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ORIGINAL PAPER

Feral dual spaces and (strongly) distinguished spaces $C(X)$

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Abstract
Following Dieudonné and Schwartz a locally convex space is *distinguished* if its strong dual is barrelled. The distinguished property for spaces $C_p(X)$ of continuous real-valued functions over a Tychonoff space X is a peculiar (although applicable) property. It is known that $C_p(X)$ is distinguished if and only if $C_p(X)$ is large in $\mathbb{R}^X$ if and only if X is a Δ-space (in sense of Reed) if and only if the strong dual of $C_p(X)$ carries the finest locally convex topology. Our main results about spaces whose strong dual has only finite-dimensional bounded sets (see Theorems 2, 7 and Proposition 4) are used to study distinguished spaces $C_k(X)$ with the compact-open topology. We also put together several known facts (Theorem 6) about distinguished spaces $C_p(X)$ with self-contained full proofs.

Keywords Distinguished space · Bidual space · Fundamental family of bounded sets · Point-finite family

Mathematics Subject Classification 54C35 · 46A03

1 Introduction

Recall that a locally convex space E is *distinguished* if its strong dual E'_β is barrelled (i.e. any absolutely convex, absorbing and closed subset of E'_β is a neighbourhood of zero). In fact (an equivalent condition [27, 23.7]), E is distinguished if and only if E is large in $\left(E'', \sigma\left(E'', E'\right)\right)$. Recall also that a subspace F of a locally convex space G is *large* in G if every bounded subset of G is contained in the closure in G of some bounded subset of F,

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[34, Definition 8.3.22]. All semi-reflexive locally convex spaces are distinguished ([38, IV 5.5]).

If X is a Tychonoff space then $C_p(X)$ and $C_k(X)$ denote the linear space $C(X)$ of all real-valued continuous functions defined on X equipped with the pointwise convergence and the compact-open topology, respectively.

The simplest examples of distinguished $C_p(X)$ spaces which are not semi-reflexive are those with X any countable nondiscrete Tychonoff space (see [12]). Several recently obtained results about distinguished spaces $C_p(X)$ have been proved in articles [12, 15, 16, 24, 25, 32]. We know that $C_p(X)$ is distinguished if and only if the strong dual $L_\beta(X)$ of $C_p(X)$ (i.e. the topological dual $C_p(X)'$ of $C_p(X)$ with the strong topology $\beta(C_p(X)', C_p(X))$) carries the finest locally convex topology, see [12, 15]. In [24] we proved that $C_p(X)$ is distinguished if and only if X is a Δ-space (in the sense of Reed [35]). This theorem apparently provides a nice connection with problems from the set theory and related with Δ-sets, λ-sets, and Q-sets X, and corresponding distinguished spaces $C_p(X)$.

We proved, among others, that: (i) For each metrizable scattered space X the space $C_p(X)$ is distinguished ([24, Proposition 4.1]); (ii) For each compact Eberlein scattered space X the space $C_p(X)$ is distinguished ([24, Theorem 3.7] or [15, Theorem 49]); (iii) If $X = [0, \omega_1]$ then $C_p(X)$ is not distinguished ([24, Theorem 3.12]).

A similar characterization of distinguished spaces $C_p(X)$ in term of the space X has been proved in [16].

In Sect. 2 we put together (Theorem 6) several equivalent conditions for $C_p(X)$ to be distinguished with self-contained proofs.

Being motivated by results from papers mentioned above we propose

Definition 1 A locally convex space E is *strongly distinguished* if its strong dual E'_β caries the finest locally convex topology (i.e. any absolutely convex and absorbing subset of E'_β is a neighbourhood of zero).

Following [22] a locally convex space is *feral* if every bounded set in E is finite-dimensional. Hence a locally convex space E carries the finest locally convex topology if and only if E is feral and E is bornological (i.e. every locally bounded linear map from E into any locally convex space is continuous). Recall that a linear map between locally convex spaces is *locally bounded* if it maps bounded sets into bounded sets; clearly, any continuous linear map is locally bounded.

In [12] we showed that $C_p(X)$ *is strongly distinguished if and only if it is distinguished*, see also [15]; in [13, page 392] we proved that for each Tychonoff space X the strong dual $L_\beta(X)$ of $C_p(X)$ is feral. Consequently, $C_p(X)$ is strongly distinguished if and only if its strong dual $L_\beta(X)$ is bornological.

In [26] we proved the following

Theorem 1 ([26, Theorem 1])
For a Tychonoff space X the following are equivalent:

(1) *The space $C_k(X)$ is strongly distinguished.*
(2) *X is a Δ-space and every compact subset of X is finite.*
(3) *The space $C_k(X)$ is large in $\mathbb{R}^X$.*

Since a linear space with the finest locally convex topology is feral, Theorem 1 suggests the following natural

Problem 1 *Characterize locally convex spaces whose strong dual is feral.*

This problem is solved in Theorem 7 providing some applications, see Corollary 7 (extending [11, Theorem 3.3]). The main application of Theorem 7 extends Theorem 1, essentially for (1) $\Rightarrow$ (2), and is formulated below:

Theorem 2 *For a Tychonoff space X the following are equivalent:*

(1) *The strong dual of $C_k(X)$ is feral.*
(2) *Every compact subset of X is finite.*

If $C_k(X)$ is quasi-barrelled, then the strong bidual of $C_k(X)$ is feral if and only if X is finite.

Recall that a locally convex space E is *quasi-barrelled* if every absolutely convex closed set in E absorbing bounded sets in E is a neighbourhood of zero. It is not clear if the quasi-barrelledness of $C_k(X)$ in Theorem 2 can be omitted. Note however that for each Tychonoff space X the strong bidual $M_\beta(X)$ of $C_p(X)$ is feral if and only if X is finite, see Corollary 4.

Recall that $C_k(X)$ is quasi-barrelled if and only if X is a W-space i.e. every b-bounding subset B of X is relatively compact, see [34, Theorem 10.1.21]; the b-boundedness of B means that for every bounded subset M of $C_k(X)$ we have $\sup\{|f(x)| : x \in B, f \in M\} < \infty$.

Observe that the item (2) in Theorem 2 does not guarantee that the strong dual of $C_k(X)$ is bornological, see Example 1.

Krupski and Marciszewski proved that for any infinite compact spaces X and Y the spaces $C_p(X)$ and $C_w(Y)$ are not isomorphic, where $C_w(Y)$ is the Banach space $C(Y)$ with its weak topology i.e. $C_w(Y) = (C(Y), \sigma(C(Y), C(Y)'))$ ([29, Corollary 3.2], see also [28] for motivations and results around this line of research).

Applying Theorem 2 we extend this result by proving the following.

Corollary 1 *Let X and Y be Tychonoff spaces. Let $C_w(Y)$ be the space $C_k(Y)$ with its weak topology $\sigma(C_k(Y), C_k(Y)')$. If there exists a continuous linear map from $C_p(X)$ onto $C_w(Y)$, then every compact subset of Y is finite. In particular, the spaces $C_p(X)$ and $C_w(Y)$ are not isomorphic, if Y contains an infinite compact subset.*

In fact we prove a more general statement, see Corollary 5.

A collection $\{U_t : t \in T\}$ of subsets of a topological space X is called

(a) *point-finite* if for every $x \in X$ the set $\{t \in T : x \in U_t\}$ is finite;

(b) *compact-finite* if for every compact subset F of X the set $\{t \in T : U_t \cap F \neq \emptyset\}$ is finite;

(c) an *open expansion* of a collection $\{X_t : t \in T\}$ of subsets of X if U_t is open and $U_t \supset X_t$ for every $t \in T$.

Clearly, a decreasing sequence (X_n) of subsets of X is point-finite if and only if $\bigcap_{n=1}^\infty X_n = \emptyset$.

A topological space X is called

(a) a Δ-*space* if every decreasing point-finite sequence (X_n) of subsets of X admits a decreasing point-finite open expansion (U_n), see [24];

(b) a *strong* Δ-*space* if every decreasing point-finite sequence (X_n) of subsets of X admits a decreasing compact-finite open expansion (U_n).

Theorems 1, 2 and 6 imply the following

Corollary 2 *For a Tychonoff space X the following are equivalent:*

(1) *The space $C_k(X)$ is strongly distinguished.*
(2) *The space $C_k(X)$ is large in $\mathbb{R}^X$.*

(3) *The strong dual* $C_k(X)'_\beta$ *of* $C_k(X)$ *is feral and the strong dual* $L_\beta(X)$ *of* $C_p(X)$ *is bornological.*
(4) X *is a* Δ*-space and every compact subset of* X *is finite.*
(5) X *is a strong* Δ*-space.*
(6) *Any countable disjoint collection of subsets of* X *admits a compact-finite open expansion in* X.
(7) *Any countable partition of* X *admits a compact-finite open expansion in* X.

Proof (1)$\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) follow by Theorems 1, 2 and 6.
(4) $\Rightarrow$ (5) $\Rightarrow$ (6) $\Rightarrow$ (7) $\Rightarrow$ (4) are clear (see the Proof of Theorem 6). □

Note however that there exist (non-complete) Montel spaces (hence distinguished) whose strong dual is not bornological, see [34, Example 4.7.8]. Clearly, for every discrete space X and for every countable Tychonoff space X the space $C_p(X)$ is distinguished, since every discrete space and every countable Tychonoff space are Δ-spaces.

On the other hand, we show that

Example 1 There exists an uncountable pseudocompact Haydon space X with all compact sets finite such that $C_k(X)$ $(= C_p(X))$ is not distinguished.

Recall that any Haydon space X is a subspace of $\beta\mathbb{N}$ with $X \supset \mathbb{N}$.

Another example is related with the paper [20].

Example 2 The space $\omega^* = (\beta\mathbb{N} \setminus \mathbb{N})$ contains a dense countably compact subspace X such that every compact subset of X is finite and $C_k(X)$ $(= C_p(X))$ is not distinguished.

The subspace $(\ell_\infty)_p = \{(x_n) \in \mathbb{R}^\mathbb{N} : \sup_n |x_n| < \infty\}$ of $\mathbb{R}^\mathbb{N}$ is distinguished. For all Haydon spaces X there exists a continuous open linear map from $C_p(X)$ onto $(\ell_\infty)_p$ (see [23, Theorem 1.3]), but for some Haydon spaces X the space $C_p(X)$ is not distinguished (by Example 1).

We note also the following more general fact involving the case $C_k(X)$.

Proposition 1 *Let* X *be a subspace of* $\beta\mathbb{N}$ *with* $X \supset \mathbb{N}$. *Then*

(1) $C_k(X)$ *admits a continuous open linear map onto* $\mathbb{R}^\mathbb{N}$ *or* c_0 *or* ℓ_2 *or* $(\ell_\infty)_p$.
(2) $C_p(X)$ *admits a continuous open linear map onto* $\mathbb{R}^\mathbb{N}$ *or* $(\ell_\infty)_p$.

In particular, the spaces $C_p(X)$ *and* $C_k(X)$ *have infinite-dimensional quotients that are distinguished although these spaces can be not distinguished.*

We assume that all locally convex spaces are Hausdorff and over the field $\mathbb{R}$ of real numbers.

2 A few facts about distinguished spaces

A locally convex space E is called *quasi-normable* [34, Definition 8.3.34] if for every neighbourhood of zero U in E there exists a neighbourhood of zero V such that for every $\lambda > 0$ there exists a bounded set B in E with $V \subseteq B + \lambda U$.

The class of quasi-normable spaces contains, for example, (DF)-spaces, metrizable spaces $C_k(X)$, spaces $C^n(\Omega)$ for open subsets Ω of $\mathbb{R}^\mathbb{N}$, as well as all Fréchet-Montel spaces (see [19, 33]).

Grothendieck showed that a metrizable locally convex space E is distinguished if and only if its strong dual $E'_\beta = (E', \beta(E', E))$ is bornological, [34, Theorem 8.3.44]. Heinrich [18] observed that each metrizable quasi-normable locally convex space satisfies the density condition what implies that *every metrizable quasi-normable locally convex space is distinguished.* In particular, the strong dual of a distinguished metrizable locally convex space can be described as a regular (LB)-space, see [34, Observation 8.5.14 (e)].

We refer to [5, 6, 15, 18, 19, 27] for several information about distinguished metrizable spaces. The most interesting example of a nondistinguished Fréchet (i.e. a metrizable and complete locally convex) space is the K öthe's echelon space from [27, 31.7].

Since each space $C_k(X)$ is quasi-normable (see [19, 10.8.2 Theorem]) and any metrizable quasi-normable space is distinguished, we have

Proposition 2 *Any metrizable space $C_k(X)$ is distinguished.*

Note however that non-metrizable and distinguished spaces $C_k(X)$ which are not strongly distinguished do exist, see Example 7.

Proposition 2 implies the following

Theorem 3 *$C_k(X)$ is distinguished for any locally compact paracompact space X.*

Proof It is known that X is the direct sum $\oplus_{t\in T} X_t$ of locally compact σ-compact spaces, see [9]. Then $C_k(X)$ is isomorphic to the product $\prod_{t\in T} C_k(X_t)$ of Fréchet spaces $C_k(X_t)$. Hence the strong dual $C_k(X)'_\beta$ of $C_k(X)$ is isomorphic to the direct sum $\bigoplus_{t\in T} C_k(X_t)'_\beta$, see [38, Exc. 8, p.192]. By Proposition 2 each space $C_k(X_t)'_\beta$ is barrelled, so $C_k(X)'_\beta$ is barrelled, too ([34, Corollary 4.2.7]). □

Note that the strong dual $C_k(X)'_\beta$ of $C_k(X)$ is a complete strict (LB)-space, i.e., complete strict inductive limit of a sequence of Banach spaces what is a consequence of the fact that $C_k(X)'_\beta$ is bornological (by applying the Grothendieck theorem, see [27, 23.7, 29.3]).

Below we provide two concrete examples of metrizable dense subspaces of $\mathbb{R}^{\mathbb{N}}$ which are strongly distinguished.

Example 3 The spaces $(\ell_\infty)_p$ and $(c_0)_p = \{(x_n) \in \mathbb{R}^{\mathbb{N}} : x_n \to 0\}$ with the topologies induced from $\mathbb{R}^{\mathbb{N}}$ are dense large subspaces of $\mathbb{R}^{\mathbb{N}}$, so they are strongly distinguished. On the other hand, does not exist a Tychonoff space X such that $(\ell_\infty)_p$ is isomorphic to $C_p(X)$. In fact, as easily seen, $(\ell_\infty)_p$ is σ-compact, while the space $C_p(X)$ is σ-compact if and only if X is finite, see [1, Theorem I.2.1].

Distinguished spaces $C_p(X)$ are totally different from distinguished spaces $C_k(X)$, see Theorem 6. Recall first the following general

Theorem 4 [15, 36] *The following assertions are equivalent for any locally convex space E.*

(1) *E has the finest locally convex topology.*
(2) *Every absolutely convex absorbing subset of E is a neighbourhood of zero.*
(3) *E is the strong dual of the product $\mathbb{R}^{\dim E}$.*
(4) *E is barrelled and admits a continuous basis [36].*

Clearly, any linear functional on a linear space E is continuous in the finest locally convex topology ξ on E; if E is infinite-dimensional, then ξ is non metrizable (since every metrizable infinite dimensional locally convex space admits a discontinuous linear functional).

On the other hand, in [15, Theorem 9] we proved

Theorem 5 *The homeomorphic copy of X in the dual $C_p(X)'$ with the weak*-topology is a continuous basis in $L_\beta(X)$.*

Recall (see [22]) that a locally convex space E is called *primitive* if for any increasing sequence (E_n) of linear subspaces of E with $\bigcup_{n=1}^{\infty} E_n = E$, a linear functional f on E is continuous if and only if its restrictions to E_n, $n \in \mathbb{N}$, are continuous.

Using [12, 15, 16, 24, 36] we put together several equivalent conditions for $C_p(X)$ to be distinguished.

Proofs of some implications are original.

Theorem 6 *For a Tychonoff space X the following are equivalent:*

(1) *$C_p(X)$ is distinguished.*
(2) *$C_p(X)$ is strongly distinguished.*
(3) *$C_p(X)$ and $\mathbb{R}^X$ have the same strong duals.*
(4) *The strong dual $L_\beta(X)$ of $C_p(X)$ is the direct sum of $|X|$-many lines.*
(5) *$L_\beta(X)$ is reflexive.*
(6) *$L_\beta(X)$ is bornological.*
(7) *$L_\beta(X)$ is quasi-barrelled.*
(8) *$L_\beta(X)$ is primitive.*
(9) *The strong bidual $M_\beta(X)$ of $C_p(X)$ is the product space $\mathbb{R}^X$.*
(10) *$M(X) = \mathbb{R}^X$ (as sets) i.e. $L_\beta(X)' = L_\beta(X)^*$.*
(11) *$M_\beta(X)$ is reflexive.*
(12) *$M_\beta(X)$ is quasi-complete.*
(13) *$C_p(X)$ is large in $\mathbb{R}^X$.*
(14) *For each $f \in \mathbb{R}^X$ there exists a bounded subset B of $C_p(X)$ such that $f \in cl_{\mathbb{R}^X}(B)$.*
(15) *X is a Δ-space.*
(16) *Any countable disjoint collection of subsets of X admits a point-finite open expansion in X.*
(17) *X is coverable i.e. any countable partition of X admits a point-finite open expansion in X.*

Proof (1) $\Rightarrow$ (2) $\Rightarrow$ (3) follows from Theorem 4 and Theorem 5.

(3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (1) and (2) $\Rightarrow$ (6) are obvious.

(6) $\Rightarrow$ (1) since every quasi-complete bornological locally convex space is a barrelled space ([7, Theorem 3.6.19]) and the space $L_\beta(X)$ is quasi-complete.

(1) $\Leftrightarrow$ (7) since every barrel in $L_\beta(X)$ is bornivorous.

(1) $\Rightarrow$ (8) since any barrelled locally convex space is primitive ([34, Proposition 4.1.6]).

(3) $\Leftrightarrow$ (13) follows by the bipolar theorem ([38, Theorem 1.5]).

(9) $\Rightarrow$ (11) $\Rightarrow$ (12) are obvious. (12) $\Rightarrow$ (9) $\mathbb{R}^X$ is the quasi-completion of the subspace G consisting of all functions $f \in \mathbb{R}^X$ with finite support. By Theorem 5, $G \subset M_\beta(X) \subset \mathbb{R}^X$. Thus $M_\beta(X) = \mathbb{R}^X$.

(3) $\Rightarrow$ (9) follows by reflexivity of $\mathbb{R}^X$. (9) $\Rightarrow$ (10) is obvious.

(10) $\Leftrightarrow$ (14) Put $F = C_p(X)$ and $E = M_\beta(X)$. Then $F'^* = L(X)^* = \mathbb{R}^X$ and $\sigma(F'^*, F') = \sigma(L(X)^*, L(X)) = \sigma(\mathbb{R}^X, L(X))$ is the product topology of $\mathbb{R}^X$. By [38, Theorem 5.4, IV], $f \in E$ if and only if there exists a bounded subset B of F such that $f \in \mathrm{cl}_{\sigma(F'^*, F')}(B) = \mathrm{cl}_{\mathbb{R}^X}(B)$.

(8) $\Rightarrow$ (10) Let $f \in \mathbb{R}^X$. Put $X_n = \{x \in X : |f(x)| \le n\}$ and $L_n = \{\mu \in L(X) : \mathrm{supp}(\mu) \subset X_n\}$ for $n \in \mathbb{N}$. Clearly, (L_n) is an increasing sequence of linear subspaces of $L(X)$ with $\bigcup_{n=1}^{\infty} L_n = L(X)$. Let $n \in \mathbb{N}$ and $f_n = f\chi_{X_n}$. The set $B_n = \{g \in C(X) :$

$|g(x)| \leq n$ for all $x \in X\}$ is bounded in $C_p(X)$ and $f_n \in \mathrm{cl}_{\mathbb{R}^X}(B_n)$. By [38, Theorem 5.4, IV], $\mathrm{cl}_{\mathbb{R}^X}(B_n) \subset M(X)$, so $f_n \in M(X)$. Clearly, $f|_{L_n} = f_n|_{L_n}$, so $f|_{L_n}$ is continuous for any $n \in \mathbb{N}$. Hence $f \in M(X)$. Thus $\mathbb{R}^X = M(X)$.

(13) $\Rightarrow$ (14) is obvious.

(14) $\Rightarrow$ (15) Let (X_n) be a decreasing sequence of subsets of X with empty intersection. Put $X_0 = X$. Let $f \in \mathbb{R}^X$ with $f(x) = n+1$ for all $x \in [X_{n-1} \setminus X_n]$, $n \in \mathbb{N}$. Let B be a bounded subset of $C_p(X)$ with $f \in \mathrm{cl}_{\mathbb{R}^X}(B)$. Let $x \in X$. Then $x \in [X_{n-1} \setminus X_n]$ for some $n \in \mathbb{N}$.

The set $W = \{g \in \mathbb{R}^X : |g(x) - f(x)| < 1\}$ is a neighbourhood of f in $\mathbb{R}^X$, so $W \cap B \neq \emptyset$. Let $g_x \in W \cap B$. Then $g_x(x) > n$, so the set $V_x = g_x^{-1}((n,\infty))$ is an open neighbourhood of x in X. Put $U_n = \bigcup_{x \in X_n} V_x$ for $n \in \mathbb{N}$. Clearly, (U_n) is a decreasing sequence of open subsets of X and $X_n \subset U_n$ for $n \in \mathbb{N}$. Let $n \in \mathbb{N}$. For $x \in X_n$ and $z \in V_x$ we have $g_x(z) > n$, so $\sup_{g \in B} |g(z)| > n$ for $z \in U_n$. Let $y \in X$.

For some $m \in \mathbb{N}$ we have $\sup_{g \in B} |g(y)| \leq m$, since B is bounded in $C_p(X)$. It follows that $y \notin U_m$, so $y \notin \bigcap_{n=1}^{\infty} U_n$. Thus $\bigcap_{n=1}^{\infty} U_n = \emptyset$.

(15) $\Rightarrow$ (13) Let A be a non-empty bounded subset of $\mathbb{R}^X$. Let $\psi : X \to \mathbb{R}$, $\psi(x) = \sup_{f \in A} |f(x)|$. Put $X_n = \{x \in X : |\psi(x)| \geq n-1\}$ for $n \in \mathbb{N}$. Then (X_n) is a decreasing sequence of subsets of X with empty intersection. Thus there exists a decreasing sequence (U_n) of open subsets of X with an empty intersection such that $X_n \subset U_n$ for $n \in \mathbb{N}$. For any $x \in X$ there exists $\varphi(x) \in \mathbb{N}$ with

$$x \in [U_{\varphi(x)} \setminus U_{\varphi(x)+1}].$$

We have $\psi(x) < \varphi(x)$ for every $x \in X$. Indeed, let $x \in X$ and $n \in \mathbb{N}$ with $n-1 \leq |\psi(x)| < n$. Then $x \in X_n \subset U_n$ and $x \notin U_{\varphi(x)+1}$, so $n < \varphi(x)+1$. Thus $\psi(x) < n \leq \varphi(x)$.

Clearly, the set $B = \{g \in C_p(X) : |g| \leq \varphi\}$ is bounded in $C_p(X)$. We shall prove that $A \subset \mathrm{cl}_{\mathbb{R}^X}(B)$. Let $f \in A$. Let W be a neighbourhood of f in $\mathbb{R}^X$. Then there exists a finite subset K of X such that

$$\{g \in \mathbb{R}^X : g|_K = f|_K\} \subset W.$$

Let $\{V_x : x \in K\}$ be a family of pairwise disjoint open subsets of X with

$$x \in V_x \subset U_{\varphi(x)}, x \in K.$$

For every $x \in K$ there exists a continuous function

$$h_x : X \to [-\varphi(x), \varphi(x)]$$

with $h_x(x) = f(x)$ and $h_x(y) = 0$ for $y \in V_x^c$ (see [9, Theorem 3.1.7]). The function $h : X \to \mathbb{R}$, $h = \sum_{x \in K} h_x$ is continuous and $h|_K = f|_K$, so $h \in W$.

We shall prove that $h \in B$. Clearly, $h(x) = 0$ for $x \in (\bigcup_{x \in K} V_x)^c$. Let $x \in K$. For $y \in [K \setminus \{x\}]$ we have $V_x \subset V_y^c$, so $h_y|_{V_x} = 0$.
Thus $h|_{V_x} = h_x|_{V_x}$. For $t \in V_x$ we have

$$|h(t)| = |h_x(t)| \leq \varphi(x) \leq \varphi(t),$$

since $V_x \subset U_{\varphi(x)}$. Thus $|h| \leq \varphi$, so $h \in B$. It follows that $W \cap B \neq \emptyset$, so $f \in \mathrm{cl}_{\mathbb{R}^X}(B)$. Hence $A \subset \mathrm{cl}_{\mathbb{R}^X}(B)$.

(15) $\Rightarrow$ (16) Let (S_n) be a disjoint collection of subsets of X. Put $X_n = \bigcup_{m=n}^{\infty} S_m$ for $n \in \mathbb{N}$. Then (X_n) is a decreasing point-finite sequence of subsets of X. Thus there exists a decreasing point-finite open expansion (U_n) of (X_n); clearly (U_n) is an open expansion of (S_n).

(16) $\Rightarrow$ (17) is obvious.

(17) $\Rightarrow$ (15) Let (X_n) be a decreasing point-finite sequence of subsets of $X_0 = X$. Put $P_n = [X_{n-1} \backslash X_n]$, $n \in \mathbb{N}$. Then (P_n) is a partition of X. Let (U_n) be a point-finite open expansion of (P_n) and $V_n = \bigcup_{m=n+1}^{\infty} U_m$, $n \in \mathbb{N}$. Then (V_n) is a decreasing point-finite open expansion of (X_n), since $X_n = \bigcup_{m=n+1}^{\infty} P_m \subset \bigcup_{m=n+1}^{\infty} U_m = V_n$, $n \in \mathbb{N}$. □

On the other hand, notice the following fact implying that the item (11) in Theorem 6 cannot be replaced by $M_\beta(X)$ is Baire.

Remark 1 Ferrando and Kąkol [12, Theorem 3.9] proved that the strong dual $L_\beta(X)$ of $C_p(X)$ is always distinguished (i.e. $M_\beta(X)$ is always barrelled) and then Ferrando and Saxon asked [16, Problem 11] if the Baire property of $M_\beta(X)$ implies that $C_p(X)$ is distinguished. The answer is negative (as noticed in [16, Addendum]) since $M(\omega_1)$ is a Baire space [16, Corollary 21] but $C_p(\omega_1)$ is not distinguished [24].

For spaces $C_k(X)$ note the following simple

Proposition 3 *Let X be a Tychonoff space. The space $C_k(X)$ is feral if and only if X is finite.*

Proof Clearly, $C_k(X)$ is feral if X is finite. Assume that X is infinite. If X is pseudocompact, then $C_k(X)$ admits a stronger normed topology. If X is not pseudocompact, then $C_k(X)$ contains an isomorphic copy of $\mathbb{R}^{\mathbb{N}}$ ([21, Theorem 2.12]). Thus $C_k(X)$ is not feral. □

3 Feral dual spaces

Two locally convex spaces E and F are called *bornologically isomorphic* if there exists a linear bijective map $T : E \to F$ such that T and T^{-1} are locally bounded; E is a *free locally convex space* if E carries the finest locally convex topology.

In order to prove Theorem 2 we need the following simple fact which is probably known.

Lemma 1 *Let E and F be locally convex spaces. Assume that there exists a continuous linear surjection $T : E \to F$ which is bounded covering, i.e. for every bounded set B in F there exists a bounded set A_B in E with $T(A_B) = B$. Then the strong dual E'_β of E contains an isomorphic copy of the strong dual F'_β of F.*

Proof The adjoint map $T^* : F'_\beta \to E'_\beta$ is injective. Let B be a bounded subset of F. The seminorms $p_B : F'_\beta \to [0, \infty)$, $\xi \to \sup_{f \in B} |\xi(f)|$ and $p_{A_B} : E'_\beta \to [0, \infty)$, $\eta \to \sup_{e \in A_B} |\eta(e)|$ are continuous and

$$p_B(\xi) = \sup_{f \in B} |\xi(f)| = \sup_{e \in A_B} |\xi(Te)| = \sup_{e \in A_B} |(T^*\xi)(e)| = p_{A_B}(T^*\xi)$$

for any $\xi \in F'_\beta$. It follows that T^* is an isomorphism onto its range, so E'_β contains an isomorphic copy of F'_β. □

Dealing with locally convex spaces we have the following characterization for quasi-barrelled spaces carrying the weak topology.

Theorem 7 *For a locally convex space E the following are equivalent:*

(1) *E'_β is feral.*

(2) *E'_β is bornologically isomorphic to a free locally convex space.*
(3) *E is quasi-barrelled and carries the weak topology.*
(4) *Every compact set in E'_β has finite topological dimension.*

Proof (1) $\Leftrightarrow$ (2) is easy.

The equivalence (1) $\Leftrightarrow$ (4) has been proved in [4, Theorem 1.2].

(1) $\Rightarrow$ (3) First we show that $E_\mu = (E, \mu(E, E'))$ is quasi-barrelled. Let B be a bounded subset of E'_β. Then B is finite-dimensional, so the closure W of the absolutely convex hull of B in $(E', \sigma(E', E))$ is compact. Thus the set ${}^\circ W = \{x \in E : |f(x)| \le 1 \text{ for all } f \in W\}$ is a neighbourhood of zero in E_μ and $B \subset W \subset ({}^\circ W)^\circ \subset E'$; so B is $\mu(E, E')$-equicontinuous. Thus E_μ is quasi-barrelled.

Let E'_ν be the space E' with the finest locally convex topology. The identity map $I : E'_\nu \to E'_\beta$ is a continuous linear surjection. Since every bouned subset of E'_β is finite-dimensional, I is bounded covering. By Lemma 1 we derive that the strong bidual $(E'_\beta)'_\beta$ is isomorphic to a subspace of $(E'_\nu)'_\beta$. On the other hand, E'_ν is the direct sum $\bigoplus_{t\in X} \mathbb{R}$, where X is a Hamel basis of E'; so its strong dual $(E'_\nu)'_\beta$ is isomorphic to the product $\mathbb{R}^X$, which clearly carries the weak topology. Hence $(E'_\beta)'_\beta = (E'', \beta(E'', E')$ carries the weak topology and $\beta(E'', E')|_E = \sigma(E', E)$, since every subspace of a locally convex space with the weak topology has the weak topology. Since E_μ is quasi-barrelled, $\beta(E'', E')|_E = \mu(E, E')$, see [19, Proposition 11.2.2].

Thus $\sigma(E, E') = \mu(E, E')$, so E is quasi-barrelled and carries the weak topology.

(3) $\Rightarrow$ (1) Let X be a Hamel basis of E' with the topology induced from $(E', \sigma(E', E))$. The linear map $\psi : E \to C_p(X)$, $z \to \psi_z$, where $\psi_z(x) = x(z)$ for $x \in X$, is an isomorphism between E and $\psi(E)$, and $\psi(E)$ is dense in $C_p(X)$. Thus we can identify E with a dense subspace of the product $\mathbb{R}^X$ and E' with $(\mathbb{R}^X)'$.

Let B be a bounded subset of E'_β. Since E is quasi-barrelled, B is equicontinuous, so there exists a neighbourhood U of zero in E such that $B \subset U^\circ \subset E' = (\mathbb{R}^X)'$. Clearly, the closure V of U in $\mathbb{R}^X$ is a neighbourhood of zero in $\mathbb{R}^X$. Let $f \in B$ and $v \in V$. Then there exists a net $(u_t)_{t\in T}$ in U convergent to v in $\mathbb{R}^X$. Hence the net $(f(u_t))_{t\in T}$ is convergent to $f(u)$ in $\mathbb{R}$ and $|f(u_t)| \le 1, t \in T$, so $|f(u)| \le 1$. Thus $B \subset V^\circ \subset (\mathbb{R}^X)'$. The set V° is bounded in $(\mathbb{R}^X)'_\beta$. Indeed, let A be a bounded subset of $\mathbb{R}^X$. Then $sA \subset V$ for some $s > 0$, so $sV^\circ \subset A^\circ$. Hence V° is bounded in $(\mathbb{R}^X)'_\beta = \oplus_{x\in X}\mathbb{R}$. Thus V° is finite-dimensional, since the direct sum $\oplus_{x\in X}\mathbb{R}$ is feral; so $B \subset V^\circ$ is finite-dimensional. Hence E'_β is feral. □

Since $C_p(X)$ is quasi-barrelled ([19, Corollary 11.7.3]) and carries the weak topology, Theorem 7 applies the following known result (see [13]).

Corollary 3 *$L_\beta(X)$ is feral for any Tychonoff space X.*

Corollary 4 *Let X be a Tychonoff space. The strong bidual $M_\beta(X)$ of $C_p(X)$ is feral if and only if X is finite.*

Proof ($\Rightarrow$) The strong dual $L_\beta(X)$ of $C_p(X)$ is quasi-barrelled (by Theorem 7) and complete (by [12, Proposition 3.10]). Consequently, $L_\beta(X)$ is barrelled, so $M_\beta(X)$ is the product $\mathbb{R}^X$ (by Theorem 6). Since $M_\beta(X)$ is feral, X is finite. ($\Leftarrow$) is obvious. □

The following proposition provides a stronger version of Corollary 1.

Proposition 4 *Let E be a locally convex space.*
If there exists a Tychonoff space X and a locally bounded linear map T from $C_p(X)$ onto E, then the strong dual E'_β of E is feral.

Proof Let υX be the realcompactification of the space X.

The restriction map $\pi : C_p(\upsilon X) \to C_p(X)$, $f \to f|_X$ is a continuous linear surjection, see [21, Lemma 9.1].

The composition map $\hat{T} : C_p(\upsilon X) \to E$, $\hat{T} = T \circ \pi$ is a locally bounded linear surjection. $C_p(\upsilon X)$ is bornological by Buchwalter–Schmets theorem ([8]). Applying [34, Proposition 6.1.8] we deduce that $\hat{T}$ is continuous. Then the adjoint map $\hat{T}^* : E'_\beta \to L_\beta(\upsilon X)$ is injective and continuous. Hence the range $\hat{T}^*(E')$ admits a locally convex topology ξ, stronger than the topology restricted from $L_\beta(\upsilon X)$ and such that E'_β is isomorphic with $(\hat{T}^*(E'), \xi)$. By [14, page 392] the space $L_\beta(\upsilon X)$ is feral. Hence the strong dual E'_β of E is feral. □

Corollary 5 *Let X and Y be Tychonoff spaces.*

If there exists a locally bounded linear map T from $C_p(X)$ onto $C_k(Y)$, then every compact subset of Y is finite.

Proof By Proposition 4 the strong dual of $C_k(Y)$ is feral. Hence, by Theorem 2, each compact subset of Y is finite. □

Proposition 4 suggests the following.

Problem 2 *Does there exist a locally convex space E whose strong dual is feral such that E is not continuous [not locally bounded] linear image of $C_p(X)$ for any Tychonoff space X?*

Recall that the strongly distinguished space $(\ell_\infty)_p$ is not isomorphic to any space $C_p(X)$ (Example 3), although if X is a Tychonoff space containing a copy of $\beta\mathbb{N}$, then $C_p(X)$ admits a continuous open linear map onto $(\ell_\infty)_p$, see [3, Theorem 1].

Now we prove Theorem 2

Proof of Theorem 2 (I). ($\Rightarrow$) Assume that X contains an infinite compact subset K. The restriction map $R : C_k(X) \to C_k(K)$, $R(f) = f|K$ is an open continuous surjection, see [23, Proposition 2.9]. Put $M = \ker R$. Let $Q : C_k(X) \to C_k(X)/M$ be the quotient map. Then the map $\overline{R} : C_k(X)/M \to C_k(K)$, $f + M \to f|_K$ is an isomorphism, since $\overline{R} \circ Q = R$. By Theorem 7 the space $C_k(X)$ carries the weak topology, so $C_k(X)/M$ carries the weak topology, too. We have a contradiction, since $C_k(X)/M$ is isomorphic to the infinite-dimensional Banach space $C_k(K)$.

($\Leftarrow$) follows by Corollary 3, since $C_k(X) = C_p(X)$.

(II). Assume that $E = C_k(X)$ is quasi-barrelled. Then E is a subspace of $(E'_\beta)'_\beta$ ([19, Proposition 11.2.2]). Hence, if $(E'_\beta)'_\beta$ is feral, then E is feral, too; so X is finite, by Proposition 3. The converse is obvious. □

Another consequence of Theorem 7 yields the following

Corollary 6 *An infinite-dimensional metrizable locally convex space E is strongly distinguished if and only if it is isomorphic to a dense subspace of $\mathbb{R}^{\mathbb{N}}$. Hence a Fréchet space F is strongly distinguished if and only if F is finite-dimensional or F is isomorphic to $\mathbb{R}^{\mathbb{N}}$.*

Proof ($\Rightarrow$) By Theorem 7 and its proof, E carries the weak topology and it is isomorphic to a dense subspace of $\mathbb{R}^X$ for some infinite set X. E is metrizable, so X is countable.

($\Leftarrow$) By [34, Observation 8.3.23 (b)] E is large in $\mathbb{R}^{\mathbb{N}}$, so the strong duals of E and $\mathbb{R}^{\mathbb{N}}$ coincide and are isomorphic to φ, the $\aleph_0$-dimensional linear space with the finest locally convex topology. □

Recall that a locally convex space admits a fundamental bounded resolution if there exists a family $\{B_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ of bounded sets such that $B_\alpha \subset B_\beta$ for all $\alpha \leq \beta$ and each bounded set in E is contained in some B_α. Clearly every metrizable locally convex space admits such resolution, see [21, Lemma 15.2]. A locally convex space E has a $\mathbb{N}^\mathbb{N}$-base if there exists a base $\{U_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ of neighborhoods of zero such that $U_\beta \subset U_\alpha$ for all elements $\alpha \leq \beta$ in $\mathbb{N}^\mathbb{N}$. Clearly, every metrizable locally convex space has an $\mathbb{N}^\mathbb{N}$-base, [21].

In [11, Theorem 3.3] we proved that $C_p(X)$ *admits a fundamental bounded resolution if and only if* X *is countable* (if and only if $C_p(X)$ is metrizable).

Using Theorem 7 we extend the above result and supplement Corollary 6.

Corollary 7 *An infinite-dimensional quasi-barrelled locally convex space E is isomorphic to a dense subspace of $\mathbb{R}^\mathbb{N}$ if and only if E has a fundamental bounded resolution and E caries the weak topology.*

Proof ($\Rightarrow$) is clear. ($\Leftarrow$) By the proof of Theorem 7, E is isomorphic to a dense subspace of $\mathbb{R}^X$ for some infinite set X. Assume that X is uncountable. Then E' is uncountable-dimensional, since $\dim E' = \dim(\mathbb{R}^X)' = \dim(\oplus_{x\in X}\mathbb{R})$. Let $\{B_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ be a fundamental bounded resolution of E. Then $\{B_\alpha^\circ : \alpha \in \mathbb{N}^\mathbb{N}\}$ is a $\mathbb{N}^\mathbb{N}$-base of neighbourhoods of zero in the strong dual E'_β of E. By [4, Theorem 1.2], E'_β contains an infinite-dimensional metrizable compact set; a contradiction, since E'_β is feral (by Theorem 7). Thus X is countable. □

4 Proof of Proposition 1 and Examples

Proof of Proposition 1 (1) Consider three cases: (1.1). X is not pseudocompact. Then the space $C_k(X)$ contains a complemented copy of the space $\mathbb{R}^\mathbb{N}$, see [21, Theorem 2.12] and [34, Corollary 2.6.5]. Hence there exists a continuous open linear map from $C_k(X)$ onto $\mathbb{R}^\mathbb{N}$.

(1.2). X is pseudocompact and every compact subset of X is finite. Since $\mathbb{N} \subset X \subset \beta\mathbb{N}$, then $\mathbb{N}$ is C*-embedded into X. Applying [3, Theorem 1] we get that $C_p(X)$ has a quotient $C_p(X)/W$ isomorphic to the subspace $(\ell_\infty)_p$ of $\mathbb{R}^\mathbb{N}$ (endowed with the product topology), where $W = \bigcap_n \{f \in C_p(X) : \sum_{x\in F_n} f(x) = 0\}$ and (F_n) is a sequence of non-empty, finite and pairwise disjoint subsets of $\mathbb{N}$ with $\lim_n |F_n| = \infty$. Hence $C_k(X)(= C_p(X))$ admits a continuous open linear map onto $(\ell_\infty)_p$.

(1.3). X is pseudocompact and contains an infinite compact subset K. By [23, Proposition 2.9] the restriction map $T : C_k(X) \to C_k(K)$, $f \to f|_K$, is a continuous open linear surjection. By [37] the Banach space $C(K)$ admits a continuous open linear map onto ℓ_2 or c_0. Thus $C_k(X)$ admits a continuous open linear map onto ℓ_2 or c_0.

(2) If X is not pseudocompact, then $C_p(X)$ contains a complemented copy of $\mathbb{R}^\mathbb{N}$ ([2, Sect. 4]), so $C_p(X)$ admits a continuous open linear map onto $\mathbb{R}^\mathbb{N}$.

If X is pseudocompact, then $C_p(X)$ has a quotient isomorphic to $(\ell_\infty)_p$ (see (1.2)), so $C_p(X)$ admits a continuous open linear map onto $(\ell_\infty)_p$. □

Below we provide concrete situations where $C_k(X)$ is distinguished but its strong dual $C_k(X)'_\beta$ is not feral.

Example 4 Let X be an uncountable hemicompact space. Then the space $C_k(X)$ is distinguished but its strong dual $C_k(X)'_\beta$ is not feral.

Proof X is hemicompact, i.e. X is covered by a sequence of compact sets (K_n) such that each compact set in X is contained in some K_n, so $C_k(X)$ is metrizable. Applying Proposition 2 we infer that $C_k(X)$ is distinguished. Since X is uncountable, for some $n \in \mathbb{N}$ the set K_n is infinite. By Theorem 2, $C_k(X)'_\beta$ is not feral. □

A topological space X is called a *Q-space* if each subset of X is G_δ. Recall that a normal space X is a Q-space if and only if X is *strongly splittable*, i.e. for every $f \in \mathbb{R}^X$ there exists a sequence $(f_n)_n$ in $C_p(X)$ such that $f_n \to f$ in $\mathbb{R}^X$, see [39, Problems 445, 447]. Using Theorem 2 one gets the following

Proposition 5 *For a normal space X the assertions are equivalent.*

(1) *X is a Q-space and every compact subset of X is finite.*
(2) *For each $f \in \mathbb{R}^X$ there exists a bounded sequence (f_n) in $C_k(X)$ such that $f_n \to f$ in $\mathbb{R}^X$.*

Proof (1) ⇒ (2). Then $C_k(X) = C_p(X)$. Let $f \in \mathbb{R}^X$. X is a Q-space, so there exists a sequence $(f_n) \subset C_p(X)$ with $f_n \to f$ in $\mathbb{R}^X$. Then (f_n) is bounded in $C_p(X)$ $(= C_k(X))$.

(2) ⇒ (1). Then X is a Q-space and $C_k(X)$ is dense in $\mathbb{R}^X$. It is known that $C_k(X)$ has a base $\{U_t : t \in T\}$ of neighbourhoods of zero such that U_t, $t \in T$, are closed in $C_p(X)$. For any $t \in T$, the set $\mathrm{cl}_{\mathbb{R}^X}(U_t)$ is a neighbourhoods of zero in $\mathbb{R}^X$, so $U_t = C_p(X) \cap \mathrm{cl}_{\mathbb{R}^X}(U_t)$ is a neighbourhood of zero in $C_p(X)$. Thus the topological spaces $C_k(X)$ and $C_p(X)$ are equal, so any compact subset of X is finite. □

Example 1 uses the following scheme of constructing uncountable pseudocompact spaces without infinite compact subsets due to Haydon, [17]. Let $\omega^* = (\beta\mathbb{N} \setminus \mathbb{N})$. For each infinite subset A of $\mathbb{N}$, choose a cluster point u_A of A in $\beta\mathbb{N}$. Let $X = (\mathbb{N} \cup \{u_A : A \text{ is an infinite subset of } \mathbb{N}\})$ be topologized as a subspace of $\beta\mathbb{N}$. To simplify the notation we will call such spaces the Haydon spaces. It is known that each Haydon space X is an uncountable pseudocompact space and each compact subset of X is finite, see [17].

A point x of a topological space X is said to be a *a weak P-point* if for any countable subset F of $(X \setminus \{x\})$ we have $x \notin \overline{F}$. Clearly, any countable set of weak P-points of X is discrete. By [30], see also [20], the set of all weak P-points of the space ω^* is dense in ω^*.

Thus for any infinite subset A of $\mathbb{N}$ there exists an element $u_A \in \overline{A} \cap \omega^*$, that is a weak P-point of X. Then any countable subset of the set $Z := \{u_A : A \text{ is an infinite subset of } \mathbb{N}\}$ is discrete. Thus any countable subset of the Haydon space $Y = (\mathbb{N} \cup Z)$ is scattered.

We will use the following two results.

In [24] Kąkol and Leiderman proved the following

Theorem 8 [24, Theorem 4.7] *If X is countably compact and $C_p(X)$ is distinguished, then X is scattered.*

A related result has been proved by Leiderman and Tkachuk in [32].

Theorem 9 [32, Theorem 3.1] *If X is pseudocompact and $C_p(X)$ is distinguished then every countable subset of X is scattered.*

Proof of Example 1 The compact space ω^* has no isolated points. It is well-known (and easy to prove), that every compact space without isolated points contains a non-empty countable subset without isolated points, too. Thus there exists a non-empty countable subset P of ω^* without isolated points; clearly, P is not scattered.

Let $P = \{p_n : n \in \mathbb{N}\}$ and $A_n = \{k \in \mathbb{N} : k > n\}$ for $n \in \mathbb{N}$. Since $p_n \in (\overline{A_n} \setminus \mathbb{N})$, $n \in \mathbb{N}$, there exists a Haydon space X, that contains P. By Theorem 9, the space $C_k(X)$ $(= C_p(X))$ is not distinguished. □

Proof of Example 2 In [20] Juhasz and van Mill proved that the space ω^* contains a dense subspace X such that X is countably compact and non-scattered but all countable subsets of X are scattered. Observe that every compact subset of X is finite. Indeed, if a compact subset A of $X \subset \beta N$ is infinite, then A contains a copy of $\beta\mathbb{N}$, but βN contains a countable subset which is not scattered; a contradiction. By Theorem 8, the space $C_k(X)$ $(= C_p(X))$ is not distinguished. □

Example 5 Let $\mathfrak{D}'(\Omega)$ be the space of all distributions over an open set $\Omega \subset \mathbb{R}^n$. Then every uncountable-dimensional subspace of $\mathfrak{D}'(\Omega)$ contains an infinite-dimensional metrizable compact set. In particular, $\mathfrak{D}'(\Omega)$ is not feral.

Proof The space $\mathfrak{D}(\Omega)$ is a strict countable inductive limit of Fréchet Montel spaces ([19, Example 4.6.3]), so $\mathfrak{D}(\Omega)$ admits a fundamental bounded resolution (see [21] and the proof of Proposition 16.7 (ii) there). Therefore $\mathfrak{D}'(\Omega)$ admits an $\mathbb{N}^{\mathbb{N}}$-base. Now the conclusion follows from [4, Theorem 1.2]. □

Example 6 Every uncountable-dimensional subspace of $C_k(\mathbb{R}^{\mathbb{N}})$ contains a metrizable compact infinite-dimensional set and the strong dual of $C_k(\mathbb{R}^{\mathbb{N}})$ admits an infinite-dimensional compact set. In particular, $C_k(\mathbb{R}^{\mathbb{N}})$ and its strong dual are not feral.

Proof The first claim follows from the fact that $C_k(\mathbb{R}^{\mathbb{N}})$ admits an $\mathbb{N}^{\mathbb{N}}$-base (by [10]) and then we apply [4, Theorem 1.2]. The other claim follows from Theorem 2. □

By the theorem of Heinrich (see Sect. 2), we know that every quasi-normable metrizable locally convex space is distinguished.

Problem 3 *Is every quasi-normable locally convex space with a $\mathbb{N}^{\mathbb{N}}$-base a distinguished space?*

Note that every (LB)-space is quasi-normable ([34]) and each (LB)-space has a $\mathbb{N}^{\mathbb{N}}$-base ([21]). Moreover, each space $C_p(X)$ is quasi-normable ([15]) and $C_p(X)$ has an $\mathbb{N}^{\mathbb{N}}$-base if and only if $C_p(X)$ is metrizable ([21]). Recall also that $C_k(\mathbb{R}^{\mathbb{N}})$ is quasi-normable and has an $\mathbb{N}^{\mathbb{N}}$-base by applying the main theorem of [10].

Example 7 There exists a non-metrizable distinguished space $C_k(X)$ which is not strongly distinguished.

Proof By [21, Example 2.4], there exists a Tychonoff space X such that $C_k(X)$ is a (df)-space but not (DF)-space. Then $C_k(X)$ is not metrizable, since any metrizable (df)-space is a (DF)-space. By [21, Theorem 2.14], the strong dual of $C_k(X)$ is a Fréchet space, so $C_k(X)$ is distinguished but not strongly distinguished. □

Note also that $C_k(\mathbb{R}^{\mathbb{N}})$ is not a (df)-space; it is even not covered by a sequence of bounded sets. Indeed, this follows directly from [23, Lemma 2.3].

Problem 4 *Is the space $C_k(X)$ distinguished when X is metrizable? In particular, are the spaces $C_k(\mathbb{R}^{\mathbb{N}})$ and $C_k(\mathbb{Q})$ distinguished?*

Problem 5 *Characterize distinguished spaces $C_k(X)$ in terms of X.*

Data Availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study

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