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ORIGINAL PAPER

Algebraic structures in the set of sequences of independent random variables

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Abstract
The space whose subsets we analyse with respect to lineability is $L_0(\Omega, P)^{\mathbb{N}}$ consisting of random variables sequences on probability space (Ω, P) with atomless probability measure P. We study lineability and algebrability of $L_0(\Omega, P)^{\mathbb{N}}$-subsets of independent random variables with additional properties connected with various types of convergence, laws of large numbers, and Markov and Kolgomorov conditions.

Keywords Lineability · Algebrability · Independent random variables · Laws of large numbers

Mathematics Subject Classification Primary 46B87; Secondary 15A03

1 Introduction

The last 20 years brought numerous papers devoted to the existence of large and rich algebraic structures inside subsets of linear spaces, function algebras and their Cartesian products. The topic has gained such popularity that the monograph devoted to it has been released [2] and a few surveys appeared [5], [3]. Recently the subject has obtained its place in Mathematical Subject Classification— 46B87. The custom name for problems in this area are lineability or algebrability problems. A large number of sets in function and sequence spaces naturally arising in many branches of mathematics were studied from this perspective. Probability, however, is rather scarcely represented in lineability theory. Only three of the papers on line-

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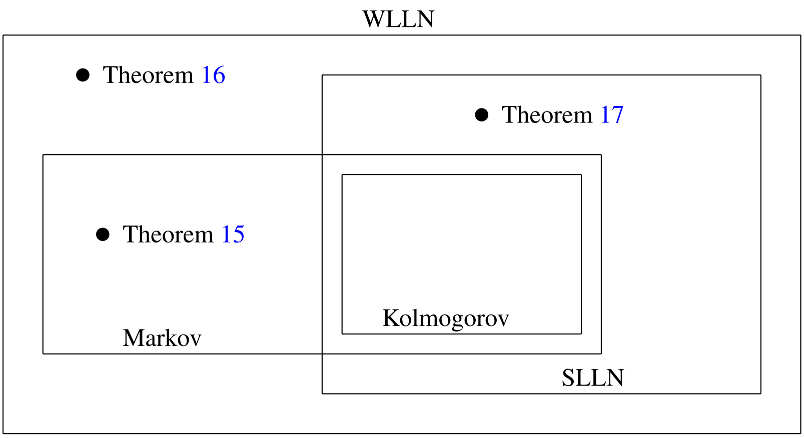
ability published so far address probability theory—[6], [4] and [8]. The first of them [6] was published in RACSAM in 2017. The main feature distinguishing the probability theory from the measure theory is the independence or some kind of dependence for random variables. The other are theorems typical for probability theory, like Borel-Cantelli Lemma, the Laws of Large Numbers and so-called Markov and Kolmogorov conditions related to them, Central Limit Theorem, etc. In the mentioned papers independence or martingale dependence was assumed or obtained in [6, Theorem 5] (pairwise independence), [4, Theorem 10 and Theorem 11] (martingales), [4, Theorem 13] and [8, Theorem 2.11] (independence).

In present paper we have set the following goals. Firstly, to select the assumptions on the probability spaces so that there is no need to define spaces for particular theorems. This is discussed in Sect. 3. Secondly, to consider exclusively sequences of independent random variables. The space whose subsets we analyse with respect to lineability is the space $L_0(\Omega, P)^{\mathbb{N}}$, i.e. the space of sequences of random variables on probability space (Ω, P) with atomless probability measure P.

Throughout the paper [8] the Authors considered the following condition on probability space: $(\Omega, \mathcal{F})$ is not isomorphic to $([N], 2^{[N]})$ for any N, where $[N] = \{0, 1, \dots, N-1\}$. This condition says that probability space contains infinitely many events. Let us compare that condition to the following: in $(\Omega, \mathcal{F}, P)$ there is a sequence of independent events with probabilities in $(0, 1)$. Clearly, the latter condition implies that $(\Omega, \mathcal{F})$ cannot be isomorphic to finite probability spaces. However, there are infinite probability spaces which do not contain any proper pair of events (that is both with probabilities in $(0, 1)$) which are independent, see [7]. In our results we will construct sequences of independent random variables, which is possible in non-atomic probability space. Any probabilistic measure can be written as a sum of two measures: an atomic and an atomless. Our results hold true if the atomless part of a probabilistic measure is non-void. However, we decided to assume that probabilistic spaces are atomless to make our results more clear for the first reading.

The paper is organized as follows. In Sect. 2 we remain the basic definitions and facts concerning lineability, algebrability, and that of probability theory. In Sect. 3 we analyse the properties of probability spaces with atomless measures. These properties are likely to be known to the probability experts, however we were not able to find them in a single source. In Sect. 4 we consider various kinds of probability convergence. In Theorem 11 we consider the sequences of independent random variables convergent in probability but not almost everywhere. In Theorem 12 we consider the ones for which the Cesaro means are divergent in probability. In Theorem 13 we consider uniformly bounded sequences convergent in probability but not almost surely. Our theorems generalize known ones or are essentially different of them —compare Theorem 11 with [8, Theorem 2.2] and [6, Theorem 1], and Theorem 12 with [8, Theorem 2.11 and Theorem 2.15]. The inspiration, particularly for the last part of the paper come from the Stoyanov monograph [10]. Without the knowledge of these examples the paper would not have appeared in the present shape. Section 5 is devoted to the Laws of Large Numbers. In this part we frequently use the fact that the zero random variable is independent with respect to any other, even to itself. This and the use of almost disjoint families of subsets of $\mathbb{N}$ allows us to construct the necessary $\mathfrak{c}$-dimensional linear spaces consisting of sequences of independent random variables. The following diagram summarize the most of the results from Sect. 5.



2 Preliminaries

2.1 Lineability and algebrability

Definition 1 Let κ be a cardinal number.

(1) Let $\mathcal{L}$ be a vector space and $A \subseteq \mathcal{L}$. We say that A is κ-lineable if $A \cup \{0\}$ contains a κ-dimensional subspace of $\mathcal{L}$.
(2) Let $\mathcal{L}$ be a commutative algebra and $A \subseteq \mathcal{L}$. We say that A is κ-algebrable if $A \cup \{0\}$ contains a κ-generated subalgebra B of $\mathcal{L}$ (i.e. the minimal cardinality of the system of generators of B is κ).
(3) Let $\mathcal{L}$ be a commutative algebra and $A \subseteq \mathcal{L}$. We say that A is strongly κ-algebrable if $A \cup \{0\}$ contains a κ-generated subalgebra B that is isomorphic to a free algebra.

Proposition 2 *$X = \{x_\alpha : \alpha < \kappa\}$ is the set of free generators of some free algebra if and only if the set $\tilde{X}$ of elements of the form $x_{\alpha_1}^{k_1} x_{\alpha_2}^{k_2} \cdots x_{\alpha_n}^{k_n}$ is linearly independent; equivalently for any $k \in \mathbb{N}$, any nonzero polynomial P in k variables without a constant term and any distinct $x_{\alpha_1}, \ldots, x_{\alpha_k} \in X$, we have that $P\left(x_{\alpha_1}, \ldots, x_{\alpha_k}\right)$ is nonzero.*

2.2 Borel–Cantelli lemma and various kinds of convergence

Let X be a random variable on a probability space $(\Omega, \mathcal{F}, P)$. By EX and $VarX$ we denote the expected value of X and variation of X, respectively.

Lemma 3 [Borel–Cantelli] *Let (A_n) be a sequence of events in the probability space $(\Omega, \mathcal{F}, P)$. Let $\limsup A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$ be a set of those $\omega \in \Omega$ which belong to infinitely many A_n's. Then*

(a) *if $\sum_{n=1}^{\infty} P(A_n) < \infty$, then $P(\limsup A_n) = 0$;*
(b) *if $\sum_{n=1}^{\infty} P(A_n) = \infty$ and $A_1, A_2, \ldots$ are independent, then $P(\limsup A_n) = 1$.*

Let (X_n) be a sequence of random variables on $(\Omega, \mathcal{F}, P)$. Then

(p-w) $X_n \to X$ means that (X_n) converges point-wise to X on Ω,

(a.s.) $X_n \xrightarrow{as} X$ means that (X_n) converges to X almost surely,
(prob) $X_n \xrightarrow{P} X$ means that (X_n) converges to X in probability,
(dist) $X_n \xrightarrow{d} X$ means that (X_n) converges to X in distribution.

The following implications are well-known: (p-w) $\Longrightarrow$ (a.s.) $\Longrightarrow$ (prob) $\Longrightarrow$ (dist)

2.3 Laws of large numbers

Let (X_n) be a sequence of random variables defined on the probability space $(\Omega, \mathcal{F}, P)$. Let $S_n = X_1 + \cdots + X_n$. The sequence (X_n) satisfies the strong law of large numbers (or SLLN) if

$$\frac{1}{n} S_n - \frac{1}{n} E S_n \xrightarrow{as} 0.$$

The sequence (X_n) satisfies the weak law of large numbers (or WLLN) if

$$\frac{1}{n} S_n - \frac{1}{n} E S_n \xrightarrow{P} 0$$

that is

$$\lim_{n\to\infty} P\left(\left|\frac{1}{n} S_n - \frac{1}{n} E S_n\right| \geq \varepsilon\right) = 0$$

for every positive ε.

Theorem 4 [Markov] *Suppose that (X_n) is a sequence of random variables such that*

$$\lim_{n\to\infty} \frac{1}{n^2} Var(X_1 + \cdots + X_n) = 0. \tag{1}$$

Then (X_n) satisfies the WLLN.

We will refer to condition (1) as the Markov condition.

Theorem 5 [Kolmogorov] *Let (X_n) be a sequence of independent random variables. If*

$$\sum_{n=1}^{\infty} \frac{Var X_n}{n^2} < \infty, \tag{2}$$

then (X_n) satisfies SLLN.

We will refer to condition (2) as the Kolmogorov condition.

Note that the Kolmogorov condition implies the Markov condition for independent random variables. Indeed, suppose that $\sum_{n=1}^{\infty} \frac{Var X_n}{n^2} < \infty$. Let $\varepsilon > 0$. There is N such that $\sum_{n=N+1}^{\infty} \frac{Var X_n}{n^2} < \varepsilon/2$. Let $n_0 > N$ be such that

$$\frac{Var X_1 + \cdots + Var X_N}{m^2} < \frac{\varepsilon}{2}$$

for every $m > n_0$. Then

$$\begin{aligned}\frac{Var X_1 + \cdots + Var X_m}{m^2} &= \frac{Var X_1 + \cdots + Var X_N}{m^2} + \frac{Var X_{N+1} + \cdots + Var X_m}{m^2} \\ &< \frac{\varepsilon}{2} + \frac{Var X_{N+1}}{(N+1)^2} + \cdots + \frac{Var X_m}{m^2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

Thus $\lim_{n\to\infty} \frac{1}{n^2} Var(X_1 + \cdots + X_n) = 0$.

Note that the proved implication is a statement about real sequences and has no probabilistic nature. The following diagram briefly summarizes relations between considered notions.

$$\begin{tikzcd}
\text{WLLN} & \text{SLLN} \arrow[l] \\
\text{Markov} \arrow[u] & \text{Kolmogorov} \arrow[l] \arrow[u]
\end{tikzcd}$$

None of the above arrows, or implications, can be reversed.

2.4 Almost disjoint families

A family $\mathcal{A}$ of infinite subsets of $\mathbb{N}$ is called almost disjoint if any two distinct members of $\mathcal{A}$ have a finite intersection. It is well-known that there is an almost disjoint family of cardinality continuum. A family $\mathcal{I}$ of subsets of $\mathbb{N}$ is called ideal, provided that it is closed under taking subsets and finite unions, and does not contain $\mathbb{N}$. A particular example, which will be of our interest, is a so-called summable ideal. Let (a_n) be a sequence of non-negative reals with $\sum_{n=1}^{\infty} a_n = \infty$. Then $\mathcal{I}_{(a_n)} := \{S \subseteq \mathbb{N} : \sum_{n\in S} a_n < \infty\}$ is called a summable ideal. A family $\mathcal{A}$ is called $\mathcal{I}_{(a_n)}$-almost disjoint if it is almost disjoint and $A \notin \mathcal{I}_{(a_n)}$ for every $A \in \mathcal{A}$.

Lemma 6 *Suppose that a series of positive real numbers $\sum_{n=1}^{\infty} a_n$ is divergent. Then there exists an $\mathcal{I}_{(a_n)}$-almost disjoint family of cardinality continuum.*

Proof Let $\{A_\alpha : \alpha < \mathfrak{c}\}$ be an almost disjoint family. Since the series $\sum_{n=1}^{\infty} a_n$ is divergent, we find indices $n_1 < n_2 < \ldots$ such that

$$\sum_{k=n_i}^{n_{i+1}-1} a_k \geq 1.$$

Sets of the form $B_\alpha := \bigcup_{i\in A_\alpha} \{n_i, n_i + 1, \ldots, n_{i+1} - 1\}$ constitute the desired family. $\square$

3 Atomless probability measure

We say that a measure μ on $(\Omega, \mathcal{F})$ is atomless if for any μ-measurable set A with $\mu(A) > 0$ there is a μ-measurable set B with $0 < \mu(B) < \mu(A)$. All considered subsets of Ω are in $\mathcal{F}$.

Lemma 7 *Let μ be an atomless probability measure and let A be μ-measurable. Then for any $0 < t < \mu(A)$ there is $B \subseteq A$ with $\mu(B) = t$.*

Proof Firstly let us observe that for any $0 < t < \mu(A)$ there is $B \subseteq A$ with $\mu(B) \leq t$. Since μ is atomless, there is $C \subseteq A$ with $0 < \mu(C) < \mu(A)$. Let $B_0 = C$ if $\mu(C) \leq \mu(A \setminus C)$, and $B_0 = C \setminus A$ otherwise. Then $\mu(B_0) \leq \frac{1}{2}\mu(A)$. Proceeding inductively we find a sequence (B_n) with $B_{i+1} \subseteq B_i$ and $0 < \mu(B_i) \leq \frac{1}{2^{i+1}}\mu(A)$. Thus there is i with $0 < \mu(B_i) \leq t$.

Now, we are ready to prove the assertion. By a transfinite induction we define $B_\alpha \subseteq A$ such that $B_{\alpha+1} \subseteq A \setminus \bigcup_{\beta\leq\alpha} B_\beta$ and $0 < \mu(B_{\alpha+1}) \leq t - \sum_{\beta\leq\alpha} \mu(B_\beta)$. Clearly after countably many steps, say after $\eta < \omega_1$ many, the construction stops. Then $\sum_{\beta\leq\eta} \mu(B_\beta) = t$ and B_α's are pairwise disjoint. Therefore putting $B = \bigcup_{\beta\leq\eta} B_\beta$ we have $\mu(B) = t$. $\square$

Lemma 8 *Let μ be an atomless probability measure. Let $n_i \geq 2$ for $i \in \mathbb{N}$. Assume that $(p^i_j)_{i\in\mathbb{N}, j\leq n_i}$ are such that*

(1) $0 < p^i_j < 1$
(2) $\sum_{j\leq n_i} p^i_j = 1$.

Then there exists $(A^i_j)_{i\in\mathbb{N}, j\leq n_i}$ such that

(i) $\mu(A^i_j) = p^i_j$
(ii) $A^i_1, A^i_2, \ldots, A^i_{n_i}$ *are pairwise disjoint*
(iii) *The σ-fields $\mathcal{F}_i := \sigma(\{A^i_j : j \leq n_i\})$ are independent.*

Proof Using Lemma 7 we find pairwise disjoint $A^1_1, A^1_2, \ldots, A^1_{n_1}$ with $\mu(A^1_j) = p^1_j$. Now, assume that for every $i \leq k$ we have already defined A^i_j's fulfilling (i)–(iii). Let $j \leq n_{k+1}$. Fix $j_1 \leq n_1, j_2 \leq n_2, \ldots, j_k \leq n_k$. By the inductive assumption $A^1_{j_1} \cap A^2_{j_2} \cap \cdots \cap A^k_{j_k} = p^1_{j_1} p^2_{j_2} \cdots p^k_{j_k}$. Using Lemma 7 again we find $C_{j_1 j_2 \ldots j_k j} \subseteq A^1_{j_1} \cap A^2_{j_2} \cap \cdots \cap A^k_{j_k}$ such that $\mu(C_{j_1 j_2 \ldots j_k j}) = p^1_{j_1} p^2_{j_2} \cdots p^k_{j_k} p^{k+1}_j$. Put

$$A^{k+1}_j := \bigcup_{j_1\leq n_1} \bigcup_{j_2\leq n_2} \cdots \bigcup_{j_k\leq n_k} C_{j_1 j_2 \ldots j_k j}.$$

Then $(A^i_j)_{i\leq k+1, j\leq n_i}$ fulfills (i)–(iii). □

Lemma 9 *Let μ be an atomless probability measure. Assume that $\{B^1_k : k \leq K_1\}, \ldots, \{B^t_k : k \leq K_t\}$ and $\{A_i : i \leq I\}$ are partitions of Ω into sets of positive μ-measure and such that $\sigma(\{B^1_k : k \leq K_1\}), \ldots, \sigma(\{B^t_k : k \leq K_t\})$ and $\sigma(\{A_i : i \leq I\})$ are independent σ-fields. Let $\{p^\ell_i : \ell = 1, 2, \ldots, n_i\}$, $i \leq I$, be positive real numbers with $p^1_i + p^2_i + \cdots + p^{n_i}_i = \mu(A_i)$. Then there exist $A^\ell_i \subseteq \Omega$, $i \leq I$, $\ell = 1, 2, \ldots, n_i$ such that*

(i) $A^1_i, A^2_i, \ldots, A^{n_i}_i$ *are pairwise disjoint and* $\bigcup_{\ell=1}^{n_i} A^\ell_i = A_i$;
(ii) $\mu(A^\ell_i) = p^\ell_i$ *for every* $i \leq I$ *and* $\ell \leq n_i$;
(iii) *σ-fields $\sigma(\{B^1_k : k \leq K_1\}), \ldots, \sigma(\{B^t_k : k \leq K_t\})$ and $\sigma(\{A^\ell_i : i \leq I, \ell = 1, 2, \ldots, n_i\})$ are independent.*

Proof Fix $i \leq I$. By the independence assumption

$$\mu(A_i \cap B_{k_1} \cap \cdots \cap B_{k_t}) = \mu(A_i)\mu(B_{k_1}) \cdots \mu(B_{k_t})$$

for every $\bar{k} = (k_1, \ldots, k_t) \in \{1, \ldots, K_1\} \times \cdots \times \{1, \ldots, K_t\}$. Using Lemma 7 we find $C^1_{i,\bar{k}}, C^2_{i,\bar{k}}, \ldots, C^{n_i}_{i,\bar{k}}$ such that

- $C^1_{i,\bar{k}}, C^2_{i,\bar{k}}, \ldots, C^{n_i}_{i,\bar{k}}$ are pairwise disjoint and $\bigcup_{\ell=1}^{n_i} C^\ell_{i,k} = A_i \cap B_k$;
- $\mu(C^\ell_{i,\bar{k}}) = p^\ell_i \mu(B_{k_1}) \cdots \mu(B_{k_t})$.

Having this we define A^ℓ_i as a union $\bigcup_{\bar{k}} C^\ell_{i,\bar{k}}$. Note that

$$\mu(A^\ell_i) = \mu(\bigcup_{\bar{k}} C^\ell_{i,\bar{k}}) = \sum_{\bar{k}} \mu(C_{i,\bar{k}}) = \sum_{\bar{k}} p^\ell_i \mu(B_{k_1}) \cdots \mu(B_{k_t}) = p^\ell_i.$$

Clearly conditions (i)–(iii) are fulfilled. □

Let us define partitions $\mathcal{X}_k$ of $\mathbb{R}$, $k = 0, 1, \ldots$ as follows: $\mathcal{X}_0 = \{(-\infty, -1], (-1, 0], (0, 1], (1, \infty)\}$ and

$$\mathcal{X}_k = \left\{ \left(\frac{n}{2^k}, \frac{n+1}{2^k} \right] : -k2^k \leq n < k2^k \right\} \cup \{(-\infty, -k], (k, \infty)\}.$$

Then $\mathcal{X}_{k+1}$ arises from $\mathcal{X}_k$ by

- dividing each $(\frac{n}{2^k}, \frac{n+1}{2^k}]$ into two dyadic sub-intervals $(\frac{2n}{2^{k+1}}, \frac{2n+1}{2^{k+1}}]$ and $(\frac{2n+1}{2^{k+1}}, \frac{2n+2}{2^{k+1}}]$;
- adding $2 \cdot 2^{k+1}$ dyadic intervals $(\frac{n}{2^{k+1}}, \frac{n+1}{2^{k+1}}]$ for $-(k+1)2^{k+1} \leq n < -k2^{k+1}$ and $k2^{k+1} \leq n < (k+1)2^{k+1}$;
- replacing $(-\infty, -k]$ and (k, ∞) by $(-\infty, -(k+1)]$ and $(k+1, \infty)$.

Note that $\sigma(\mathcal{X}_0) \subseteq \sigma(\mathcal{X}_1) \subseteq \ldots$ If X is a random variable, then by μ_X we denote its distribution.

Theorem 10 *Let P be an atomless probability measure on $(\Omega, \mathcal{F})$. Let $\mu_0, \mu_1, \ldots$ be a sequence of distributions on $\mathbb{R}$. Then there are independent random variables $X_0, X_1, \ldots$ defined on $(\Omega, \mathcal{F})$ with $\mu_{X_i} = \mu_i$ for every $i \in \mathbb{N}$.*

Proof We will inductively define partitions $\mathcal{Y}_\ell^k$ of $\mathbb{N}$ and random variables X_ℓ^k such that

(i) $\mathcal{Y}_\ell^k = \{A_{I,\ell} : I \in \mathcal{X}_k\}$, in other words partition $\mathcal{Y}_\ell^k$ is indexed by elements of partition $\mathcal{X}_k$;
(ii) $\mu_\ell(I) = P(A_{I,\ell})$ for every $I \in \mathcal{X}_k$;
(iii) If $I_1, \ldots, I_p \in \mathcal{X}_{k+1}$ are pairwise disjoint, $I = I_1 \cup \cdots \cup I_p$ and $I \in \mathcal{X}_k$, then $A_{I,\ell} = A_{I_1,\ell} \cup \cdots \cup A_{I_p,\ell}$;
(iv) Let $\omega \in \Omega$. If $\omega \in A_{I,\ell}$ where $I = (-\infty, -k]$, then $X_\ell^k(\omega) = -k$; otherwise $X_\ell^k(\omega) = \inf I$ for $\omega \in A_{I,\ell}$;
(v) For every $\ell_1 < \ell_2 < \cdots < \ell_t$ and $k_1, k_2, \ldots, k_t$, σ-fields $\sigma(\mathcal{Y}_{\ell_1}^{k_1}), \sigma(\mathcal{Y}_{\ell_2}^{k_2}), \ldots, \sigma(\mathcal{Y}_{\ell_t}^{k_t})$ are independent.

Suppose that we have defined $\mathcal{Y}_\ell^k$ and X_ℓ^k for every $k, \ell \in \mathbb{N}$. Note that $(X_\ell^k)_{k=1}^\infty$ converges in measure to some X_ℓ with $\mu_{X_\ell} = \mu_\ell$. To show that $X_1, X_2, \ldots$ are independent, it is enough to prove that

$$X_1^{-1}(I_1), X_2^{-1}(I_2), \ldots, X_\ell^{-1}(I_\ell)$$

are independent for every (left-open and right-closed) dyadic interval. Note that $I_i \in \sigma(\mathcal{X}_{k_i})$ for some k_i. Then $X_i^{-1}(I_i) = (X_i^{k_i})^{-1}(I_i) = A_{I_i,i} \in \mathcal{Y}_i^{k_i}$. So by (v) we obtain that $X_1, X_2, \ldots$ are independent.

We will construct $\mathcal{Y}_\ell^k$ and X_ℓ^k fulfilling (i)–(v) by induction. In first step we construct $\mathcal{Y}_0^0$ and X_0^0 as follows. Using Lemma 8 we find a partition $\mathcal{Y}_0^0 = \{A_{I,0} : I \in \mathcal{X}_0\}$ of Ω with $P(A_{I,0}) = \mu_0(I)$. This gives us (i) and (ii). Put

$$X_0^0(\omega) = \begin{cases} -1 & \text{if} \quad \omega \in A_{(-\infty,-1],0} \cup A_{(-1,0],0} \\ 0 & \text{if} \quad \omega \in A_{(0,1],0} \\ 1 & \text{if} \quad \omega \in A_{(1,\infty),0}. \end{cases}$$

This just (iv). Conditions (iii) and (v) are also satisfied.

Now, assume that we have already constructed $\mathcal{Y}_\ell^k$ and X_ℓ^k for $k, \ell \leq n$. In one step we construct $\mathcal{Y}_\ell^{n+1}$, X_ℓ^{n+1} for $\ell = 0, 1, \ldots, n$, and $\mathcal{Y}_{n+1}^k$, X_{n+1}^k for $k = 0, 1, \ldots, n+1$. Using $2n + 3$ times Lemma 9 we define $\mathcal{Y}_\ell^{n+1}$ and $\mathcal{Y}_{n+1}^k$ fulfilling (i), (ii), (iii) and (v). Then we define X_ℓ^{n+1} and X_{n+1}^k using the formula given in (iv). $\square$

4 Various kinds of convergence

Let $(\Omega, \mathcal{F}, P)$ be a measurable space. By $L_0(\Omega)$ we denote the space of all random variables. The following should be compared with [1, Theorem 7.1] where the Authors proved maximal dense lineability of random variables sequences defined on [0, 1] which tend to 0 in measure, but not almost everywhere. Note that strong algebrability and dense lineability are incomparable notions—none of them implies the other.

Theorem 11 *Let P be an atomless probability measure on $(\Omega, \mathcal{F})$. There exists a $\mathfrak{c}$-generated free algebra $\mathcal{A} \subseteq L_0(\Omega)^{\mathbb{N}}$ such that any $(X_n) \in \mathcal{A} \setminus \{\mathbf{0}\}$ is a sequence of independent random variables such that*

(i) $X_n \xrightarrow{P} 0$
(ii) $\limsup_{n\to\infty} |X_n(\omega)| = \infty$ *for every* $\omega \in \Omega$.
(iii) $\lim_{n\to\infty} E|X_n| = \infty$.

Proof By Lemma 8 there are $(A_j^n)_{n\in\mathbb{N}, j\le n+1}$ such that $P(A_j^n) = \frac{1}{n+1}$ and sigma fields $\sigma(\{A_j^n : j \le n+1\})$ are independent. Let $B_n = A_1^n$ for $n \in \mathbb{N}$. Since B_n are independent and $\sum_{n=1}^{\infty} \frac{1}{n+1} = \infty$, by Borel-Cantelli lemma we obtain that $P(\limsup B_n) = 1$. Then $C := \Omega \setminus \limsup B_n$ has probability 0. Put $I_n := C \cup B_n$. Then $P(I_n) = P(B_n) = \frac{1}{n+1}$ and every $\omega \in \Omega$ belongs to infinitely many I_n's.

Let $U \subseteq \mathbb{R}$ be a set of cardinality $\mathfrak{c}$ linearly independent over $\mathbb{Q}$. For any $\alpha \in U$ and $n \in \mathbb{N}$ we define a random variable $X_n^{(\alpha)} := e^{\alpha n}\mathbf{1}_{I_n}$ where $\mathbf{1}_{I_n}$ is a characteristic function of I_n. The sequence $(X_n^{(\alpha)})$ consists of independent random variables.

We will show that any non-trivial algebraic combination of elements from $\{(X_n^{(\alpha)}) : \alpha \in U\}$ is either a null sequence or it is a sequence of independent random variables fulfilling (i) and (ii). Let $(k_{il} : i \le m, l \le j)$ be a matrix of non-negative integers with non-zero and distinct rows, and assume that $c_1, \ldots, c_m \in \mathbb{R}$ do not vanish simultaneously. Consider the following algebraic combination of $X_n^{(\alpha_1)}, X_n^{(\alpha_2)}, \ldots, X_n^{(\alpha_j)}$

$$Y_n = \sum_{i=1}^{m} c_i e^{(k_{i1}\alpha_1 + \ldots + k_{ij}\alpha_j)n} \mathbf{1}_{I_n}. \tag{3}$$

Since the set U is linearly independent, the numbers $k_{11}\alpha_1 + \ldots + k_{1j}\alpha_j, \ldots, k_{m1}\alpha_1 + \ldots + k_{mj}\alpha_j$ are distinct. To simplify the notation, put $\gamma_i := k_{i1}\alpha_1 + \ldots + k_{ij}\alpha_j$ for every $i = 1, \ldots, m$. We may assume that $\gamma_1 > \gamma_2 > \cdots > \gamma_m$. Then for any $n \in \mathbb{N}$ and $\omega \in I_n$,

$$Y_n(\omega) = \sum_{i=1}^{m} c_i e^{\gamma_i n} = e^{\gamma_1 n}\left(c_1 + \sum_{i=2}^{m} c_i e^{(\gamma_i - \gamma_1)n}\right).$$

Thus $Y_n(\omega) \approx c_1 e^{\gamma_1 n}$ for every $\omega \in I_n$ for large enough n. By (3), we obtain (Y_n) is a sequence of independent random variables. Since every $\omega \in \Omega$ belongs to infinitely many I_n's and $\gamma_i > 1$, then

$$\limsup_{n\to\infty} |Y_n(\omega)| = \infty.$$

Again by (3), we obtain $P(Y_n \neq 0) \le \frac{1}{n+1}$, and therefore $Y_n \xrightarrow{P} 0$. Since $E(Y_n) = \frac{e^{\gamma_1 n}}{n}$, then $\lim_{n\to\infty} E(Y_n) = \infty$ $\square$

Theorem 12 *Let* $(\Omega, \mathcal{F}, P)$ *be an atomless probability space. Then the set*

$$\left\{(X_n) \in L_0(\Omega)^{\mathbb{N}} : X_n \xrightarrow{P} 0 \text{ but } \frac{X_1 + \cdots + X_n}{n} \text{ does not converge in probability to } 0\right\}$$

is strongly $\mathfrak{c}$*-algebrable.*

Proof Let (A_n) be a sequence of independent events defined on $(\Omega, \mathcal{F}, P)$ such that $P(A_n) = \frac{1}{n+1}$. Let $U \subseteq (1, 2)$ be linearly independent over $\mathbb{Q}$. Fix $\alpha \in U$. Put $X_n^{(\alpha)} = 2^{\alpha n}\mathbf{1}_{A_n}$. Let $(k_{il} : i \leq m, l \leq j)$ be a matrix of non-negative integers with non-zero and distinct rows, and assume that $c_1, \ldots, c_m \in \mathbb{R}$ do not vanish simultaneously. Consider the following algebraic combination of $X_n^{(\alpha_1)}, X_n^{(\alpha_2)}, \ldots, X_n^{(\alpha_j)}$

$$Y_n = \sum_{i=1}^{m} c_i 2^{(k_{i1}\alpha_1 + \ldots + k_{ij}\alpha_j)n}\mathbf{1}_{A_n}. \tag{4}$$

Since the set U is linearly independent, the numbers $k_{11}\alpha_1 + \ldots + k_{1j}\alpha_j, \ldots, k_{m1}\alpha_1 + \ldots + k_{mj}\alpha_j$ are distinct. As in the proof of Theorem 11 we put $\gamma_i := k_{i1}\alpha_1 + \ldots + k_{ij}\alpha_j$ for every $i = 1, \ldots, m$. We may assume that $\gamma_1 > \gamma_2 > \cdots > \gamma_m$. Then for any $n \in \mathbb{N}$ and $\omega \in A_n$,

$$Y_n(\omega) = \sum_{i=1}^{m} c_i 2^{\gamma_i n} = 2^{\gamma_1 n}\left(c_1 + \sum_{i=2}^{m} c_i 2^{(\gamma_i - \gamma_1)n}\right).$$

There is n_0 such that for $n \geq n_0$,

(a) $|Y_n| \geq \frac{1}{2}|c_1|2^{\gamma_1 n} > \frac{1}{2}|c_1|2^n$ on A_n and $Y_n = 0$ on the complement A_n^c of A_n;
(b) Y_n has the same sign as c_1, and therefore $|Y_{n_0} + \cdots + Y_n| = |Y_{n_0}| + \cdots + |Y_n|$

Let $k_0 \geq n_0$ be such that $P(|Y_1| + \cdots + |Y_{n_0-1}| < \frac{1}{4}|c_1|2^{k_0}) = 1$. The aim of the following reasoning is to show that $P\left(\frac{|Y_1|+\cdots+|Y_{2^k}|}{2^k} < \frac{|c_1|}{4}\right)$ tends to zero, and consequently $\frac{|Y_1|+\cdots+|Y_n|}{n}$ does not converge in probability to zero.

For $k > k_0$ we have

$$\begin{aligned}
&P\left(\frac{|Y_1 + \cdots + Y_{2^k}|}{2^k} < \frac{1}{4}|c_1|\right)\\
&= P\left(|Y_1 + \cdots + Y_{2^k}| < \frac{1}{4}|c_1|2^k\right)\\
&\leq P\left(|Y_{n_0} + \cdots + Y_{2^k}| - |Y_1 + \cdots + Y_{n_0-1}| < \frac{1}{4}|c_1|2^k\right)\\
&= P\left(|Y_{n_0} + \cdots + Y_{2^k}| < \frac{1}{4}|c_1|2^k + |Y_1 + \cdots + Y_{n_0-1}|\right)\\
&= P\left(|Y_{n_0}| + \cdots + |Y_{2^k}| < \frac{1}{4}|c_1|2^k + |Y_1 + \cdots + Y_{n_0-1}|\right)\\
&\leq P\left(|Y_{n_0}| + \cdots + |Y_{2^k}| < \frac{1}{4}|c_1|2^k + |Y_1| + \cdots + |Y_{n_0-1}|\right)\\
&\leq P\left(|Y_{n_0}| + \cdots + |Y_{2^k}| < \frac{1}{2}|c_1|2^k\right).
\end{aligned}$$

If $\omega \in A_n$ for some $n \geq k$, then $Y_n(\omega) \geq \frac{1}{2}|c_1| \cdot 2^n \geq \frac{1}{2}|c_1| \cdot 2^k$. Hence $|Y_{n_0}| + \cdots + |Y_{2^k}| \geq |Y_n| \geq \frac{1}{2}|c_1| \cdot 2^k$. Thus

$$P\left(|Y_{n_0}| + \cdots + |Y_{2^k}| < \frac{1}{2}|c_1|2^k\right) \leq P(A_k^c \cap \cdots \cap A_{2^k}^c) = P(A_k^c) \cdots P(A_{2^k}^c)$$
$$= \frac{2^k}{2^k+1} \cdot \frac{2^k-1}{2^k} \cdots \frac{k}{k+1} = \frac{k}{2^k+1} \to 0.$$

Thus $\frac{Y_1+\cdots+Y_n}{n}$ does not converge in probability to 0. Since $P(Y_n = 0) = \frac{n}{n+1}$, then $Y_n \xrightarrow{P} 0$. □

Theorem 13 *Let $(\Omega, \mathcal{F}, P)$ be an atomless probability space. Let $\mathbb{U}$ be a set of all sequences $(X_n) \in L_0(\Omega)^{\mathbb{N}}$ such that*

(i) *(X_n) is independent*
(ii) $X_n \xrightarrow{P} 0$
(iii) *there is a constant $C > 0$ with $|X_n| < C$ for every $n \in \mathbb{N}$*
(iv) *$X_n \nrightarrow 0$ almost surely.*

Then $\mathbb{U}$ is $\mathfrak{c}$-lineable.

Proof Let (A_n) be a sequence of independent events in Ω with $P(A_n) = \frac{1}{n}$. Let $\mathcal{A} = \{B_\alpha : \alpha < \mathfrak{c}\}$ be an $\mathcal{I}_{(1/n)}$-almost disjoint family. We define $(X_n^{(\alpha)})$ as follows:

$$X_n^{(\alpha)} = \begin{cases} \mathbf{1}_{A_n} & \text{if} \quad n \in B_\alpha \\ 0 & \text{if} \quad n \notin B_\alpha. \end{cases}$$

Let $c_1, \ldots, c_m \in \mathbb{R} \setminus \{0\}$ and $\alpha_1 < \cdots < \alpha_m < \mathfrak{c}$. Consider $Y_n = c_1 X_n^{(\alpha_1)} + \cdots + c_m X_n^{(\alpha_m)}$. Since $B := B_{\alpha_1} \setminus (B_{\alpha_2} \cup \cdots \cup B_{\alpha_m})$ is infinite and $c_1 \neq 0$, by Borel–Cantelli lemma we obtain that $P(\limsup_{n \in B} A_n) = 1$, and therefore $Y_n \nrightarrow 0$ a.s. for $n \in B$, which implies (iv). Clearly (Y_n) is independent and $Y_n \xrightarrow{P} 0$; thus (i) and (ii) holds. Note also that $|Y_n| \leq |c_1| + \cdots + |c_m|$ for every $n \in \mathbb{N}$ which gives (iii). Therefore $(Y_n) \in \mathbb{U}$, which shows that $\mathbb{U}$ is $\mathfrak{c}$-lineable. □

It is well known that if $(\Omega, \mathcal{F}, P)$ is atomic probability space, then for any (X_n) defined there, $X_n \xrightarrow{P} 0$ is equivalent to $X_n \xrightarrow{a.s.} 0$. This shows that Theorems 11, 12 and 13 do not hold for atomic spaces.

Problem 14 *Is the set $\mathbb{U}$ defined in Theorem 13 strongly $\mathfrak{c}$-algebrable?*

5 Laws of large numbers

Theorem 15 *Let $(\Omega, \mathcal{F}, P)$ be an atomless probability space. Then the set*

$\{(X_n) \in L_0(\Omega)^{\mathbb{N}} : X_n$ *are independent and* (X_n) *satisfies Markov condition but not SLLN*$\}$

is $\mathfrak{c}$-lineable.

Proof Let $a_n = \frac{1}{(n+1)\log(n+1)}$. Note that $\sum_{n=1}^{\infty} a_n$ is divergent, so there is an almost disjoint family $\{B_\alpha \subseteq \mathbb{N} : \alpha < \mathfrak{c}\}$ such that $B_\alpha \notin \mathcal{I}_{(a_n)}$.

Using Lemma 8 for

$$p_1^n = p_2^n = \frac{1}{2(n+1)\log(n+1)} \quad \text{and} \quad p_3^n = 1 - \frac{1}{(n+1)\log(n+1)}$$

we obtain sets $\{A_i^n : n \in \mathbb{N}, i = 1, 2, 3\}$ such that $P(A_i^n) = p_i^n$ and σ-fields $\sigma(\{A_i^n : i = 1, 2, 3\})$ are independent. For $n \in \mathbb{N}$ let Z_n be a random variable given by

$$Z_n(\omega) = \begin{cases} -n & \text{if} \quad \omega \in A_1^n \\ n & \text{if} \quad \omega \in A_2^n \\ 0 & \text{if} \quad \omega \in A_3^n. \end{cases}$$

(Note that Z_n can be defined shortly as $-n\mathbf{1}_{A_1^n} + n\mathbf{1}_{A_2^n}$.) Then Z_n are independent such that

$$P(Z_n = n) = P(Z_n = -n) = \frac{1}{2(n+1)\log(n+1)} \quad \text{and}$$
$$P(Z_n = 0) = 1 - \frac{1}{(n+1)\log(n+1)}.$$

Now, for $\alpha < \mathfrak{c}$ and $n \in \mathbb{N}$ we define: $X_n^{(\alpha)} = Z_n$ for indexes n from B_α, and $X_n^{(\alpha)} = 0$ otherwise. Consider a linear subspace $\mathcal{V}$ of $L_0(\Omega)^{\mathbb{N}}$ spanned by $\{(X_n^{(\alpha)}) : \alpha < \mathfrak{c}\}$. Let $(X_n) \in \mathcal{V}$ be a non-null sequence. Then there are $c_1, \ldots, c_m \in \mathbb{R}\setminus\{0\}$ and $\alpha_1 < \cdots < \alpha_m < \mathfrak{c}$ such that $X_n = \sum_{i=1}^m c_i X_n^{(\alpha_i)}$ for every $n \geq 1$. Note that X_n are independent.

Observe that $X_n = Z_n \cdot \sum_{i=1}^m c_i \mathbf{1}_{B_{\alpha_i}}(n)$. Thus

$$Var X_n = \left| \sum_{i=1}^m c_i \mathbf{1}_{B_{\alpha_i}}(n) \right|^2 Var Z_n \leq \left(\sum_{i=1}^m |c_i| \right)^2 Var Z_n.$$

Since X_n are independent, Z_n are independent and (Z_n) satisfies Markov condition (see [10]), that is $\frac{1}{n^2} Var(Z_1 + \cdots + Z_n) \to 0$,

$$0 \leq \frac{1}{n^2} Var(X_1 + \cdots + X_n) = \frac{1}{n^2} \sum_{j=1}^n Var X_j \leq \frac{1}{n^2} \sum_{j=1}^n \left(\sum_{i=1}^m |c_i| \right)^2$$
$$Var Z_j = \frac{1}{n^2} \left(\sum_{i=1}^m |c_i| \right)^2 Var(Z_1 + \cdots + Z_n) \to 0.$$

Thus (X_n) satisfies Markov condition, and consequently the weak law of large numbers.

Put $B := B_{\alpha_1} \setminus \bigcup_{i=2}^m B_{\alpha_i}$. Then $B \notin \mathcal{I}_{(a_n)}$. Note that $n \in B$ implies that $X_n = c_1 X_n^{(\alpha_1)} = c_1 Z_n$, and therefore

$$P(|X_n| \geq |c_1| n) = P(|Z_n| \geq n) = \frac{1}{(n+1)\log(n+1)} = a_n.$$

Since $B \notin \mathcal{I}_{(a_n)}$, then

$$\sum_{n \in B} P(|X_n| \geq |c_1| n) = \sum_{n \in B} a_n = \infty.$$

By Borel–Cantelli lemma $P(\limsup\{|X_n| \geq |c_1| n\}) = 1$. Since $\frac{X_n}{n} = \frac{S_n}{n} - \frac{S_{n-1}}{n}$, then $\frac{S_n}{n} \nrightarrow 0$, where $S_n = X_1 + \cdots + X_n$. On the other hand $EX_n = 0$, which implies $\frac{1}{n} ES_n = 0$. That means that (X_n) does not satisfy strong law of large numbers. □

Theorem 16 *Let P be an atomless probability measure on a measure space $(\Omega, \mathcal{F})$. The set of all sequences $(X_n) \in L_0(\Omega)^{\mathbb{N}}$ of random variables satisfying the weak law of large numbers but neither the strong law of large numbers nor Markov condition is $\mathfrak{c}$-lineable.*

Proof By Theorem 10 there is a sequence (Z_n) of independent random variables defined on $(\Omega, \mathcal{F}, P)$ whose distribution functions are absolutely continuous and their densities f_n are given by the formula

$$f_n(x) = \frac{1}{\sqrt{2}\sigma_n} \exp\left(-\frac{\sqrt{2}|x|}{\sigma_n}\right), \text{ where } \sigma_n = \frac{2n^2}{(\log n)^2}.$$

Then $EZ_n = 0$ and $VarZ_n = \sigma_n^2$. Let $\{B_\alpha \subseteq \mathbb{N} : \alpha < \mathfrak{c}\}$ be an almost disjoint family. We define

$$X_n^{(\alpha)}(\omega) = \begin{cases} Z_n(\omega) & \text{if } \ n \in B_\alpha \\ 0 & \text{if } \ n \notin B_\alpha. \end{cases}$$

Let (X_n) be a non-zero sequence contained in a vector subspace of $(L_2(\Omega))^{\mathbb{N}}$ spanned by $\{(X_n^{(\alpha)}) : \alpha < \mathfrak{c}\}$. Then $X_n = \sum_{i=1}^m c_i X_n^{(\alpha_i)}$ for some $m \in \mathbb{N}$, $c_1, \ldots, c_m \in \mathbb{R} \setminus \{0\}$ and $\alpha_1 < \cdots < \alpha_m < \mathfrak{c}$.

Now, we will check that (X_n) does not satisfy SLLN. Let $n \in B_{\alpha_1} \setminus \bigcup_{i>1} B_{\alpha_i}$. Then $X_n = c_1 X_n^{(\alpha_1)} = c_1 Z_n$ and

$$P(|X_n| \geq |c_1|n) = P(|Z_n| \geq n) = \exp\left(-\frac{\sqrt{2}(\log n)^2}{2n}\right).$$

The set $B_{\alpha_1} \setminus \bigcup_{i>1} B_{\alpha_i}$ is infinite and $\exp(-\frac{\sqrt{2}(\log n)^2}{2n}) \to 1$. Therefore the series

$$\sum_{n \in B_{\alpha_1} \setminus \bigcup_{i>1} B_{\alpha_i}} P(|X_n| \geq |c_1|n) \text{ is divergent.}$$

Since X_n are independent, then Borel-Cantelli lemma implies that $P(\limsup\{|X_n| \geq |c_1|n\}) = 1$. Consequently (X_n) does not satisfy SLLN.

Now we show that the Markov condition does not hold for (X_n). Since X_n are independent, then

$$\frac{1}{n^2} Var(X_1 + \cdots + X_n) = \frac{1}{n^2}(VarX_1 + \cdots + VarX_n) \geq \frac{1}{n^2} VarX_n$$

For $n \in B_{\alpha_1} \setminus \bigcup_{i>1} B_{\alpha_i}$

$$\frac{1}{n^2} VarX_n = \frac{c_1^2}{n^2} VarZ_n = \frac{c_1^2}{n^2}\sigma_n^2 = \frac{4c_1 n^2}{(\log n)^4}.$$

Since $\frac{4c_1 n^2}{(\log n)^4} \to \infty$ and $B_{\alpha_1} \setminus \bigcup_{i>1} B_{\alpha_i}$ is infinite, (X_n) does not fulfill the Markov condition.

Stoyanov proved in [10, Sect. 15.4] using Feller Theorem that (Z_n) fulfills the weak law of large numbers. It can be easily shown that any linear combination of $(X_n^{(\alpha)})$'s satisfies the weak law of large numbers as well. □

Recall that two sequences of random variables $\{\xi_n\}$ and $\{\eta_n\}$ are said to be equivalent in the sense of Khintchine if $\sum_{n=1}^{\infty} P[\xi_n \neq \eta_n] < \infty$. According to [9, Theorem 1.2.4] two such sequences simultaneously satisfy or do not satisfy the SLLN.

Theorem 17 *Let P be an atomless probability measure on a measure space $(\Omega, \mathcal{F})$. The set*

$$\{(X_n) \in L_0(\Omega)^{\mathbb{N}} : (X_n) \textit{ fulfills SLLN but not the Kolmogorov condition}\}$$

is $\mathfrak{c}$-lineable.

Proof Let (Z_n) be a sequence of independent random variables defined on Ω such that $P(Z_n = 1) = P(Z_n = -1) = \frac{1}{2} - \frac{1}{2^{n+1}}$ and $P(Z_n = 2^n) = P(Z_n = -2^n) = \frac{1}{2^{n+1}}$. Then $EZ_n = 0$ and $VarZ_n = 1 - \frac{1}{2^n} + 2^n$. Let $\{B_\alpha : \alpha \in [0, 1]\}$ be an almost disjoint family of subsets of $\mathbb{N}$. Put

$$X_n^{(\alpha)}(\omega) = \begin{cases} Z_n(\omega) & \text{if } \ n \in B_\alpha \\ 0 & \text{if } \ n \notin B_\alpha. \end{cases}$$

Let $0 \le \alpha_1 < \alpha_2 < \cdots < \alpha_m \le 1$, and $c_1, c_2 \ldots, c_m \in \mathbb{R} \setminus \{0\}$. Let $Y_n = \sum_{i=1}^m c_i X_n^{(\alpha_i)}$. Since $B := B_{\alpha_1} \setminus (B_{\alpha_2} \cup \cdots \cup B_{\alpha_m})$ is infinite and $Y_n = c_1 Z_n$ if $n \in B$, then $EY_n = 0$ for every n, and $VarY_n = 1 - \frac{1}{2^n} + 2^n$ for $n \in B$. Thus

$$\sum_{n=1}^{\infty} \frac{VarY_n}{n^2} \ge \sum_{n \in B} \frac{2^n}{n^2} = \infty$$

which means that (Y_n) does not fulfill the Kolmogorov condition.

Let us define $(\hat{Z}_n)$ as follows

$$\hat{Z}_n = \pm 1 \iff Z_n = \pm 1 \text{ and } \hat{Z}_n = 0 \iff |Z_n| = 2^n.$$

Then $(\hat{Z}_n)$ and (Z_n) are equivalent in the sense of Khintchine as $P(\hat{Z}_n \neq Z_n) = \frac{1}{2^n}$. Morevoer $E\hat{Z}_n = 0$ and $Var\hat{Z}_n = 1 - \frac{1}{2^n}$. Thus $(\hat{Z}_n)$ satisfies the Kolmogorov condition. Put

$$\hat{X}_n^{(\alpha)}(\omega) = \begin{cases} \hat{Z}_n(\omega) & \text{if } \ n \in B_\alpha \\ 0 & \text{if } \ n \notin B_\alpha \end{cases}$$

and $\hat{Y}_n = \sum_{i=1}^m c_i \hat{X}_n^{(\alpha_i)}$. Then $E\hat{Y}_n = 0$ and

$$Var\hat{Y}_n = E(\hat{Y}_n)^2 \le \left(\sum_{n=1}^m |c_n|\right)^2 E(\hat{Z}_n)^2 = \left(\sum_{n=1}^m |c_n|\right)^2 Var\hat{Z}_n.$$

Therefore

$$\sum_{n=1}^{\infty} \frac{Var\hat{Y}_n}{n^2} \le \left(\sum_{n=1}^m |c_n|\right)^2 \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.$$

Thus $(\hat{Y}_n)$ satisfies the Kolmogorov condition, and consequently (Y_n) fulfills SLLN. □

Let us recall here the big-O and the little-o notation. Having two sequences (a_n) and (b_n) of positive reals we write $a_n = O(b_n)$, if there is a constant $C > 0$ such that $a_n \le Cb_n$ for every $n \in \mathbb{N}$; we write $a_n = o(b_n)$ if $\lim_{n\to\infty} a_n/b_n = 0$. Although SLLN does not imply Kolmogorov condition, the latter cannot be improved in the sense that $\sum_{n=1}^{\infty} \frac{\sigma_n^2}{n^2} < \infty$ would be replaced by $\sum_{n=1}^{\infty} a_n \sigma_n^2 < \infty$ for some sequence (a_n) of positive reals with $a_n = o(1/n^2)$

Theorem 18 *Let P be an atomless probability measure on a measure space $(\Omega, \mathcal{F})$. Assume that $\sum_{n=1}^{\infty} \frac{\sigma_n^2}{n^2} = \infty$. Then the set*

$$\{(X_n) \in L_0(\Omega)^{\mathbb{N}} : EX_n = 0,\ VarX_n = O(\sigma_n^2) \text{ and } (X_n) \text{ does not obey the SLLN}\}$$

is $\mathfrak{c}$-lineable.

Proof Let $\mathcal{I} := \mathcal{I}_{(\sigma_n^2/n^2)} = \{A \subseteq \mathbb{N} : \sum_{n\in A} \frac{\sigma_n^2}{n^2} < \infty\}$. Let $\{B_\alpha : \alpha \in [0, 1]\}$ be an $\mathcal{I}$-almost disjoint family. Let A_n^i for $n \in \mathbb{N}$ and $i = 1, 2, 3$ be such that

(i) $P(A_n^1) = P(A_n^2) = \frac{\sigma_n^2}{2n^2}$ and $P(A_n^3) = 1 - \frac{\sigma_n^2}{n^2}$, if $\frac{\sigma_n^2}{n^2} \le 1$
(ii) $P(A_n^1) = P(A_n^2) = \frac{1}{2}$ and $P(A_n^3) = 0$, if $\frac{\sigma_n^2}{n^2} > 1$
(iii) the families $\{A_n^i : i = 1, 2, 3\}$ are independent.

Let $\alpha \in [0, 1]$. If $n \notin B_\alpha$, we put $X_n^{(\alpha)} \equiv 0$. If $n \in B_\alpha$ and $\frac{\sigma_n^2}{n^2} \le 1$, we put

$$X_n^{(\alpha)}(\omega) = \begin{cases} -n & \text{if } \omega \in A_n^1 \\ n & \text{if } \omega \in A_n^2 \\ 0 & \text{if } \omega \in A_n^3. \end{cases}$$

If $n \in B_\alpha$ and $\frac{\sigma_n^2}{n^2} > 1$, we put

$$X_n^{(\alpha)}(\omega) = \begin{cases} -\sigma_n & \text{if } \omega \in A_n^1 \\ \sigma_n & \text{if } \omega \in A_n^2. \end{cases}$$

Then $EX_n^{(\alpha)} = 0$ for every $n \in \mathbb{N}$, $VarX_n = \sigma_n^2$ iff $n \in B_\alpha$. Let $Y_n = \sum_{i=1}^m c_i X_n^{(\alpha_i)}$ be a linear combination of $X_n^{(\alpha_1)}, \ldots, X_n^{(\alpha_m)}$ where $c_i \neq 0$, $0 \le \alpha_1 < \cdots < \alpha_m \le 1$. Since $B := B_{\alpha_1} \setminus \bigcup_{i=2}^m B_{\alpha_i} \notin \mathcal{I}$, then $VarY_n = |c_1|\sigma_n^2$ for $n \in B$. Since $B \notin \mathcal{I}$, then $\sum_{n=1}^{\infty} \frac{VarY_n}{n^2} = \infty$. Moreover, for $n \in B$ and $\varepsilon \in (0, 1)$:

$$P(\frac{|Y_n|}{n} \ge \varepsilon) = P(Y_n \neq 0) = \begin{cases} \frac{\sigma_n^2}{n^n} & \text{if } \frac{\sigma_n^2}{n^2} \le 1 \\ 1 & \text{if } \frac{\sigma_n^2}{n^2} > 1. \end{cases}$$

Then $\sum_{n=1}^{\infty} P(|Y_n| > \varepsilon n) = \infty$ and by Borel-Cantelli lemma $\frac{Y_n}{n} \nrightarrow 0$ almost surely. Thus (Y_n) does not obey the SLLN. □

Lemma 19 *Let $(b_n) \in \ell_1$. Suppose that $a_n = o(b_n)$. Then there exists (x_n) such that $\sum_{n=1}^{\infty} a_n x_n < \infty$ and $\sum_{n=1}^{\infty} b_n x_n = \infty$.*

Proof Since $a_n = o(b_n)$, there is $c_n \to 0$ with $a_n = c_n b_n$. If $(c_n) \in \ell_1$, then we put $d_n = 1$ for every $n \in \mathbb{N}$. Otherwise there is an infinite set $A \in \mathcal{I}_{(c_n)}$ and then we put

$$d_n = \begin{cases} 1 & \text{if } n \in A \\ \frac{1}{n^2} & \text{if } n \notin A. \end{cases}$$

Finally we define x_n as d_n/b_n. Then

$$\sum_{n=1}^{\infty} a_n x_n = \sum_{n=1}^{\infty} c_n b_n \cdot \frac{d_n}{b_n} = \sum_{n=1}^{\infty} c_n d_n < \infty$$

and

$$\sum_{n=1}^{\infty} b_n x_n = \sum_{n=1}^{\infty} b_n \cdot \frac{d_n}{b_n} = \sum_{n=1}^{\infty} d_n = \infty.$$

□

We say that a sequence (X_n) of independent random variables fulfills (a_n)-Kolmogorov condition provided that $\sum_{n=1}^{\infty} a_n Var(X_n) < \infty$.

Corollary 20 *Let P be an atomless probability measure on a measure space $(\Omega, \mathcal{F})$. Let $a_n = o(\frac{1}{n^2})$. The set of all $(X_n) \in L_0(\Omega)^{\mathbb{N}}$ such that*

- *$EX_n = 0$,*
- *(X_n) fulfills (a_n)-Kolmogorov condition,*
- *(X_n) does not obey the SLLN*

is $\mathfrak{c}$-lineable.

Proof Using Lemma 19 for $b_n = \frac{1}{n^2}$, we find (x_n) such that $\sum_{n=1}^{\infty} a_n x_n < \infty$ and $\sum_{n=1}^{\infty} x_n/n^2 = \infty$. Then using Theorem 18 for $\sigma_n^2 = x_n$, we obtain that the set of all $(X_n) \in L_0(\Omega)^{\mathbb{N}}$ such that

- $EX_n = 0$,
- $VarX_n = O(\sigma_n^2)$,
- (X_n) does not obey the SLLN

is $\mathfrak{c}$-lineable. The equality $VarX_n = O(\sigma_n^2)$ means that there is a constant $C > 0$ such that $VarX_n \le C\sigma_n^2$. Thus

$$\sum_{n=1}^{\infty} a_n VarX_n \le C \sum_{n=1}^{\infty} a_n \sigma_n^2 = C \sum_{n=1}^{\infty} a_n x_n < \infty.$$

□

Data Availability Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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