \mathcal{S}$, logarithmic coefficients $\gamma_n := \gamma_n(f)$ of f are defined by

$$F_f(z) := \log \frac{f(z)}{z} = 2 \sum_{n=1}^{\infty} \gamma_n(f) z^n, \quad z \in \mathbb{D}, \ \log 1 := 0. \tag{2}$$

and play a crucial role in the theory of univalent functions, and in articular to prove the Milin conjecture ([19], see also [7, p. 155]). We note that for the class $\mathcal{S}$ sharp estimates are known only for γ_1 and γ_2, namely,

$$|\gamma_1| \le 1, \quad |\gamma_2| \le \frac{1}{2} + \frac{1}{\mathrm{e}^2} = 0.635\ldots$$

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Estimating the modulus of logarithmic coefficients for $f \in \mathcal{S}$ and various subclasses has been considered recently by several authors (e.g., [1, 2, 5, 8, 12, 24]).

For $q, n \in \mathbb{N}$, the Hankel determinant $H_{q,n}(f)$ of $f \in \mathcal{A}$ of the form (1) is defined as

$$H_{q,n}(f) := \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2(q-1)} \end{vmatrix},$$

and in particular many authors have examined the second and the third Hankel determinants $H_{2,2}(f)$ and $H_{3,1}(f)$ over selected subclasses of $\mathcal{A}$, (see e.g., [4, 11] with further references). We note that $H_{2,1}(f) = a_3 - a_2^2$ is the well known coefficient functional which for $\mathcal{S}$ was studied first in 1916 by Bieberbach (see e.g., [9, Vol. I, p. 35]).

Based on the these ideas, in this paper and in [10] we propose research study of the Hankel determinants $H_{q,n}(F_f/2)$ which entries are logarithmic coefficients of f. We are therefore concerned with

$$H_{q,n}(F_f/2) = \begin{vmatrix} \gamma_n & \gamma_{n+1} & \cdots & \gamma_{n+q-1} \\ \gamma_{n+1} & \gamma_{n+2} & \cdots & \gamma_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ \gamma_{n+q-1} & \gamma_{n+q} & \cdots & \gamma_{n+2(q-1)} \end{vmatrix}.$$

Differentiating (2) and using (1) we obtain

$$\gamma_1 = \frac{1}{2}a_2, \quad \gamma_2 = \frac{1}{2}\left(a_3 - \frac{1}{2}a_2^2\right), \quad \gamma_3 = \frac{1}{2}\left(a_4 - a_2 a_3 + \frac{1}{3}a_2^3\right), \tag{3}$$

and so

$$H_{2,1}(F_f/2) = \gamma_1\gamma_3 - \gamma_2^2 = \frac{1}{4}\left(a_2 a_4 - a_3^2 + \frac{1}{12}a_2^4\right). \tag{4}$$

Note that when $f \in \mathcal{S}$, then for $f_\theta(z) := \mathrm{e}^{-\mathrm{i}\theta} f(\mathrm{e}^{\mathrm{i}\theta} z),\ \theta \in \mathbb{R}$,

$$H_{2,1}(F_{f_\theta}/2) = \frac{\mathrm{e}^{4\mathrm{i}\theta}}{4}\left(a_2 a_4 - a_3^2 + \frac{1}{12}a_2^4\right) = \mathrm{e}^{4\mathrm{i}\theta} H_{2,1}(F_f/2), \tag{5}$$

so $|H_{2,1}(F_{f_\theta}/2)|$ is rotationally invariant.

In this paper we find sharp upper bounds for $H_{2,1}(F_f/2)$ in the case when f is strongly starlike or strongly convex function of order α, defined respectively as follows. Given $\alpha \in (0, 1]$, a function $f \in \mathcal{A}$ is called strongly starlike of order α if

$$\left|\arg \frac{zf'(z)}{f(z)}\right| < \alpha\frac{\pi}{2}, \quad z \in \mathbb{D},\ \arg 1 := 0. \tag{6}$$

Also, a function $f \in \mathcal{A}$ is called strongly convex of order α if

$$\left|\arg\left\{1 + \frac{zf''(z)}{f'(z)}\right\}\right| < \alpha\frac{\pi}{2}, \quad z \in \mathbb{D},\ \arg 1 := 0. \tag{7}$$

We denote these classes by $\mathcal{S}_\alpha^*$ and $\mathcal{S}_\alpha^c$ respectively, noting that $\mathcal{S}_1^* =: \mathcal{S}^*$ and $\mathcal{S}_1^c =: \mathcal{S}^c$ are the classes of starlike and convex functions, respectively.

The class of strongly starlike functions was introduced by Stankiewicz [21, 22], and independently by Brannan and Kirwan [3] (see also [9, Vol. I, pp. 137-142]). Stankiewicz [22] found an external geometrical characterization of strongly starlike functions and Brannan and Kirwan gave a geometrical condition called δ-visibility, which is sufficient for functions to be strongly starlike. Subsequently Ma and Minda [16] proposed an internal characterization of functions in $\mathcal{S}^*_\alpha$ based on the concept of k-starlike domains. Further results regarding the geometry of strongly starlike functions were given in [14, Chapter IV], [15] and [23].

In view of (6) and (7) both classes $\mathcal{S}^*_\alpha$ and $\mathcal{S}^c_\alpha$ can be represented using the Carathéodory class $\mathcal{P}$, i.e., the class of analytic functions p in $\mathbb{D}$ of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \quad z \in \mathbb{D}, \tag{8}$$

having a positive real part in $\mathbb{D}$. Thus the coefficients of functions in $\mathcal{S}^*_\alpha$ and $\mathcal{S}^c_\alpha$ have a convenient representation in terms of the coefficients of functions in $\mathcal{P}$. Therefore obtaining the upper bound of $H_{2,1}(F_f/2)$, we base our analysis on well-known expressions for c_2 (e.g., [20, p. 166]), and c_3 (Libera and Zlotkiewicz [17, 18]), and c_4 obtained recently in [13], all of which are contained in the following lemma [13]. Let $\overline{\mathbb{D}} := \{z \in \mathbb{C} : |z| \leq 1\}$ and $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$.

Lemma 1 *If $p \in \mathcal{P}$ and is given by (6) with $c_1 \geq 0$, then*

$$c_1 = 2\zeta_1, \tag{9}$$

$$c_2 = 2\zeta_1^2 + 2(1 - \zeta_1^2)\zeta_2 \tag{10}$$

and

$$c_3 = 2\zeta_1^3 + 4(1 - \zeta_1^2)\zeta_1\zeta_2 - 2(1 - \zeta_1^2)\zeta_1\zeta_2^2 + 2(1 - \zeta_1^2)(1 - |\zeta_2|^2)\zeta_3. \tag{11}$$

for some $\zeta_1 \in [0, 1]$ and $\zeta_2, \zeta_3 \in \overline{\mathbb{D}}$.

For $\zeta_1 \in \mathbb{T}$, there is a unique function $p \in \mathcal{P}$ with c_1 as in (9), namely,

$$p(z) = \frac{1 + \zeta_1 z}{1 - \zeta_1 z}, \quad z \in \mathbb{D}.$$

For $\zeta_1 \in \mathbb{D}$ and $\zeta_2 \in \mathbb{T}$, there is a unique function $p \in \mathcal{P}$ with c_1 and c_2 as in (9)–(10), namely,

$$p(z) = \frac{1 + \left(\overline{\zeta_1}\zeta_2 + \zeta_1\right) z + \zeta_2 z^2}{1 + \left(\overline{\zeta_1}\zeta_2 - \zeta_1\right) z - \zeta_2 z^2}, \quad z \in \mathbb{D}. \tag{12}$$

For $\zeta_1, \zeta_2 \in \mathbb{D}$ and $\zeta_3 \in \mathbb{T}$, there is a unique function $p \in \mathcal{P}$ with c_1, c_2 and c_3 as in (9)–(11), namely,

$$p(z) = \frac{1 + \left(\overline{\zeta_2}\zeta_3 + \overline{\zeta_1}\zeta_2 + \zeta_1\right) z + \left(\overline{\zeta_1}\zeta_3 + \zeta_1\overline{\zeta_2}\zeta_3 + \zeta_2\right) z^2 + \zeta_3 z^3}{1 + \left(\overline{\zeta_2}\zeta_3 + \overline{\zeta_1}\zeta_2 - \zeta_1\right) z + \left(\overline{\zeta_1}\zeta_3 - \zeta_1\overline{\zeta_2}\zeta_3 - \zeta_2\right) z^2 - \zeta_3 z^3}, \quad z \in \mathbb{D}. \tag{13}$$

We will also use the following lemma.

Lemma 2 [6] *Given real numbers A, B, C, let*

$$Y(A, B, C) := \max \left\{ |A + Bz + Cz^2| + 1 - |z|^2 : z \in \overline{\mathbb{D}} \right\}.$$

I. *If $AC \geq 0$, then*

$$Y(A, B, C) = \begin{cases} |A| + |B| + |C|, & |B| \geq 2(1 - |C|), \\ 1 + |A| + \dfrac{B^2}{4(1 - |C|)}, & |B| < 2(1 - |C|). \end{cases}$$

II. *If* $AC < 0$, *then*

$$Y(A,B,C) = \begin{cases} 1 - |A| + \dfrac{B^2}{4(1-|C|)}, & -4AC(C^{-2}-1) \le B^2 \wedge |B| < 2(1-|C|), \\ 1 + |A| + \dfrac{B^2}{4(1+|C|)}, & B^2 < \min\left\{4(1+|C|)^2, -4AC(C^{-2}-1)\right\}, \\ R(A,B,C), & \text{otherwise}, \end{cases}$$

where

$$R(A,B,C) := \begin{cases} |A| + |B| - |C|, & |C|(|B|+4|A|) \le |AB|, \\ -|A| + |B| + |C|, & |AB| \le |C|(|B|-4|A|), \\ (|C|+|A|)\sqrt{1 - \dfrac{B^2}{4AC}}, & \text{otherwise}. \end{cases}$$

2 Strongly starlike functions

We prove the following sharp inequality for $|H_{2,1}(F_f/2)|$ for the class $\mathcal{S}^*_\alpha$.

Theorem 1 *If* $f \in \mathcal{S}^*_\alpha$, $\alpha \in (0,1]$, *then*

$$|H_{2,1}(F_f/2)| = |\gamma_1\gamma_3 - \gamma_2^2| \le \frac{1}{4}\alpha^2. \tag{14}$$

The inequality is sharp.

Proof Fix $\alpha \in (0,1]$ and let $f \in \mathcal{S}^*_\alpha$ be given by (1). Then by (6),

$$zf'(z) = (p(z))^\alpha f(z), \quad z \in \mathbb{D}, \tag{15}$$

for some $p \in \mathcal{P}$ given by (8). Substituting (1) and (8) into (15) and equating coefficients gives

$$\begin{aligned} a_2 &= \alpha c_1, \quad a_3 = \frac{\alpha}{4}\left[2c_2 + (3\alpha-1)c_1^2\right], \\ a_4 &= \frac{\alpha}{36}\left[12c_3 + 6(5\alpha-2)c_1c_2 + (17\alpha^2 - 15\alpha + 4)c_1^3\right]. \end{aligned} \tag{16}$$

Since the class $\mathcal{S}^*_\alpha$ is invariant under the rotations and (5) holds, we may assume that $a_2 \ge 0$, so by (16) that $c_1 \ge 0$, i.e., in view of (9) that $\zeta_1 \in [0,1]$. Hence from (4) and (9)–(11) we obtain

$$\begin{aligned} \gamma_1\gamma_3 - \gamma_2^2 &= \frac{1}{4}\left(a_2a_4 - a_3^2 + \frac{1}{12}a_2^4\right) \\ &= \frac{\alpha^2}{576}\left[48c_1c_3 - 12(1-\alpha)c_1^2c_2 - 36c_2^2 + (7+\alpha)(1-\alpha)c_1^4\right] \\ &= \frac{\alpha^2}{36}\left[(4-\alpha^2)\zeta_1^4 + 6\alpha(1-\zeta_1^2)\zeta_1^2\zeta_2 - 3(3+\zeta_1^2)(1-\zeta_1^2)\zeta_2^2\right. \\ &\quad \left. +12(1-\zeta_1^2)(1-|\zeta_2|^2)\zeta_1\zeta_3\right]. \end{aligned} \tag{17}$$

A. Suppose that $\zeta_1 = 1$. Then by (17), for $\alpha \in (0,1]$,

$$|\gamma_1\gamma_3 - \gamma_2^2| = \frac{\alpha^2(4-\alpha^2)}{36} \le \frac{\alpha^2}{4}.$$

B. Suppose that $\zeta_1 = 0$. Then by (17), for $\alpha \in (0, 1]$,

$$|\gamma_1\gamma_3 - \gamma_2^2| = \frac{\alpha^2}{4}|\zeta_2|^2 \leq \frac{\alpha^2}{4}.$$

C. Suppose that $\zeta_1 \in (0, 1)$. Then since $|\zeta_3| \leq 1$ from (17) we obtain

$$\begin{aligned} &|\gamma_1\gamma_3 - \gamma_2^2| \\ &\leq \frac{\alpha^2}{36}\left[\left|(4-\alpha^2)\zeta_1^4 + 6\alpha(1-\zeta_1^2)\zeta_1^2\zeta_2 - 3(3+\zeta_1^2)(1-\zeta_1^2)\zeta_2^2\right|\right. \\ &\quad \left.+12(1-\zeta_1^2)(1-|\zeta_2|^2)\zeta_1\right] \\ &\leq \frac{\alpha^2}{3}\zeta_1(1-\zeta_1^2)\left[|A + B\zeta_2 + C\zeta_2^2| + 1 - |\zeta_2|^2\right], \end{aligned} \tag{18}$$

where

$$A := \frac{(4-\alpha^2)\zeta_1^3}{12(1-\zeta_1^2)}, \quad B := \frac{1}{2}\alpha\zeta_1, \quad C := -\frac{3+\zeta_1^2}{4\zeta_1}.$$

Since $AC < 0$, we now apply Lemma 2 only for the case II.

C1. Note that the inequality

$$-4AC\left(\frac{1}{C^2} - 1\right) - B^2 = \frac{(4-\alpha^2)\zeta_1^2(3+\zeta_1^2)}{12(1-\zeta_1^2)}\left(\frac{16\zeta_1^2}{(3+\zeta_1^2)^2} - 1\right) - \frac{\alpha^2}{4}\zeta_1^2 \leq 0$$

is equivalent to

$$-\frac{(4-\alpha^2)(9-\zeta_1^2)}{3(3+\zeta_1^2)} - \alpha^2 \leq 0,$$

which evidently holds for $\zeta_1 \in (0, 1)$.

However, the inequality $|B| < 2(1 - |C|)$ is equivalent to $\alpha\zeta_1^2 < -(1-\zeta_1^2)(3-\zeta_1^2)$, which is false for $\zeta_1 \in (0, 1)$.

C2. Since

$$4(1+|C|)^2 = \frac{(\zeta_1^2 + 4\zeta_1 + 3)^2}{4\zeta_1^2} > 0$$

and

$$-4AC\left(\frac{1}{C^2} - 1\right) = -\frac{(4-\alpha^2)\zeta_1^2(9-\zeta_1^2)}{12(3+\zeta_1^2)} < 0,$$

a simple calculation shows that the inequality

$$\frac{\alpha^2\zeta_1^2}{4} = B^2 < \min\left\{4(1+|C|)^2, -4AC\left(\frac{1}{C^2} - 1\right)\right\} = -\frac{(4-\alpha^2)\zeta_1^2(9-\zeta_1^2)}{12(3+\zeta_1^2)}$$

is false for $\zeta_1 \in (0, 1)$.

C3. Next note that the inequality

$$|C|(|B| + 4|A|) - |AB| = \frac{3+\zeta_1^2}{4\zeta_1}\left(\frac{1}{2}\alpha\zeta_1 + \frac{(4-\alpha^2)\zeta_1^3}{3(1-\zeta_1^2)}\right) - \frac{\alpha(4-\alpha^2)\zeta_1^4}{24(1-\zeta_1^2)} \leq 0$$

is equivalent to $(\alpha-1)(\alpha^2-\alpha-8)\zeta_1^4-6(\alpha^2+\alpha-4)\zeta_1^2+9\alpha\le 0$. However the last inequality is false for $\zeta_1\in(0,1)$ since $(\alpha-1)(\alpha^2-\alpha-8)\ge 0$ and $\alpha^2+\alpha-4<0$ for $\alpha\in(0,1]$.

C4. Note that the inequality

$$\begin{aligned}&|AB|-|C|(|B|-4|A|)\\&=\frac{\alpha(4-\alpha^2)\zeta_1^4}{24(1-\zeta_1^2)}-\frac{3+\zeta_1^2}{4\zeta_1}\left(\frac{1}{2}\alpha\zeta_1-\frac{(4-\alpha^2)\zeta_1^3}{3(1-\zeta_1^2)}\right)\le 0\end{aligned}\tag{19}$$

is equivalent to

$$\delta(\zeta_1^2)\ge 0,\tag{20}$$

where

$$\delta(t):=9\alpha-3(8+2\alpha-2\alpha^2)t-(8+7\alpha-2\alpha^2-\alpha^3)t^2,\quad t\in(0,1).$$

We see that for $\alpha\in(0,1]$,

$$8+2\alpha-2\alpha^2>0,\quad 8+7\alpha-2\alpha^2-\alpha^3>0,\tag{21}$$

and the discriminant $\Delta:=144(4+4\alpha-\alpha^3)>0$ for $\alpha\in(0,1]$. Thus we consider

$$t_{1,2}:=\frac{3(8+2\alpha-2\alpha^2)\mp 12\sqrt{4+4\alpha-\alpha^3}}{-2(8+7\alpha-2\alpha^2-\alpha^3)}.$$

From (21) it follows that $t_2<0$ and so it remains to check if $0<t_1<1$. The inequality $t_1>0$ is equivalent to $8\alpha+7\alpha^2-2\alpha^3-\alpha^4>0$ which is true for $\alpha\in(0,1]$. Further, the inequality $t_1<1$ can be written as

$$256+256\alpha-100\alpha^2-104\alpha^3+5\alpha^4+10\alpha^5+\alpha^6>0$$

which is true since

$$\begin{aligned}&256+256\alpha-100\alpha^2-104\alpha^3+5\alpha^4+10\alpha^5+\alpha^6\\&>52+256\alpha+5\alpha^4+10\alpha^5+\alpha^6>0,\quad \alpha\in(0,1].\end{aligned}$$

Therefore (20), and so (19) is valid for $0<\zeta_1\le\zeta':=\sqrt{t_1}$. Then by (19), Lemma 2 and the fact that φ decreases, we obtain

$$\begin{aligned}|\gamma_1\gamma_3-\gamma_2^2|&\le\frac{\alpha^2}{3}\zeta_1(1-\zeta_1^2)(-|A|+|B|+|C|)\\&=\frac{\alpha^2}{36}\varphi(\zeta_1)\le\frac{\alpha^2}{36}\varphi(0)=\frac{\alpha^2}{4},\end{aligned}\tag{22}$$

where

$$\varphi(u):=9-6(1-\alpha)u^2-(1+\alpha)(7-\alpha)u^4,\quad 0\le u\le\zeta'.$$

C5. It remains to consider the last case in Lemma 2, which in view of C4, holds for $\zeta'<\zeta_1<1$. Then by (18),

$$\begin{aligned}|\gamma_1\gamma_3-\gamma_2^2|&\le\frac{\alpha^2}{3}\zeta_1(1-\zeta_1^2)(|C|+|A|)\sqrt{1-\frac{B^2}{4AC}}\\&=\frac{\alpha^2}{18}\psi(\zeta_1)\le\frac{\alpha^2}{18}\psi(\zeta'),\end{aligned}\tag{23}$$

where

$$\psi(t) := \left[9 - 6t^2 + (1-\alpha^2)t^4\right]\sqrt{\frac{3+(1-\alpha^2)t^2}{(4-\alpha^2)(3+t^2)}}, \quad \zeta' \le t < 1.$$

To see that the last inequality in (23) is true, note that the function ψ is decreasing, since

$$\psi'(t) = -\frac{t}{(4-\alpha^2)(3+t^2)^2}\sqrt{\frac{(4-\alpha^2)(3+t^2)}{3+(1-\alpha^2)t^2}} \times \left[4(9-(1-\alpha^2)^2t^4)(3+t^2) + 3\alpha^2(3-(1-\alpha)t^2)(3-(1+\alpha)t^2)\right] < 0$$

for $\zeta' < t < 1$.

Simple but tedious computations show that

$$\varphi(\zeta') = \psi(\zeta').$$

Hence from (22) and (23) we see that

$$\frac{\alpha^2}{18}\psi(\zeta') \le \frac{\alpha^2}{4}.$$

D. Summarizing from parts A-C we see that inequality (14) follows.

Equality holds for the function $f \in \mathcal{A}$ given by (15), where

$$p(z) := \frac{1+z^2}{1-z^2}, \quad z \in \mathbb{D}. \tag{24}$$

Then $c_1 = c_3 = 0$ and $c_2 = 2$, so by (16), $a_2 = a_4 = 0$ and $a_3 = \alpha$, and therefore by (3), $\gamma_1 = \gamma_3 = 0$ and $\gamma_2 = \alpha/2$, which completes the proof of the theorem. □

For $\alpha = 1$ we obtain the following result for the class $\mathcal{S}^*$ of starlike functions [10].

Corollary 1 *If $f \in \mathcal{S}^*$, then*

$$|\gamma_1\gamma_3 - \gamma_2^2| \le \frac{1}{4}.$$

The inequality is sharp.

3 Strongly convex functions

We prove the following sharp inequality for $|H_{2,1}(F_f/2)|$ in the class $\mathcal{S}^c_\alpha$.

Theorem 2 *If $f \in \mathcal{S}^c_\alpha$, $\alpha \in (0,1]$, then*

$$|\gamma_1\gamma_3 - \gamma_2^2| \le \begin{cases} \dfrac{\alpha^2}{36}, & 0 < \alpha \le \dfrac{1}{3}, \\ \dfrac{\alpha^2(17+18\alpha+13\alpha^2)}{144(4+6\alpha+\alpha^2)}, & \dfrac{1}{3} < \alpha \le 1. \end{cases} \tag{25}$$

Both inequalities are sharp.

Proof Fix $\alpha \in (0,1]$ and let $f \in \mathcal{S}^c_\alpha$ be given by (1). Then by (7),

$$f'(z) + zf''(z) = f'(z)(p(z))^\alpha, \quad z \in \mathbb{D}, \tag{26}$$

for some $p \in \mathcal{P}$ given by (8). Substituting (1) and (8) into (26) and equating coefficients we obatin

$$\begin{aligned} a_2 &= \frac{1}{2}\alpha c_1, \quad a_3 = \frac{\alpha}{12}\left[2c_2 + (3\alpha - 1)c_1^2\right], \\ a_4 &= \frac{\alpha}{144}\left[12c_3 + 6(5\alpha - 2)c_1c_2 + (17\alpha^2 - 15\alpha + 4)c_1^3\right]. \end{aligned} \tag{27}$$

As in the proof of Theorem 1 we may assume that $c_1 \geq 0$, i.e., in view of (9) that $\zeta_1 \in [0,1]$. Hence from (4) and (9)–(11) we have

$$\begin{aligned} \gamma_1\gamma_3 - \gamma_2^2 &= \frac{\alpha^2}{2304}\left[24c_1c_3 + 4(3\alpha - 2)c_1^2c_2 - 16c_2^2 + (\alpha^2 - 6\alpha + 4)c_1^4\right] \\ &= \frac{\alpha^2}{144}\left[(2+\alpha^2)\zeta_1^4 + 6\alpha(1-\zeta_1^2)\zeta_1^2\zeta_2 - 2(1-\zeta_1^2)(2+\zeta_1^2)\zeta_2^2\right. \\ &\quad \left. + 6(1-\zeta_1^2)(1-|\zeta_2|^2)\zeta_1\zeta_3\right]. \end{aligned} \tag{28}$$

A. Suppose that $\zeta_1 = 1$. Then by (28), for $\alpha \in (0,1]$,

$$|\gamma_1\gamma_3 - \gamma_2^2| = \frac{\alpha^2(2+\alpha^2)}{144}. \tag{29}$$

B. Suppose that $\zeta_1 = 0$. Then from (28), for $\alpha \in (0,1]$,

$$|\gamma_1\gamma_3 - \gamma_2^2| = \frac{\alpha^2}{36}|\zeta_2|^2 \leq \frac{\alpha^2}{36}. \tag{30}$$

C. Suppose that $\zeta_1 \in (0,1)$. Since $|\zeta_3| \leq 1$ from (28) we obtain

$$\begin{aligned} &|\gamma_1\gamma_3 - \gamma_2^2| \\ &\leq \frac{\alpha^2}{144}\left[\left|(2+\alpha^2)\zeta_1^4 + 6\alpha(1-\zeta_1^2)\zeta_1^2\zeta_2 - 2(1-\zeta_1^2)(2+\zeta_1^2)\zeta_2^2\right|\right. \\ &\quad \left. + 6(1-\zeta_1^2)(1-|\zeta_2|^2)\zeta_1\right] \\ &= \frac{\alpha^2}{24}\zeta_1(1-\zeta_1^2)\left[|A + B\zeta_2 + C\zeta_2^2| + 1 - |\zeta_2|^2\right], \end{aligned} \tag{31}$$

where

$$A := \frac{(2+\alpha^2)\zeta_1^3}{6(1-\zeta_1^2)}, \quad B := \alpha\zeta_1, \quad C := -\frac{2+\zeta_1^2}{3\zeta_1}.$$

Since $AC < 0$, we apply Lemma 2 only in the case II.

C1. Note that the inequality

$$-4AC\left(\frac{1}{C^2} - 1\right) - B^2 = \frac{2(2+\alpha^2)\zeta_1^2(2+\zeta_1^2)}{9(1-\zeta_1^2)}\left(\frac{9\zeta_1^2}{(2+\zeta_1^2)^2} - 1\right) - \alpha^2\zeta_1^2 \leq 0$$

is equivalent to $-2(2+\alpha^2)(4-\zeta_1^2) \leq 9\alpha^2(2+\zeta_1^2)$, which evidently holds for $\zeta_1 \in (0,1)$.

Moreover, the inequality $|B| < 2(1-|C|)$ is equivalent to $3\alpha\zeta_1^2 < -2(1-\zeta_1)(2-\zeta_1)$, which is false for $\zeta_1 \in (0,1)$.

C2. Since

$$4(1+|C|)^2 = \frac{4(\zeta_1^2+3\zeta_1+2)^2}{9\zeta_1^2} > 0$$

and

$$-4AC\left(\frac{1}{C^2}-1\right) = -\frac{2(2+\alpha^2)\zeta_1^2(4-\zeta_1^2)}{9(2+\zeta_1^2)} < 0,$$

we see that the inequality

$$\alpha^2\zeta_1^2 = B^2 < \min\left\{4(1+|C|)^2, -4AC\left(\frac{1}{C^2}-1\right)\right\} = -\frac{2(2+\alpha^2)\zeta_1^2(4-\zeta_1^2)}{9(2+\zeta_1^2)}$$

is false for $\zeta_1 \in (0,1)$.

C3. Next observe that the inequality

$$|C|(|B|+4|A|) - |AB| = \frac{2+\zeta_1^2}{3\zeta_1}\left(\alpha\zeta_1 + \frac{2(2+\alpha^2)\zeta_1^3}{3(1-\zeta_1^2)}\right) - \frac{(2+\alpha^2)\alpha\zeta_1^4}{6(1-\zeta_1^2)} \leq 0$$

is equivalent to

$$\phi(\zeta_1^2) \leq 0, \tag{32}$$

where

$$\phi(t) := (-3\alpha^3+4\alpha^2-12\alpha+8)t^2 + (8\alpha^2-6\alpha+16)t + 12\alpha, \quad t \in (0,1).$$

Note that $8\alpha^2-6\alpha+16 > 0$ for $\alpha \in (0,1]$ and $-3\alpha^3+4\alpha^2-12\alpha+8 \geq 0$ for $\alpha \in (0,\alpha_0]$, where $\alpha_0 \approx 0.74858\ldots$. Thus for $\alpha \in (0,\alpha_0]$ inequality (32) is evidently false. If $\alpha \in (\alpha_0,1]$, then $\Delta := 4\left(52\alpha^4-72\alpha^3+217\alpha^2-144\alpha+64\right) > 0$, and so we consider

$$t_{1,2} := \frac{-4\alpha^2+3\alpha-8 \mp \sqrt{52\alpha^4-72\alpha^3+217\alpha^2-144\alpha+64}}{-3\alpha^3+4\alpha^2-12\alpha+8}.$$

Observe now that $t_1 > 1$. Indeed, the inequality $t_1 > 1$ is equivalent to the evidently true inequality

$$\sqrt{52\alpha^4-72\alpha^3+217\alpha^2-144\alpha+64} > 3\alpha^3-8\alpha^2+15\alpha-16,$$

since the right hand side is negative for all $\alpha \in (\alpha_0,1]$. Further, $t_2 < 0$. Indeed this inequality is equivalent to $-3\alpha^3+4\alpha^2-12\alpha+8 < 0$ which clearly holds for $\alpha \in (\alpha_0,1]$. Thus we deduce that the inequality (32) is false.

C4. Note next that the inequality

$$|AB| - |C|(|B|-4|A|) = \frac{(2+\alpha^2)\alpha\zeta_1^4}{6(1-\zeta_1^2)} - \frac{2+\zeta_1^2}{3\zeta_1}\left(\alpha\zeta_1 - \frac{2(2+\alpha^2)\zeta_1^3}{3(1-\zeta_1^2)}\right) \leq 0 \tag{33}$$

is equivalent to

$$\delta(\zeta_1^2) \leq 0, \tag{34}$$

where

$$\delta(s) := (3\alpha^3+4\alpha^2+12\alpha+8)s^2 + 2(4\alpha^2+3\alpha+8)s - 12\alpha, \quad s \in (0,1),$$

so that $\Delta := 4\left(52\alpha^4 + 72\alpha^3 + 217\alpha^2 + 144\alpha + 64\right) > 0$ for $\alpha \in (0, 1]$. Therefore $s_1 < 0$, where

$$s_{1,2} := \frac{-(4\alpha^2 + 3\alpha + 8) \mp \sqrt{52\alpha^4 + 72\alpha^3 + 217\alpha^2 + 144\alpha + 64}}{3\alpha^3 + 4\alpha^2 + 12\alpha + 8}.$$

Moreover $0 < s_2 < 1$ holds. Indeed, both inequalities $s_2 > 0$ and $s_2 < 1$ are equivalent to the evidently true inequalities

$$36\alpha^4 + 48\alpha^3 + 144\alpha^2 + 96\alpha > 0,$$

and

$$9\alpha^6 + 48\alpha^5 + 102\alpha^4 + 264\alpha^3 + 264\alpha^2 + 336\alpha + 192 > 0,$$

respectively. Thus (34), and so (33) is valid only when

$$0 < \zeta_1 \le \sqrt{s_2} =: \zeta'.$$

Then by (31) and Lemma 2,

$$|\gamma_1\gamma_3 - \gamma_2^2| \le \frac{1}{24}\alpha^2\zeta_1(1 - \zeta_1^2)(-|A| + |B| + |C|) = \varphi(\zeta_1),$$

where

$$\varphi(u) := \frac{\alpha^2}{144}\left[-(\alpha^2 + 6\alpha + 4)u^4 + 2(3\alpha - 1)u^2 + 4\right], \quad 0 \le u \le \zeta'.$$

Since

$$\varphi'(u) = -\frac{\alpha^2 u}{36}\left[(\alpha^2 + 6\alpha + 4)u^2 + 1 - 3\alpha\right], \quad 0 < u < \zeta',$$

we see that for $0 < \alpha \le 1/3$, the function φ decreases and so

$$\varphi(u) \le \varphi(0) = \frac{\alpha^2}{36}, \quad 0 \le u \le \zeta'. \tag{35}$$

In the case $1/3 < \alpha \le 1$,

$$0 < u_0 := \sqrt{\frac{3\alpha - 1}{\alpha^2 + 6\alpha + 4}} < \zeta_1 \tag{36}$$

is a unique critical point of φ, which is a maximum.

It remains therefore to establish the second inequality, i.e., $u_0 < \zeta_1$, which is equivalent to

$$\begin{aligned} r(\alpha) :=& 117\alpha^8 + 240\alpha^7 - 149\alpha^6 - 1212\alpha^5 - 4344\alpha^4 \\ & - 6288\alpha^3 - 4464\alpha^2 - 1920\alpha - 448 < 0, \quad \alpha \in (0, 1], \end{aligned}$$

and since

$$r(\alpha) \le -149\alpha^6 - 1212\alpha^5 - 4344\alpha^4 - 6288\alpha^3 - 4464\alpha^2 - 1920\alpha - 91 < 0$$

for $\alpha \in (0, 1]$, we deduce that $u_0 < \zeta_1$.

Thus for $1/3 < \alpha \le 1$, we have

$$\varphi(u) \le \varphi(u_0) = \frac{\alpha^2(17 + 18\alpha + 13\alpha^2)}{144(4 + 6\alpha + \alpha^2)}, \quad 0 \le u \le \zeta'. \tag{37}$$

C5. We now consider the last case in Lemma 2, which in view of C4 holds for $\zeta' < \zeta_1 < 1$. Then by (31),

$$|\gamma_1\gamma_3 - \gamma_2^2| \le \frac{\alpha^2}{24}\zeta_1(1-\zeta_1^2)(|C|+|A|)\sqrt{1-\frac{B^2}{4AC}} = \psi(\zeta_1) \le \psi(\zeta'), \tag{38}$$

where

$$\psi(u) := \frac{\alpha^2}{144}(\alpha^2u^4 - 2u^2 + 4)\sqrt{\frac{13\alpha^2+8+(4-7\alpha^2)u^2}{2(2+\alpha^2)(2+u^2)}}, \quad \zeta' \le u \le 1.$$

To show that the last inequality in (38) holds, observe that ψ is decreasing. Indeed, by a simple computation,

$$\begin{aligned}\psi'(u) = &-\frac{\alpha^2 x}{288(2+\alpha^2)(2+x^2)^2}\sqrt{\frac{2(2+\alpha^2)(2+u^2)}{13\alpha^2+8+(4-7\alpha^2)u^2}}\\ &\times\left[4(1-\alpha^2u^2)(2+u^2)\left(13\alpha^2+8+(4-7\alpha^2)u^2\right)\right.\\ &\left.+27\alpha^2(\alpha^2u^4-2u^2+4)\right],\end{aligned}$$

for $\zeta' < u < 1$. Note that

$$13\alpha^2+8+(4-7\alpha^2)u^2 > 0, \quad \zeta' < u < 1, \tag{39}$$

which is clearly true for $0 < \alpha \le 2/\sqrt{7}$. If $2/\sqrt{7} < \alpha \le 1$, then

$$13\alpha^2+8+(4-7\alpha^2)u^2 = 13\alpha^2+8-(7\alpha^2-4)u^2 \ge 6\alpha^2+12 > 0$$

for $\zeta' < u < 1$. Further

$$\alpha^2u^4 - 2u^2 + 4 \ge \alpha^2u^4 + 2 > 0, \quad \zeta' < u < 1. \tag{40}$$

Thus from (39) and (40) it follows that $\psi'(u) < 0$ for $\zeta' < u < 1$, so ψ decreases and hence

$$\psi(u) \le \psi(\zeta'), \quad \zeta' \le u \le 1. \tag{41}$$

Simple but tedious computations show that

$$\varphi(\zeta') = \psi(\zeta'),$$

and so from (41), (35) and (37) we deduce that for $\alpha \in (0, 1/3]$,

$$\psi(u) \le \frac{\alpha^2}{36}, \quad \zeta' \le u \le 1,$$

and for $\alpha \in (1/3, 1]$,

$$\psi(u) \le \varphi(u_0), \quad \zeta' \le u \le 1.$$

D. It remains to compare the bounds in (29), (30), (35) and (37). The inequality

$$\frac{\alpha^2(2+\alpha^2)}{144} \le \frac{\alpha^2}{36}, \quad \alpha \in (0, 1],$$

is trivial, and the inequality

$$\frac{\alpha^2(2+\alpha^2)}{144} \le \frac{\alpha^2(17+18\alpha+13\alpha^2)}{144(4+6\alpha+\alpha^2)}, \quad \alpha \in (1/3, 1],$$

is equivalent to

$$-\alpha^4 - 6\alpha^3 + 7\alpha^2 + 6\alpha + 9 \leq 0, \quad \alpha \in (1/3, 1],$$

which is clearly true, and the inequality

$$\frac{\alpha^2}{36} \leq \frac{\alpha^2(17 + 18\alpha + 13\alpha^2)}{144(4 + 6\alpha + \alpha^2)}, \quad \alpha \in (1/3, 1],$$

is equivalent to the evidently true inequality $(3\alpha - 1)^2 \geq 0$.

Thus summarizing the results in parts A-C we see that (25) is established.

We finally show that the inequalities in (25) are sharp. When $\alpha \in (0, 1/3]$, equality holds for the function $f \in \mathcal{A}$ given by (26) with p given by (24). In this case $c_1 = c_3 = 0$ and $c_2 = 2$, so by (27), $a_2 = a_4 = 0$ and $a_3 = \alpha/3$ and therefore $\gamma_1 = \gamma_3 = 0$ and $\gamma_2 = \alpha/6$.

When $\alpha \in (1/3, 1]$, equality holds for the function $f \in \mathcal{A}$ given by (26), where p is given by (12) with $\zeta_1 = u_0 =: \tau$, and u_0 given by (36), $\zeta_2 = -1$ and $\zeta_3 = 1$, i.e.,

$$p(z) := \frac{1 - z^2}{1 - 2\tau z + z^2}, \quad z \in \mathbb{D},$$

which completes the proof of the theorem. □

For $\alpha = 1$ we obtain the sharp inequality for the class $\mathcal{S}^c$ of convex functions [10].

Corollary 2 *If $f \in \mathcal{S}^c$, then*

$$|\gamma_1\gamma_3 - \gamma_2^2| \leq \frac{1}{33}.$$

The inequality is sharp.

Acknowledgements The authors would like to express their thanks to the referees for their constructive advices and comments that helped to improve this paper.

Availability of data and material The manuscript has no associated data.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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