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ORIGINAL PAPER

Thom condition and monodromy

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Abstract
We give the definition of the Thom condition and we show that given any germ of complex analytic function $f: (X, x) \to (\mathbb{C}, 0)$ on a complex analytic space X, there exists a geometric local monodromy without fixed points, provided that $f \in \mathfrak{m}^2_{X,x}$, where $\mathfrak{m}_{X,x}$ is the maximal ideal of $\mathcal{O}_{X,x}$. This result generalizes a well-known theorem of the second named author when X is smooth and proves a statement by Tibar in his PhD thesis. It also implies the A'Campo theorem that the Lefschetz number of the monodromy is equal to zero. Moreover, we give an application to the case that X has maximal rectified homotopical depth at x and show that a family of such functions with isolated critical points and constant total Milnor number has no coalescing of singularities.

Keywords Monodromy · Milnor fibration · Relative polar curves

Mathematics Subject Classification Primary 32S40; Secondary 32S50 · 32S55

Introduction

In [21] Milnor proved that for any germ of complex function:

$$f : (\mathbb{C}^{n+1}, x) \to (\mathbb{C}, 0)$$

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one can associate a smooth locally trivial fibration for $1 \gg \varepsilon > 0$:

$$\varphi_\varepsilon : \mathbb{S}_\varepsilon(x) \backslash f^{-1}(0) \to \mathbb{S}^1$$

induced by $f/|f|$, where $\mathbb{S}_\varepsilon(x)$ is the sphere centered at x with radius ε and $\mathbb{S}^1$ is the circle of radius 1 of $\mathbb{C}$ centered at the origin.

In [8] Hamm made the observation that, when x is an isolated critical point of f, the fibration of Milnor is isomorphic to the local fibration, for $1 \gg \varepsilon \gg \eta > 0$:

$$\psi_{\varepsilon,\eta} : \mathring{B}_\varepsilon(x) \cap f^{-1}(\mathbb{S}_\eta) \to \mathbb{S}_\eta$$

induced by f, where $\mathring{B}_\varepsilon(x)$ is the open ball centered at the point x with radius ε.

From the work of [10] (Théorème 1.2.1 p. 322) the hypothesis of isolated singularity can be lifted. Moreover the proper map:

$$\overline{\psi}_{\varepsilon,\eta} : B_\varepsilon(x) \cap f^{-1}(\mathbb{S}_\eta) \to \mathbb{S}_\eta$$

is a locally trivial fibration.

Milnor's fibration leads to a notion of monodromy associated to f at x. Precisely let $\varphi : X \to \mathbb{S}^1$ be a proper locally trivial smooth fibration. One can build on X a smooth vector field v which lifts the unit vector field tangent to $\mathbb{S}^1$. The integration of this vector field defines a smooth morphism $h : F \to F$ of a fiber of φ onto itself that we call *a geometric monodromy of* φ. A geometric monodromy is not uniquely defined, but one can prove that its isotopy class is unique. Therefore there is an isomorphism induced by a geometric monodromy of φ on the homology (or cohomology) of the fiber F called *the monodromy of* φ.

In the case of Milnor's fibration one often use the terminology of *local geometric monodromy* and *local monodromy* of f at the point x.

In [16] the second named author gave a proof of the fact that for any germ of complex analytic function:

$$f : (\mathbb{C}^{n+1}, x) \to (\mathbb{C}, 0)$$

having a critical point at x, there is a local geometric monodromy of f at x without fixed points.

By a well-known theorem of Lefschetz (see e.g. [11, p. 179]) this result implies that the local monodromy of f at x has a Lefschetz number equal to 0. In fact, in [1], A'Campo showed that the Lefschetz number is zero in a more general situation: let (X, x) be any germ of complex analytic space and denote by $\mathfrak{m}_{X,x}$ the the maximal ideal of the local ring $\mathcal{O}_{X,x}$ of germs of analytic functions of X at x.

Theorem 0.1 (cf. [1]) *Let* $f : (X, x) \to (\mathbb{C}, 0)$ *be a germ of complex analytic function such that* $f \in \mathfrak{m}^2_{X,x}$. *Then the local monodromy of* f *at* x *has Lefschetz number equal to* 0.

A'Campo used heavy mathematical machinery to prove this result in [1] and attributed its proof to P. Deligne.

In this work we give the following generalization of Lê's theorem, which in particular implies Theorem 0.1:

Theorem 0.2 *Let* $f : (X, x) \to (\mathbb{C}, 0)$ *be a germ of complex analytic function such that* $f \in \mathfrak{m}^2_{X,x}$. *Then there is a local geometric monodromy of* f *at* x *which does not fix any point.*

A big part of the argument in [16] relies strongly on the fact that for a sufficiently generic linear form $\ell : \mathbb{C}^{n+1} \to \mathbb{C}$, the map $\Phi = (\ell, f) : (X, x) \to (\mathbb{C}^2, 0)$ satisfies the Thom condition (see definition below) with respect to some convenient stratification. This allows to lift and integrate the plane vector field given by the carrousel in order to construct a local geometric monodromy in which we can apply an induction argument. It is well known that any complex analytic function $f : (X, x) \to (\mathbb{C}, 0)$ satisfies the Thom condition with respect to some stratification (see for instance [3, 12]). However, this is not true in general when we consider maps $(X, x) \to (\mathbb{C}^p, 0)$, with $p > 1$ (see Example 2.8).

Unfortunately, sometimes the Thom condition used to be ignored and some authors use it but without an explicit mention. So, we feel that it is important to emphasize this aspect of the theory. In Sect. 2, we show that given a map $\Phi = (g, f) : (X, x) \to (\mathbb{C}^2, 0)$ and a Whitney stratification of X, then Φ satisfies the Thom condition provided that:

1. g is the restriction of a submersion $\tilde{g} : U \to \mathbb{C}$;
2. $f^{-1}(0)$ is union of strata;
3. $\Gamma := \overline{C(\Phi)\setminus f^{-1}(0)}$ is empty or a curve (i.e., it has dimension one), for $C(\Phi)$ the critical locus of Φ;
4. $\Phi^{-1}(0) \cap \Gamma \subseteq \{x\}$;
5. for each stratum $S \in \mathcal{S}$ such that $\dim S \geq 1$, $\ker D_x\tilde{g} \notin \nu^{-1}_{\bar{S},f}(x)$, where $\nu_{\bar{S},f} : C(\bar{S}, f) \to \bar{S}$ is the relative conormal bundle of $\bar{S}$;

see Theorem 2.6. In particular, all these conditions are easily satisfied when we consider g as the restriction of a sufficiently generic linear form $\ell : \mathbb{C}^{n+1} \to \mathbb{C}$.

Another important contribution of [16] is the notion of privileged polydisks. This gives a fundamental system of neighbourhoods which is more convenient than the Euclidean balls if we want to proceed by induction on the dimension of X. We show in Section 4 how to adapt this notion in the case that X is not smooth.

We remark that a statement of Theorem 0.2 already appeared in [29] (see also [28]), following the ideas of [16] about relative polar curves and the carrousel construction. However, the technical details about the Thom condition, the lifting and integration of the vector field or the construction of the privileged polydisks are not mentioned in [29]. Here, we offer a complete and detailed explanation of all the steps in the proof.

As in [16], the proof of Theorem 0.2 uses the notion of relative polar curve, which is due essentially to R. Thom. When $X = \mathbb{C}^{n+1}$ we first choose a sufficiently small open neighbourhood U of x. For almost all linear function $\ell : \mathbb{C}^{n+1} \to \mathbb{C}$, one has that the critical space of the restriction $(\ell, f)|_{U\setminus\{f=0\}}$ is either always empty or a non-singular curve. When it is non-empty, we call the closure of the critical space of $(\ell, f)|_{U\setminus\{f=0\}}$ the relative polar curve $\Gamma_\ell(f, x)$ of f at x with respect to ℓ.

The remarkable property of the relative polar curve is that, when f has a critical point at x, its image by $(\ell, f)|_U$ is empty or a curve that Thom called the Cerf's diagram, which has as tangent cone the axis of values of ℓ (see *e.g.* [19, Proposition 6.7.5]). We show in Section 1 how to adapt this construction to the case that X is singular at x by taking a Whitney stratification. The condition that $f \in \mathfrak{m}^2_{X,x}$ is used here in order to prove that the tangent cone of the Cerf's diagram is the ℓ-axis.

Associated with the Cerf's diagram we have the *carrousel*, a construction which again appears in [16]. This is a vector field ω over a small enough solid torus $D \times \partial D_\eta$ centered at the origin in $\mathbb{C} \times \mathbb{C}$ such that:

(i) its projection onto the second component gives a tangent vector field over ∂D_η of length η and positive direction (called in [16] the *unitary vector field* of ∂D_η),
(ii) its restriction to $\{0\} \times \partial D_\eta$ is indeed the unitary vector field,
(iii) for every component of the Cerf's diagram with reduced equation $\delta_\alpha = 0$, ω is tangent to every $\delta_\alpha = \epsilon$ with ϵ small enough, and

(iv) the only integral curve that is closed after a loop in ∂D_η is $\{0\} \times \partial D_\eta$.

Now we can use techniques of stratification theory to lift the carrousel ω and obtain a stratified vector field on X which is globally integrable. The integral curves of this vector field define a local geometric monodromy of f at x and of its restriction to $X \cap \{\ell = 0\}$, defined on section 2. By condition (iv), the fixed points of the monodromy of f can appear only on $X \cap \{\ell = 0\}$. Thus, the proof of Theorem 0.2 follows by induction on the dimension of X at x.

We give an example that the condition that $f \in \mathfrak{m}^2_{X,x}$ is necessary, even if f has critical point at x in the stratified sense. In the last section, we also extend a well-known theorem of the second named author (see [15]) about no coalescing of families of functions with isolated critical points and constant total Milnor number. The extension works when we consider functions on spaces with maximal rectified homotopical depth (also called spaces with Milnor property in [9]).

1 Relative polar curves

Let $f : (X, x) \to (\mathbb{C}, 0)$ be the germ of a complex analytic function. We still call $f : X \to \mathbb{C}$ a representative of this germ. Let $\mathcal{S} = (S_\alpha)_{\alpha \in A}$ be a Whitney stratification of a sufficiently small representative X of (X, x). By Whitney stratification we mean a regular complex analytic stratification defined by Whitney in [30, §19, p. 540]. In particular the strata S_α and their closures $\overline{S}_\alpha$ are complex analytic spaces. We can assume that x is in the closure $\overline{S}_\alpha$ of all the strata S_α. So, the set of indices A is finite.

Using [18, Lemma 21 §3] one can prove that there is a non-empty open Zariski subset Ω_α of the space of affine functions such that, for every ℓ in Ω_α, $\ell(x) = 0$ and the critical locus C_α of $(\ell, f)|_{S_\alpha \setminus f^{-1}(0)}$, the function induced by (ℓ, f) on the space $S_\alpha \setminus f^{-1}(0)$, is either always empty or a non-singular curve. Then, the closure Γ_α of C_α in X is either empty or a reduced curve. Furthermore, we can also show that one can choose the Ω_α's such that the restriction $(\ell, f)|_{\Gamma_\alpha}$ is finite for any $\alpha \in A$. We define (see for instance [17, p. 310]);

Definition 1.1 For $\ell \in \bigcap_{\alpha \in A} \Omega_\alpha$ the union $\bigcup_{\alpha \in A} \Gamma_\alpha$ is either empty or a reduced curve. This curve is called the *relative polar curve* $\Gamma_\ell(f, \mathcal{S}, x)$ of f at x relatively to ℓ and the stratification $\mathcal{S}$ of X.

Remark 1.2 Notice that if the stratum S_α has dimension one, the whole stratum S_α is critical and Γ_α is the closure $\overline{S}_\alpha$. In this case, since S_α is connected, Γ_α is a branch of the curve $\Gamma_\ell(f, \mathcal{S}, x)$ at x, *i.e.* an analytically irreducible curve at x.

A theorem of Remmert implies that the image of Γ_α by (ℓ, f) is either empty or a curve Δ_α, for any $\alpha \in A$ (see, for example, [2, p. 5]).

We define:

Definition 1.3 The union $\bigcup_{\alpha \in A} \Delta_\alpha$ is either empty or a reduced curve. When it is a curve, it is called the *Cerf's diagram* $\Delta_\ell(f, \mathcal{S}, x)$ *of* f *at* x *relatively to* ℓ and the stratification $\mathcal{S}$. Otherwise we say that the Cerf's diagram of f at x relatively to ℓ and the stratification $\mathcal{S}$ is empty.

When the stratification $\mathcal{S}$ is fixed, we shall speak of the relative polar curve $\Gamma_\ell(f, x)$ and the Cerf's diagram $\Delta_\ell(f, x)$ without mentioning the stratification $\mathcal{S}$. But the reader must be aware that the notion of polar curve and Cerf's diagram depends on the choice of the stratification.

We shall go back and forth between the case $(\mathbb{C}^{n+1}, x)$ and the general case of germs of reduced analytic spaces (X, x) and compare them to generalize what we have in [16]. For example, if $(X, x) = (\mathbb{C}^{n+1}, x)$, we can consider a Whitney stratification which has only one stratum. In [16], we have seen that the emptiness of $\Gamma_\ell(f, x)$ means that the Milnor fiber of f at x is diffeomorphic to the product of the Milnor fiber of $f|_{\{\ell=0\}}$ at x with an open disc, hence the local geometric monodromy of f at x is induced by the product of the local geometric monodromy of $f|_{\{\ell=0\}}$ at x and the identity of the open disc.

Also, for a germ of complex analytic function $f : (X, x) \to (\mathbb{C}, 0)$, in general, we may suppose that the hyperplane $\{\ell = 0\}$ is transverse to all the strata of the Whitney stratification $\mathcal{S}$ and it induces a Whitney stratification on $X \cap \{\ell = 0\}$. Then, using the same arguments of [18], we can prove the following:

Proposition 1.4 *If, for a general linear form ℓ at x, the relative polar curve $\Gamma_\ell(f, x)$ is empty, there is a stratified homeomorphism of the Milnor fiber of f at x and the product with an open disc with the Milnor fiber of the restriction $f|_{X\cap\{\ell=0\}}$ at x.*

The proof of this proposition is based on the techniques Mather used to prove the Thom-Mather first isotopy lemma, cf. [7, 20]. We will outline these techniques in the next section and use them later.

Now observe that when $(X, x) = (\mathbb{C}^{n+1}, x)$ the point x is a critical point of f if and only if $f \in \mathfrak{m}^2_{\mathbb{C}^{n+1},x}$, where $\mathfrak{m}_{\mathbb{C}^{n+1},x}$ is the maximal ideal of the local ring $\mathcal{O}_{\mathbb{C}^{n+1},x}$. In the case of a germ of complex analytic function on (X, x), the hypothesis $f \in \mathfrak{m}^2_{X,x}$, where $\mathfrak{m}_{X,x}$ is the maximal ideal of $\mathcal{O}_{X,x}$, replaces the condition that f is critical at x. In fact, a key result for the proof of Theorem 0.2 is:

Proposition 1.5 *For a sufficiently general linear form ℓ, if $f \in \mathfrak{m}^2_{X,x}$, every branch of the Cerf's diagram $\Delta_\ell(f, x)$ is tangent at the point $(0, 0)$ to the first axis, the image by (ℓ, f) of $\{f = 0\}$.*

Proof Of course, we have a Whitney stratification $\mathcal{S} = (S_\alpha)_{\alpha\in A}$ on a sufficiently small representative of the germ (X, x). We may assume that x is in the closure of all the strata.

It is enough to prove the proposition for the image Δ_α of Γ_α by (ℓ, f), for each $\alpha \in A$.

In [16], it was considered that ℓ is a coordinate of $\mathbb{C}^{n+1}$ to compare easily the growth of f and ℓ along a component of the Cerf's diagram. We are going to give a similar proof for the $(\mathbb{C}^{n+1}, x)$ case for any general linear form, and generalize it twice to reach our current context.

Suppose that $(X, x) = (\mathbb{C}^{n+1}, x)$, for our purpose ℓ can be expressed as:

$$\ell(v) = \langle v, a\rangle = \sum_{i=1}^{n+1} v_i \overline{a_i},$$

and we can assume that $\|a\| = 1$. Let us define H as the kernel of ℓ and, then, any vector of $\mathbb{C}^{n+1}$ can be written as a sum of a vector of H and a multiple of the vector a (note that a is the unitary normal of H).

Now we can take a parametrization $p(t)$ of a branch of Γ_α and compare the growths of f and ℓ there. Using de l'Hôpital's rule and identifying ℓ with its differential we have:

$$\lim_{t\to 0} \frac{(f \circ p)(t)}{(\ell \circ p)(t)} = \lim_{t\to 0} \frac{(f \circ p)'(t)}{(\ell \circ p)'(t)} = \lim_{t\to 0} \frac{df_{p(t)}\big(p'(t)\big)}{\ell\big(p'(t)\big)}.$$

Now we can decompose $p'(t)$ as the sum of a vector of H, say $p'_H(t)$, and λa. Hence:

$$\lim_{t\to 0} \frac{df_{p(t)}\big(p'(t)\big)}{\ell\big(p'(t)\big)} = \lim_{t\to 0} \frac{df_{p(t)}\big(p'_H(t)\big) + \lambda df_{p(t)}\,(a)}{\ell\big(p'_H(t)\big) + \lambda \ell\,(a)}.$$

Furthermore, we know that df and ℓ are colinear along $p(t)$, and we have assumed $\ell(a) = \|a\| = 1$, therefore

$$\begin{aligned}\lim_{t\to 0} \frac{df_{p(t)}\big(p'_H(t)\big) + \lambda df_{p(t)}\,(a)}{\ell\big(p'_H(t)\big) + \lambda \ell\,(a)} &= \lim_{t\to 0} \frac{df_{p(t)}\,(a)}{\ell\,(a)} \\ &= df_x(a).\end{aligned}$$

At this point we see where the condition of $f \in \mathfrak{m}^2_{X,x}$ appears, because this last term is zero. This proves the tangency of the statement in this context.

If we want the same result on $(X, x) \subset (\mathbb{C}^N, x)$, with (X, x) regular at x, the main problem is that ℓ is defined in X, and we cannot work with such a vector a and space H. What we can do is to extend ℓ to the ambient space and work on the tangent bundle of X, hence we can choose a linear function $L : \mathbb{C}^N \to \mathbb{C}$ such that $L|_X = \ell$ and H' as the kernel of L. By genericity $T_x X$ is not contained in H' so $H' \cap T_x X$ is a hyperplane of $T_x X$, say H. This happens, nearby x, for every tangent space along a parametrization of Γ_α (in this case there is only one strata), so we can reproduce the computations we did before.

Finally, if (X, x) is general, we can still extend ℓ but we cannot work with the tangent bundle of X any more (*e.g.*, if (X, x) is a Whitney umbrella at x even x is a strata by itself). To avoid this complication, firstly we shall find a convenient hyperplane of $\mathbb{C}^N$ for the role of H' and then work with the extension of f when needed. From now on, we will work with a strata S_α, or its adherence, but for the sake of the similarity with the previous cases we will call Y the closure $\overline{S_\alpha}$.

Therefore, our first step is to find an hyperplane to work with. For this purpose consider the (projective) conormal space $C(Y)$ of Y in $\mathbb{C}^N$, this is given by the closure in $Y \times \check{\mathbb{P}}^{N-1}$ of the space:

$$\left\{(q, H') \,|\, (q, H') \in Y^{reg} \times \check{\mathbb{P}}^{N-1} : T_q Y^{reg} \subset H'\right\},$$

together with the conormal map $\nu : C(Y) \to Y$. It is a classical fact (cf. [27, II.4.1]) that $\dim \nu^{-1}(x) \le N - 2$ or, being more specific, there is a hyperplane H' outside $\nu^{-1}(x)$ and by continuity outside every fiber of ν over a neighbourhood of x in Y.

Therefore, consider a linear form $L : \mathbb{C}^N \to \mathbb{C}$ with such a hyperplane H' as kernel and define ℓ to be $L|_Y$. Furthermore, since $f \in \mathfrak{m}^2_{Y,x}$, we can take an extension $F : (\mathbb{C}^N, x) \to (\mathbb{C}, 0)$ of f, such that $F \in \mathfrak{m}^2_{\mathbb{C}^N,x}$.

Finally, as we have done before, consider a parametrization $p(t)$ of a branch of Γ_α and compare the growths of $f\big(p(t)\big)$ and $\ell\big(p(t)\big)$. To do so define a_t to be the unitary normal of the hyperplane $H_t := H' \cap T_{p(t)}Y$ in $T_{p(t)}Y$, well defined by the previous election of H', and a_0 its limit. Now we can keep proceeding as before and finish the computation with F, i.e.:

$$\begin{aligned}\lim_{t\to 0}\frac{(f\circ p)(t)}{(\ell\circ p)(t)} &= \lim_{t\to 0} df_{p(t)}(a_t)\\ &= \lim_{t\to 0} dF_{p(t)}(a_t)\\ &= dF_x(a_0)\\ &= 0.\end{aligned}$$

Note that the election of H', for S_α, was made in an open set. Since we have only a finite number of strata to which x is adherent, we can take a common H' for every S_α and repeat the computation. This finishes the proof. □

2 Thom condition for maps onto the plane

As it is well known, the Thom condition appears as hypothesis in many important results because it gives control on a map defined over stratified spaces. For example, it appears in the Thom–Mather isotopy lemmas. We recall now its definition (see also Fig. 1).

Recall the definitions of stratified map and Thom map:

Definition 2.1 A map $f : X \to X'$ between Whitney stratified sets X and X' is a *stratified map* if the restriction of the map on each stratum of X is submersive onto a stratum of X', i.e., $f(S_\alpha) \subseteq S'_\beta$ and $f|_{S_\alpha} : S_\alpha \to S'_\beta$ is submersive where S_α is a stratum of X and S'_β is a stratum of X'.

Definition 2.2 (cf. [7, II.2.5]) Let $f : N \to N'$ be a smooth map between manifolds N and N' and $X \subset N$ and $X' \subset N'$ two stratified sets such that $f(X) \subset X'$ and the induced map $f| : X \to X'$ is a stratified map. We say that S_t is *Thom regular over* S_r *at* p *relatively to* $f| : X \to X'$ if any sequence of points $\{q_n\}_n \subset S_t$ converging to $p \in S_r$ is such that

$$\ker D_p\ f|_{S_r} \subseteq \lim_n \ker D_{q_n}\ f|_{S_t}\,, \tag{1}$$

when the limit exists. If $f|$ is Thom regular for any pair of strata we simply say that the pair of stratifications is a *Thom stratification* of $f|$ and that $f|$ is a *Thom map* or that it satisfies the *Thom* $a_{f|}$ *condition*.

Not every map has a stratification such that it is a Thom map, for example the map $f : \mathbb{R}^2 \to \mathbb{R}^2$ so that $f(x, y) = (x, xy)$ does not admit a Thom stratification (see [7, page 24]). However, if we consider complex functions $f : (X, x) \to (\mathbb{C}, 0)$, with (X, x) a germ of a complex space, we have a result of existence of stratifications providing the Thom regularity condition given by Hironaka in [12, Corollary 1]. Furthermore, Briançon, Maisonobe and Merle, in [3, Theorem 4.2.1], gave a result that assures the Thom condition provided the stratification is Whitney regular.

Let $(X, x) \subseteq (\mathbb{C}^N, x)$ be a germ of a complex analytic space. This contrast suggests that the case of a map germ $\Phi = (g, f) : (X, x) \to (\mathbb{C}^2, 0)$ is harder to study, as there could be maps that do not admit any Thom stratification (see also Example 2.8 below). In order to study this case we need another definition.

Definition 2.3 Given a complex space $S \subseteq \mathbb{C}^N$ and a function $f : S \to \mathbb{C}$, the *conormal bundle of* $\bar{S}$ *relative to* f or, simply, the *relative conormal bundle of* $\bar{S}$ is the closed space

$$C(\bar{S}, f) := \overline{\left\{(p, H) \in S^{reg} \times \check{\mathbb{P}}^{N-1} \,|\, \ker D_q f \subseteq H\right\}}$$

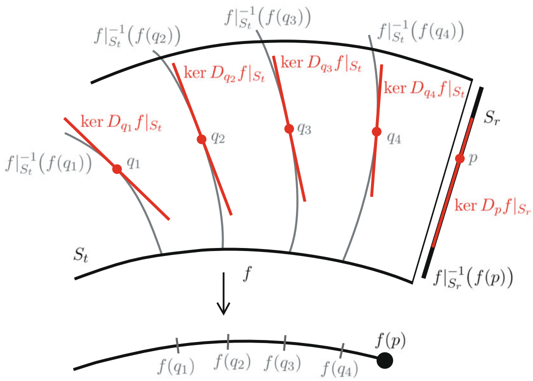

Fig. 1 Representation of the Thom condition

together with the projection $\nu_{\bar{S},f} : C(\bar{S}, f) \to \bar{S}$.

Remark 2.4 Observe that $C(\bar{S}, f)$ coincides with $C(\bar{S})$ given in the proof of Proposition 1.5 when f is constant.

Furthermore, we will need to refine some stratifications to provide the Thom condition:

Lemma 2.5 *Let $\varphi : V \to W$ be a smooth map, where $V \subset \mathbb{R}^n$ and $W \subset \mathbb{R}^m$ are open subsets. Assume that $\mathcal{S}$ is a Whitney stratification of a subset $X \subset V$ such that for all $S \in \mathcal{S}$, $\varphi|_S : S \to W$ is a submersion and that $\mathcal{T}$ is a Whitney stratification of W. Then,*

$$\mathcal{S}' = \left\{ S \cap \varphi^{-1}(T) : \ S \in \mathcal{S}, \ T \in \mathcal{T} \right\}$$

is also a Whitney stratification of X.

Proof Take a pair of strata $A = S \cap \varphi^{-1}(T)$ and $B = S' \cap \varphi^{-1}(T')$ such that $A \subseteq \bar{B}$, with $S, S' \in \mathcal{S}$ and $T, T' \in \mathcal{T}$. We factorize φ as the composition:

$$V \xrightarrow{\ i\ } G(\varphi) \xrightarrow{\ \pi_2\ } W$$

where $G(\varphi) \subset V \times W$ is the graph of φ, i is the diffeomorphism given by $i(v) = \big(v, \varphi(v)\big)$ and $\pi_2(v, w) = w$. It follows that B is Whitney regular over A in V if and only if $i(B)$ is Whitney regular over $i(A)$ in $G(\varphi)$, or equivalently, in $V \times W$. To prove this observe that we can write $i(A)$ and $i(B)$ in the form

$$i(A) = i(S) \cap (S \times T), \quad i(B) = i(S') \cap (S' \times T'). \tag{2}$$

Moreover, we know that $i(S')$ is Whitney regular over $i(S)$ and $S' \times T'$ is Whitney regular over $S \times T$.

Let $\{x_n\}$ and $\{y_n\}$ be sequences in $i(A)$ and $i(B)$ respectively, both converging to $x \in i(A)$. We also assume that $\overline{x_n y_n}$ converges to a line L and that $T_{y_n} i(B)$ converges to a plane E in the corresponding Grassmannians of $\mathbb{R}^n \times \mathbb{R}^m$. We have to show that $L \subset E$.

By taking subsequences if necessary, we can assume that $T_{y_n} i(S')$ converges to a plane E_1 and that $T_{y_n}(S' \times T')$ converges to another plane $E_2 \times E_3$, again in the corresponding Grassmannians of $\mathbb{R}^n \times \mathbb{R}^m$. Since $i(S')$ is Whitney regular over $i(S)$ and $S' \times T'$ is Whitney regular over $S \times T$, we have $L \subset E_1 \cap (E_2 \times E_3)$.

From (2) it follows that $E \subset E_1 \cap (E_2 \times E_3)$. Furthermore, $\varphi|_{S'} : S' \to W$ is a submersion, which factors as the composition

$$S' \xrightarrow{i} i(S') \xrightarrow{\pi_2} W \ .$$

This implies that $T_{y_n} i(S')$ and $T_{y_n}(S' \times T')$ are transverse in $\left(T_{\pi_1(y_n)} S'\right) \times \mathbb{R}^m$. Therefore, E_1 and $E_2 \times E_3$ are also transverse in $E_2 \times \mathbb{R}^m$. Thus, $\dim E = \dim E_1 \cap (E_2 \times E_3)$ and hence $E = E_1 \cap (E_2 \times E_3)$. □

Theorem 2.6 *Consider the complex analytic germ*

$$\Phi = (g, f) : (X, x) \to (\mathbb{C}^2, 0)$$

and take a representative X of (X, x) such that it is a closed analytic subset of some open set $U \subseteq \mathbb{C}^N$, and consider also a Whitney stratification $\mathcal{S}$ of X. Assume that:

1. *g is the restriction of a submersion $\tilde{g} : U \to \mathbb{C}$;*
2. *$f^{-1}(0)$ is union of strata;*
3. *$\Gamma := \overline{C(\Phi)\backslash f^{-1}(0)}$ is empty or a curve (i.e., it has dimension one), for $C(\Phi)$ the critical locus of Φ;*
4. *$\Phi^{-1}(0) \cap \Gamma \subseteq \{x\}$;*
5. *for each stratum $S \in \mathcal{S}$ such that $\dim S \geq 1$, $\ker D_x \tilde{g} \notin \nu^{-1}_{\bar{S}, f}(x)$, where $\nu_{\bar{S}, f} : C(\bar{S}, f) \to \bar{S}$ is given by the relative conormal bundle of $\bar{S}$.*

Then, with a possibly smaller representative, Φ is a Thom map with a stratification $\{\mathcal{S}', \mathcal{T}\}$ such that $\mathcal{S}'$ refines $\mathcal{S}$.

Proof First of all, observe that f is a Thom map with the provided stratification $\mathcal{S}$ of X, together with the stratification $\{\mathbb{C}\backslash 0, 0\}$ in the target, by [3, Theorem 4.2.1]. Now, we want a refinement of $\mathcal{S}$, say $\mathcal{S}'$, that makes Φ a Thom map with a convenient stratification $\mathcal{T}$ in the target.

As we show below, this is attained if we refine $\mathcal{S}$ so that Γ, $f^{-1}(0)\backslash\Gamma$ and $X\backslash\left(f^{-1}(0) \cup \Gamma\right)$ are union of strata and the refinement is Whitney regular. We can use Lemma 2.5 to achieve this on $X\backslash\left(f^{-1}(0) \cup \Gamma\right)$, with a restriction of Φ and the stratification $\mathcal{T}$ on $\mathbb{C}^2$ given by 0, $\Delta\backslash 0$, $L\backslash 0$ and $\mathbb{C}^2\backslash\left(\Delta \cup L\right)$, where $\Delta := \Phi(\Gamma)$ and $L := \Phi\left(f^{-1}(0)\right)$. Observe that we already have that $f^{-1}(0)$ is a union of strata, by Assumption 2, so we only need to check that adding $\Gamma\backslash x$ as stratum does not change the Whitney condition. However, as Γ is a curve, the set where the Whitney condition could fail (sometimes called *bad set*) is of dimension zero (see, for example, [7, Proposition 2.6]). This implies that we can take a smaller representative of (X, x) if needed where the result holds.

Now we show that the Thom condition holds for Φ and the stratifications $\mathcal{S}'$ and $\mathcal{T}$ as above. We consider two strata $S_r, S_t \in \mathcal{S}'$ such that $S_r \subseteq \bar{S}_t$ and a sequence $\{q_n\}_n \subset S_t$ converging to $p \in S_r$. We want to check Thom's regularity condition given in Equation (1) for Φ. To do so, we separate by cases: considering strata in Γ, $f^{-1}(0)\backslash\Gamma$ or $X\backslash\left(f^{-1}(0) \cup \Gamma\right)$.

If S_r is contained in Γ, the Thom condition is always satisfied trivially because Φ is a local diffeomorphism when restricted to Γ except, perhaps, at x. This comes from the fact that there is a representative of the restriction $\Phi|_\Gamma$ that is finite, by Assumption 4, so $\Delta = \Phi(\Gamma)$ is a curve or empty.

If both S_r and S_t are contained in $f^{-1}(0)\backslash\Gamma$, we have that the Thom condition (recall Eq. 1) for Φ is equivalent to

$$T_p S_r \cap \ker D_p \tilde{g} \subseteq \lim_n \left(T_{q_n} S_t \cap \ker D_{q_n} \tilde{g}\right). \tag{3}$$

However, as $\ker D_\bullet \tilde{g}$ is a hyperplane (by Assumption 1) and by Assumption 5, we have that

$$\lim_n \left(T_{q_n} S_t \cap \ker D_{q_n} \tilde{g}\right) = \lim_n \left(T_{q_n} S_t\right) \cap \ker D_p \tilde{g}$$

and Eq. 3 is satisfied, as $\mathcal{S}$ is a Whitney stratification and

$$\lim_n \left(T_{q_n} S_t\right) \supseteq T_p S_r.$$

In a similar fashion, if S_t is contained in $X \backslash \left(f^{-1}(0) \cup \Gamma\right)$ and S_r is contained in $X \backslash \left(f^{-1}(0) \cup \Gamma\right)$ or $f^{-1}(0) \backslash \Gamma$, we have that

$$\begin{aligned} \ker D_p f|_{S_r} \cap \ker D_p \tilde{g} &\subseteq \lim_n \ker D_{q_n} f|_{S_t} \cap \ker D_p \tilde{g} \\ &= \lim_n \left(D_{q_n} f|_{S_t} \cap \ker D_p \tilde{g}\right), \end{aligned}$$

where the first inclusion is given by the Thom condition on f and the second equality is given by Assumptions 1 and 5. □

Remark 2.7 Finally, observe that we can apply Theorem 2.6 with

$$\Phi = (\ell, f) : (X, x) \to (\mathbb{C}^2, 0)$$

given in Section 1, provided ℓ is generic enough, to make Φ a Thom map with certain stratification. More precisely, the objects Γ and Δ of Theorem 2.6 are, in this case, the polar curves and the Cerf's diagram of f relatively to ℓ.

Example 2.8 In [26, p. 286] it is shown that the germ

$$\begin{aligned} \Phi = (g, f) : (\mathbb{C}^3, 0) &\to (\mathbb{C}^2, 0) \\ (x, y, z) &\mapsto (y, x^2 - y^2 z) \end{aligned}$$

does not have a representative with the Thom condition. Indeed, it is Assumption 5 from Theorem 2.6 what fails (observe that, in this case, $\Gamma = \varnothing$), showing its importance.

3 Lifting vector fields

The construction of the Milnor fibration of a complex analytic function $f : (X, x) \to (\mathbb{C}, 0)$ when X is a complex analytic space is a consequence of the Thom-Mather first isotopy lemma. The strategy to prove Theorem 0.2 is to take a generic linear form ℓ on the ambient space of (X, x) and consider the map $\Phi = (\ell, f) : (X, x) \to (\mathbb{C}^2, 0)$. We want to trivialize this map in such a way that its composition with the projection onto the second component $\pi_2 : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ gives the Milnor fibration of f and its restriction to $\left(X \cap \ell^{-1}(0), x\right)$ gives the Milnor fibration of the restriction $f : \left(X \cap \ell^{-1}(0), x\right) \to (\mathbb{C}, 0)$. This would allow us to use an induction process, as in [16].

One could think that this could be done just by using the Thom-Mather second isotopy lemma. Unfortunately, this seems not possible and we are forced to use some of the ingredients in the proof of the isotopy lemmas, like controlled tube systems or controlled stratified vector fields, in order to construct a lifting of the vector field which fits into our problem. For the sake of completeness, we include in this section all the definitions and main results that we need for that purpose. Instead of the original proof of the isotopy lemmas by Mather [20], we follow the notations and statements of [7, Chapter II], where the reader can find more details and the proofs of all the results.

We recall that a *stratified vector field* on a stratified set X of a smooth manifold N is a map $\xi : X \to TN$ tangent to each stratum S_α of X and smooth on S_α, but ξ might not be continuous. We now give the definitions of controlled tube system and controlled stratified vector field:

Definition 3.1 (cf. [7, II.1.4]) If X is a submanifold of N, a *tube at* X is a quadruple $T = (E, \pi, \rho, e)$ where $\pi : E \to X$ is a smooth vector bundle, $\rho : E \to \mathbb{R}$ is a quadratic function of a Riemann metric on E that vanishes on the zero section and $e : \big(E, \zeta(X)\big) \to (N, X)$ is a germ along $\zeta(X)$ of a local diffeomorphism, commuting with the zero section $\zeta : X \to E$ so that $e \circ \zeta$ along X is the inclusion $X \subset N$.

If X is a Whitney stratified subset of a manifold N, a *tube system for the stratification* consists of a tube for every strata.

Definition 3.2 (cf. [7, II.2.5]) A tube system $\mathcal{T} = \{T_\alpha\}_{\alpha \in A}$, with $T_\alpha = (E_\alpha, \pi_\alpha, \rho_\alpha, e_\alpha)$, for a Whitney stratification $\{S_\alpha\}_{\alpha \in A}$ of some subset X of a manifold N is *weakly controlled* if the relation

$$(\pi_\alpha \circ e_\alpha^{-1}) \circ (\pi_{\alpha'} \circ e_{\alpha'}^{-1}) = (\pi_\alpha \circ e_\alpha^{-1}), \qquad \alpha, \alpha' \in A$$

holds for every pair of tubes $(T_\alpha, T_{\alpha'})$ where the composition makes sense.

We remark that the notion of weakly controlled tube system for a stratification is not a strange thing to ask, actually any given Whitney stratification admits a weakly controlled tube system (cf. [7, II.2.7]).

Definition 3.3 (cf. [7, II.3.1]) If we have a tube system for a Whitney stratification of X and ξ is a stratified vector field on X we shall say that ξ is *a weakly controlled vector field* if:

$$D(\pi_\alpha \circ e_\alpha^{-1}) \circ \xi = \xi \circ (\pi_\alpha \circ e_\alpha^{-1})$$

holds for every tube, using the notation of Definition 3.2.

Next, we give the control conditions relative to a stratified map (recall Definition 2.1).

Definition 3.4 (cf. [7, II.2.5]) Let $f : N \to N'$ be a smooth map and $X \subset N$ and $X' \subset N'$ two stratified sets such that $f(X) \subset X'$ and the induced map $f| : X \to X'$ is a stratified map. Assume also that we have a tube system $\mathcal{T} = \{T_\alpha\}_{\alpha \in A}$ for the stratification $\{S_\alpha\}_{\alpha \in A}$ of X and a tube system $\mathcal{T}' = \{T'_\beta\}_{\beta \in B}$ for the stratification $\{S'_\beta\}_{\beta \in B}$ of X'. Then, we say that $\mathcal{T}$ is *controlled* over $\mathcal{T}'$ if

(i) $\mathcal{T}$ is weakly controlled,
(ii) $f \circ (\pi_\alpha \circ e_\alpha^{-1}) = (\pi_\beta \circ e_\beta^{-1}) \circ f$, for every S_α mapping into S'_β, and
(iii) $(\rho_\alpha \circ e_\alpha^{-1}) \circ (\pi_{\alpha'} \circ e_{\alpha'}^{-1}) = (\rho_\alpha \circ e_\alpha^{-1})$ holds for every pair $(T_\alpha, T_{\alpha'})$ such that $f(S_\alpha \cup S_{\alpha'}) \subseteq S'_\beta$ for some S'_β in X'.

In addition to the control conditions we also need a regularity condition for stratified maps of the same nature as the Whitney condition for stratified sets. This is the Thom condition given in Definition 2.2. In fact, the Thom condition ensures the existence of a controlled tube system as follows:

Theorem 3.5 (cf. [7, II.2.6]) *Let N, N', X, X', f be as in Definition 3.4 and assume $f| : X \to X'$ is a Thom map. Then, each weakly controlled tube system $\mathcal{T}'$ of X' has a tube system $\mathcal{T}$ of X controlled over $\mathcal{T}'$.*

We also have control conditions relative to a stratified map for stratified vector fields.

Definition 3.6 (cf. [7, II.3.1]) Let X, X', $\mathcal{T}$,$\mathcal{T}'$, and $f| : X \to X'$ be as in Definition 3.4. Assume that we have ξ and ξ' stratified vector fields on X and X', respectively. We say that ξ *lifts* ξ' if $Df|_{S_\alpha} \circ \xi = \xi' \circ f|_{S_\alpha}$, for every stratum S_α of X. Furthermore, we say that ξ is *controlled* over ξ' if

(i) ξ lifts ξ',
(ii) ξ is weakly controlled,
(iii) $D(\rho_\alpha \circ e_\alpha^{-1}) \circ \xi|_{f^{-1}S'_\beta} = 0$ for every S_α mapping into S'_β.

Again, the Thom condition is the key point to lift any weakly controlled vector field in the target to a controlled vector field in the source:

Theorem 3.7 (cf. [7, II.3.2]) *Let N, N', X, X', f be as in Definition 3.4 and assume $f| : X \to X'$ is a Thom map. Let $\mathcal{T}$ and $\mathcal{T}'$ be tube systems of the stratifications of X and X', respectively, such that $\mathcal{T}$ is controlled over $\mathcal{T}'$. Then, any weakly controlled vector field ξ' on X' lifts to a stratified vector field ξ which is controlled over ξ'.*

The last ingredient is about integrability of stratified vector fields. Specifically, if we have a stratified vector field ξ on X and we integrate it on every stratum S_α we have a smooth flow $\theta_\alpha : D_\alpha \to S_\alpha$, where $D_\alpha \subseteq \mathbb{R} \times S_\alpha$ is the maximal domain of the integration, which contains $\{0\} \times S_\alpha$. Setting D as the union of every D_α, we obtain a map $\theta : D \to X$ that is not necessarily continuous.

Definition 3.8 (cf. [7, II.4.3]) With the notation above, if θ is continuous on a neighbourhood of $\{0\} \times X$ we say that ξ is *locally integrable*. Furthermore, if $D = \mathbb{R} \times X$ we say that ξ is *globally integrable*.

It is here where the control conditions over the vector fields play their role:

Theorem 3.9 (cf. [7, II.4.6]) *Let N, N', X, X', f be as in Definition 3.4. Assume also that X is locally closed in N. If ξ and ξ' are stratified vector fields on X and X', respectively, and ξ is controlled over ξ' with respect to some tube system $\mathcal{T}$ of X, then ξ is locally integrable if ξ' is so.*

Theorem 3.10 (cf. [7, II.4.8] *Let N, N', X, X', f be as in Definition 3.4. Assume also $f| : X \to X'$ is proper. If ξ and ξ' are stratified vector fields on X and X', respectively, and ξ is locally integrable, then ξ is globally integrable if ξ' is so.*

So, to summarize, if we combine Theorems 3.7, 3.5, 3.9 and 3.10 we get:

Corollary 3.11 *Let N, N', X, X', f be as in Definition 3.4 and assume $f| : X \to X'$ is a Thom proper map. If we have a weakly controlled tube system with a weakly controlled vector field ξ' on X' such that it is globally integrable, it lifts to a globally integrable vector field on X.*

Corollary 3.12 *Let $f : N \to N'$ be a smooth map and let $X \subset N$ be a Whitney stratified subset such that $f| : X \to N'$ is a proper stratified submersion. If ξ' is a globally integrable smooth vector field on N', it lifts to a globally integrable vector field on X.*

Corollary 3.12 is a consequence of Corollary 3.11 since the Thom condition is satisfied in this case (see [7, II.3.3]). We also remark that these two corollaries, among other things, are used in [7] to prove the Thom-Mather isotopy lemmas.

Finally, we show how Corollary 3.12 can be used to construct a local geometric monodromy of a function in a specific way. Let $\varphi\colon X \to \mathbb{S}^1$ be a locally trivial C^0- fibration with fiber $F = \varphi^{-1}(t_0)$. It is well known that $\varphi\colon X \to \mathbb{S}^1$ is C^0- equivalent to the fibration $\pi : {}^{F\times[0,2\pi]}/{\sim} \to \mathbb{S}^1$, where the relation is given by $(x, 0) \sim \big(h(x), 2\pi\big)$ for some homeomorphism $h : F \to F$ and $\pi([x, t]) = t$. As we have mentioned in the introduction, such homeomorphism h is called a *geometric monodromy* of φ, although it is not smooth in general. Since there are some choices, a geometric monodromy is not unique. However one can prove that its isotopy class is well defined, so the induced map on homology (or cohomology) is uniquely given by φ and it is simply called *the monodromy* of φ.

In our case, given a complex analytic function $f\colon (X, x) \to (\mathbb{C}, 0)$ there exist ϵ and η with $0 < \eta \ll \epsilon \ll 1$ such that

$$f\colon X \cap B_\epsilon \cap f^{-1}(\partial D_\eta) \to \partial D_\eta \tag{4}$$

is a proper stratified submersion, for some Whitney stratification on the source and the trivial stratification on ∂D_η (see [17]). By the Thom-Mather first isotopy lemma, (4) is a locally trivial C^0- fibration with fiber F.

In fact, we have something more. We take on ∂D_η the vector field of constant length η and positive direction. By Corollary 3.12, this vector field can be lifted to a stratified vector field ξ on the source which is globally integrable.

The flow of ξ provides the local trivialisations of (4) and it follows that the geometric monodromy obtained in this way $h : F \to F$ is a stratified homeomorphism (that is, it preserves strata and the restriction on each stratum is a diffeomorphism). We call $h : F \to F$ the *local geometric monodromy of* f *at* x *induced by* ξ.

In the next section we show that instead of a Euclidean ball B_ϵ we can take a convenient polydisc, which is better to proceed with the induction hypothesis.

4 Privileged polydiscs

In [16], instead of a usual Milnor ball B for a complex analytic function $f : (\mathbb{C}^{n+1}, x) \to (\mathbb{C}, 0)$, it is considered a privileged polydisc $\Delta = D_1 \times \cdots \times D_{n+1}$ with respect to some generic choice of coordinates $z_1, \ldots, z_{n+1}$ in $\mathbb{C}^{n+1}$. Here we show how to adapt this notion to the case of a function $f : (X, x) \to (\mathbb{C}, 0)$ on a complex analytic set X.

Assume that $\dim(X, x) = n + 1$ and that (X, x) is embedded in $(\mathbb{C}^N, 0)$. We take a representative $f : X \to \mathbb{C}$ and complex analytic Whitney stratifications in X and $\mathbb{C}$ such that $f : X \to \mathbb{C}$ is a stratified function. We say that $z_1, \ldots, z_N$ are *generic coordinates* if for each $i = 0, \ldots, n$, the $(N - i)$-plane H^i through the origin given by $\{z_1 = \cdots = z_i = 0\}$ is transverse to all the strata of X except, perhaps, the stratum $\{x\}$.

We consider the set $X^i = \pi_i(X \cap H^i) \subset \mathbb{C}^{N-i}$, where π_i is the projection onto the last $N - i$ coordinates, with the induced stratification and the function $f^i : X^i \to \mathbb{C}$ given by

$$f^i(z_{i+1}, \ldots, z_N) = f(0, \ldots, 0, z_{i+1}, \ldots, z_N).$$

A *polydisc* centered at 0 in $\mathbb{C}^N$ is a set of the form $\Delta = D_1 \times \cdots \times D_n \times B$ where $D_1, \ldots, D_n \subset \mathbb{C}$ are discs and $B \subset \mathbb{C}^{N-n}$ is a ball centered at x. We also denote by $\Delta^i = D_{i+1} \times \cdots \times D_n \times B$ the corresponding polydisc in $\mathbb{C}^{N-i}$. Each polydisc Δ^i is considered with the obvious Whitney stratification given by taking all combinations of products of interiors and boundaries on the discs and the ball (see [16, 1.3]).

Observe that a polydisc has a ball in the product of its definition. This ball is necessary to have control on the codimension of X. More precisely, as we will work with the sets X^i and $X^i \cap \Delta^i$, we need to stop taking sections as soon as X^i is a curve and, at that point, we take a ball that completes the product structure we want to find (the so-called polydiscs).

Definition 4.1 We say that Δ is a *privileged* polydisc if for any smaller polydisc $\Delta' \subset \Delta$ centered at 0 in $\mathbb{C}^N$, all the strata of $\left(\Delta'\right)^i$ are transverse to all the strata of X^i, for all $i = 0, \dots, n$.

For each privileged polydisc Δ, the set $X^i \cap \Delta^i$ has an induced Whitney stratification. By the curve selection lemma, the function $f^i : X^i \cap \Delta^i \to \mathbb{C}$ has an isolated critical value in the stratified sense at the origin in $\mathbb{C}$. So, $(f^i)^{-1}(b)$ is transverse to all the strata of $X^i \cap \Delta^i$, for all $b \in \mathbb{C}$ small enough. In particular, there exists $\eta > 0$ small enough such that

$$f^i : X^i \cap \Delta^i \cap (f^i)^{-1}(\partial D_\eta) \longrightarrow \partial D_\eta$$

is a proper stratified submersion and hence, a locally C^0-trivial fibration homotopic to a Milnor fibration with a homotopy which preserves the fibres. This follows from the Thom-Mather first isotopy lemma and the fact that privileged polydiscs are good neighbourhoods relatively to $\{f = 0\}$ in Prill's sense (cf. [25]), see the end of [16, Section 1] for more details. In fact, this is the original definition of privileged polydisc in [16] in the case $X = \mathbb{C}^{n+1}$. The existence of privileged polydiscs is proved in the next lemma:

Lemma 4.2 *Any small enough polydisc is privileged.*

Proof We show by induction on $i = 0, \dots, n$ that f^i has a privileged polydisc Δ^i. The case $i = n$ is obvious since a privileged polydisc is nothing but a Milnor ball. Assume f^i has a privileged polydisc Δ^i. We shall find a disc D_ϵ such that $D_\epsilon \times \Delta^i$ is a privileged polydisc for f^{i-1}. We use the function $\rho : \mathbb{C}^{N-i+1} \to \mathbb{R}$ given by $\rho(z) = |z_i|$. By the curve selection lemma we can find $\epsilon > 0$ such that for any $0 < \epsilon' \leq \epsilon$, $\partial D_{\epsilon'} \times \mathbb{C}^{N-i}$ is transverse to each stratum of X^{i-1}.

Consider the polydisc $D_{\epsilon'} \times \left(\Delta^i\right)'$, for a polydisc $\left(\Delta^i\right)'$ contained in Δ^i and $\epsilon' \leq \epsilon$. We have two types of strata: $\mathring{D}_{\epsilon'} \times R_\alpha$ and $\partial D_{\epsilon'} \times R_\alpha$, for some stratum R_α of $\left(\Delta^i\right)'$. On the other hand, if we consider a stratum S_β of X^{i-1}, and we take the hyperplane section to get X^i, it gives the stratum S'_β of X^i.

By induction hypothesis, R_α is transverse to S'_β, that is,

$$T_z R_\alpha + T_z S'_\beta = \mathbb{C}^{N-i}, \tag{5}$$

for all $z \in R_\alpha \cap S'_\beta$. This obviously implies that

$$\mathbb{C} \times T_z R_\alpha + T_{(t,z)} S_\beta = \mathbb{C} \times \mathbb{C}^{N-i},$$

which gives the transversality between $\mathring{D}_{\epsilon'} \times R_\alpha$ and S_β at (t, z).

Moreover, the choice of ϵ implies that

$$T_t \partial D_{\epsilon'} \times \mathbb{C}^{N-i} + T_{(t,z)} S_\beta = \mathbb{C} \times \mathbb{C}^{N-i},$$

for all $(t, z) \in (\partial D_{\epsilon'} \times \mathbb{C}^{N-i}) \cap (V \times S_\beta)$. Therefore, any vector $(u, v) \in \mathbb{C} \times \mathbb{C}^{N-i}$ can be written as $(u, v) = (u_1, v_1) + (u_2, v_2)$, for some $(u_1, v_1) \in T_t \partial D_{\epsilon'} \times \mathbb{C}^{N-i}$ and

$(u_2, v_2) \in T_{(t,z)} S_\beta$. If $z \in R_\alpha$, we also have by (5) that $v_1 = w_1 + w_2$, with $w_1 \in T_z R_\alpha$ and $w_2 \in T_z S'_\beta$. We get

$$(u, v) = (u_1, w_1) + (0, w_2) + (u_2, v_2),$$

with $(u_1, w_1) \in T_t \partial D_{\epsilon'} \times T_z R_\alpha$ and $(0, w_2) + (u_2, v_2) \in T_{(t,z)} S_\beta$. This shows that $\partial D_{\epsilon'} \times R_\alpha$ is also transverse to S_β. □

Remark 4.3 We see in the proof of Lemma 4.2 that the choice of the radius of each disc of Δ is independent of the radii of the other discs. The reason of this independence is that we were asking that $\partial D_{\epsilon'} \times \mathbb{C}^{N-i}$ has to be transverse to each stratum of X^{i-1} at any point instead of being transverse only at points on $X^{i-1} \cap \mathbb{C} \times \Delta^i$, which would have given $D_{\varepsilon'}$ a relation with Δ^i that restricts it. However, as presented in the proof, there could be a relation between the radii of the discs and the radius of the ball.

5 Proof of the main theorem

In this section we give the proof of Theorem 0.2. The proof is by induction on the dimension of (X, x). To do this, we need the carrousel construction in [16] of the second named author. We also refer to [19] for a detailed construction of the carrousel for a general complex analytic germ of plane curve $(C, 0)$. In our case, we apply this construction for the Cerf's diagram $C = \Delta_\ell(f, x)$ of a holomorphic function $f : (X, x) \to (\mathbb{C}, 0)$ with respect to a generic linear form ℓ. The key point here is that if $f \in \mathfrak{m}^2_{X,x}$, then all the branches of C are tangent to the axis $\{v = 0\}$ at the origin, where u, v are the coordinates of the plane $\mathbb{C}^2$ (see Proposition 1.5).

Lemma 5.1 (cf. [16, 3.2.2]) *Let $(C, 0)$ be a germ of complex analytic plane curve whose tangent cone is the axis $\{v = 0\}$. There exist small enough discs D and D_η centered at the origin in $\mathbb{C}$ and a smooth vector field ω on the solid torus $D \times \partial D_\eta$ such that:*

(i) The projection onto the second component of ω gives the unit tangent vector field over ∂D_η (i.e., the tangent field of length η, the radius of D_η, in the positive direction),
(ii) the restriction to $\{0\} \times \partial D_\eta$ is indeed the unit vector field,
(iii) the vector field ω is tangent to $(D \times \partial D_\eta) \cap \{\delta = \epsilon\}$ for all $\epsilon \in \mathbb{C}$ small enough, where $\delta = 0$ is a reduced equation of C, and
(iv) the only integral curve that is closed after a loop in ∂D_η is $\{0\} \times \partial D_\eta$.

The discs D and D_η in Lemma 5.1 are chosen small enough so that there is a disc D_1 containing D strictly such that $\big(D_1 \times \{0\}\big) \cap C = \{0\}$ and such that $\{v = t\}$, for $\eta \geq |t| > 0$, intersects the curve C in $\big(D \times \{0\}, C\big)_0$ points in $D \times D_\eta$ where $(\bullet, \bullet)_0$ is the local intersection number at 0 (see Fig. 2).

The geometrical meaning of the carrousel is the following (see Fig. 3): we first take a representative C of the plane curve on some open neighbourhood W of the origin in the plane $\mathbb{C}^2$. Let L be the intersection of W with the axis $\{u = 0\}$. We consider W with the Whitney stratification given by the strata $W \setminus (C \cup L)$, $C \setminus \{0\}$, $L \setminus \{0\}$ and $\{0\}$ and the function germ $\pi_2 : (W, 0) \to (\mathbb{C}, 0)$ given by $\pi_2(u, v) = v$. The choice of D and D_η is made so that

$$\pi_2 : D \times \partial D_\eta \to \partial D_\eta$$

is a proper stratified submersion with the induced stratification in $D \times \partial D_\eta$. By Items (i) to (iii) in Lemma 5.1, ω is a stratified vector field on $D \times \partial D_\eta$ which is a lifting of the

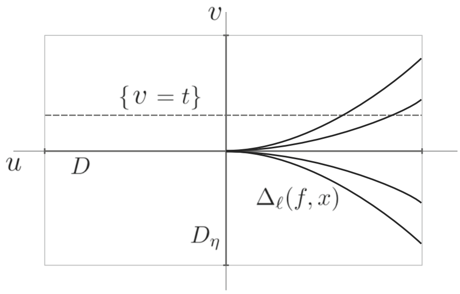

Fig. 2 Cerf's diagram and the setting to construct the carrousel

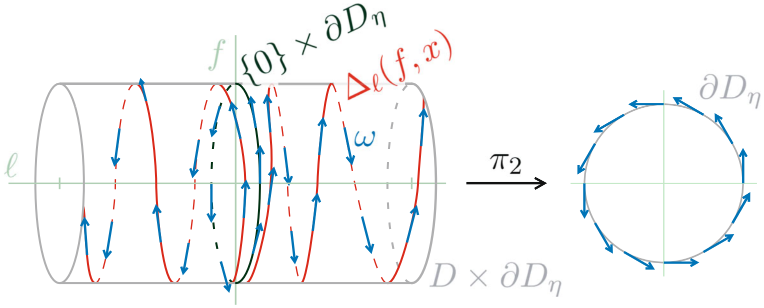

Fig. 3 Representation of a carrousel ω

unit tangent vector field on ∂D_η. Hence, its flow provides a local geometric monodromy $h : D \times \{t\} \to D \times \{t\}$ for some $t \in \partial D_\eta$, which preserves the point $(0, t)$ and the finite set $C \cap \big(D \times \{t\}\big)$. By condition (iv), the only fixed point of h is $(0, t)$.

Now we can give the proof of our main result:

Proof of Theorem 0.2 Assume that $(X, x) \subset (\mathbb{C}^N, x)$. We take a privileged polydisc Δ in $\mathbb{C}^N$ at x and a small disc D_η in $\mathbb{C}$ at 0 such that the restriction

$$f : X \cap \Delta \cap f^{-1}(\partial D_\eta) \to \partial D_\eta$$

is a proper stratified submersion. We claim that there exists a stratified vector field ξ on $X \cap \Delta \cap f^{-1}(\partial D_\eta)$ which is a lifting of the unit vector field θ on ∂D_η whose flow provides a local geometric monodromy with no fixed points. We prove this by induction on the dimension of X at x.

Assume first that $\dim(X, x) = 1$. Let $X_1, \dots, X_r$ be the analytic branches of X at x. Then $X \cap \Delta \cap f^{-1}(\partial D_\eta)$ is the disjoint union of all the sets $X_i \cap \Delta \cap f^{-1}(\partial D_\eta)$, $i = 1, \dots, r$. Hence, it is enough to show the claim in the case that X is irreducible at x. Let $n : \tilde{X} \to X$ be the normalization of X at x.

Since $f \in \mathfrak{m}^2_{X,x}$, we can take an analytic extension $\overline{f} : (\mathbb{C}^N, x) \to (\mathbb{C}, 0)$ such that $F \in \mathfrak{m}^2_N$. After a reparametrization, we can assume that $\tilde{X}$ is an open neighbourhood of 0 in $\mathbb{C}$, $\{0\} = n^{-1}(x)$ and $F \circ n(s) = s^k$, for some $k \geq 2$. In this case, θ lifts in a unique way by the map $F \circ n$ and has a local geometric monodromy with no fixed points. But n induces a diffeomorphism on $\tilde{X} \setminus \{0\}$ onto $X \setminus \{x\}$, so we have also a unique lifting on $X \cap \Delta \cap f^{-1}(\partial D_\eta)$ whose geometric monodromy has no fixed points.

Now we assume the claim is true when $\dim(X, x) = n$ and prove it in the case that $\dim(X, x) = n + 1$. Let $\ell : \mathbb{C}^N \to \mathbb{C}$ be a generic linear form and consider the map

$\Phi = (\ell, f)$. We have a commutative diagram as follows:

$$\begin{array}{ccccc} X \cap \Delta \cap \Phi^{-1}(D \times \partial D_\eta) & \xrightarrow{\Phi} & D \times \partial D_\eta & \xrightarrow{\pi_2} & \partial D_\eta \\ \uparrow & & \uparrow & \nearrow_{\pi_2} & \\ X \cap \Delta \cap \ell^{-1}(0) \cap f^{-1}(\partial D_\eta) & \xrightarrow{(0,f)} & \{0\} \times \partial D_\eta & & \end{array},$$

where the vertical arrows are the inclusions and π_2 is the projection onto the second component. Here we choose the polydiscs Δ and $D \times D_\eta$ small enough such that Φ is a Thom proper map (see Lemma 4.2, Theorem 2.6 and Remark 2.7). The stratification in $D \times \partial D_\eta$ is given by the strata $D \times \partial D_\eta \backslash (C \cup L)$, $(D \times \partial D_\eta) \cap C$ and L, where $L = \{0\} \times \partial D_\eta$ and $C = \Delta_\ell(f, x)$ is the Cerf's diagram.

By induction hypothesis, there exists a stratified vector field ξ_1 on $X \cap \Delta \cap \ell^{-1}(0) \cap f^{-1}(\partial D_\eta)$ which is a lifting of θ and whose geometric monodromy has no fixed points. If C is empty, the claim is obvious by Proposition 1.4, so we can assume that C is not empty.

By the carrousel of Lemma 5.1, there exists a stratified vector field ω on $D \times \partial D_\eta$ which satisfies Items (i) to (iv) of the lemma. Since ω is a lifting of θ, it is globally integrable by Theorem 3.10. Moreover, ω is not zero along L and $(D \times \partial D_\eta) \cap C$, so we can use the flow of ω to construct a weakly controlled tube system $\mathcal{T}'$ of $D \times \partial D_\eta$ such that ω is weakly controlled. By Corollary 3.11, ω lifts to a stratified vector field ξ on $X \cap \Delta \cap \Phi^{-1}(D \times \partial D_\eta)$ which is globally integrable. Moreover, by using a partition of unity, we can construct ξ in such a way that it coincides with ξ_1 on $X \cap \Delta \cap \ell^{-1}(0) \cap f^{-1}(\partial D_\eta)$.

Let $F = X \cap \Delta \cap f^{-1}(t)$, with $t \in \partial D_\eta$ and consider the geometric monodromy $h : F \to F$ induced by ξ. On one hand, ξ is an extension of ξ_1, so $h\big(F \cap \ell^{-1}(0)\big) = F \cap \ell^{-1}(0)$ and h has no fixed points on $F \cap \ell^{-1}(0)$. On the other hand, Item (iv) of Lemma 5.1 implies that h does not have fixed points on $F \backslash \ell^{-1}(0)$ either. This completes the proof. □

The proof relied on the hypothesis of f being in $\mathfrak{m}^2_{X,x}$, and actually this hypothesis is necessary. Here we give a couple of examples which illustrate this fact.

Example 5.2 Let $(C, 0)$ be the ordinary triple point singularity in $(\mathbb{C}^3, 0)$. This is equal to the union of the three coordinate axis in $\mathbb{C}^3$ and the defining equations are given by the 2×2-minors of the matrix

$$M = \begin{pmatrix} x & y & z \\ y & z & x \end{pmatrix}$$

This gives to $(C, 0)$ a structure of isolated determinantal singularity in the sense of [24]. According also to [24], we can construct a determinantal smoothing of $(C, 0)$ by taking the 2×2-minors of $M_t = M + tA$, where A is a generic 2×3-matrix with coefficients in $\mathbb{C}$ and $t \in \mathbb{C}$.

In fact, let

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and let $(X, 0)$ be the surface in $(\mathbb{C}^3 \times \mathbb{C}, 0)$ defined as the zero set of the 2×2-minors of M_t. The projection $f : (X, 0) \to (\mathbb{C}, 0)$, $f(x, y, z, t) = t$ provides a flat deformation whose special fibre is $(C, 0)$ and whose generic fibre $F = f^{-1}(t)$, for $t \neq 0$, is a smooth curve. We can see F as a kind of "determinantal Milnor fibre" of $(C, 0)$.

It follows from [4, p. 279] that F is diffeomorphic to a disk with two holes (as in Fig. 4) and that the monodromy $h_* : H_1(F; \mathbb{Z}) \to H_1(F; \mathbb{Z})$ is the identity. Since $H_1(F; \mathbb{Z}) \cong \mathbb{Z}^2$,

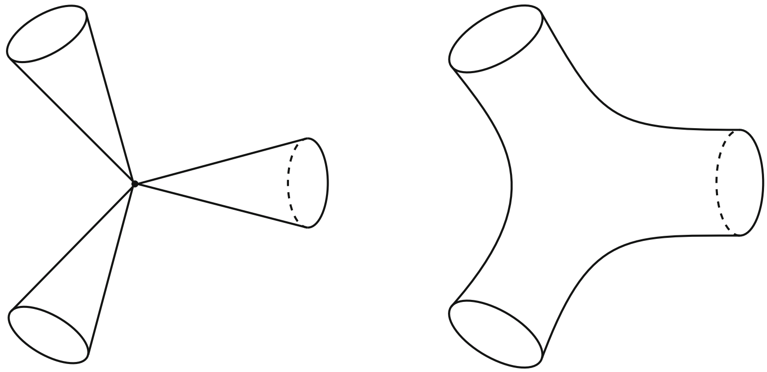

Fig. 4 The ordinary triple point singularity and its determinantal Milnor fibre

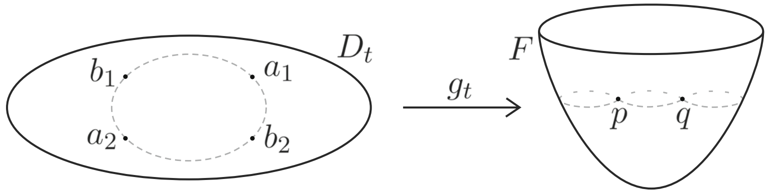

Fig. 5 The map g_t and the double points, a_1, b_1, a_2 and b_2

the Lefschetz number is -1, and hence any local geometric monodromy must have a fixed point. A simple computation shows that $f \notin \mathfrak{m}_{X,x}^2$ in this example.

Example 5.3 Consider the A_4 plane curve singularity $(C, 0)$ whose equation in $(\mathbb{C}^2, 0)$ is $x^5 - y^2 = 0$. The monodromy of the classical Milnor fibre of $(C, 0)$ is well known and we will not discuss it. Instead, we look at the monodromy of the disentanglement of $(C, 0)$ in Mond's sense (see [22, Chapter 7]). We see $(C, 0)$ as the image of the map germ $g_0 \colon (\mathbb{C}, 0) \to (\mathbb{C}^2, 0)$ given by $g_0(s) = (s^2, s^5)$, which has an isolated instability at the origin.

Since we are in the range of Mather's nice dimensions, we can take a stabilisation, that is, a 1-parameter unfolding $G \colon (\mathbb{C} \times \mathbb{C}, 0) \to (\mathbb{C}^2 \times \mathbb{C}, 0)$, $G(s, t) = (g_t(s), t)$ such that for any $t \neq 0$, g_t has only stable singularities. By definition, the *disentanglement* F is the image of the map g_t intersected with a small enough ball B in $\mathbb{C}^2$ centered at the origin and t small enough. Since F is 1-dimensional and connected, it has the homotopy type of a bouquet of spheres (this is true also in higher dimensions by a theorem due to Lê) of dimension 1. The number of such spheres is called the *image Milnor number* and is denoted by $\mu_I(g_0)$.

Observe that F is also the generic fibre of the function $f : (X, 0) \to (\mathbb{C}, 0)$ where $(X, 0)$ is the image of G in $(\mathbb{C}^2 \times \mathbb{C}, 0)$ and $f(x, y, t) = t$. We are interested in the local monodromy of f at the origin.

In our case, we take $g_t(s) = (s^2, s^5 + ts)$. It is easy to see that for $t \neq 0$, g_t is an immersion with two transverse double points $p = g_t(a_1) = g_t(a_2)$ and $q = g_t(b_1) = g_t(b_2)$ where a_1, a_2, b_1, b_2 are the four roots of $s^4 + t = 0$, with $a_1 = -a_2$ and $b_1 = -b_2$. Hence, g_t defines a stabilisation of g_0. Observe that the number of double points coincides with the delta invariant $\delta(C, 0) = 2$. The disentanglement F is the image of g_t and is homeomorphic to the quotient of a closed 2-disk D_t under the relations $a_1 \sim a_2$ and $b_1 \sim b_2$ (see Fig. 5). Thus, F has the homotopy type of $S^1 \vee S^1$ and $\mu_I(g_0) = 2$.

The locally C^0-trivial fibration is the restriction $f : X \cap (B \times S^1_\eta) \to S^1_\eta$, for a small enough $\eta > 0$.

In order to construct a geometric monodromy $h\colon F \to F$ it is enough to find a 1-parameter group of stratified homeomorphisms $h_\theta\colon X \cap (B \times S^1_\eta) \to X \cap (B \times S^1_\eta)$, with $\theta \in \mathbb{R}$, which make the following diagram commutative

$$\begin{array}{ccc} X \cap (B \times S^1_\eta) & \xrightarrow{f} & S^1_\eta \\ {\scriptstyle h_\theta}\downarrow & & \downarrow{\scriptstyle r_\theta} \\ X \cap (B \times S^1_\eta) & \xrightarrow{f} & S^1_\eta \end{array},$$

where $r_\theta(t) = e^{i\theta}t$. In this situation, $h : F \to F$ is obtained as the restriction of $h_{2\pi}$.

Since $(C, 0)$ is weighted homogeneous with weights $(5, 2)$, instead of a Euclidean ball in $\mathbb{C}^2$ it is better to consider the (non-Euclidean) ball B given by $|x|^5 + |y|^2 \le 1$. Thus, $C \cap B = g_0(D)$, where D is the disk in $\mathbb{C}$ given by $|s|^{10} \le 1/2$. For $t \neq 0$, $F = g_t(D_t)$, where now $D_t = g_t^{-1}(B)$ is the disk in $\mathbb{C}$ given by

$$|s|^{10} + |s|^2|s^4 + t|^2 \le 1.$$

Given a point $(x, y, t) \in X$, we have $(x, y, t) = G(s, t)$ for some $s \in \mathbb{C}$. We define $h_\theta\colon X \to X$ as

$$h_\theta(G(s, t)) = G\left(e^{\frac{i\theta}{4}}s, e^{i\theta}t\right).$$

Now, we have to check that, indeed, this gives a group of stratified homeomorphisms. We consider in X the stratification given by $\{X\setminus Y, Y\}$, where Y is the double point curve with equations $x^2 + t = 0$, $y = 0$. Since G is an embedding on $X\setminus Y$, h_θ is well defined and is a diffeomorphism on $X\setminus Y$. When $(x, y, t) \in Y$ we have $G(s, t) = (s^2, 0, t)$, with $s^2 = x$ and $s^4 + t = 0$. It follows that

$$h_\theta(x, 0, t) = G\left(e^{\frac{i\theta}{4}}s, e^{i\theta}t\right) = \left(e^{\frac{i\theta}{2}}s^2, 0, e^{i\theta}t\right) = \left(e^{\frac{i\theta}{2}}x, 0, e^{i\theta}t\right),$$

and $\left(e^{i\theta/2}x\right)^2 + e^{i\theta}t = e^{i\theta}(x^2 + t) = 0$. Thus, h_θ is also well defined on Y, $h(Y) = Y$ and the restriction $h\colon Y \to Y$ is a diffeomorphism. It is also clear that $h_\theta\colon X \to X$ and its inverse are both continuous, so it is a stratified homeomorphism. It only remains to show that $h_\theta\big(X \cap (B \times S^1_\eta)\big) = X \cap (B \times S^1_\eta)$, because we have to work with a specific representative, but this a consequence of the equality:

$$\left|e^{\frac{i\theta}{4}}s\right|^{10} + \left|e^{\frac{i\theta}{4}}s\right|^2 \left|\left(e^{\frac{i\theta}{4}}s\right)^4 + e^{i\theta}t\right|^2 = |s|^{10} + |s|^2|s^4 + t|^2.$$

The geometric monodromy $h\colon F \to F$ is now the restriction of $h_{2\pi}$, which gives $h(g_t(s)) = g_t\left(e^{i\pi/2}s\right)$, that is, it is obtained by a $\pi/2$-rotation in the disk D_t.

To finish, we compute $h_*\colon H_1(F; \mathbb{Z}) \to H_1(F; \mathbb{Z})$. We recall that F is homeomorphic to the quotient of D_t under the relations $a_1 \sim a_2$ and $b_1 \sim b_2$. The four points a_1, a_2, b_1, b_2 are on a square contained in the interior of D_t and centered at the origin, which is obviously invariant under the $\pi/2$-rotation. We denote by a, b, c, d the four edges of the square as in Fig. 6.

We take the cycles $a + b$ and $c + d$ as a basis of $H_1(F; \mathbb{Z})$. Obviously, $h_*(a + b) = b + c$ and $h_*(b + c) = c + d = -(a + b)$ so the matrix of h_* with respect to this basis is:

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

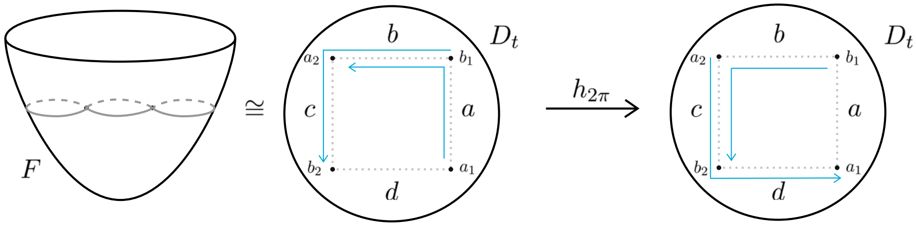

Fig. 6 Monodromy of the fiber F

The Lefschetz number is 1 and hence, any local geometric monodromy must have a fixed point. In fact, in our construction there is exactly one fixed point, namely, the origin of the disk D_t which is invariant under the $\pi/2$-rotation. As in Example 5.2, it is not difficult to check that $f \notin \mathfrak{m}^2_{X,x}$.

6 Applications

The first application of Theorem 0.2 is a new proof of the following result, originally given by A'Campo. It can be used to show that any hypersurface (X, x) in $\mathbb{C}^{n+1}$ with smooth topological type, must be smooth.

Corollary 6.1 (cf. [1, Theorem 3]) *Let $X \subset \mathbb{C}^{n+1}$ be a germ of a hypersurface, not necessarily smooth at $x \in X$. If the Milnor fiber F_x of X at x has trivial reduced homology with complex coefficients, $\tilde{H}_i(F_x; \mathbb{C}) \cong 0$, then x is a smooth point of X.*

Proof Let $f : (\mathbb{C}^{n+1}, x) \to (\mathbb{C}, 0)$ be the holomorphic germ which gives a reduced equation of (X, x). The Lefschetz number of the local monodromy of f is 1 and, hence, $f \notin \mathfrak{m}^2_{\mathbb{C}^{n+1},x}$, by Theorem 0.2. □

We recall that two germs of complex spaces (X, x) and (Y, y) in $\mathbb{C}^{n+1}$ have the same topological type if there exists a homeomorphism $\varphi : (\mathbb{C}^{n+1}, x) \to (\mathbb{C}^{n+1}, y)$ such that $\varphi(X, x) = (Y, y)$.

Corollary 6.2 *Let (X, x) be a germ of hypersurface in $\mathbb{C}^{n+1}$. If (X, x) has the topological type of a smooth hypersurface, then (X, x) is smooth.*

Proof If (X, x) has the topological type of a smooth hypersurface then its Milnor fibre has trivial reduced homology by [14, Proposition, p. 261]. This implies that (X, x) is smooth by Corollary 6.1. □

This corollary is related to Zariski's multiplicity conjecture [31] which claims that two hypersurfaces in $\mathbb{C}^{n+1}$ with the same topological type have the same multiplicity. Since a hypersurface is smooth if and only if it has multiplicity 1, Corollary 6.2 is just a particular case of the conjecture. Zariski showed the conjecture for plane curves but it remains still open in higher dimensions. Another related result is Mumford's theorem [23] which states that if X is a normal surface and X is a topological manifold at $x \in X$, then X is smooth at x.

Our second application is a no coalescing theorem for families of functions defined on spaces with Milnor property. In [15], the second named author showed the following interesting application of A'Campo's theorem (see also [6, 13]). Let $\{H_t\}_{t\in\mathbb{C}}$ be an analytic family of hypersurfaces defined on some open subset $U \subset \mathbb{C}^n$ with only isolated singularities. Take B a Milnor ball for H_0 around a singular point $x_0 \in H_0$ and assume for all t small enough, the sum of the Milnor numbers of all the singular points of H_t in B is constant, that is,

$$\sum_{x \in H_t \cap B} \mu(H_t, x) = \mu(H_0, x_0).$$

Then $H_t \cap B$ contains a unique singular point x of H_t. The purpose of this section is to prove an adapted version of this result in a more general context, namely, for Milnor spaces in the sense of [9]:

Definition 6.3 A *Milnor space* is a reduced complex space X such that at each point $x \in X$, the rectified homotopical depth $\operatorname{rhd}(X, x)$ is equal to $\dim(X, x)$.

We refer to [9] for the definition of the rectified homotopical depth and basic properties of Milnor spaces. In general, $\operatorname{rhd}(X, x) \leq \dim(X, x)$, so Milnor spaces are those whose rectified homotopical depth is maximal at any point. Some important properties are the following:

(1) any smooth space X is a Milnor space,
(2) any Milnor space X is equidimensional,
(3) if X is a Milnor space and Y is a hypersurface in X (i.e. Y has codimension one and is defined locally in X by one equation), then Y is also a Milnor space.

As a consequence, any local complete intersection X (not necessarily with isolated singularities) is a Milnor space. Our setting is motivated by the following theorem due to Hamm and Lê (see [9, Theorem 9.5.4]):

Theorem 6.4 *Let (X, x) be a germ of Milnor space and assume that $f : (X, x) \to (\mathbb{C}, 0)$ has an isolated critical point in the stratified sense. Then the general fibre F of f has the homotopy type of a bouquet of spheres of dimension* $\dim(X, x) - 1$.

Corollary 6.5 *With the hypothesis and notation of Theorem 6.4, if $f \in \mathfrak{m}_{X,x}^2$, then the trace of the induced map $h_* : H_{n-1}(F; \mathbb{Z}) \to H_{n-1}(F; \mathbb{Z})$ by the monodromy $h : F \to F$ is $(-1)^n$, where $n =$* $\dim(X, x)$.

Definition 6.6 With the hypothesis and notation of Theorem 6.4, the number of spheres of F is called the *Milnor number* of f and is denoted by $\mu(f)$. We say that the critical point is *non-trivial* if $\mu(f) > 0$.

We want to generalize the non-coalescing theorem of the second author for families of hypersurfaces $\{H_t\}_{t \in \mathbb{C}}$ in [15]. As we want to generalize it in the setting of fibers inside Milnor spaces, we obviously need a convenient concept of *family of Milnor spaces* and its corresponding *family of complex functions that give the equations of the fibers*. This is covered in Definition 6.7.

Consider a germ of complex analytic space (X_0, x_0). Let $f_0 : (X_0, x_0) \to (\mathbb{C}, 0)$ be a germ of holomorphic function. Let X_0 be a small representative of (X_0, x_0) and let $\mathcal{S}$ be a Whitney stratification of X_0. We assume that a representative f_0 has an isolated critical point in the stratified sense at x_0. We define:

Definition 6.7 A *stratified deformation* of (X_0, x_0) is a flat deformation $\pi : (\mathfrak{X}, x_0) \to (\mathbb{C}, 0)$, where $\mathfrak{X}$ is an analytic space with an analytic Whitney stratification such that, for a representative π:

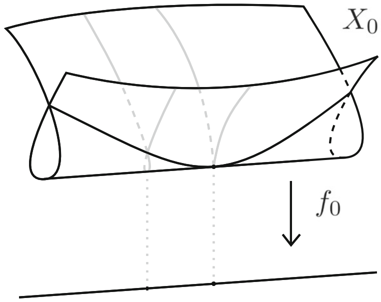

Fig. 7 The function f_0 with a critical point

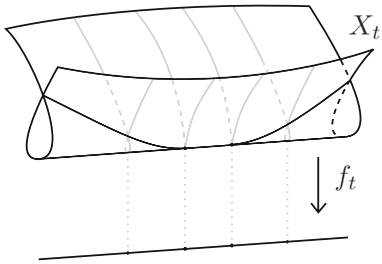

Fig. 8 The function f_t with two critical points

1. $\pi^{-1}(0) = X_0$ as analytic spaces,
2. π has isolated critical points in the stratified sense,
3. the stratification of X_0 coincides with the induced stratification by $\mathfrak{X}$ on $\pi^{-1}(0)$.

Given a stratified deformation $\pi\colon (\mathfrak{X}, x_0) \to (\mathbb{C}, 0)$, a *stratified unfolding* of a germ f_0 as above is a holomorphic map $\mathcal{F}\colon (\mathfrak{X}, x_0) \to (\mathbb{C} \times \mathbb{C}, 0)$ such that $p_1 \circ \mathcal{F}|_{X_0} = f_0$ and $p_2 \circ \mathcal{F} = \pi$, where $p_i : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$, $i = 1, 2$, is the ith-projection.

We can always assume that $\mathfrak{X}$ is embedded in $\mathbb{C}^N \times \mathbb{C}$ and choose coordinates in such a way that $\pi(x, t) = t$. So, we can write the stratified unfolding as $\mathcal{F}(x, t) = (f_t(x), t)$. For each $t \in \mathbb{C}$, we have a function $f_t\colon X_t \to \mathbb{C}$, where $X_t = \pi^{-1}(t)$. Here we consider in X_t the stratification induced by $\mathfrak{X}$ and denote by $\Sigma(f_t)$ the set of stratified critical points of f_t.

Example 6.8 We consider the function $f_0 : (X_0, 0) \to (\mathbb{C}, 0)$, where X_0 is the surface in $\mathbb{C}^3$ given by $z^2 - y(x^2 + y)^2 = 0$ and $f_0(x, y, z) = x$. The stratification in X_0 is $\big\{\{0\}, C_0 - \{0\}, X_0\big\}$, where C_0 is the curve $z = x^2 + y = 0$. It is easy to see f_0 has only one critical point in the stratified sense at the origin and that $\mu(f_0) = 1$ (see Fig. 7).

Now we define a stratified deformation $\pi : (\mathfrak{X}, 0) \to (\mathbb{C}, 0)$ and a stratified unfolding $\mathcal{F}\colon (\mathfrak{X}, 0) \to (\mathbb{C} \times \mathbb{C}, 0)$ as follows: $\mathfrak{X}$ is the hypersurface in $\mathbb{C}^3 \times \mathbb{C}$ with equation $z^2 - y(x^2 + y + t)^2 = 0$, $\pi(x, y, z, t) = t$ and $\mathcal{F}(x, y, z, t) = (x, t)$. The stratification in $\mathfrak{X}$ is $\big\{\{0\}, \mathcal{D}\backslash\{0\}, \mathcal{C}\backslash\mathcal{D}, \mathfrak{X}\backslash\mathcal{C}\big\}$, where $\mathcal{D}$ is the curve $z = y = x^2 + t = 0$ and $\mathcal{C}$ is the surface $z = x^2 + y + t = 0$. Again it is not difficult to check that all conditions of Definition 6.7 hold.

For $t \neq 0$, $f_t : X_t \to \mathbb{C}$ has two critical points in the stratified sense at $\big(\pm\sqrt{-t}, 0, 0\big)$, which are the points in $D_t := X_t \cap \mathcal{D}$. We see that f_t has also Milnor number 1 at each critical point $\big(\pm\sqrt{-t}, 0, 0\big)$ (see Fig. 8).

The following theorem could seem very restrictive due to the length of the hypotheses. On the contrary, its statement only says that, with a *general notion of family of ambient spaces*

($\mathfrak{X}$) and a *general notion of equation of the fibers* ($\mathcal{F}$), if it happens what we have proven in some cases (Theorem 0.2 or Corollary 6.5), then we have a non-coalescing result.

Theorem 6.9 *Let $f_0\colon (X_0, x_0) \to (\mathbb{C}, 0)$ be a function with a non-trivial isolated critical point and let $\mathcal{F}\colon (\mathfrak{X}, x_0) \to (\mathbb{C}\times\mathbb{C}, 0)$ be a stratified unfolding of f_0 such that $\mathfrak{X}$ is a Milnor space. We set $Y_t = f_t^{-1}(0)$ and assume that for any $x \in \Sigma(f_t) \cap Y_t$, the trace of the local monodromy of f_t at x in dimension $n-1$ is $(-1)^n$, where $\dim(X_0, x_0) = n$. Let B_0 be a Milnor ball for f_0 at x_0 and assume that for any $t \in \mathbb{C}$ small enough,*

$$\sum_{x\in\Sigma(f_t)\cap Y_t\cap B_0} \mu_x(f_t) = \mu_{x_0}(f_0), \tag{6}$$

where $\mu_x(f_t)$ is the Milnor number of f_t at x. Then $Y_t \cap B_0$ contains a unique non-trivial critical point x of f_t.

Proof Denote by $\Sigma(\mathcal{F})$ the set of stratified critical points of $\mathcal{F}$ and assume that $\dim(X_0, x_0) = n > 2$. It follows that $(x, t) \in \Sigma(\mathcal{F})$ if and only if $x \in \Sigma(f_t)$. Since the stratification of $\mathfrak{X}$ is analytic, $\Sigma(\mathcal{F})$ is also analytic. On one hand, we have that

$$\dim \Sigma(\mathcal{F}) \cap \{t = 0\} = \dim \Sigma(f_0) = 0$$

and thus, $\dim \Sigma(\mathcal{F}) \leq 1$. On the other hand, by (6), we obtain that

$$\Sigma(\mathcal{F}) \cap \{t = t_0\} = \Sigma(f_{t_0}) \neq \emptyset,$$

for $t_0 \neq 0$, so $\dim \Sigma(\mathcal{F}) = 1$. Moreover, $\mathcal{F}^{-1}(0) \cap \Sigma(\mathcal{F}) = \{0\}$, hence its image $\Delta = \mathcal{F}\big(\Sigma(\mathcal{F})\big)$ is also analytic of dimension 1 in $(\mathbb{C}\times\mathbb{C}, 0)$ by Remmert's proper map theorem.

We fix a small enough open polydisc $D_\eta \times D_\rho$ in $\mathbb{C}\times\mathbb{C}$ such that the restriction

$$\mathcal{F} : (B_0 \times D_\rho)\backslash\mathcal{F}^{-1}(\Delta) \to (D_\eta \times D_\rho)\backslash\Delta \tag{7}$$

is a proper stratified submersion and such that $\Delta \cap (D_\eta \times \{0\}) = \{0\}$. By the Thom–Mather first isotopy lemma, (7) is a locally C^0-trivial fibration. Given $s \in D_\eta\backslash\{0\}$, we have $(s, 0) \in (D_\eta \times D_\rho)\backslash\Delta$ and hence the fibre,

$$\mathcal{F}^{-1}(s, 0) \cap (B_0 \times D_\rho) = (f_0^{-1}(s) \cap B_0) \times \{0\},$$

coincides with the general fibre of f_0.

Let $t \in D_\rho$ and assume that $\Sigma(f_t) \cap f_t^{-1}(0) \cap B_0 = \{x_1, \ldots, x_k\}$. For each $i = 1, \ldots, k$, we take a Milnor ball B_i for f_t at x_i such that B_i is contained in the interior of B_0 and $B_i \cap B_j = \emptyset$ if $i \neq j$. Now we choose $0 < \eta' < \eta$ such that for all s, with $0 < |s| < \eta'$, $(s, t) \notin \Delta$.

Fix a point $s \in D_{\eta'}$ and consider the loop $\gamma(\theta) = se^{i\theta}$, $\theta \in [0, 2\pi]$. This loop induces a monodromy $h : f_t^{-1}(s) \cap B_0 \to f_t^{-1}(s) \cap B_0$ which coincides, up to isotopy, with the geometric monodromy of f_0 at x_0. Moreover, by adding the boundaries of the balls B_i as strata in the domain of (7), we can assume that:

(1) $h(f_t^{-1}(s) \cap B_i) = f_t^{-1}(s) \cap B_i$ and $h_i = h|_{f_t^{-1}(s)\cap B_i}$ is the monodromy of f_t at x_i, for each $1 = 1, \ldots, k$;
(2) h is the identity outside the interior of $B_1 \cup \cdots \cup B_k$.

Let $U = f_t^{-1}(s) \cap \left(B_0 \setminus \bigcup_{i=1}^k \mathring{B}_i \right)$ and $V = f_t^{-1}(s) \cap \bigcup_{i=1}^k B_i$. By considering the Mayer-Vietoris sequence of the pair (U, V) we get a diagram whose rows are exact sequences:

$$\begin{array}{ccccccc} 0 \longrightarrow & H_{n-1}(U \cap V) & \longrightarrow & H_{n-1}(U) \oplus H_{n-1}(V) & \longrightarrow & H_{n-1}(U \cup V) & \longrightarrow \\ & \downarrow id & & \downarrow id \oplus \left(\oplus_{i=1}^k (h_i)_* \right) & & \downarrow h_* & \\ 0 \longrightarrow & H_{n-1}(U \cap V) & \longrightarrow & H_{n-1}(U) \oplus H_{n-1}(V) & \longrightarrow & H_{n-1}(U \cup V) & \longrightarrow \end{array}$$

$$\begin{array}{ccccc} \longrightarrow & H_{n-2}(U \cap V) & \longrightarrow & H_{n-2}(U) & \longrightarrow 0 \\ & \downarrow id & & \downarrow id & \\ \longrightarrow & H_{n-2}(U \cap V) & \longrightarrow & H_{n-2}(U) & \longrightarrow 0 \end{array}$$

By the exactness in one of the rows of the sequence we get

$$a - \left(b + \sum_{i=1}^k \mu_{x_i}(f_t) \right) + \mu_{x_0}(f_0) - c + d = 0,$$

where $a = \text{rk } H_{n-1}(U \cap V)$, $b = \text{rk } H_{n-1}(U)$, $c = \text{rk } H_{n-2}(U \cap V)$ and $d = \text{rk } H_{n-2}(U)$. Our hypothesis implies that

$$a - b - c + d = 0.$$

Now we use the fact that the trace is additive, which gives:

$$a - \left(b + \sum_{i=1}^k \text{tr} \left((h_i)_* \right) \right) + \text{tr}(h_*) - c + d = 0,$$

and hence

$$\sum_{i=1}^k \text{tr} \left((h_i)_* \right) = \text{tr}(h_*).$$

Again by hypothesis, $\text{tr} \left((h_i)_* \right) = \text{tr}(h_*) = (-1)^n$, for all $i = 1, \ldots, k$, so necessarily $k = 1$.

We can use the same ideas if $n = 1$ or $n = 2$, with a diagram similar to the one we use above. □

Observe that the hypothesis of having trace equal to $(-1)^n$ at any point can be relaxed to having trace $k \neq 0$ that does not depend on the point. Also, the hypothesis of $\mathfrak{X}$ being a Milnor space is given to assure that the generic fibers of f_t have homology only in middle dimension (by Theorem 6.4). One can prove something similar if, in general, the non-trivial homology is sparse.

Remark 6.10 The proof of Theorem 6.9 is an adaptation of the proof given in [15] for the case $X = \mathbb{C}^n$. A similar argument appears also in the paper [5], where it is showed that any family of ICIS with constant total Milnor number has no coalescence of singularities.

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