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ORIGINAL PAPER

On the spectrum of weighted shifts

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Abstract

In Linear Dynamics, the most studied class of linear operators is certainly that of weighted shifts, on the separable Banach spaces c_0 and ℓ^p, $1 \leq p < \infty$. Over the last decades, the intensive study of such operators has produced an incredible number of versatile, deep and beautiful results that are applicable in various areas of Mathematics; and the relationships between various important notions, especially concerning chaos and hyperbolic properties, as well as spectrum of weighted shifts, have been investigated. In this paper, we investigate the point spectrum of weighted shifts and, under some regularity hypotheses on the weight sequence, we deduce the spectrum.

Keywords Weighted shifts · Linear dynamics · Spectrum · Point spectrum

Mathematics Subject Classification Primary 47B37; Secondary 47A10 · 47A05 · 47A16

Contents

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1 Introduction

In 1929, Birkhoff obtained an example of a linear operator having a dense orbit [5]. Some years later, the same has been shown for other two types of fundamental operators in analysis: the differentiation operators [22] and the shifts [27]. Motivated by these three examples and by the fact that density of orbits is one of the main ingredients of chaos, in the twentieth century, many mathematicians began to focus on the dynamical properties of linear operators starting from the analysis of the three ones mentioned above, thus leading to the birth of the fascinating area of Mathematics known as *Linear Dynamics*. Over the years, the usefulness of the Birkhoff operators, of the differentiation operators and of the shifts, has become undoubted, making them a model for understanding the behaviors of more complex operators. This is particularly illustrated by the shifts or, more generally, the weighted shifts, whose flexibility and "simplicity of action" have made them not only the most studied class of linear operators, but also an excellent tool to clearly illustrate definitions and create crucial counterexamples in Linear Dynamics. In particular, weighted shifts are known to be essential for the comprehension of some classes of linear operators, among which *composition operators* [11], just to cite one of the most famous and versatile examples. Hence, starting in 1995 with the characterization of hypercyclic weighted shifts [29], their dynamical properties have been so studied that, nowadays, the most of the fundamentals of Linear Dynamics (like hypercyclicity, mixing, chaos, Li-Yorke chaos, expansivity, hyperbolicity and shadowing) are completely characterized for such operators [3, 4, 10, 12]. See [1, 18] and [23] for a global overview on the subject.

As it is often the case in Operator Theory, it may be useful to know the structure of the spectrum and to impose some conditions on it, to ensure that a certain dynamical property manifests. For instance, it is known that an invertible operator T on $\mathbb{C}^n$ is expansive if and only if the point spectrum of T does not intersect the unit circle $\mathbb{T}$ [15, Theorem 1]. Subsequently, as it is well-known that an invertible operator T on a Banach space X is hyperbolic if its spectrum does not intersect $\mathbb{T}$, it was shown in [3, Theorem D] and [24, Theorem 1] that T is uniformly expansive if and only if the same condition is satisfied by the approximate point spectrum, providing, as a corollary, that invertible hyperbolic operators are uniformly expansive. Moreover, the above requirement on the spectrum is a necessary condition to get Li-Yorke chaos [2, Corollary 6]. As hyperbolicity, also generalized hyperbolicity is defined, for an invertible operator T on a Banach space X, in terms of the spectrum of certain restrictions of the operators. These results are just a glimpse of the close relationship between the spectrum of an operator T and its dynamical properties and, also in the specific case of weighted shifts, it turns out that the spectrum plays a key role in the related theory.

In this paper we focus on weighted shifts on ℓ^p-spaces: we generalize some results, proved on the Hilbert space ℓ^2, to weighted shifts defined on the Banach space ℓ^p, $1 \leq p < \infty$. The present article aims, after recalling basic and fundamental results about the spectrum of shifts and weighted shifts (Sect. 2), to investigate the point spectrum of weighted shifts and, under some regularity hypotheses on the weight sequence, to deduce the spectrum (Sect. 3).

Throughout the article, as usual, $\mathbb{N}$ denotes the set of all positive integers; $\mathbb{Z}$ denotes the set of all integers; $\mathbb{D}$ and $\mathbb{T}$ are, respectively, the open unit disk and the unit circle in the complex plane $\mathbb{C}$.

2 Definitions and background results

In this section, we fix some terminology and recall some basic notions and results concerning the spectral theory of, first, a bounded linear operator on a complex Banach space X and, then, more specifically, of shifts [13, 19, 21]. In the sequel, $\mathcal{L}(X)$ denotes the algebra of all bounded linear operators from the complex Banach space X into itself.

2.1 Spectrum and spectral radius

Definition 2.1 Let $T \in \mathcal{L}(X)$. The *spectrum* $\sigma(T)$ of the operator T is the set

$$\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible in } \mathcal{L}(X)\}.$$

Definition 2.2 Let $T \in \mathcal{L}(X)$. The *spectral radius* $r(T)$ of the operator T is defined by

$$r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}.$$

By the Gelfand formula (see, e.g., [28, Chap. 10]), the spectral radius $r(T)$ satisfies the *spectral radius formula* $r(T) = \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}}$ and hence $r(T) \leq \|T\|$, and that $\sigma(T)$ is a non-empty compact subset of $\mathbb{C}$ contained in the ball $\{\lambda \in \mathbb{C} : |\lambda| \leq \|T\|\}$. Moreover, if T is invertible then

$$\sigma(T^{-1}) = \left\{\frac{1}{\lambda} : \lambda \in \sigma(T)\right\},$$

and so

$$\frac{1}{r(T^{-1})} = \inf\{|\lambda| : \lambda \in \sigma(T)\},$$

and hence, in this case the spectrum $\sigma(T)$ of T lies in the annulus $\{\lambda \in \mathbb{C} : \frac{1}{r(T^{-1})} \leq |\lambda| \leq r(T)\}$.

We now recall the definitions of some subsets of the spectrum which will be used in the sequel.

Definition 2.3 Let $T \in \mathcal{L}(X)$.

(1) The set of all $\lambda \in \sigma(T)$ such that $T - \lambda I$ is not one-to-one is called the *point spectrum* of T and is denoted by $\sigma_p(T)$. It follows that $\sigma_p(T)$ consists of all the *eigenvalues* of T.
(2) The set of all $\lambda \in \sigma(T)$ such that $T - \lambda I$ is not Fredholm (an operator is Fredholm if its range is closed and both its kernel and its cokernel are finite-dimensional) is called the *essential spectrum* of T and is denoted by $\sigma_e(T)$.

The essential spectrum $\sigma_e(T)$ is a closed subset of $\sigma(T)$ and, clearly, it is empty when the space X is finite-dimensional [20]. Moreover, it is invariant under compact perturbation, as the following result shows:

Proposition 2.4 [14, Theorem 4.1] *Let $T \in \mathcal{L}(X)$. If S is a compact operator then*

$$\sigma_e(T + S) = \sigma_e(T).$$

We recall the following useful result:

Proposition 2.5 [19, Proposition 1.F] *Let T be a bounded operator on a Banach space X. If there exists an invertible operator $S : X \to X$ for which $\|S - T\| < \|S^{-1}\|^{-1}$, then T is itself invertible.*

We conclude this paragraph by recalling that the above definitions of spectrum can be extended to any bounded linear operator T acting on a real Banach space X via its complexification $T_{\mathbb{C}}$, in the sense that we define the spectrum of T as the set of all $\lambda \in \mathbb{C}$ such that $T_{\mathbb{C}} - \lambda I$ is not invertible as an operator acting on $X_{\mathbb{C}}$, i.e. we define $\sigma(T) = \sigma(T_{\mathbb{C}})$.

2.2 Weighted shifts

We recall some preliminary definitions and results.

Definition 2.6 Let $X = \ell^p(\mathbb{N})$, $1 \le p < \infty$ or $X = c_0(\mathbb{N})$. Let $w = \{w_n\}_{n\in\mathbb{N}}$ be a bounded sequence of scalars, called *weight sequence*. Then,

- the *unilateral weighted forward shift* $F_w : X \to X$ is defined by

$$F_w(\{x_n\}_{n\in\mathbb{N}}) = \{w_{n-1}x_{n-1}\}_{n\in\mathbb{N}},$$

  meaning

$$F_w(\{x_1, x_2, ...\}) = \{0, w_1x_1, w_2x_2, ...\};$$

- the *unilateral weighted backward shift* $B_w : X \to X$ is defined by

$$B_w(\{x_n\}_{n\in\mathbb{N}}) = \{w_{n+1}x_{n+1}\}_{n\in\mathbb{N}},$$

  meaning

$$B_w(\{x_1, x_2, ...\}) = \{w_2x_2, w_3x_3, ...\}.$$

If, instead of $\mathbb{N}$, we consider $\mathbb{Z}$, the shift is called *bilateral*.

Clearly, a weighted shift (unilateral or bilateral) is injective if and only if none of the weights is zero, and a bilateral weighted shift is invertible if and only if $\inf_{n\in\mathbb{Z}} |w_n| > 0$. Of course, a unilateral weighted shift is never invertible.

Remark 2.7 Let $w = \{w_n\}_{n\in\mathbb{Z}}$ be a weight sequence with $\inf_{n\in\mathbb{Z}} |w_n| > 0$. Obviously, $B_w^{-1} = F_{\tilde{w}}$ and $F_w^{-1} = B_{\tilde{w}}$, where $\tilde{w} = \{\frac{1}{w_n}\}_{n\in\mathbb{Z}}$.

Remark 2.8 Let $w = \{w_n\}_{n\in\mathbb{Z}}$, be a weight sequence. Let T be the bilateral weighted shift F_w or B_w on $X = \ell^p(\mathbb{Z})$, $1 \le p < \infty$, or $X = c_0(\mathbb{Z})$. It will be useful for the sequel to note that, for every $n \in \mathbb{N}$,

(1) $\|T^n\| = \sup_{k\in\mathbb{Z}} |w_k w_{k+1} \cdots w_{k+n-1}|$, and
(2) if T is invertible, then $\|T^{-n}\| = \sup_{k\in\mathbb{Z}} |w_k w_{k+1} \cdots w_{k+n-1}|^{-1}$.

Proof We only show (1) for $T = F_w$, as the rest follows in a similar fashion. Given F_w on $X = \ell^p(\mathbb{Z})$, $1 \le p < \infty$, then

$$\begin{aligned}\|F_w^n(\{x_k\}_{k\in\mathbb{Z}})\|_p &= \|\{w_{k-n}\cdots w_{k-1}x_{k-n}\}_{k\in\mathbb{Z}}\|_p = \left(\sum_{k\in\mathbb{Z}} |w_{k-n}\cdots w_{k-1}x_{k-n}|^p\right)^{\frac{1}{p}} \\ &\le \sup_{k\in\mathbb{Z}} |w_{k-n}\cdots w_{k-1}| \left(\sum_{k\in\mathbb{Z}} |x_{k-n}|^p\right)^{\frac{1}{p}} \\ &= \sup_{k\in\mathbb{Z}} |w_k\cdots w_{k+n-1}| \left(\sum_{k\in\mathbb{Z}} |x_k|^p\right)^{\frac{1}{p}} = \sup_{k\in\mathbb{Z}} |w_k\cdots w_{k+n-1}|\|\{x_k\}_{k\in\mathbb{Z}}\|_p,\end{aligned}$$

and, on $X = c_0(\mathbb{Z})$,

$$\begin{aligned}\|F_w^n(\{x_k\}_{k\in\mathbb{Z}})\|_\infty &= \|\{w_{k-n}\cdots w_{k-1}x_{k-n}\}_{k\in\mathbb{Z}}\|_\infty = \sup_{k\in\mathbb{Z}} |w_{k-n}\cdots w_{k-1}x_{k-n}| \\ &\le \sup_{k\in\mathbb{Z}} |w_{k-n}\cdots w_{k-1}| \sup_{k\in\mathbb{Z}} |x_{k-n}| = \sup_{k\in\mathbb{Z}} |w_k\cdots w_{k+n-1}| \sup_{k\in\mathbb{Z}} |x_k| \\ &= \sup_{k\in\mathbb{Z}} |w_k\cdots w_{k+n-1}|\|\{x_k\}_{k\in\mathbb{Z}}\|_\infty.\end{aligned}$$

Hence

$$\begin{aligned}\|F_w^n\| &= \inf_{\{x_k\}\in X}\{c \ge 0 \ : \ \|F_w^n(\{x_k\}_{k\in\mathbb{Z}})\|_X \le c\|\{x_k\}_{k\in\mathbb{Z}}\|_X\} \\ &\le \sup_{k\in\mathbb{Z}} |w_k\cdots w_{k+n-1}|.\end{aligned}$$

By computing F_w^n at $e_k = \{\ldots 0, 0, \underbrace{1}_{k\text{-th position}}, 0, 0, \ldots\}$, the reverse of the above inequality is obtained. Hence

$$\|F_w^n\| = \sup_{k\in\mathbb{Z}} |w_k w_{k+1}\cdots w_{k+n-1}|.$$

Moreover, if F_w is invertible, an analogous computation gives

$$\|F_w^{-n}\| = \sup_{k\in\mathbb{Z}} |w_k w_{k+1}\cdots w_{k+n-1}|^{-1}.$$

□

Remark 2.9 By replacing $\mathbb{Z}$ by $\mathbb{N}$ in (1) of Remark 2.8, an analogous argument gives the norms of F_w and B_w in the unilateral case.

Proposition 2.10 [20, Proposition 1.6.15]; [30, Theorem 4]; [3, Remark 35] *Let $X = \ell^p(\mathbb{N})$, $1 \le p < \infty$, or $X = c_0(\mathbb{N})$. Let T be the unilateral weighted shift F_w or B_w, on X. Then, the spectrum of T is the disk*

$$\sigma(T) = \{\lambda \in \mathbb{C} : |\lambda| \le r(T)\}.$$

Proposition 2.11 [6, Theorem 2.1]; [30, Theorem 5]; [3, Remark 35] *Let $X = \ell^p(\mathbb{Z})$, $1 \le p < \infty$, or $X = c_0(\mathbb{Z})$. Let T be the bilateral weighted shift F_w or B_w, on X. Then, the followings hold.*

(a) If T is non-invertible, then its spectrum is the disk

$$\sigma(T) = \{\lambda \in \mathbb{C} : |\lambda| \le r(T)\}.$$

(b) If T is invertible, then its spectrum is the annulus

$$\sigma(T) = \left\{\lambda \in \mathbb{C} : \frac{1}{r(T^{-1})} \le |\lambda| \le r(T)\right\}.$$

The followings are well-known results which will be useful in the sequel.

Proposition 2.12 [18, Exercise 5.2.10]; [20, Proposition 1.6.14]; [30, Proposition 4] *Let $X = \ell^p$, $1 \le p < \infty$, or $X = c_0$. Let T be the unilateral (resp. bilateral) weighted shift F_w or B_w. Then, T is compact if and only if* $\lim_{n\to\infty} w_n = 0$ $(resp. \lim_{|n|\to\infty} w_n = 0)$.

When the weights $w = \{w_n\}_{n\in A}$, with $A = \mathbb{N}$ or $A = \mathbb{Z}$, are such that $w_n = 1$ for each $n \in A$, then F_w and B_w reduce to the (unweighted) forward and backward shifts denoted by F and B, respectively. Clearly, $\|F\| = \|B\| = 1$, and for this simple case the spectrum and its part were completely analyzed, as partially summarized in the following result (for a detailed description of all the parts of the spectrum, see [31] and [19, Proposition 2.M], for the case $p = 2$; and [17, Corollary 3.2]; [20, Example 3.7.7]; [7, Proposition 10.2.8], for the case $1 \le p < \infty$).

Proposition 2.13 *Let B and F be the unilateral backward and the unilateral forward shift, respectively. Then, the followings hold:*

(1) $\sigma_p(F) = \emptyset$; $\sigma_e(F) = \mathbb{T}$;
(2) $\sigma_p(B) = \mathbb{D}$; $\sigma_e(B) = \mathbb{T}$;
(3) $\sigma(F) = \sigma(B) = \{\lambda \in \mathbb{C} : |\lambda| \le 1\}$.

If B and F denote the bilateral backward and the bilateral forward shift, respectively, then

(4) $\sigma_p(F) = \emptyset$; $\sigma_e(F) = \mathbb{T}$;
(5) $\sigma_p(B) = \emptyset$; $\sigma_e(B) = \mathbb{T}$;
(6) $\sigma(F) = \sigma(B) = \mathbb{T}$.

The definition of an adjoint operator is given, in general, for operators defined on a Hilbert space. If one tries to transfer this definition to operators on Banach spaces, it immediately appears an obstacle: the absence of an inner product. Hence, given a Banach space X, it is necessary to introduce the dual space X^*, and to define a new product on $X \times X^*$, $\langle \cdot, \cdot \rangle$, as in the following definition.

Definition 2.14 Let X be a Banach space and let X^* be its dual. Let $x^* \in X^*$. Then, for each $x \in X$, we define

$$\langle x, x^* \rangle = x^*(x).$$

Definition 2.15 (*Adjoint operator*) Let X be a Banach space and let $T \in \mathcal{L}(X)$. The operator $T^* \in \mathcal{L}(X^*)$ defined by $T^*x^* = x^* \circ T$, that is

$$\langle x, T^*x^* \rangle = \langle Tx, x^* \rangle, \quad x \in X, x^* \in X^*,$$

is the *adjoint of T*. When T is an operator on a Hilbert space X, then it is said to be *unitary* if $T^* = T^{-1}$.

The following example will be useful in the sequel.

Example 2.16 Let $A = \mathbb{N}$ or $A = \mathbb{Z}$. As it is well-known, the dual space of $\ell^p(A)$, $1 \leq p < \infty$, is given by $\ell^q(A)$, where $\frac{1}{p} + \frac{1}{q} = 1$ for $1 < p < \infty$, and $q = \infty$ for $p = 1$. The continuous linear functionals x^* on $\ell^p(A)$ are precisely the maps of the form

$$x^*(x) = \langle x, x^* \rangle = \sum_{n \in A} x_n \overline{y_n}, \qquad (\spadesuit)$$

with $y = \{y_n\}_{n \in A} \in \ell^q(A)$.

Now, let $A = \mathbb{Z}$ and let F_w be a bilateral weighted forward shift on $\ell^p(\mathbb{Z})$, $1 \leq p < \infty$. Then, according to Definition 2.15, its adjoint is a bilateral weighted backward shift on $\ell^q(\mathbb{Z})$ given by

$$F_w^*(e_n) = \overline{w_{n-1}} e_{n-1},$$

for each $n \in \mathbb{Z}$ (see, for instance, [25]). In fact, given any $x \in \ell^p(\mathbb{Z})$ and $x^* \in \ell^q(\mathbb{Z})$, it is

$$\begin{aligned}
\langle x, F_w^* x^* \rangle = \langle F_w x, x^* \rangle &= \sum_{n=-\infty}^{+\infty} (F_w x)_n \overline{y_n} = \sum_{n=-\infty}^{+\infty} w_{n-1} x_{n-1} \overline{y_n} \\
&= \sum_{n=-\infty}^{+\infty} w_n x_n \overline{y_{n+1}} = \sum_{n=-\infty}^{+\infty} x_n \overline{(\overline{w_n} y_{n+1})} \\
&= \sum_{n=-\infty}^{+\infty} x_n B_{\tilde{w}}(\overline{y_n}) = \langle x, B_{\tilde{w}} x^* \rangle, \quad (\text{where } \tilde{w} = \{\tilde{w}_n\}_n \text{ with } \tilde{w}_n = \overline{w_{n-1}})
\end{aligned}$$

and hence $F_w^*(e_n) = B_{\tilde{w}}(e_n) = \tilde{w}_n e_{n-1} = \overline{w_{n-1}} e_{n-1}$, for each $n \in \mathbb{Z}$.

Analogously, if $A = \mathbb{N}$ and F_w is a unilateral weighted forward shift on $\ell^p(\mathbb{N})$, $1 \leq p < \infty$, then its adjoint is the unilateral weighted backward shift on $\ell^q(\mathbb{N})$ given by $F_w^*(e_0) = 0$ and $F_w^*(e_n) = \overline{w_{n-1}} e_{n-1}$, for each $n \geq 1$. Similarly, $B_w^*(e_n) = \overline{w_{n+1}} e_{n+1}$, for each $n \geq 1$.

2.3 Similarity of operators

The notion of similarity plays an important role in the theory of bounded linear operators (see [19, 20]).

Definition 2.17 (*Similarity*) Let X be a Banach space. Two operators $T, S \in \mathcal{L}(X)$ are called *similar* if there exists an invertible operator $W \in \mathcal{L}(X)$ such that $S = W^{-1} T W$.

In Linear Dynamics, the word "similarity" is replaced by "conjugation", a word originally borrowed from Topological Dynamics. A special similarity on inner product spaces, i.e., on Hilbert spaces, is given by the unitarily equivalence:

Definition 2.18 (*Unitarily Equivalence*) Let $\mathcal{H}$ be a Hilbert space. Two operators $T, S \in \mathcal{L}(\mathcal{H})$ are said to be *unitarily equivalent* if there exists a unitary operator $U \in \mathcal{L}(\mathcal{H})$ such that $S = U^{-1} T U$.

Note that similarity is a less severe definition than the one of unitarily equivalence. The importance of similarity and unitarily equivalence follows from the fact that they preserve many properties of an operator. It is known that similarity preserves invariant subspaces, that is, if two operators are similar, and if one has a nontrivial invariant subspace, then so does the other [19, Proposition 1.J]. Moreover, among the invariants of similarity the most important are the spectrum and the spectral radius, as the following result shows:

Proposition 2.19 [19, Proposition 2.B] *Similarity preserves the spectrum and its parts, and so it preserves the spectral radius also. That is, in particular, if X is a Banach space and $T, S \in \mathcal{L}(X)$ are two similar operators, then:*

(1) $\sigma_p(T) = \sigma_p(S)$;
(2) $\sigma(T) = \sigma(S)$;
(3) $r(T) = r(S)$.

Of course, if two operators on an Hilbert space are unitarily equivalent, then they are also similar. The converse, in general, only holds for normal operators [19, Proposition 3.I]. Hence, unitarily equivalent operators on Hilbert spaces have, in particular, the same spectrum and the same point spectrum [26, Theorem A.11]. Moreover, it is well-known that unitary equivalence preserves the operator norm [19, Proposition 2.B]. The following result about unitarily equivalent weighted shifts on ℓ^2 is well-known.

Proposition 2.20 [8, Proposition 6.2]; [30, Corollary 1] *Let $A = \mathbb{N}$ or $A = \mathbb{Z}$ and $X = \ell^2(A)$. Let T be a weighted shift F_w (resp. B_w), with weights $w = \{w_n\}_{n \in A}$. Then, T is unitarily equivalent to the weighted shift $F_{\tilde{w}}$ (resp. $B_{\tilde{w}}$), where $\tilde{w} = \{|w_n|\}_{n \in A}$.*

As the following proposition shows, the same result holds for weighted shifts on the Banach space ℓ^p, $1 \leq p < \infty$, if we consider similarity instead of unitarily equivalence.

Proposition 2.21 *Let $A = \mathbb{N}$ or $A = \mathbb{Z}$ and $X = \ell^p(A)$, $1 \leq p < \infty$. Let T be a weighted shift F_w (resp. B_w), with weights $w = \{w_n\}_{n \in A}$. Then, T is similar to the weighted shift $F_{\tilde{w}}$ (resp. $B_{\tilde{w}}$), where $\tilde{w} = \{|w_n|\}_{n \in A}$.*

Proof The proof is showed only for $T = F_w$, as small changes give it for $T = B_w$. Hence, let $T = F_w$. According to Definition 2.17, we need to find an invertible operator $W \in \mathcal{L}(X)$ with $F_w = W^{-1} F_{\tilde{w}} W$, i.e. such that

$$W F_w = F_{\tilde{w}} W. \tag{$\bullet$}$$

In order to do that, let us consider the operator $T : X \to X$ given by $T(e_n) = \gamma_n e_n$, where $\gamma = \{\gamma_n\}_{n \in A}$ is a bounded sequence of scalars. Then T acts as a diagonal operator, namely, $T(\{x_n\}_{n \in A}) = \{\gamma_n x_n\}_{n \in A}$.

Note that, for each $\{x_n\}_{n \in A} \in X$,

$$T F_w(\{x_n\}_{n \in A}) = T(\{w_{n-1} x_{n-1}\}_{n \in A}) = \{\gamma_n w_{n-1} x_{n-1}\}_{n \in A},$$

and

$$F_{\tilde{w}} T(\{x_n\}_{n \in A}) = F_{\tilde{w}}(\{\gamma_n x_n\}_{n \in A}) = \{\tilde{w}_{n-1} \gamma_{n-1} x_{n-1}\}_{n \in A}.$$

Therefore, $T F_w = F_{\tilde{w}} T$ if and only if the sequence $\gamma = \{\gamma_n\}_{n \in A}$ satisfies

$$\gamma_n w_{n-1} x_{n-1} = \tilde{w}_{n-1} \gamma_{n-1} x_{n-1}, \forall n \in A. \tag{$\clubsuit$}$$

Therefore, choosing $\gamma_0 = 1$ and taking, for $n > 0$

$$\gamma_n = \begin{cases} 1 & \text{if } w_{n-1} = 0 \\ \frac{\tilde{w}_{n-1}}{w_{n-1}} \gamma_{n-1} & \text{if } w_{n-1} \neq 0 \end{cases}$$

and, for $n \leq 0$

$$\gamma_{n-1} = \begin{cases} 1 & \text{if } \tilde{w}_{n-1} = 0 \\ \frac{w_{n-1}}{\tilde{w}_{n-1}} \gamma_n & \text{if } \tilde{w}_{n-1} \neq 0, \end{cases}$$

we have that such a sequence $\{\gamma_n\}_{n\in A}$ satisfies (♣), that is $TF_w = F_{\tilde{w}}T$.

Moreover, note that:

- T is linear;
- T is bounded since, for each n, $|\gamma_n| = 1$ (as, by hypothesis, $|w_{n-1}| = |\tilde{w}_{n-1}|$), and then $\|T\| = \sup_{n\in A} |\gamma_n| = 1$;
- T is invertible, as $\inf_{n\in A} |\gamma_n| = 1 > 0$. In particular, $T^{-1}(e_n) = \frac{1}{\gamma_n} e_n$.

Taking $W = T$ we obtain an invertible operator $W \in \mathcal{L}(X)$ such that $WF_w = F_{\tilde{w}}W$, and hence, by Definition 2.17, F_w and $F_{\tilde{w}}$ are similar. □

3 On the spectrum of weighted shifts

Up to now, we have not made assumptions about the sign of the weights. It is well-known (see [8, p. 54]; [30, p. 56]) that if a weighted shift has a finite number of zero weights, then it is the direct sum of a finite number of finite dimensional operators and a weighted shift with nonzero weights. Hence, weighted shifts with a finite number of zero weights lead back to weighted shifts with all weights non-zero. For this reason, and by Proposition 2.20 and Proposition 2.21, from now on we only consider positive weights $\{w_n\}_{n\in A}$.

In the previous section, Proposition 2.11 describes the spectrum of a weighted shifts with $w = \{w_n\}_{n\in A}$, $A = \mathbb{N}$ or $A = \mathbb{Z}$, a bounded sequence of scalars. In this section, we will see more detailed results on the spectrum of F_w and B_w, both in the unilateral case and in the bilateral case.

3.1 The unilateral case

Proposition 3.1 *Let T be a unilateral weighted shift F_w or B_w on $\ell^p(\mathbb{N})$, $1 \le p < \infty$, with $\{w_n\}_{n\in\mathbb{N}}$ a bounded sequence of positive reals. Let*

$$\lim_{n\to\infty} w_n = w.$$

Then, the spectral radius of T is

$$r(T) = w.$$

Proof The proof is showed for $T = F_w$, as small changes provide the case $T = B_w$. Hence, consider $T = F_w$. Let $\epsilon > 0$ and $\overline{n} \in \mathbb{N}$ such that $|w_n - w| < \epsilon$ for each $n \ge \overline{n}$. Then, in particular, for each $n > \overline{n}$,

$$0 < w_{\overline{n}} \cdots w_{n-1} < (w+\epsilon)^{n-\overline{n}}.$$

Therefore, for each $k \in \mathbb{N}$ and for each $n > \overline{n}$,

$$0 < w_{k+\overline{n}} \cdots w_{k+n-1} < (w+\epsilon)^{n-\overline{n}},$$

and, hence,

$$0 < w_k \cdots w_{k+\overline{n}-1} w_{k+\overline{n}} \cdots w_{k+n-1} < w_k \cdots w_{k+\overline{n}-1}(w+\epsilon)^{n-\overline{n}}.$$

Then, it follows that

$$0 \le r(F_w) = \lim_{n\to\infty} \|F_w^n\|^{\frac{1}{n}} = \lim_{n\to\infty} \left(\sup_{k\in\mathbb{N}} |w_k \cdots w_{k+n-1}|\right)^{\frac{1}{n}}$$

$$
\begin{aligned}
&\leq \lim_{n\to\infty}\left[\sup_{k\in\mathbb{N}}(w_k\cdots w_{k+\overline{n}-1})\right]^{\frac{1}{n}}(w+\epsilon)^{1-\frac{\overline{n}}{n}}\\
&\leq \lim_{n\to\infty}(\sup_{k\in\mathbb{N}} w_k)^{\frac{\overline{n}}{n}}(w+\epsilon)^{1-\frac{\overline{n}}{n}}\\
&\leq \lim_{n\to\infty} M^{\frac{\overline{n}}{n}}(w+\epsilon)^{1-\frac{\overline{n}}{n}} \qquad (M:=\max\{\sup_{k\in\mathbb{N}} w_k,\, w+\epsilon\})\\
&= w+\epsilon,
\end{aligned}
$$

If $w=0$, then, as ϵ is arbitrary, it follows that $r(F_w)=0=w$.

Now, assume that $w>0$. From the previous computation we obtain that $r(F_w)\leq w$. We want to show that $r(F_w)\geq w$. Let F denote the unilateral forward shift on $\ell^p(\mathbb{N})$, $1\leq p<\infty$, i.e.

$$
F(\{x_n\}_{n\in\mathbb{N}})=\{x_{n-1}\}_{n\in\mathbb{N}}.
$$

Consider the operator $F_{\widetilde{w}}=F_w-wF$ defined by:

$$
F_{\widetilde{w}}(\{x_n\}_{n\in\mathbb{N}})=(F_w-wF)(\{x_n\}_{n\in\mathbb{N}})=\{(w_{n-1}-w)x_{n-1}\}_{n\in\mathbb{N}}.
$$

Hence, $F_{\widetilde{w}}$ is the unilateral weighted forward shift on $\ell^p(\mathbb{N})$, $1\leq p<\infty$, with weights $\widetilde{w}=\{w_n-w\}_{n\in\mathbb{N}}$. In our case, the hypothesis

$$
\lim_{n\to\infty} w_n=w
$$

implies that

$$
\lim_{n\to\infty}\widetilde{w_n}=0,
$$

which means, by Proposition 2.12, that $F_{\widetilde{w}}$ is compact. Therefore, from Proposition 2.4 it follows that:

$$
\begin{aligned}
\sigma_e(F_w)=\sigma_e(wF+(F_w-wF))&=\sigma_e(wF+F_{\widetilde{w}})\\
&=\sigma_e(wF)\quad(\text{as } F_{\widetilde{w}} \text{ is compact})\\
&=w\mathbb{T}\\
&=\{\lambda:|\lambda|=w\}.
\end{aligned}
$$

As $\sigma_e(F_w)$ is a closed subset of $\sigma(F_w)$, then it follows

$$
r(F_w)=\sup\{|\lambda|,\lambda\in\sigma(F_w)\}\geq w.
$$

Hence, the spectral radius of F_w is $r(F_w)=w$. □

Remark 3.2 Let $X=\ell^p(\mathbb{N})$, $1\leq p<\infty$, and let T denote a unilateral weighted shift F_w or B_w, on X, with $\{w_n\}_{n\in\mathbb{N}}$ a positive weight sequence. Let

$$
\lim_{n\to\infty} w_n=w.
$$

Then, by (1) of Remark 2.8 and Proposition 3.1, together with the spectral radius formula, we get

$$
r(T)=w=\lim_{n\to\infty}w_n=\lim_{n\to\infty}\|T^n\|^{\frac{1}{n}}=\lim_{n\to\infty}\left(\sup_{k\in\mathbb{N}}|w_kw_{k+1}\cdots w_{k+n-1}|\right)^{\frac{1}{n}}
$$

$$= \lim_{n\to\infty}\left[\sup_{k\in\mathbb{N}}(w_k w_{k+1}\cdots w_{k+n-1})\right]^{\frac{1}{n}}.$$

Take $w_n = 3^{(-1)^n}$. Then

$$\lim_{n\to\infty}\left[\sup_{k\in\mathbb{N}}(w_k w_{k+1}\cdots w_{k+n-1})\right]^{\frac{1}{n}} = 1$$

but, clearly, $\{w_n\}_{n\in\mathbb{N}}$ is not regular, with $\overline{\lim}_{n\to\infty} w_n = 3$ and $\underline{\lim}_{n\to\infty} w_n = \frac{1}{3}$. In particular, Proposition 3.1 does not hold anymore if we replace lim with $\overline{\lim}$.

We point out that, in the Hilbert case $p = 2$, Proposition 3.1 is a consequence of [30, Proposition 15] and [8, Proposition 6.8 (a)]. As in [16, Remark 1.2] and [30, Theorem 8] for the Hilbert case $p = 2$, the point spectrum of a unilateral forward weighted shift on $\ell^p(\mathbb{N})$, $1 \le p < \infty$, turns out to be empty, and the point spectrum of a unilateral backward weighted shift on $\ell^p(\mathbb{N})$, $1 \le p < \infty$, satisfies

$$\{0\} \cup \{\lambda \in \mathbb{C} : |\lambda| < \tilde{r}\} \subseteq \sigma_p(B_w) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \le \tilde{r}\}$$

where $\tilde{r} = \underline{\lim}_{n\to\infty}(w_2\cdots w_{n+1})^{\frac{1}{n}}$. The proofs for the general case $1 \le p < \infty$ are analogous to the ones for $p = 2$.

Proposition 3.3 *Let $F_w : \ell^p(\mathbb{N}) \to \ell^p(\mathbb{N})$, $1 \le p < \infty$, be a unilateral weighted forward shift with $\{w_n\}_{n\in\mathbb{N}}$ a bounded positive weight sequence. Then $\sigma_p(F_w) = \emptyset$.*

Proof By contradiction, let $\lambda \in \sigma_p(F_w)$. Then, there exists $\{x_n\}_{n\in\mathbb{N}} \in \ell^p(\mathbb{N})\setminus\{0\}$ eigenvector of F_w corresponding to the eigenvalue λ, i.e., $F_w(\{x_n\}_{n\in\mathbb{N}}) = \{\lambda x_n\}_{n\in\mathbb{N}}$. By definition, $F_w(\{x_n\}_{n\in\mathbb{N}}) = \{w_{n-1}x_{n-1}\}_{n\in\mathbb{N}}$ and, hence, the coordinates of $\{x_n\}_{n\in\mathbb{N}}$ are such that

$$\lambda x_1 = 0 \quad\text{and}\quad w_{n-1}x_{n-1} = \lambda x_n \text{ for each } n \ge 2. \tag{$\star$}$$

By hypothesis, $w_n > 0$ for each $n \in \mathbb{N}$, therefore F_w is injective and then $\lambda \ne 0$. Hence, it follows from ($\star$) that $x_n = 0$ for each $n \in \mathbb{N}$. This is a contradiction and so it must be $\sigma_p(F_w) = \emptyset$. □

Proposition 3.4 *Let $B_w : \ell^p(\mathbb{N}) \to \ell^p(\mathbb{N})$, $1 \le p < \infty$, be a unilateral weighted forward shift with $\{w_n\}_{n\in\mathbb{N}}$ a bounded positive weight sequence. Then*

$$\{0\} \cup \{\lambda \in \mathbb{C} : |\lambda| < \tilde{r}\} \subseteq \sigma_p(B_w) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \le \tilde{r}\}$$

where $\tilde{r} = \underline{\lim}_{n\to\infty}(w_2\cdots w_{n+1})^{\frac{1}{n}}$.

Proof Let $\lambda \in \sigma_p(B_w)$. Then, there exists $\{x_n\}_{n\in\mathbb{N}} \in \ell^p(\mathbb{N}) \setminus \{0\}$ eigenvector of B_w corresponding to the eigenvalue λ, i.e., $B_w(\{x_n\}_{n\in\mathbb{N}}) = \{\lambda x_n\}_{n\in\mathbb{N}}$. By definition, $B_w(\{x_n\}_{n\in\mathbb{N}}) = \{w_{n+1}x_{n+1}\}_{n\in\mathbb{N}}$ and, hence, the coordinates of $\{x_n\}_{n\in\mathbb{N}}$ are such that

$$w_{n+1}x_{n+1} = \lambda x_n \text{ for each } n \ge 1. \tag{$\star$}$$

Therefore, assuming without loss of generality $x_1 = 1$, for each $n \in \mathbb{N}$, we have

$$x_{n+1} = \frac{\lambda^n}{w_2\cdots w_{n+1}}.$$

Then

$$\|x\|_p^p = \sum_{n=1}^{\infty} |x_n|^p = 1 + \sum_{n=2}^{\infty} \left(\frac{|\lambda|^{n-1}}{w_2 \cdots w_n} \right)^p.$$

By applying the Cauchy-Hadamard criterion, as

$$\overline{\lim_{n\to\infty}} \sqrt[n]{\frac{|\lambda|^{n-1}}{w_2 \cdots w_n}} = \frac{|\lambda|}{\tilde{r}},$$

we obtain the conclusion. □

3.2 The bilateral case

In this section we focus on bilateral weighted shifts. We first generalize a result on the point spectrum proved, for the Hilbert case $p = 2$, in [8, Proposition 6.8 (b)] and [30, Theorem 9], to $1 \le p < \infty$. Then, under some regularity hypotheses on the weight sequence, we deduce the spectrum.

Proposition 3.5 *Let $\{w_n\}_{n\in\mathbb{Z}}$ be a bounded positive weight sequence. Assume that*

$$\lim_{n\to\infty} w_n = w_+ \text{ and } \lim_{n\to\infty} w_{-n} = w_-.$$

Let F_w and B_w be the bilateral weighted forward shift and the bilateral weighted backward shift, respectively, on $\ell^p(\mathbb{Z})$, $1 \le p < \infty$. Then the followings hold:

(1) *If $w_+ \le w_-$, then $\{\lambda : w_+ < |\lambda| < w_-\} \subseteq \sigma_p(F_w) \subseteq \{\lambda : w_+ \le |\lambda| \le w_-\}$.*
(2) *If $w_- < w_+$, then $\sigma_p(F_w) = \emptyset$.*
(3) *If $w_- \le w_+$, then $\{\lambda : w_- < |\lambda| < w_+\} \subseteq \sigma_p(B_w) \subseteq \{\lambda : w_- \le |\lambda| \le w_+\}$.*
(4) *If $w_+ < w_-$, then $\sigma_p(B_w) = \emptyset$.*

Proof (1). We separate the two cases, Case 1.A: $w_+ > 0$ and Case 1.B: $w_+ = 0$.

Case 1.A Assume $w_+ > 0$. We start by showing that $\sigma_p(F_w) \subseteq \{\lambda : w_+ \le |\lambda| \le w_-\}$. Let $\lambda \in \sigma_p(F_w)$. Let $x = \{x_n\}_{n\in\mathbb{Z}}$ be an eigenvector corresponding to λ, i.e. $F_w(\{x_n\}_{n\in\mathbb{Z}}) = \{\lambda x_n\}_{n\in\mathbb{Z}}$. By definition, $F_w(\{x_n\}_{n\in\mathbb{Z}}) = \{w_{n-1}x_{n-1}\}_{n\in\mathbb{Z}}$ and hence the coordinates of $x = \{x_n\}_{n\in\mathbb{Z}}$ are such that

$$w_{n-1}x_{n-1} = \lambda x_n \text{ for each } n \in \mathbb{Z}.$$

Therefore, for each $n \in \mathbb{N}$, we have

$$x_n = \frac{w_0 \cdots w_{n-1}}{\lambda^n} x_0 \quad \text{and} \quad x_{-n} = \frac{\lambda^n}{w_{-n} \cdots w_{-1}} x_0.$$

Then

$$\begin{aligned}
\|x\|_p^p &= \sum_{n=-\infty}^{+\infty} |x_n|^p \\
&= \sum_{n=1}^{+\infty} |x_{-n}|^p + |x_0|^p + \sum_{n=1}^{+\infty} |x_n|^p \\
&= \sum_{n=1}^{+\infty} |\lambda|^{pn} |x_0|^p (w_{-n} \cdots w_{-1})^{-p} + |x_0|^p + \sum_{n=1}^{+\infty} |\lambda|^{-pn} |x_0|^p (w_0 \cdots w_{n-1})^p
\end{aligned}$$

$$= |x_0|^p \left[\sum_{n=1}^{+\infty} |\lambda|^{pn}(w_{-n}\cdots w_{-1})^{-p} + 1 + \sum_{n=1}^{+\infty} |\lambda|^{-pn}(w_0\cdots w_{n-1})^p\right]. \quad (\spadesuit)$$

By contradiction, assume $|\lambda| < w_+$. Choose $\epsilon > 0$ such that $|\lambda| < w_+ - \epsilon$. As $\lim_{n\to\infty} w_n = w_+$, let $\overline{n}$ be so large that $w_n > w_+ - \epsilon$ for each $n \geq \overline{n}$. Then, for each $n > \overline{n}$,

$$w_{\overline{n}}\cdots w_{n-1} > (w_+ - \epsilon)^{n-1-\overline{n}+1} = (w_+ - \epsilon)^{n-\overline{n}}.$$

Therefore

$$\begin{aligned}
\infty > \|x\|_p^p &\geq \sum_{n=1}^{+\infty} |\lambda|^{-pn}(w_0\cdots w_{n-1})^p \\
&= \sum_{n=1}^{\overline{n}} |\lambda|^{-pn}(w_0\cdots w_{n-1})^p + \sum_{n=\overline{n}+1}^{+\infty} |\lambda|^{-pn}(w_0\cdots w_{\overline{n}-1}\cdots w_{n-1})^p \\
&> \sum_{n=1}^{\overline{n}} |\lambda|^{-pn}(w_0\cdots w_{n-1})^p + (w_0\cdots w_{\overline{n}-1})^p \sum_{n=\overline{n}+1}^{+\infty} |\lambda|^{-pn}(w_+ - \epsilon)^{p(n-\overline{n})} \\
&= \sum_{n=1}^{\overline{n}} |\lambda|^{-pn}(w_0\cdots w_{n-1})^p + (w_0\cdots w_{\overline{n}-1})^p (w_+ - \epsilon)^{-p\overline{n}} \\
&\quad \times \sum_{n=\overline{n}+1}^{+\infty} |\lambda|^{-pn}(w_+ - \epsilon)^{pn}.
\end{aligned}$$

As $|\lambda| < w_+ - \epsilon$, then the geometric series on the right diverges: this is a contradiction. Hence, it must be $|\lambda| \geq w_+$.

Analogously, by contradiction, assume $|\lambda| > w_-$. Choose $\epsilon > 0$ such that $|\lambda| > w_- + \epsilon$. As $\lim_{n\to\infty} w_{-n} = w_-$, let $\tilde{n}$ be so large that $w_{-n} < w_- + \epsilon$ for each $n \geq \tilde{n}$. Then, for each $n > \tilde{n}$,

$$w_{-\tilde{n}-1}\cdots w_{-n} < (w_- + \epsilon)^{n-\tilde{n}}.$$

Therefore

$$\begin{aligned}
\infty > \|x\|_p^p &\geq \sum_{n=1}^{+\infty} |\lambda|^{pn}(w_{-n}\cdots w_{-1})^{-p} \\
&= \sum_{n=1}^{\tilde{n}} |\lambda|^{pn}(w_{-n}\cdots w_{-1})^{-p} + \sum_{n=\tilde{n}+1}^{+\infty} |\lambda|^{pn}(w_{-n}\cdots w_{-\tilde{n}}\cdots w_{-1})^{-p} \\
&> \sum_{n=1}^{\tilde{n}} |\lambda|^{pn}(w_{-n}\cdots w_{-1})^{-p} + (w_{-\tilde{n}}\cdots w_{-1})^{-p} \sum_{n=\tilde{n}+1}^{+\infty} |\lambda|^{pn}(w_- + \epsilon)^{-p(n-\tilde{n})} \\
&= \sum_{n=1}^{\tilde{n}} |\lambda|^{pn}(w_{-n}\cdots w_{-1})^{-p} + (w_{-\tilde{n}}\cdots w_{-1})^{-p}(w_- + \epsilon)^{p\tilde{n}} \\
&\quad \times \sum_{n=\tilde{n}+1}^{+\infty} |\lambda|^{pn}(w_- + \epsilon)^{-pn}.
\end{aligned}$$

As $|\lambda| > w_- + \epsilon$, then the geometric series on the right diverges. This is a contradiction. So, it must be $|\lambda| \leq w_-$. Hence, we have just proved that, in the case $w_+ > 0$, it is

$$\sigma_p(F_w) \subseteq \{\lambda : w_+ \leq |\lambda| \leq w_-\}.$$

Now, we prove the first inclusion in (1). That is, we show that if λ is such that $w_+ < |\lambda| < w_-$, then λ is an eigenvalue of the vector $x = \{x_n\}_{n\in\mathbb{Z}}$ defined by choosing $x_0 \neq 0$ and $w_{n-1}x_{n-1} = \lambda x_n$ for each $n \in \mathbb{Z}$.

Indeed, choose $\epsilon > 0$ such that $w_+ + \epsilon < |\lambda| < w_- - \epsilon$.

Let $\overline{n}$ be so large that $w_- - \epsilon < w_{-n}$ and $w_n < w_+ + \epsilon$ for each $n \geq \overline{n}$. Then, for each $n > \overline{n}$,

$$w_{\overline{n}} \cdots w_{n-1} < (w_+ + \epsilon)^{n-1-\overline{n}+1} = (w_+ + \epsilon)^{n-\overline{n}}$$

and

$$w_{-\tilde{n}-1} \cdots w_{-n} > (w_- - \epsilon)^{n-\tilde{n}}.$$

Therefore

$$\begin{aligned}
\sum_{n=1}^{+\infty} |\lambda|^{-pn}(w_0 \cdots w_{n-1})^p &= \sum_{n=1}^{\overline{n}} |\lambda|^{-pn}(w_0 \cdots w_{n-1})^p + \sum_{n=\overline{n}+1}^{+\infty} |\lambda|^{-pn}(w_0 \cdots w_{\overline{n}-1} \cdots w_{n-1})^p \\
&< \sum_{n=1}^{\overline{n}} |\lambda|^{-pn}(w_0 \cdots w_{n-1})^p + (w_0 \cdots w_{\overline{n}-1})^p \sum_{n=\overline{n}+1}^{+\infty} |\lambda|^{-pn}(w_+ + \epsilon)^{p(n-\overline{n})} \\
&= \sum_{n=1}^{\overline{n}} |\lambda|^{-pn}(w_0 \cdots w_{n-1})^p + \\
&\quad +(w_0 \cdots w_{\overline{n}-1})^p (w_+ + \epsilon)^{-p\overline{n}} \sum_{n=\overline{n}+1}^{+\infty} |\lambda|^{-pn}(w_+ + \epsilon)^{pn} \\
&< \infty \quad (\text{as } |\lambda| > w_+ + \epsilon)
\end{aligned}$$

and

$$\begin{aligned}
\sum_{n=1}^{+\infty} |\lambda|^{pn}(w_{-n} \cdots w_{-1})^{-p} &= \sum_{n=1}^{\tilde{n}} |\lambda|^{pn}(w_{-n} \cdots w_{-1})^{-p} + \sum_{n=\tilde{n}+1}^{+\infty} |\lambda|^{pn}(w_{-n} \cdots w_{-\tilde{n}} \cdots w_{-1})^{-p} \\
&< \sum_{n=1}^{\tilde{n}} |\lambda|^{pn}(w_{-n} \cdots w_{-1})^{-p} + (w_{-\tilde{n}} \cdots w_{-1})^{-p} \sum_{n=\tilde{n}+1}^{+\infty} |\lambda|^{pn}(w_- - \epsilon)^{-p(n-\tilde{n})} \\
&= \sum_{n=1}^{\tilde{n}} |\lambda|^{pn}(w_{-n} \cdots w_{-1})^{-p} + (w_{-\tilde{n}} \cdots w_{-1})^{-p}(w_- - \epsilon)^{p\tilde{n}} \sum_{n=\tilde{n}+1}^{+\infty} |\lambda|^{pn}(w_- - \epsilon)^{-pn} \\
&< \infty \quad (\text{as } |\lambda| < w_- - \epsilon).
\end{aligned}$$

It follows from (♠) that $\|x\|_p < \infty$. We conclude that the vector $x = \{x_n\}_{n\in\mathbb{Z}}$ is such that

- $w_{n-1}x_{n-1} = \lambda x_n$ for each $n \in \mathbb{Z}$;
- $x \in \ell^p(\mathbb{Z}) \setminus \{0\}$;

i.e. it is an eigenvector of λ. Hence

$$\{\lambda : w_+ < |\lambda| < w_-\} \subseteq \sigma_p(F_w).$$

Case 1.B Assume $w_+ = 0$. We distinguish the cases $w_- \neq 0$ and $w_- = 0$. If $w_- \neq 0$, and if $\lambda \in \sigma_p(F_w)$, from the previous computation and the fact that $|\lambda| \geq 0 = w_+$, it follows that $\sigma_p(F_w) \subseteq \{\lambda : 0 \leq |\lambda| \leq w_-\}$. Moreover, a similar computation shows that $\{\lambda : 0 < |\lambda| < w_-\} \subseteq \sigma_p(F_w)$.

If $w_- = 0$, note that $\{\lambda : w_+ < |\lambda| < w_-\} = \emptyset \subseteq \sigma_p(F_w)$. Moreover, if $\lambda \in \sigma_p(F_w)$, a computation similar to above shows that $|\lambda| = 0$, i.e. $\sigma_p(F_w) \subseteq \{0\}$.

For the proof of (2), just note that if $w_- < w_+$, then, arguing as above, we obtain $\sigma_p(F_w) = \emptyset$.

(3) Assume $w_- \leq w_+$. As in (1), $\lambda \in \sigma_p(B_w)$ if and only if there exists $\{x_n\}_{n\in\mathbb{Z}} \in \ell^p(\mathbb{Z}) \setminus \{0\}$ such that $B_w(\{x_n\}_{n\in\mathbb{Z}}) = \{\lambda x_n\}_{n\in\mathbb{Z}}$. On the other hand, $B_w(\{x_n\}_{n\in\mathbb{Z}}) = \{w_{n+1}x_{n+1}\}_{n\in\mathbb{Z}}$. Hence, the components of $x = \{x_n\}_{n\in\mathbb{Z}}$ are such that

$$w_{n+1}x_{n+1} = \lambda x_n \text{ for each } n \in \mathbb{Z}.$$

Therefore, for each $n \in \mathbb{N}$ we have:

$$x_n = \frac{\lambda^n}{w_1 \cdots w_n} x_0 \quad \text{and} \quad x_{-n} = \frac{w_{-n+1} \cdots w_0}{\lambda^n} x_0.$$

We want to determine λ such that $\|x\|_p < \infty$. Arguing as in (1), with a similar computation we obtain that, if $w_- \leq w_+$ then

$$\{\lambda : w_- < |\lambda| < w_+\} \subseteq \sigma_p(B_w) \subseteq \{\lambda : w_- \leq |\lambda| \leq w_+\}.$$

The implication (4) follows noting that if $w_- > w_+$, then, by arguing as in (3), we obtain $\sigma_p(B_w) = \emptyset$. □

Proposition 3.6 *Let $\{w_n\}_{n\in\mathbb{Z}}$ be a bounded positive weight sequence. Let*

$$\lim_{n\to+\infty} w_n = w_+ \text{ and } \lim_{n\to\infty} w_{-n} = w_-.$$

Let T be the bilateral weighted shift F_w or B_w, on $X = \ell^p(\mathbb{Z})$, $1 \leq p < \infty$. Then the followings hold:

(a) *If T is invertible, then*

$$\sigma(T) = \{\lambda : \min\{w_-, w_+\} \leq |\lambda| \leq \max\{w_-, w_+\}\}.$$

(b) *If T is not invertible, then*

$$\sigma(T) = \{\lambda : |\lambda| \leq \max\{w_-, w_+\}\}.$$

Proof By using Proposition 2.11, as $\sigma(F_w) = \sigma(B_w)$, hence without loss of generality we can consider $T = F_w$. We may assume that $\min\{w_-, w_+\} \neq 0$, otherwise, by Proposition 2.12, F_w is compact and hence every nonzero $\lambda \in \sigma(F_w)$ is an eigenvalue of F_w, i.e., $\sigma(F_w) = \{0\} \cup \sigma_p(F_w)$ [9, Proposition 7.1].

Hence, let $\min\{w_-, w_+\} \neq 0$. Note that, by the hypotheses, for each $\epsilon > 0$, there exist $\overline{n}, \overline{\overline{n}} \in \mathbb{N}$ such that

$$w_+ - \epsilon < w_n < w_+ + \epsilon, \forall n \geq \overline{n}$$

and

$$w_- - \epsilon < w_{-n} < w_- + \epsilon, \forall n \geq \overline{\overline{n}}.$$

Let $N = \max\{\overline{n}, \overline{\overline{n}}\}$. Note that

$$\begin{aligned} |w_k \dots w_{k+n-1}|^{\frac{1}{n}} &= (w_k \dots w_{k+n-1})^{\frac{1}{n}} \\ &< \begin{cases} \max\{w_+ + \epsilon, w_- + \epsilon\}^{\frac{n}{n}} & \text{if } |k| > N \\ (\sup_{k \in [-N,N]} w_k)^{\frac{h+1}{n}} \cdot \max\{w_+ + \epsilon, w_- + \epsilon\}^{\frac{n-h-1}{n}} & \text{if } |k| \leq N, k+h = N \end{cases} \\ &= \begin{cases} \max\{w_+ + \epsilon, w_- + \epsilon\} & \text{if } |k| > N \\ (\sup_{k \in [-N,N]} w_k)^{\frac{h+1}{n}} \cdot \max\{w_+ + \epsilon, w_- + \epsilon\}^{1-\frac{h+1}{n}} & \text{if } |k| \leq N, k+h = N \end{cases}. \end{aligned}$$

Hence,

$$\begin{aligned} r(F_w) &= \lim_{n\to\infty} \|F_w^n\|^{\frac{1}{n}} = \lim_{n\to\infty} \sup_{k\in\mathbb{Z}} |w_k \dots w_{k+n-1}|^{\frac{1}{n}} = \lim_{n\to\infty} \sup_{k\in\mathbb{Z}} (w_k \dots w_{k+n-1})^{\frac{1}{n}} \\ &\leq \lim_{n\to\infty} \max \left\{ \max\{w_+ + \epsilon, w_- + \epsilon\}; (\sup_{k\in[-N,N]} w_k)^{\frac{h+1}{n}} \right. \\ &\quad \left. \cdot \max\{w_+ + \epsilon, w_- + \epsilon\}^{1-\frac{h+1}{n}} \right\} \\ &= \max\{w_+ + \epsilon, w_- + \epsilon\} \\ &= \max\{w_+, w_-\} + \epsilon. \end{aligned}$$

Analogously,

$$\begin{aligned} |w_k \dots w_{k+n-1}|^{\frac{1}{n}} &= (w_k \dots w_{k+n-1})^{\frac{1}{n}} \\ &> \begin{cases} \min\{w_+ - \epsilon, w_- - \epsilon\}^{\frac{n}{n}} & \text{if } |k| > N \\ (\inf_{k \in [-N,N]} w_k)^{\frac{h+1}{n}} \cdot \min\{w_+ - \epsilon, w_- - \epsilon\}^{\frac{n-h-1}{n}} & \text{if } |k| \leq N, k+h = N \end{cases} \\ &= \begin{cases} \min\{w_+ - \epsilon, w_- - \epsilon\} & \text{if } |k| > N \\ (\inf_{k \in [-N,N]} w_k)^{\frac{h+1}{n}} \cdot \min\{w_+ - \epsilon, w_- - \epsilon\}^{1-\frac{h+1}{n}} & \text{if } |k| \leq N, k+h = N \end{cases}. \end{aligned}$$

Hence,

$$\begin{aligned} r(F_w^{-1}) &= \lim_{n\to\infty} \|F_w^{-n}\|^{\frac{1}{n}} = \lim_{n\to\infty} \sup_{k\in\mathbb{Z}} \left| \frac{1}{w_k \dots w_{k+n-1}} \right|^{\frac{1}{n}} = \lim_{n\to\infty} \sup_{k\in\mathbb{Z}} \left(\frac{1}{w_k \dots w_{k+n-1}} \right)^{\frac{1}{n}} \\ &\leq \lim_{n\to\infty} \max \left\{ \frac{1}{\min\{w_+ - \epsilon, w_- - \epsilon\}}; \right. \end{aligned}$$

$$\left.\frac{1}{(\inf\limits_{k\in[-N,N]} w_k)^{\frac{h+1}{n}}\cdot \min\{w_+-\epsilon, w_--\epsilon\}^{1-\frac{h+1}{n}}}\right\}$$
$$=\frac{1}{\min\{w_+-\epsilon, w_--\epsilon\}}$$
$$=\frac{1}{\min\{w_+, w_-\}-\epsilon}.$$

From the above computations it follows, by the arbitrariness of $\epsilon > 0$, that $r(F_w) \leq \max\{w_+, w_-\}$ and $r(F_w^{-1})^{-1} \geq \min\{w_+, w_-\}$.
Hence, by using Proposition 2.11, one can get that

- if F_w is invertible, then
$$\sigma(F_w) = \{\lambda : r(F_w^{-1})^{-1} \leq |\lambda| \leq r(F_w)\} \subseteq \{\lambda : \min\{w_-, w_+\} \leq |\lambda| \leq \max\{w_-, w_+\}\},$$
- if F_w is not invertible, then
$$\sigma(F_w) = \{\lambda : |\lambda| \leq r(F_w)\} \subseteq \{\lambda : |\lambda| \leq \max\{w_-, w_+\}\}.$$

As $\sigma(F_w)$ ($\sigma(B_w)$) is compact and $\sigma_p(F_w) \subseteq \sigma(F_w)$ ($\sigma_p(B_w) \subseteq \sigma(B_w)$), then the conclusion follows by applying statements (1) and (3) of Proposition 3.5. □

Example 3.7 As an application of the last proposition, consider the sequence $\{w_n\}_{n\in\mathbb{Z}}$ defined as:

$$w_n = \begin{cases} 3 + \frac{1}{1+n} & \text{if } n \geq 0 \\ \frac{1}{2} - \frac{1}{n} & \text{if } n \leq -1 \end{cases}$$

Let $T = F_w$ or B_w. Then, clearly, T is invertible and $\max\{w_-, w_+\} = 3$, $\min\{w_-, w_+\} = \frac{1}{2}$.
Hence, by Proposition 3.6,

$$\sigma(T) = \left\{\lambda : \frac{1}{2} \leq |\lambda| \leq 3\right\}.$$

We conclude with the following remark.

Remark 3.8 Let $X = \ell^p(\mathbb{Z})$, $1 \leq p < \infty$, and let T denote a bilateral weighted shift F_w or B_w, on X, with $\{w_n\}_{n\in\mathbb{Z}}$ a bounded positive weight sequence. Let

$$\overline{\lim_{n\to\infty}}\, w_n = w_+^+;\ \overline{\lim_{n\to\infty}}\, w_{-n} = w_-^+$$

and

$$\underline{\lim_{n\to\infty}}\, w_n = w_+^-;\ \underline{\lim_{n\to\infty}}\, w_{-n} = w_-^-.$$

Note that the Proposition 3.6 cannot be improved by replacing $\min\{w_-, w_+\}$ with $\min\{w_-^-, w_+^-\}$ and $\max\{w_-, w_+\}$ with $\max\{w_-^+, w_+^+\}$.
To see this, take for example

$$w_n = \begin{cases} 3^{(-1)^n} & \text{if } n \geq 0 \\ 2^{(-1)^n} & \text{if } n \leq -1. \end{cases}$$

Then, by Remark 2.8 and by the spectral radius formula,

$$r(T) = \lim_{n\to\infty} \left[\sup_{k\in\mathbb{N}} (w_k w_{k+1} \cdots w_{k+n-1}) \right]^{\frac{1}{n}} = 1 = r(T^{-1})^{-1}.$$

Clearly, $\{w_{-n}\}_{n\in\mathbb{N}}$ and $\{w_n\}_{n\in\mathbb{N}}$ are not regular, with $w_-^- = \frac{1}{2}$, $w_-^+ = 2$, $w_+^- = \frac{1}{3}$, $w_+^+ = 3$, and hence $\max\{w_-^+, w_+^+\} = 3$ and $\min\{w_+^-, w_-^-\} = \frac{1}{3}$.

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