Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat. (2022) 116: 141
https://doi.org/10.1007/s13398-022-01284-8

ORIGINAL PAPER

On diagonal operators between the sequence (LF)-spaces $l_p(\mathcal{V})$

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Received: 31 January 2022 / Accepted: 31 May 2022 / Published online: 27 June 2022

Abstract
Diagonal (multiplication) operators acting between a particular class of countable inductive spectra of Fréchet sequence spaces, called sequence (LF)-spaces, are investigated. We prove results concerning boundedness, compactness, power boundedness, and mean ergodicity. Furthermore, we determine when a diagonal operator is Montel and reflexive. We analyze the spectra in terms of the system of weights defining the spaces.

Keywords Sequence (LF)-spaces · Multiplication operator · Power bounded operator · Mean ergodic operator · Montel operator · Reflexive operator · Compact operator · Spectrum

Mathematics Subject Classification Primary 47B37 · 46E99 · 47A10; Secondary 47A35 · 47B07

1 Introduction

In this paper, we are concerned with studying diagonal (multiplication) operators acting on a particular class of sequence (LF)-spaces. Many authors have investigated these mappings, for instance, in the setting of weighted spaces of (vector-valued) continuous functions by Manhas [17, 18] and Oubbi [20] among others. Recently many authors have focused on studying the ergodic properties of the diagonal operators acting on (LB)-spaces of functions and sequences, for instance, [10, 22]. In the context of Köthe echelon spaces, Crofts [12] investigated diagonal operators. The case of multiplication operators on weighted spaces of analytic functions on the complex unit disc was studied by Bonet and Ricker [11]. Albanese and the author in [5] have studied the spectra and the ergodic properties of the multiplication operators on the space $\mathcal{S}(\mathbb{R})$ of rapidly decreasing functions and the (PLB)-space of its multipliers. Echelon and co-echelon spaces were studied by Köthe and Toeplitz. In a paper [24] published in 1992, Vogt characterized the regularity, completeness, and (weak) acyclicity of Köthe (LF)-sequence spaces E_p, $1 \leq p \leq \infty \cup \{0\}$.

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In this note, we treat different aspects of diagonal operators acting between the sequence (LF)-spaces $l_p(\mathcal{V})$, extending the recent results of Rodríguez–Arenas obtained in [22] for Köthe echelon spaces. The sequence (LF)-spaces $l_p(\mathcal{V})$ are defined as an inductive limit of a spectrum of echelon spaces. Vogt in [24] has obtained important results concerning regularity and completeness of sequence (LF)-spaces. We focus our attention on the action of the diagonal operator on them. The purpose of this work is to characterize in terms of the weight sequences the boundedness, compactness, being Montel, and reflexivity of the diagonal operator $M_\varphi \colon l_p(\mathcal{V}) \to l_p(\mathcal{W})$, $(x_i)_{i\in\mathbb{N}} \mapsto (x_i\varphi_i)_{i\in\mathbb{N}}$, with $1 \leq p \leq \infty \cup \{0\}$, $\mathcal{V}, \mathcal{W}$ two systems of weights and $\varphi \in \omega$. Furthermore, we determine the spectra of the diagonal operators and study their ergodic properties. Our notation for functional analysis is standard. We refer the reader to [6, 16, 19].

The article is divided into six sections. In Sect. 2, we establish the notation and recall some of the most fundamental definitions concerning (LF)-spaces. Moreover, we characterize the boundedness, compactness, being Montel, and reflexivity for operators acting between (LF)-spaces. Section 3 is devoted to the definitions and the main properties of the sequence (LF)-spaces $l_p(\mathcal{V})$. While in Sect. 4 we study the diagonal operators acting between the spaces $l_p(\mathcal{V})$. We show when a diagonal operator $M_\varphi \colon l_p(\mathcal{V}) \to l_p(\mathcal{W})$ is continuous, bounded, compact, Montel, and reflexive in terms of the weight sequences that form the countable inductive spectra of Fréchet sequence spaces. In Sect. 5, we analyze the spectrum and the Waelbroeck spectrum of the diagonal operators. A complete discussion concerning power boundedness and (uniform) mean ergodicity for diagonal operators is given in Sect. 6.

2 Definitions and results on (LF)-spaces

The purpose of this section is to recall some definitions and fundamental results of the theory of the (LF)-spaces.

2.1 (LF)-spaces

An *(LF)-space* is a locally convex Hausdorff space (lcHs, briefly) E which is an inductive limit $E = \operatorname{ind}_n E_n$ of an inductive sequence $(E_n)_{n\in\mathbb{N}}$ of Fréchet spaces, i.e., $E = \cup_n E_n$ and $E_n \hookrightarrow E_{n+1}$ continuously for all $n \in \mathbb{N}$ (see [14] for more details). In the following, we denote by t the lc-topology of E and by t_n the Fréchet topology of each E_n, $n \in \mathbb{N}$. The lc-topology of the (LF)-space $E = \operatorname{ind}_n E_n$ is the finest lc-topology that makes the inclusions $E_n \hookrightarrow E$ continuous for all $n \in \mathbb{N}$.

Let $E = \operatorname{ind}_n E_n$ be an (LF)-space. E is called:

- *regular*, if every bounded subset of E is contained and bounded in some step E_n;
- *(pre)compactly retractive*, if for every (pre)compact subset K of E, there exists $m \in \mathbb{N}$ such that $K \subset E_m$ and it is (pre)compact there;
- *strongly boundedly retractive*, if it is regular and for all $k \in \mathbb{N}$, there exists $l \geq k$ such that (E, t) and (E_l, t_l) induce the same topology on each bounded set of (E_k, t_k);
- *boundedly retractive*, if every bounded subset B of E is contained in some step E_n and the topologies of E and E_n coincide on B;
- *sequentially retractive*, if every convergent sequence in E is contained in some step E_n and converges there.

It is well-known that every complete (LF)-space is regular, but whether the converse holds seems to be an open problem (mentioned by Grothendieck), even for (LB)-spaces. We refer the reader to [6, 24, 25] for more details.

The space (E, t) is said to satisfy the *condition (M)* (resp. *(M_0)*) *of Retakh* [21] if there exists an increasing sequence $(U_n)_{n\in\mathbb{N}}$ of subsets of E such that for all $n \in \mathbb{N}$ U_n is an absolutely convex 0-neighborhood of E_n such that

$$\forall n \in \mathbb{N}\, \exists m \geq n\, \forall \mu \geq m : t_\mu \text{ and } t_m \text{ induce the same topology on } U_n.$$

$$(\text{resp. } \forall n \in \mathbb{N}\, \exists m \geq n\ \forall \mu \geq m : \sigma(E_\mu, E'_\mu) \text{ and } \sigma(E_m, E'_m) \text{ induce the same topology on } U_n).$$

(LF)-spaces with condition (M) (resp. (M_0)) are called *acyclic* (resp. *weak-acyclic*).

The following important theorem gives some equivalence of the concepts mentioned above. This theorem is due to Wengenroth for (LF)-spaces. See [25, Theorem 6.4].

Theorem 2.1 *For an (LF)-space $E = \operatorname{ind}_n E_n$ the following conditions are equivalent:*

(1) *There is an increasing sequence $(U_n)_{n\in\mathbb{N}}$ of subsets of E such that for all $n \in \mathbb{N}$ U_n is an absolutely convex 0-neighborhood of E_n for which for all $n \in \mathbb{N}$ there exists $m \geq n$ such that t and t_m induce the same topology on U_n;*
(2) *E satisfies the condition (M);*
(3) *E is boundedly retractive;*
(4) *E is (pre)compactly retractive;*
(5) *E is sequentially retractive.*

Furthermore, the condition (M) implies the completeness of the (LF)-spaces (see [25, Corollary 6.5]). Therefore, from the above considerations, we have that if an (LF)-space $E = \operatorname{ind}_n E_n$ satisfies the condition (M), then E is strongly boundedly retractive.

Valdivia in [24, Page 161] showed that an (LF)-space $E = \operatorname{ind}_n E_n$ satisfies the condition (M_0) if, and only if, for all $m \in \mathbb{N}$ there is an absolutely convex 0-neighborhood U_m of E_m with $U_m \subseteq U_{m+1}$ such that, given any $n \in \mathbb{N}$ there is an integer $\mu > n$ such that $\sigma(E, E')$ and $\sigma(E_\mu, E'_\mu)$ coincide on U_m.

2.2 Operators acting on (LF)-spaces

A linear operator between the lcHs X and Y is called *bounded* if it maps some 0-neighborhood of X into a bounded subset of Y, while it is said to be *compact* if it maps some 0-neighborhood of X into a relatively compact subset of Y. In the following, we characterize the boundedness and compactness of operators acting between (LF)-spaces. For this issue, we denote by $\mathcal{B}(X)$ the set of the bounded subsets of a lcHs X.

We start recalling a known result of Grothendieck [14] and a similar one.

Lemma 2.2 (1) *Let G be a metrizable lcHs. Then for every family of bounded subsets $(B_j)_{j\in\mathbb{N}}$ of G, there exists a sequence $(\lambda_j)_{j\in\mathbb{N}} \in (0,\infty)^{\mathbb{N}}$ such that $\bigcup_{j=1}^{\infty} \lambda_j B_j \in \mathcal{B}(G)$.*
(2) *Let G be a metrizable lcHs. Then for every family of precompact subets $(C_j)_{j\in\mathbb{N}}$ of G, there exists a sequence $(\lambda_j)_{j\in\mathbb{N}} \in (0,\infty)^{\mathbb{N}}$ such that $\bigcup_{j=1}^{\infty} \lambda_j C_j$ is precompact in G.*

We use the previous lemma to give the following characterizations.

Proposition 2.3 *Let $E = \operatorname{ind}_n E_n$ and $F = \operatorname{ind}_n F_n$ be two (LF)-spaces. The following assertions hold:*

(1) *Assume that F is regular. Then the linear operator $T : E \to F$ is bounded if, and only if, there exists $n \in \mathbb{N}$ such that for all $m \in \mathbb{N}$ we have that $T(E_m) \subset F_n$ and the restriction $T : E_m \to F_n$ is bounded.*

(2) *Assume that F satisfies the condition (M). Then the linear operator $T\colon E \to F$ is compact if, and only if, there exists $n \in \mathbb{N}$ such that for all $m \in \mathbb{N}$ we have that the restriction $T\colon E_m \to F_n$ is compact.*

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Proof Both the proofs are analogous. Hence, we show the claim for the compactness.

Suppose that $T\colon E \to F$ is compact. By assumption, we can find a 0-neighborhood U of E such that $T(U)$ is relatively compact in F. Note that in F, since it is complete, precompactness and relative compactness are the same. Since F is equivalently precompactly retractive, we can find $n \in \mathbb{N}$ such that $T(U)$ is precompact in F_n. For all $m \in \mathbb{N}$ the set $U' := U \cap E_m$ is a 0-neighborhood in E_m such that $T(U') \subset T(U) \subset F_n$. Since $T(U)$ is precompact in F_n, the same also holds for $T(U')$. This means that the map $T\colon E_m \to F_n$ is compact.

We assume now that the condition is fulfilled and prove that $T\colon E \to F$ is compact. By assumption, there exists $n \in \mathbb{N}$ such that for all $m \in \mathbb{N}$ we can find a 0-neighborhood U_m in E_m such that $T(U_m)$ is precompact in F_n. Now we apply Lemma 2.2 at the Fréchet space F_n to find a sequence $(\lambda_m)_{m\in\mathbb{N}} \in (0,\infty)^{\mathbb{N}}$ such that $\bigcup_{m=1}^{\infty} \lambda_m T(U_m)$ is precompact in F_n. Set $U := \Gamma\left(\bigcup_{m=1}^{\infty} \lambda_m U_m\right)$, where Γ denotes the absolutely convex hull of the union. Clearly U is a 0-neighborhood of E satisfying $T(U) \subset \Gamma\left(\bigcup_{m=1}^{\infty} \lambda_m T(U_m)\right)$, which is a precompact subset of F_n, since the absolutely convex hull of a precompact set in a lcHs is still precompact. Therefore, we obtain that $T(U)$ is precompact in F_n and so in F. □

Given X, Y two lcHs, a linear operator $T : X \to Y$ is called *Montel* if it maps bounded subsets of X into relatively compact subsets of Y. If X and Y are Banach spaces, then $T\colon X \to Y$ is Montel if, and only if, it is compact. For an operator between (LF)-spaces we have the following result.

Proposition 2.4 *Let $E = \operatorname{ind}_n E_n$ and $F = \operatorname{ind}_n F_n$ be two (LF)-spaces. Suppose that F satisfies the condition (M), and E is regular. Then the continuous linear operator $T\colon E \to F$ is Montel if, and only if, for all $m \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that the restriction $T\colon E_m \to F_n$ is Montel.*

Proof Suppose that $T\colon E \to F$ is Montel. Fixed $m \in \mathbb{N}$, by the continuity of T there exists $n \in \mathbb{N}$ such that $T\colon E_m \to F_n$ is continuous (see [14]). Since F is in particular strongly boundedly retractive, we choose a $n' \geq n$ as in the definition of the condition and prove that $T\colon E_m \to F_{n'}$ is Montel. If we take B a bounded subset of E_m, due to the continuity of T we have that $T(B)$ is bounded in F_n. Moreover, by assumption, $T(B)$ is relatively compact in F. Hence, the topologies on $T(B)$ induced by F and $F_{n'}$ coincide, since F is strongly boundedly retractive. This means that $T(B)$ is relatively compact also in $F_{n'}$.

We assume now that the condition is fulfilled and prove that $T\colon E \to F$ is Montel. Fix a bounded subset B of E. Due to the regularity of E, we can find $m \in \mathbb{N}$ such that B is bounded in E_m. By assumption, there exists $n \in \mathbb{N}$ such that the restriction $T\colon E_m \to F_n$ is Montel. Hence, $T(B)$ is relatively compact in F_n and so in F. □

Remark 2.5 In the proof of Proposition 2.4, to show that the condition is sufficient, we only use the assumption of the regularity of E.

We recall that given X, Y two lcHs, an operator $T : X \to Y$ is called *reflexive* if it maps bounded subsets of X into relatively weakly compact subsets of Y. If Y is reflexive, then a continuous linear operator $T\colon X \to Y$ is reflexive. We refer the reader to [16] for more details.

We give the following characterization concerning (LF)-spaces.

Proposition 2.6 *Let $E = \operatorname{ind}_n E_n$ and $F = \operatorname{ind}_n F_n$ be two (LF)-spaces. Suppose that F satisfies the condition (M_0), and E is regular. Then the continuous linear operator $T: E \to F$ is reflexive if, and only if, for all $m \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that the restriction $T: E_m \to F_n$ is reflexive.*

Proof Suppose that $T: E \to F$ is reflexive. Fixed $m \in \mathbb{N}$, by the continuity of T there exists $n \in \mathbb{N}$ such that $T: E_m \to F_n$ is continuous (see [14]). If we take B a bounded subset of E_m, due to the continuity of T we have that $T(B)$ is bounded in F_n. Since F satisfies the condition (M_0), taking into account Valdivia's result [24, Page 161] there exists an increasing sequence $(U_k)_{k\in\mathbb{N}}$ of subsets of F such that U_k is an absolutely convex 0-neighborhood of F_k for all $k \in \mathbb{N}$ and the topologies induced on U_k from $\sigma(F, F')$ and $\sigma(F_{n'}, F'_{n'})$ coincide, for some $n' > k$. From the boundedness of $T(B)$, we can find $\lambda > 0$ such that $T(B) \subset \lambda U_n$. Moreover, by assumption, $T(B)$ is relatively weakly compact in F. This implies that $T(B)$ is relatively weakly compact in $F_{n'}$, that means that $T: E_m \to F_{n'}$ is reflexive.

We assume now that the condition is fulfilled and prove that $T: E \to F$ is reflexive. Fix a bounded subset B of E. Due to the regularity of E, we can find $m \in \mathbb{N}$ such that B is bounded in E_m. By assumption, there exists $n \in \mathbb{N}$ such that $T: E_m \to F_n$ is reflexive. Hence, $T(B)$ is relatively weakly compact in F_n and so in F. □

3 The sequence (LF)-spaces $l_p(\mathcal{V})$

In this section, we introduce the sequence (LF)-spaces $l_p(\mathcal{V})$ and recall the main properties of these spaces concerning regularity and completeness.

For all $n \in \mathbb{N}$, $V_n = \left(v_{n,k}\right)_{k\in\mathbb{N}}$ is a countable family of (strictly) positive sequences, called *weights*, on $\mathbb{N}$. We denote by $\mathcal{V}$ the sequence $(V_n)_{n\in\mathbb{N}}$ and we assume that the following two conditions are satisfied:

(1) $v_{n,k}(i) \leq v_{n,k+1}(i)$ for all $n, k \in \mathbb{N}$ and $i \in \mathbb{N}$;
(2) $v_{n,k}(i) \geq v_{n+1,k}(i)$ for all $n, k \in \mathbb{N}$ and $i \in \mathbb{N}$.

Given a *system of weights* $\mathcal{V}$ as above, for $n, k \in \mathbb{N}$ and $1 \leq p \leq \infty$ we define as usual

$$l_p(v_{n,k}) := \left\{x = (x_i)_{i\in\mathbb{N}} \in \omega \mid p_{v_{n,k}}(x) := \|(x_i v_{n,k}(i))_{i\in\mathbb{N}}\|_p < \infty\right\},$$

where $\|\cdot\|_p$ denotes the usual l_p norm. For $p = 0$ we set

$$c_0(v_{n,k}) := \left\{x = (x_i)_{i\in\mathbb{N}} \in \omega \mid \lim_{i\to\infty} v_{n,k}(i)x_i = 0\right\}.$$

These spaces are Banach with the corresponding $p_{v_{n,k}}$ norms and $c_0(v_{n,k})$ is Banach with the norm inherited from $l_\infty(v_{n,k})$. Since $l_p(v_{n,k+1})$ is continuously embedded into $l_p(v_{n,k})$, the sequence $\{l_p(v_{n,k})\}_{k\in\mathbb{N}}$ of Banach spaces forms a projective spectrum. Hence, for all $n \in \mathbb{N}$ and $1 \leq p \leq \infty$, we can consider the *echelon spaces*

$$\lambda_p(V_n) := \bigcap_{k\in\mathbb{N}} l_p(v_{n,k}) \text{ and } \lambda_0(V_n) := \bigcap_{k\in\mathbb{N}} c_0(v_{n,k}).$$

Endowed with the projective topologies $\lambda_p(V_n) = \operatorname{proj}_k l_p(v_{n,k})$ (resp. $\lambda_0(V_n) = \operatorname{proj}_k c_0(v_{n,k})$), these spaces are Fréchet with the topology defined by the corresponding seminorms $p_{n,k} := p_{v_{n,k}}$, $k \in \mathbb{N}$.

Condition (2) implies that $\lambda_p(V_n)$ is continuously embedded into $\lambda_p(V_{n+1})$. Hence, the sequence $\{\lambda_p(V_n)\}_{n\in\mathbb{N}}$ for $1 \leq p \leq \infty \cup \{0\}$ of Fréchet spaces forms an inductive spectrum and the spaces

$$l_p(\mathcal{V}) := \bigcup_{n\in\mathbb{N}} \lambda_p(V_n) \text{ and } l_0(\mathcal{V}) := \bigcup_{n\in\mathbb{N}} \lambda_0(V_n)$$

endowed with the inductive topologies $l_p(\mathcal{V}) := \operatorname{ind}_n \lambda_p(V_n)$ (resp. $l_0(\mathcal{V}) := \operatorname{ind}_n \lambda_0(V_n)$) are (LF)-spaces.

To describe the inductive topology of these spaces, we can associate to $l_p(\mathcal{V})$ the *Nachbin family* in the usual way (see [8, 24]). Set

$$\overline{V} := \left\{ \overline{v} = (\overline{v}_i)_{i\in\mathbb{N}} \in (0,\infty)^{\mathbb{N}} \mid \forall n \in \mathbb{N}\, \exists k = k(n) \in \mathbb{N} : \sup_{i\in\mathbb{N}} \frac{\overline{v}_i}{v_{n,k}(i)} < \infty \right\}.$$

A sequence $\overline{v}$ belongs to $\overline{V}$ if, and only if, $\overline{v}_i = \inf_{n\in\mathbb{N}} \alpha_n v_{n,k(n)}(i)$, for some $\alpha_n > 0$, $k(n) \in \mathbb{N}$ and for all $i \in \mathbb{N}$. For $1 \le p \le \infty$ we define in the usual way the Banach spaces

$$l_p(\overline{v}) := \{x = (x_i)_{i\in\mathbb{N}} \mid \|(\overline{v}_i x_i)_{i\in\mathbb{N}}\|_p < \infty\} \text{ and } c_0(\overline{v}) := \left\{ x = (x_i)_{i\in\mathbb{N}} \mid \lim_{i\to\infty} \overline{v}_i x_i = 0 \right\}.$$

For $1 \le p \le \infty$ or $p = 0$ we denote by

$$K_p(\overline{V}) := \operatorname*{proj}_{\overleftarrow{\overline{v}\in\overline{V}}} l_p(\overline{v}) \text{ and } K_0(\overline{V}) := \operatorname*{proj}_{\overleftarrow{\overline{v}\in\overline{V}}} c_0(\overline{v}).$$

The spaces $K_p(\overline{V})$, for $1 \le p \le \infty \cup \{0\}$, are complete, being a projective limit of complete spaces, and their seminorms will be denoted by $p_{\overline{v}}$, $\overline{v} \in \overline{V}$, when p is fixed and no confusion shall arise.

Remark 3.1 The inclusion $l_p(\mathcal{V}) \hookrightarrow K_p(\overline{V})$, for $1 \le p < \infty \cup \{0\}$, is a topological isomorphism into as a consequence of [24, Proposition 5.1].

For $p = \infty$, the inclusion $l_\infty(\mathcal{V}) \hookrightarrow K_\infty(\overline{V})$ also holds. We need to require, in addition, that the system of weights $\overline{V}$ satisfies the following condition (see [8, Theorem 7]): $\forall (k(n))_{n\in\mathbb{N}} \subset \mathbb{N}\, \exists \overline{v} \in \overline{V}\, \forall m \in \mathbb{N}, \overline{w}_m \in \overline{W}_m\, \exists M \in \mathbb{N}$ such that

$$\min\left(\overline{w}_m, \overline{v}^{-1}\right) \le \sum_{n=1}^{M} v_{n,k(n)}^{-1}, \tag{3.1}$$

where $\overline{W}_m$ denotes the system of all non-negative sequences which are dominated by sequences of the form $\inf_{k\in\mathbb{N}} \alpha_k v_{m,k(m)}^{-1}$, for some $\alpha_k > 0$, $k(m) \in \mathbb{N}$. In the (LB)-case, condition (3.1) is equivalent to condition (D) of Bierstedt and Meise (see [7]).

Furthermore, the inclusion $l_p(\mathcal{V}) \hookrightarrow K_p(\overline{V})$, for $1 \le p < \infty \cup \{0\}$, is also with dense range. This means that $K_p(\overline{V}) = \widehat{l_p(\mathcal{V})}$, where $\widehat{l_p(\mathcal{V})}$ stands for the topological completion of $l_p(\mathcal{V})$.

In the following, we recall the conditions for which we have that $l_p(\mathcal{V}) = K_p(\overline{V})$ algebraically and also topologically. Due to Remark 3.1, $l_p(\mathcal{V}) = K_p(\overline{V})$ if, and only if, the (LF)-space $l_p(\mathcal{V})$ is complete.

In our context, the characterization of the regularity is due to Vogt [24], in terms of a condition on the system of weights $\mathcal{V}$.

Definition 3.2 We say that the sequence $\mathcal{V} = \left((v_{n,k})_{k\in\mathbb{N}}\right)_{n\in\mathbb{N}}$ satisfies the *condition (WQ)* (or is of *type (WQ)*) if

$$\forall n \in \mathbb{N}\, \exists \mu, m \in \mathbb{N}\, \forall k, N \in \mathbb{N},\ \exists K \in \mathbb{N}, S > 0,\ s.t. \forall i \in \mathbb{N} :$$

$$v_{m,k}(i) \leq S(v_{n,\mu}(i) + v_{N,K}(i)).$$

Definition 3.3 We say that the sequence $\mathcal{V} = \left((v_{n,k})_{k\in\mathbb{N}}\right)_{n\in\mathbb{N}}$ satisfies the *condition (Q)* (or is of *type (Q)*) if

$$\forall n \in \mathbb{N}\, \exists \mu, m \in \mathbb{N}\, \forall k, N \in \mathbb{N}, R > 0\, \exists K \in \mathbb{N}, S > 0, \text{ s.t.} \forall i \in \mathbb{N} :$$
$$v_{m,k}(i) \leq \frac{1}{R} v_{n,\mu}(i) + S v_{N,K}(i).$$

In [24], is shown the link between the (M) and (M_0)-conditions (i.e., the acyclicity and weak-acyclicity) and the (Q) and (WQ)-conditions. We find the characterization of the regularity of the (LF)-spaces $l_p(\mathcal{V})$ in the following theorem of Vogt (see [24]).

Theorem 3.4 (1) *For $1 < p < \infty$, the following conditions are equivalent:*

(i) *$\mathcal{V}$ satisfies the condition (WQ);*
(ii) *$l_p(\mathcal{V})$ is regular;*
(iii) *$l_p(\mathcal{V})$ is complete;*
(iv) *$l_p(\mathcal{V})$ is reflexive.*

(2) *For $p = 1, \infty$, the following conditions are equivalent:*

(i) *$\mathcal{V}$ satisfies the condition (WQ);*
(ii) *$l_p(\mathcal{V})$ is regular;*
(iii) *$l_p(\mathcal{V})$ is complete.*

(3) *For $p = 0$, the following conditions are equivalent:*

(i) *$\mathcal{V}$ satisfies the condition (Q);*
(ii) *$l_0(\mathcal{V})$ is regular;*
(iii) *$l_0(\mathcal{V})$ is complete.*

By Remark 3.1 and Theorem 3.4 (see also [8]), we have

$$l_0(\mathcal{V}) = K_0(\overline{V}) \text{ alg. and top.} \;\Leftrightarrow\; \mathcal{V} \text{ satisfies the condition } (Q) \tag{3.2}$$
$$l_\infty(\mathcal{V}) = K_\infty(\overline{V}) \text{ alg. and top.} \;\Leftrightarrow\; \mathcal{V} \text{ satisfies the conditions } (WQ) + (3.1) \tag{3.3}$$
$$l_p(\mathcal{V}) = K_p(\overline{V}) \text{ alg. and top. } 1 \leq p < \infty \;\Leftrightarrow\; \mathcal{V} \text{ satisfies the condition } (WQ). \tag{3.4}$$

This result means that the inductive limit topology of the (LF)-spaces $l_p(\mathcal{V})$, under the above conditions (3.2), (3.3), (3.4), coincide with the lc'one induced by the family of seminorms $(p_{\overline{v}})_{\overline{v}\in\overline{V}}$.

4 Diagonal operators acting on $l_p(\mathcal{V})$

From now on, we work with diagonal (multiplication) operators on the sequence (LF)-spaces $l_p(\mathcal{V})$. We denote by ω the space of all the sequences $\varphi = (\varphi_i)_{i\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$. Given a sequence $\varphi = (\varphi_i)_{i\in\mathbb{N}} \in \omega$, we define the multiplication (or diagonal) operator as $M_\varphi : \omega \to \omega$ such that $(x_i)_{i\in\mathbb{N}} \mapsto (x_i\varphi_i)_{i\in\mathbb{N}}$. If M_φ acts continuously from a sequence lcHs X into a sequence lcHs Y, we say that φ is a *multiplier* from X to Y.

Firstly, we characterize the multipliers between the sequence (LF)-spaces $l_p(\mathcal{V})$.

Theorem 4.1 *Let $\mathcal{V}, \mathcal{W}$ be two systems of weights and fix $\varphi = (\varphi_i)_{i\in\mathbb{N}} \in \omega$. For $1 \leq p \leq \infty \cup \{0\}$, the following properties are equivalent:*

(1) *$M_\varphi \colon l_p(\mathcal{V}) \to l_p(\mathcal{W})$ is well-defined;*
(2) *$M_\varphi \colon l_p(\mathcal{V}) \to l_p(\mathcal{W})$ is continuous;*
(3) *For all $m \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that for all $k \in \mathbb{N}$ there exist $l \in \mathbb{N}$ for which*

$$\sup_{i\in\mathbb{N}} \frac{w_{n,k}(i)|\varphi_i|}{v_{m,l}(i)} < \infty. \tag{4.1}$$

Proof Clearly, (2) implies (1) and (1) implies (2) by the Closed Graph Theorem [16, Page 57]. Indeed, if $x = (x_i)_i \subset l_p(\mathcal{V})$ is a net convergent to $\overline{x}$ in $l_p(\mathcal{V})$ and $(M_\varphi(x_i))_i$ is convergent to $\overline{y}$ in $l_p(\mathcal{W})$, then $\overline{y} = \varphi\overline{x} = M_\varphi(\overline{x})$. This proves that the graph of M_φ is closed.

We prove the statement for $1 \le p \le \infty$. The proof in the case $p = 0$ is analogous and so is omitted.

(2)⇔(3). The diagonal operator $M_\varphi \colon l_p(\mathcal{V}) \to l_p(\mathcal{W})$ is continuous if, and only if, for all $m \in \mathbb{N}$ the diagonal operator $M_\varphi \colon \lambda_p(V_m) \to l_p(\mathcal{W})$ is continuous. From Grothendieck's Factorization Theorem [14, Page 147], M_φ is continuous if, and only if, for all $m \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that $M_\varphi \colon \lambda_p(V_m) \to \lambda_p(W_n)$ is well-defined and continuous. Now the diagonal operator between the echelon spaces is continuous if, and only if, for all $k \in \mathbb{N}$ there exists $l \in \mathbb{N}$ such that $M_\varphi \colon l_p(v_{m,l}) \to l_p(w_{n,k})$ is continuos. Consider the isometries $M_{v_{m,l}} \colon l_p(v_{m,l}) \to l_p$ such that $(x_i)_{i\in\mathbb{N}} \mapsto (v_{m,l}(i)x_i)_{i\in\mathbb{N}}$ and $M_{w_{n,k}} \colon l_p(w_{n,k}) \to l_p$ such that $(x_i)_{i\in\mathbb{N}} \mapsto (w_{n,k}(i)x_i)_{i\in\mathbb{N}}$. Setting $\phi := \left(\frac{w_{n,k}(i)\varphi_i}{v_{m,l}(i)}\right)_{i\in\mathbb{N}}$, then $M_\phi = M_{w_{n,k}} \circ M_\varphi \circ M_{v_{m,l}}^{-1}$. Taking into account that $M_\phi \colon l_p \to l_p$ is continuous if, and only if, ϕ is bounded (see [4, Lemma 15]), we get (4.1).

□

A similar characterization holds for the boundedness of multiplication operators between the sequence (LF)-spaces $l_p(\mathcal{V})$. We need to recall the characterization of boundedness for operators acting between Fréchet spaces (see [4, Lemma 25]).

Lemma 4.2 *Let $E = \mathrm{proj}_m E_m$ and $F = \mathrm{proj}_m F_m$ be Fréchet spaces such that $E = \bigcap_{m=1}^\infty E_m$, with each $(E_m, \|\cdot\|_m)$ a Banach space (resp. $F = \bigcap_{m=1}^\infty F_m$, with each $(F_m, \|\cdot\|_m)$ a Banach space). Moreover, it is assumed that E is dense in E_m and that $E_m \hookrightarrow E_{m+1}$ for all $m \in \mathbb{N}$ (resp. $F_m \hookrightarrow F_{m+1}$ for all $m \in \mathbb{N}$). Then the continuous linear operator $T \colon E \to F$ is bounded if, and only if, there exists $l \in \mathbb{N}$ such that for all $k \in \mathbb{N}$ the operator T has a unique continuous linear extension $T \colon E_l \to F_k$.*

Remark 4.3 We observe that Lemma 4.2 continues to hold even if T is not assumed to be continuous. The result follows with the same proof contained in [4, Lemma 25].

Theorem 4.4 *Let $\mathcal{V}, \mathcal{W}$ be two systems of weights and fix $\varphi = (\varphi_i)_{i\in\mathbb{N}} \in \omega$. For $1 \le p < \infty \cup \{0\}$, assume that $l_p(\mathcal{W})$ is regular. Then M_φ is bounded if, and only if, there exists $n \in \mathbb{N}$ such that for all $m \in \mathbb{N}$ there exists $l \in \mathbb{N}$ such that for all $k \in \mathbb{N}$*

$$\sup_{i\in\mathbb{N}} \frac{w_{n,k}(i)|\varphi_i|}{v_{m,l}(i)} < \infty. \tag{4.2}$$

Proof By Proposition 2.3 (1), $M_\varphi \colon l_p(\mathcal{V}) \to l_p(\mathcal{W})$ is bounded if, and only if, there exists $n \in \mathbb{N}$ such that for all $m \in \mathbb{N}$ the restriction $M_\varphi \colon \lambda_p(V_m) \to \lambda_p(W_n)$ is bounded. Using Lemma 4.2, this holds if, and only if, there exists $l \in \mathbb{N}$ such that for all $k \in \mathbb{N}$ the operator M_φ has a unique continuous linear extension $M_\varphi \colon l_p(v_{m,l}) \to l_p(w_{n,k})$. As in the proof of Theorem 4.1, this is equivalent to requiring that (4.2) is satisfied.

The same also holds for $p = 0$. □

The next target is to describe when a diagonal operator between the sequence (LF)-spaces $l_p(\mathcal{V})$ is compact. Before giving the result, we need to recall the characterization of the compactness of multiplication operators between the Banach spaces l_p and c_0. The proof is left to the reader.

Lemma 4.5 *Let $\phi \in \omega$. The following assertions hold:*

(1) *The multiplication operator $M_\phi \colon c_0 \to c_0$ is compact if, and only if, $\phi \in c_0$.*
(2) *For $1 \leq p < \infty$, the multiplication operator $M_\phi \colon l_p \to l_p$ is compact if, and only if, $\phi \in c_0$.*

As for the boundedness, we also need a characterization of the compactness for operators acting between Fréchet spaces. Again we refer to [4].

Lemma 4.6 *Let $E = \operatorname{proj}_m E_m$ and $F = \operatorname{proj}_m F_m$ be Fréchet spaces such that $E = \bigcap_{m=1}^{\infty} E_m$, with each $(E_m, \|\cdot\|_m)$ a Banach space (resp. $F = \bigcap_{m=1}^{\infty} F_m$, with each $(F_m, \|\cdot\|_m)$ a Banach space). Moreover, it is assumed that E is dense in E_m and that $E_m \hookrightarrow E_{m+1}$ for all $m \in \mathbb{N}$ (resp. $F_m \hookrightarrow F_{m+1}$ for all $m \in \mathbb{N}$). Then the linear operator $T \colon E \to F$ is compact if, and only if, there exists $l \in \mathbb{N}$ such that for all $k \in \mathbb{N}$ the operator T has a unique compact linear extension $T \colon E_l \to F_k$.*

Theorem 4.7 *Let $\mathcal{V}, \mathcal{W}$ be two systems of weights and fix $\varphi = (\varphi_i)_{i\in\mathbb{N}} \in \omega$. For $1 \leq p < \infty \cup \{0\}$, assume that $l_p(\mathcal{W})$ satisfies the condition (M). Then M_φ is compact if, and only if, there exists $n \in \mathbb{N}$ such that for all $m \in \mathbb{N}$ there exists $l \in \mathbb{N}$ such that for all $k \in \mathbb{N}$*

$$\lim_{i\to\infty} \frac{w_{n,k}(i)|\varphi_i|}{v_{m,l}(i)} = 0. \tag{4.3}$$

Proof By Proposition 2.3 (2), $M_\varphi \colon l_p(\mathcal{V}) \to l_p(\mathcal{W})$ is compact if, and only if, there exists $n \in \mathbb{N}$ such that for all $m \in \mathbb{N}$ the restriction $M_\varphi \colon \lambda_p(V_m) \to \lambda_p(W_n)$ is compact. Using Lemma 4.6, this holds if, and only if, there exists $l \in \mathbb{N}$ such that for all $k \in \mathbb{N}$ the extension $M_\varphi \colon l_p(v_{m,l}) \to l_p(w_{n,k})$ is compact. We prove that it is equivalent to the limit (4.3). Consider the isometries $M_{v_{m,l}} \colon l_p(v_{m,l}) \to l_p$ such that $(x_i)_{i\in\mathbb{N}} \mapsto (v_{m,l}(i)x_i)_{i\in\mathbb{N}}$ and $M_{w_{n,k}} \colon l_p(w_{n,k}) \to l_p$ such that $(x_i)_{i\in\mathbb{N}} \mapsto (w_{n,k}(i)x_i)_{i\in\mathbb{N}}$. Setting $\phi := \left(\frac{w_{n,k}(i)\varphi_i}{v_{m,l}(i)}\right)_{i\in\mathbb{N}}$, then $M_\phi = M_{w_{n,k}} \circ M_\varphi \circ M_{v_{m,l}}^{-1}$. Thus, M_φ is compact if, and only if, $M_\phi \colon l_p \to l_p$ is compact. Due to Lemma 4.5, this holds if, and only if, $\phi \in c_0$, i.e., if ϕ vanishes at infinity.

The same also holds for $p = 0$. □

To describe when diagonal operators between the sequence (LF)-spaces $l_p(\mathcal{V})$ are Montel, firstly, we characterize when diagonal operators between the echelon spaces are Montel.

Remark 4.8 We recall a known result about the relative compactness for l_p (see [19, Chapter 15]). For $1 \leq p < \infty$, a subset K of l_p is relatively compact if, and only if, for every $\varepsilon > 0$ there exists $j_0 \in \mathbb{N}$ such that for every $x \in K$ we have $\sum_{j=j_0+1}^{\infty} |x_j|^p < \varepsilon^p$. Analogously, for $p = 0$ a subset K of c_0 is relatively compact if, and only if, for every $\varepsilon > 0$ there exists $j_0 \in \mathbb{N}$ such that for every $x \in K$ we have $\sup_{j\geq j_0+1} |x_j| < \varepsilon$.

Let $A = (a_n)_{n\in\mathbb{N}}$ denote a *Köthe matrix*, i.e., an increasing sequence of strictly positive functions a_n. We consider

$$\lambda_\infty(A)_+ = \{x = (x_i)_{i\in\mathbb{N}} \in \omega \mid \|(a_n x)_{n\in\mathbb{N}}\|_\infty < \infty \text{ and } x_i > 0 \, \forall i \in \mathbb{N}\}.$$

The following useful description of the bounded sets in a Köthe echelon space is due to Bierstedt et al. [9].

Proposition 4.9 *Let $A = (a_n)_{n\in\mathbb{N}}$ be a Köthe matrix. For $1 \le p < \infty \cup \{0\}$, a subset B of $\lambda_p(A)$ is bounded if, and only if, there exists $\overline{a} \in \lambda_\infty(A)_+$ such that $B \subseteq B_{\overline{a}} := \left\{x \in \omega \mid \left\|\left(\frac{x_i}{\overline{a}(i)}\right)_{i\in\mathbb{N}}\right\|_p \le 1\right\}$.*

First of all, we study the case $p < \infty$. Fixed a weight v, we recall that c_0 is isomorphic to $c_0(v)$ through to the map $(x_i)_{i\in\mathbb{N}} \mapsto \left(\frac{x_i}{v(i)}\right)_{i\in\mathbb{N}}$. Hence, taking into account Remark 4.8, a subset K of $c_0(v)$ is relatively compact if, and only if, for every $\varepsilon > 0$ there exists $j_0 \in \mathbb{N}$ such that for every $x \in K$ we have $\sup_{j\ge j_0+1} v(j)|x_j| < \varepsilon$. Analogously for $1 \le p < \infty$.

Proposition 4.10 *Let $A = (a_n)_{n\in\mathbb{N}}$, $B = (b_m)_{m\in\mathbb{N}}$ be two Köthe matrices and fix $\varphi = (\varphi_i)_{i\in\mathbb{N}} \in \omega$. For $1 \le p < \infty \cup \{0\}$, assume that $M_\varphi\colon \lambda_p(A) \to \lambda_p(B)$ is continuous. The diagonal operator M_φ is Montel if, and only if, for every $\overline{a} \in \lambda_\infty(A)_+$ and for all $m \in \mathbb{N}$*

$$\lim_{i\to\infty} \overline{a}(i)|\varphi_i|b_m(i) = 0. \tag{4.4}$$

Proof Observe that from [22, Lemma 1], $M_\varphi\colon \lambda_p(A) \to \lambda_p(B)$ is continuous for $1 \le p < \infty$ if, and only if, $M_\varphi\colon \lambda_0(A) \to \lambda_0(B)$ is continuous. We prove the statement for $p = 0$. The proof in the case $1 \le p < \infty$ is analogous and so is omitted.

Suppose that $M_\varphi\colon \lambda_0(A) \to \lambda_0(B)$ is Montel. Given $\overline{a} \in \lambda_\infty(A)_+$ and $m \in \mathbb{N}$, the set $B_{\overline{a}}$ is bounded in $\lambda_0(A)$ due to Proposition 4.9. By assumption, M_φ is Montel, so $M_\varphi(B_{\overline{a}})$ is relatively compact in $\lambda_0(B)$ and hence in $c_0(b_m)$. Now it is easy to see that $\overline{a}(j)e_j \in B_{\overline{a}}$ for all $j \in \mathbb{N}$. Moreover, $M_\varphi((\overline{a}(j)e_j)_{j\in\mathbb{N}}) = (\varphi_j\overline{a}(j)e_j)_{j\in\mathbb{N}}$. Furthermore, the sequence $(\varphi_j\overline{a}(j)e_j)_{j\in\mathbb{N}}$ converges to 0 coordinatewise in $\mathbb{C}^{\mathbb{N}}$. Since $M_\varphi(B_{\overline{a}})$ is relatively compact in $c_0(b_m)$, the topology of $c_0(b_m)$ on $M_\varphi(B_{\overline{a}})$ coincides with the topology induced by $\mathbb{C}^{\mathbb{N}}$. This means that the sequence $(\varphi_j\overline{a}(j)e_j)_{j\in\mathbb{N}}$ converges to 0 in $c_0(b_m)$ and so (4.4) holds ($\|e_j\|_{c_0(b_m)} = b_m(j)$).

We assume now that the condition is fulfilled and prove that $M_\varphi\colon \lambda_0(A) \to \lambda_0(B)$ is Montel. We want to prove that for a fixed bounded subset $\mathcal{B}$ of $\lambda_0(A)$ the set $M_\varphi(\mathcal{B})$ is relatively compact in $c_0(b_m)$ for all $m \in \mathbb{N}$. To do this, since $\mathcal{B}$ is bounded, we apply Proposition 4.9 and choose $\overline{a} \in \lambda_\infty(A)_+$ such that $\mathcal{B} \subset B_{\overline{a}}$. It suffices to show that $M_\varphi(B_{\overline{a}})$ is relatively compact in $c_0(b_m)$ for all $m \in \mathbb{N}$, that is, using Remark 4.8, for every $\varepsilon > 0$ and $m \in \mathbb{N}$ there exists $j_0 \in \mathbb{N}$ such that for every $y \in M_\varphi(B_{\overline{a}})$ we have $\sup_{j\ge j_0+1} b_m(j)|y_j| < \varepsilon$. Hence, fixed $m \in \mathbb{N}$ and $\varepsilon > 0$, since (4.4) holds, we can choose $j_0 \in \mathbb{N}$ such that $\overline{a}(j)|\varphi_j|b_m(j) < \varepsilon$ for all $j \ge j_0 + 1$. If $y \in M_\varphi(B_{\overline{a}})$, then $y = (y_i)_{i\in\mathbb{N}} = (\varphi_i x_i)_{i\in\mathbb{N}}$ and $|x_i| \le \overline{a}(i)$ for all $i \in \mathbb{N}$. Then, if $j \ge j_0 + 1$ and $y \in M_\varphi(B_{\overline{a}})$, we get

$$|y_j|b_m(j) \le \overline{a}(j)|\varphi_j|b_m(j) < \varepsilon.$$

□

Therefore, the case $p < \infty$ is characterized. Now we prove that the same characterization holds for $p = \infty$.

Proposition 4.11 *Let $A = (a_n)_{n\in\mathbb{N}}$, $B = (b_m)_{m\in\mathbb{N}}$ be two Köthe matrices and fix $\varphi = (\varphi_i)_{i\in\mathbb{N}} \in \omega$. Assume that $M_\varphi\colon \lambda_\infty(A) \to \lambda_\infty(B)$ is continuous. Then M_φ is Montel if, and only if, $M_\varphi\colon \lambda_0(A) \to \lambda_0(B)$ is Montel.*

Proof First of all, let observe that $M_\varphi\colon \lambda_\infty(A) \to \lambda_\infty(B)$ is continuous if, and only if, $M_\varphi\colon \lambda_0(A) \to \lambda_0(B)$ is continuous (see [22, Lemma 1]). Moreover, if $M_\varphi\colon \lambda_\infty(A) \to \lambda_\infty(B)$ is Montel and $\mathcal{B}$ is a bounded subset of $\lambda_0(A)$, then $M_\varphi(\mathcal{B})$ is relatively compact in $\lambda_\infty(B)$ and hence in $\lambda_0(B)$, since $\lambda_0(B)$ is a closed subspace of $\lambda_\infty(B)$ and $M_\varphi(\mathcal{B}) \subset \lambda_0(B)$.

We suppose that $M_\varphi\colon \lambda_0(A) \to \lambda_0(B)$ is Montel. Applying [13, Corollary 2.3], then $M_\varphi^t\colon \lambda_0(A)'_b \to \lambda_0(B)'_b$ is Montel. Note that $\lambda_0(A)'_b$ and $\lambda_0(B)'_b$ are complete (LB)-spaces (see [6, Proposition 10]). Then applying [13, Corollary 2.4], we get that $M_\varphi^{tt}\colon (\lambda_0(A)'_b)'_b \to (\lambda_0(B)'_b)'_b$ is Montel. Since $M_\varphi^{tt} = M_\varphi$ on $\lambda_\infty(A)$ and $\lambda_\infty(A)$ is the bidual of $\lambda_0(A)$ (resp. $\lambda_\infty(B)$ is the bidual of $\lambda_0(B)$), we get the claim. □

Thus, we can characterize when a diagonal operator between the sequence (LF)-spaces $l_p(\mathcal{V})$ is Montel. The proof is an application of Propositions 2.4, 4.10 and 4.11.

Theorem 4.12 *Let $\mathcal{V}$, $\mathcal{W}$ be two systems of weights and fix $\varphi = (\varphi_i)_{i\in\mathbb{N}} \in \omega$. For $1 \leq p \leq \infty \cup \{0\}$ assume that $l_p(\mathcal{V})$ is regular and $l_p(\mathcal{W})$ satisfies the condition (M). Suppose that the diagonal operator $M_\varphi\colon l_p(\mathcal{V}) \to l_p(\mathcal{W})$ is continuous. Then M_φ is Montel if, and only if, for all $m \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that for every $\overline{v}_m \in \lambda_\infty(V_m)_+$ and for all $k \in \mathbb{N}$*

$$\lim_{i\to\infty} \overline{v}_m(i)|\varphi_i|w_{n,k}(i) = 0.$$

Now we treat the reflexivity of diagonal operators.

To study the Köthe echelon case, we have to make a distinction. For $p = 1, 0, \infty$, we want to show that a diagonal operator acting between the echelon spaces is Montel if, and only if, it is reflexive. Let $A = (a_n)_{n\in\mathbb{N}}$, $B = (b_m)_{m\in\mathbb{N}}$ be two Köthe matrices and fix $\varphi = (\varphi_i)_{i\in\mathbb{N}} \in \omega$. For $p = 1, 0, \infty$, assume that $M_\varphi\colon \lambda_p(A) \to \lambda_p(B)$ is continuous (see [22, Lemma 1]). We have:

(i) If $p = 1$, by Schur's Theorem (see [26, Page 136]) a bounded subset B of $\lambda_1(A)$ is weakly (relatively) compact if, and only if, it is (relatively) compact. So, $M_\varphi\colon \lambda_1(A) \to \lambda_1(B)$ is reflexive if, and only if, it is Montel;

(ii) If $p = 0$, we prove that $M_\varphi\colon \lambda_0(A) \to \lambda_0(B)$ is reflexive if, and only if, it is Montel. We only have to show that the condition is necessary. So, suppose that $M_\varphi\colon \lambda_0(A) \to \lambda_0(B)$ is reflexive. By a result of Grothendieck [14] (see also [16, Page 204]), this implies that $M_\varphi = M_\varphi^{tt}$ maps $\lambda_\infty(A) = (\lambda_0(A)'_b)'_b$ in $\lambda_0(B)$. In particular, we have that $M_\varphi(\lambda_\infty(A)_+) \subset \lambda_0(B)$. This is equivalent to requiring that for every $\overline{a} \in \lambda_\infty(A)_+$ and $m \in \mathbb{N}$

$$\lim_{i\to\infty} \overline{a}(i)|\varphi_i|b_m(i) = 0.$$

By Proposition 4.10, since (4.4) holds, we get the claim.

(iii) If $p = \infty$, we have that $M_\varphi\colon \lambda_\infty(A) \to \lambda_\infty(B)$ is reflexive if, and only if, it is Montel. Indeed applying [13, Corollaries 2.3, 2.4], $M_\varphi\colon \lambda_\infty(A) \to \lambda_\infty(B)$ reflexive implies $M_\varphi\colon \lambda_0(A) \to \lambda_0(B)$ reflexive. From the above case (ii), we get that this is equivalent to being Montel.

Thus, we can give a first characterization. The proof is an application of Propositions 2.4, 2.6 and the above considerations.

Theorem 4.13 *Let $\mathcal{V}$, $\mathcal{W}$ be two systems of weights and fix $\varphi = (\varphi_i)_{i\in\mathbb{N}} \in \omega$. For $p = 1, 0, \infty$, assume that $l_p(\mathcal{V})$ is regular and $l_p(\mathcal{W})$ satisfies the condition (M_0). Suppose that the diagonal operator $M_\varphi\colon l_p(\mathcal{V}) \to l_p(\mathcal{W})$ is continuous. Then M_φ is reflexive if, and only if, M_φ is Montel.*

Proof We only have to prove that if M_φ is reflexive, then M_φ is Montel. By Proposition 2.6, for all $m \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that the restriction $M_\varphi\colon \lambda_p(V_m) \to \lambda_p(W_n)$ is reflexive. But $M_\varphi\colon \lambda_p(V_m) \to \lambda_p(W_n)$ is reflexive if, and only if, it is Montel. Therefore, applying Proposition 2.4, the diagonal operator M_φ is Montel (take into account Remark 2.5). □

Now we consider the case $1 < p < \infty$. Observe that since $\lambda_p(A)$ and $\lambda_p(B)$ are reflexive spaces [6, Proposition 9], trivially the diagonal operator $M_\varphi : \lambda_p(A) \to \lambda_p(B)$ is reflexive. But in general, the diagonal operator does not necessarily have to be Montel, since it is not Montel even between the Banach spaces l_p (take $a_n = b_n = 1$).

In this case, what we have is this characterization. The proof is an obvious consequence of Theorem 3.4.

Proposition 4.14 *Let $\mathcal{V}$, $\mathcal{W}$ be two systems of weights and fix $\varphi = (\varphi_i)_{i\in\mathbb{N}} \in \omega$. For $1 < p < \infty$, assume that $l_p(\mathcal{W})$ is regular. Then the diagonal operator $M_\varphi : l_p(\mathcal{V}) \to l_p(\mathcal{W})$ is continuous if, and only if, it is reflexive.*

5 Spectrum of diagonal operators acting on $l_p(\mathcal{V})$

Let X be a lcHs and let $\mathcal{L}(X)$ denote the space of all continuous linear operators from X into itself. Given $T \in \mathcal{L}(X)$, the *resolvent set* of T is defined by

$$\rho(T, X) := \{\lambda \in \mathbb{C} \mid \lambda I - T : X \to X \text{ is bijective and } (\lambda I - T)^{-1} \in \mathcal{L}(X)\}$$

and the *spectrum* of T is defined by $\sigma(T, X) := \mathbb{C} \setminus \rho(T, X)$. The *point spectrum* is defined by

$$\sigma_p(T, X) := \{\lambda \in \mathbb{C} \mid \lambda I - T \text{ is not injective}\}.$$

Unlike for Banach spaces, it may happens that $\rho(T, X) = \emptyset$ or that $\rho(T, X)$ is not open in $\mathbb{C}$ (see, e.g., [3]). This is the reason why many authors consider the subset $\rho^*(T, X)$ of $\rho(T, X)$ consisting of all $\lambda \in \mathbb{C}$ for which there exists $\delta > 0$ such that $B_\delta(\lambda) := \{\mu \in \mathbb{C} \mid |\mu - \lambda| < \delta\} \subseteq \rho(T, X)$ and the set $\{(\mu I - T)^{-1} \mid \mu \in B_\delta(\lambda)\}$ is equicontinuous in $\mathcal{L}(X)$. If X is a Fréchet space, then it suffices that this set is bounded in $\mathcal{L}_s(X)$, where $\mathcal{L}_s(X)$ denotes $\mathcal{L}(X)$ endowed with the strong operator topology. The advantage of $\rho^*(T, X)$, whenever it is not empty, is that it is open and the resolvent map is holomorphic from $\rho^*(T, X)$ into $\mathcal{L}_b(X)$ (see, e.g., [2, Proposition 3.4]), where $\mathcal{L}_b(X)$ denotes $\mathcal{L}(X)$ endowed with the topology of the uniform convergence on bounded subsets of X). Define the *Waelbrock spectrum* $\sigma^*(T, X) := \mathbb{C} \setminus \rho^*(T, X)$, which is a closed set containing $\sigma(T, X)$.

We start studying the point spectrum of the diagonal operators. The proof of this result is standard and so is omitted (see [4] for an analogous one).

Lemma 5.1 *Let $\mathcal{V}$ be a system of weights and fix $\varphi = (\varphi_i)_{i\in\mathbb{N}} \in \omega$. For $1 \leq p \leq \infty \cup \{0\}$ we have that*

$$\sigma_p(M_\varphi, l_p(\mathcal{V})) = \{\varphi_i \mid i \in \mathbb{N}\}.$$

We determine the resolvent set of the diagonal operators acting between the sequence (LF)-spaces $l_p(\mathcal{V})$. Compare with [22, Proposition 1] for Köthe echelon spaces.

Proposition 5.2 *Let $\mathcal{V}$ be a system of weights and fix $\varphi = (\varphi_i)_{i\in\mathbb{N}} \in \omega$. For $1 \leq p \leq \infty \cup \{0\}$, the following assertions are equivalent:*

(1) $\mu \in \rho(M_\varphi, l_p(\mathcal{V}))$;
(2) *For all $m \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that for all $k \in \mathbb{N}$ there exist $l \in \mathbb{N}$ for which*

$$\sup_{i\in\mathbb{N}} \frac{v_{n,k}(i)}{v_{m,l}(i)|\varphi_i - \mu|} < \infty. \tag{5.1}$$

Proof Define $\phi := (\phi_i)_{i\in\mathbb{N}} \in \omega$ such that $\phi_i := \frac{1}{\varphi_i-\mu}$. Clearly, M_ϕ is the continuous inverse of $M_\varphi - \mu I$, whenever it exists. Applying Theorem 4.1, we obtain the equivalence of (1) and (2). □

Now we determine the Waelbrock spectrum. In contrast to what happens for the diagonal operators acting between Köthe echelon spaces [22, Theorem 1], for the sequence (LF)-spaces $l_p(\mathcal{V})$ we have to require the completeness of the inductive limit.

Theorem 5.3 *Let $\mathcal{V}$ be a system of weights and fix $\varphi = (\varphi_i)_{i\in\mathbb{N}} \in \omega$, For $1 \le p \le \infty \cup \{0\}$, suppose that $l_p(\mathcal{V}) = K_p(\overline{V})$ algebraically and topologically (see (3.2), (3.3), (3.4)). Then*

$$\sigma^*(M_\varphi, l_p(\mathcal{V})) = \overline{\sigma(M_\varphi, l_p(\mathcal{V}))} = \overline{\{\varphi_i \mid i \in \mathbb{N}\}}. \tag{5.2}$$

Proof Firstly, we prove the second equality of (5.2). By Lemma 5.1, we have that

$$\{\varphi_i \mid i \in \mathbb{N}\} = \sigma_p(M_\varphi, l_p(\mathcal{V})) \subseteq \sigma(M_\varphi, l_p(\mathcal{V})) \subseteq \overline{\sigma(M_\varphi, l_p(\mathcal{V}))},$$

and hence $\overline{\{\varphi_i \mid i \in \mathbb{N}\}} \subseteq \overline{\sigma(M_\varphi, l_p(\mathcal{V}))}$. For the other inclusion, let $\mu \notin \overline{\{\varphi_i \mid i \in \mathbb{N}\}}$. Then there is $\delta > 0$ such that $|\varphi_i - \mu| > 2\delta$ for all $i \in \mathbb{N}$. Therefore, arguing as in the proof of Proposition 5.2, we deduce that $\mu \in \rho(M_\varphi, l_p(\mathcal{V}))$. Suppose that $\mu \in \overline{\sigma(M_\varphi, l_p(\mathcal{V}))}$. This implies that $\mu \in \partial\sigma(M_\varphi, l_p(\mathcal{V}))$. Thus, there exists $\lambda \in \sigma(M_\varphi, l_p(\mathcal{V}))$ such that $|\mu - \lambda| < \delta$. Hence, for all $i \in \mathbb{N}$ we get

$$|\varphi_i - \lambda| \ge |\varphi_i - \mu| - |\mu - \lambda| > \delta.$$

Again, arguing as in the proof of Proposition 5.2, we deduce that $\lambda \in \rho(M_\varphi, l_p(\mathcal{V}))$, which is a contradiction. This prove that $\mu \notin \overline{\sigma(M_\varphi, l_p(\mathcal{V}))}$ and hence $\overline{\sigma(M_\varphi, l_p(\mathcal{V}))} \subseteq \overline{\{\varphi_i \mid i \in \mathbb{N}\}}$.

Now we prove the first equality of (5.2). It is sufficient to prove that if $\lambda \notin \overline{\{\varphi_i \mid i \in \mathbb{N}\}}$, then $\lambda \in \rho^*(M_\varphi, l_p(\mathcal{V}))$. So, fix $\lambda \notin \overline{\{\varphi_i \mid i \in \mathbb{N}\}}$. As done in the previous case, there exists $\delta > 0$ such that if $|\mu - \lambda| < 2\delta$, then $\mu \in \rho(M_\varphi, l_p(\mathcal{V}))$. Hence, we only have to show that the set $\{(M_\varphi - \mu I)^{-1} \mid |\mu - \lambda| < \delta\}$ is equicontinuous. By Lemma 5.1, we know that $\varphi_i \in \sigma(M_\varphi, l_p(\mathcal{V}))$, and thus $|\varphi_i - \mu| \ge 2\delta$ for all $i \in \mathbb{N}$. If $\mu \in \mathbb{C}$ is such that $|\mu - \lambda| < \delta$, then for all $i \in \mathbb{N}$ we get $|\varphi_i - \mu| > \delta$. For $x \in l_p(\mathcal{V})$ set $y^\mu := (M_\varphi - \mu I)^{-1}x$, i.e., $y^\mu = M_{\left(\frac{1}{\varphi_i-\mu}\right)_i} x$, for all $\mu \in \mathbb{C}$ with $|\mu - \lambda| < \delta$. Then, for all $i \in \mathbb{N}$ we have

$$|y_i^\mu| = \left|\frac{x_i}{\varphi_i - \mu}\right| \le \frac{|x_i|}{\delta}.$$

Taking into account that $l_p(\mathcal{V}) = K_p(\overline{V})$ topologically by assumption, the inductive limit topology is given by the seminorm system $(p_{\overline{v}})_{\overline{v}\in\overline{V}}$ and

$$p_{\overline{v}}\left((M_\varphi - \mu I)^{-1}x\right) \le \frac{p_{\overline{v}}(x)}{\delta},$$

for all $\overline{v} \in \overline{V}$ and $\mu \in \mathbb{C}$ with $|\mu - \lambda| < \delta$. This proves the equicontinuity of the set $\{(M_\varphi - \mu I)^{-1} \mid |\mu - \lambda| < \delta\}$ and our thesis. □

6 Ergodic properties

An operator $T \in \mathcal{L}(X)$, with X a lcHs, is called *power bounded* if $\{T^n\}_{n\in\mathbb{N}}$ is an equicontinuous subset of $\mathcal{L}(X)$.

The Cesàro means of an operator $T \in \mathcal{L}(X)$ are defined by

$$T_{[n]} := \frac{1}{n}\sum_{m=1}^{n} T^m, \quad n \in \mathbb{N}.$$

The operator T is called *mean ergodic* (resp. *uniformly mean ergodic*) if $\{T_{[n]}\}_{n\in\mathbb{N}}$ is a convergent sequence in $\mathcal{L}_s(X)$ (resp. in $\mathcal{L}_b(X)$). The Cesàro means of T satisfy the following identity

$$\frac{T^k}{k} = T_{[k]} - \frac{k-1}{k}T_{[k-1]}, \quad k \geq 2.$$

So, it is clear that $\frac{T^k}{k} \to 0$ in $\mathcal{L}_s(X)$ as $k \to \infty$, whenever T is mean ergodic. Obviously, this holds also whenever T is power bounded. If X is a Montel lcHs, then the operator T is uniformly mean ergodic whenever it is mean ergodic. Furthermore, in reflexive Fréchet spaces (or (LF)-spaces, see [1, Corollary 2.7]) every power bounded operator is necessarily mean ergodic. If the space (or the (LF)-space, see [1, Proposition 2.8]) is Montel, every power bounded operator is necessarily uniformly mean ergodic. The converse of these two statements is not true in general (see, e.g., [15, Sect. 6]).

We start giving the following result. For Köthe echelon spaces, it is well-known (see [22, Lemma 3]).

Lemma 6.1 *Let $\mathcal{V}$ be a system of weights and fix $\varphi = (\varphi_i)_{i\in\mathbb{N}} \in \omega$. For $1 \leq p \leq \infty \cup \{0\}$, suppose that $T := M_\varphi \in \mathcal{L}(l_p(\mathcal{V}))$ satisfies that $\frac{T^k x}{k} \to 0$ as $k \to \infty$ for each $x \in l_p(\mathcal{V})$. Then $\|\varphi\|_\infty \leq 1$. This holds in particular if T is power bounded or mean ergodic.*

Proof For all $j \in \mathbb{N}$, let $e_j = (\delta_{i,j})_{i\in\mathbb{N}}$. The sequence $(e_j)_{j\in\mathbb{N}}$ clearly belongs to $l_p(V)$. By assumption,

$$\lim_{k\to\infty} \frac{|\varphi_j^k|}{k} = \lim_{k\to\infty} \frac{|(T^k e_j)_j|}{k} = 0,$$

with $(T^k e_j)_j$ the j-th coordinate of $T^k e_j$. This implies $|\varphi_j| \leq 1$ and so we get the thesis.

If T is power bounded or mean ergodic, then $\frac{T^k}{k}$ converges to 0 in $\mathcal{L}_s(l_p(V))$. □

The condition of Lemma 6.1 is not only necessary, as the following theorem shows. We use the characterization of the power boundedness of the diagonal operators acting between Köthe echelon spaces, contained in [22, Proposition 2].

Lemma 6.2 *Let $A = (a_n)_{n\in\mathbb{N}}$ be a Köthe matrix and fix $\varphi = (\varphi_i)_{i\in\mathbb{N}} \in \omega$. For $1 \leq p \leq \infty \cup \{0\}$, the diagonal operator $M_\varphi \in \mathcal{L}(\lambda_p(A))$ is power bounded if, and only if, $\|\varphi\|_\infty \leq 1$.*

Theorem 6.3 *Let $\mathcal{V}$ be a system of weights and fix $\varphi = (\varphi_i)_{i\in\mathbb{N}} \in \omega$. For $1 \leq p \leq \infty \cup \{0\}$, the diagonal operator $M_\varphi \in \mathcal{L}(l_p(\mathcal{V}))$ is power bounded if, and only if, $\|\varphi\|_\infty \leq 1$.*

Proof If M_φ is power bounded, from Lemma 6.1 we get that $\|\varphi\|_\infty \leq 1$.

Suppose now that $\|\varphi\|_\infty \leq 1$. For any $x \in l_p(\mathcal{V})$, there exists $n \in \mathbb{N}$ such that $x \in \lambda_p(V_n)$. It is easy to see that under the assumption $|\varphi_i| \leq \|\varphi\|_\infty \leq 1$ for all $i \in \mathbb{N}$, the diagonal operator is defined pointwise in such a way: $M_\varphi \colon \lambda_p(V_n) \to \lambda_p(V_n)$. Applying Lemma 6.2, we get that the diagonal operator is power bounded on the step $\lambda_p(V_n)$. This implies that it is power bounded on $l_p(\mathcal{V})$. □

The same also holds for the mean ergodicity. Again, we need to recall the characterization of the mean ergodicity for the diagonal operators acting between Köthe echelon spaces given in [22, Theorem 2].

Lemma 6.4 *Let $A = (a_n)_{n\in\mathbb{N}}$ be a Köthe matrix and fix $\varphi = (\varphi_i)_{i\in\mathbb{N}} \in \omega$. For $1 \le p < \infty \cup \{0\}$, the diagonal operator $M_\varphi \in \mathcal{L}(\lambda_p(A))$ is mean ergodic if, and only if, $\|\varphi\|_\infty \le 1$.*

Theorem 6.5 *Let $\mathcal{V}$ be a system of weights and fix $\varphi = (\varphi_i)_{i\in\mathbb{N}} \in \omega$. For $1 \le p < \infty \cup \{0\}$, the diagonal operator $M_\varphi \in \mathcal{L}(l_p(\mathcal{V}))$ is mean ergodic if, and only if, $\|\varphi\|_\infty \le 1$.*

Proof If M_φ is mean ergodic, from Lemma 6.1 we get that $\|\varphi\|_\infty \le 1$.

Suppose now that $\|\varphi\|_\infty \le 1$. As done in Theorem 6.3, fixed $x \in l_p(\mathcal{V})$, the diagonal operator is defined pointwise from $\lambda_p(V_n)$ to $\lambda_p(V_n)$, for some $n \in \mathbb{N}$. Applying Lemma 6.4, we get that $\{(M_\varphi)_{[k]}x\}_{k\in\mathbb{N}}$ is a convergent sequence in $\lambda_p(V_n)$. Since $\lambda_p(V_n)$ is continuously embedded into $l_p(\mathcal{V})$, we obtain the thesis. □

Observe that in Theorems 6.3 and 6.5 completeness of $l_p(\mathcal{V})$ was not required.

Finally, we discuss the uniform mean ergodicity. In contrast to what happens for the diagonal operators acting between Köthe echelon spaces [22, Theorem 3], for the sequence (LF)-spaces $l_p(\mathcal{V})$ we have to require the completeness of the inductive limit.

Theorem 6.6 *Let $\mathcal{V}$ be a system of weights and fix $\varphi = (\varphi_i)_{i\in\mathbb{N}} \in \omega$. For all $1 \le p \le \infty \cup \{0\}$, suppose that $l_p(\mathcal{V}) = K_p(\overline{V})$ algebraically and topologically (see (3.2), (3.3), (3.4)). The following assertions are equivalent:*

(1) *$M_\varphi \in \mathcal{L}(l_\infty(\mathcal{V}))$ is mean ergodic;*
(2) *$M_\varphi \in \mathcal{L}(l_\infty(\mathcal{V}))$ is uniformly mean ergodic;*
(3) *$M_\varphi \in \mathcal{L}(l_0(\mathcal{V}))$ is uniformly mean ergodic;*
(4) *For $1 \le p < \infty$, $M_\varphi \in \mathcal{L}(l_p(\mathcal{V}))$ is uniformly mean ergodic;*
(5) *$\|\varphi\|_\infty \le 1$ and for all $n \in \mathbb{N}$, for each $\overline{v}_n \in \lambda_\infty(V_n)_+$ and $\overline{v} \in \overline{V}$*

$$\lim_{k\to\infty} \sup_{i\in\mathbb{N}\setminus J} \frac{\overline{v}_i \overline{v}_n(i)|\varphi_i||1-\varphi_i^k|}{k\,|1-\varphi_i|} = 0,$$

where $J := \{i \in \mathbb{N} : \varphi_i = 1\}$.

Proof Fix a seminorm $p_{\overline{v}}$. In the present case these seminorms define the lc-topology of $l_p(\mathcal{V})$. We may assume without loss of generality that $\varphi_i \ne 1$ for all $i \in \mathbb{N}$, that means $J = \emptyset$. Otherwise, we split the space into two sectional subspaces and observe that in the subspace in which $\varphi_i = 1$, the diagonal operator acts as the identity.

(1)⇒(5). Clearly $\|\varphi\|_\infty \le 1$, by Lemma 6.1. Fix $n \in \mathbb{N}$, $\overline{v} \in \overline{V}$ and $\overline{v}_n \in \lambda_\infty(V_n)_+$. Since M_φ is mean ergodic and $\overline{v}_n \in \lambda_\infty(V_n)_+ \subset l_\infty(\mathcal{V})$, we have

$$\lim_{k\to\infty} \sup_{i\in\mathbb{N}} \frac{\overline{v}_i \overline{v}_n(i)|\varphi_i||1-\varphi_i^k|}{k\,|1-\varphi_i|} = \lim_{k\to\infty} p_{\overline{v}}\left((M_\varphi)_{[k]}\overline{v}_n\right) = 0.$$

(5)⇒(4). We show that for a fixed $B \in \mathcal{B}(l_p(\mathcal{V}))$

$$\sup_{(x_i)_i\in B} p_{\overline{v}}\left((M_\varphi)_{[k]}(x)\right) \to 0$$

as $k \to \infty$. Observe that

$$\sup_{(x_i)_i\in B} p_{\overline{v}}\left((M_\varphi)_{[k]}(x)\right) = \sup_{(x_i)_i\in B} \left(\sum_{i\in\mathbb{N}} \left(\frac{\overline{v}_i|x_i||\varphi_i||1-\varphi_i^k|}{k\,|1-\varphi_i|}\right)^p\right)^{\frac{1}{p}}.$$

Taking into account that by assumption $l_p(\mathcal{V})$ is regular (Theorem 3.4), there exists $n \in \mathbb{N}$ such that $B \in \mathcal{B}(\lambda_p(V_n))$. Hence, we can choose $\overline{v}_n \in \lambda_\infty(V_n)_+$ as in Proposition 4.9, getting

$$\sup_{(x_i)_i \in B} p_{\overline{v}}\left((M_\varphi)_{[k]}(x)\right) \leq \sup_{i\in\mathbb{N}} \frac{\overline{v}_i \overline{v}_n(i)|\varphi_i||1-\varphi_i^k|}{k\,|1-\varphi_i|} \sup_{(x_i)_i\in B}\left(\sum_{i\in\mathbb{N}}\left(\frac{|x_i|}{\overline{v}_n(i)}\right)^p\right)^{\frac{1}{p}}$$
$$\leq \sup_{i\in\mathbb{N}} \frac{\overline{v}_i \overline{v}_n(i)|\varphi_i||1-\varphi_i^k|}{k\,|1-\varphi_i|} \to 0, \quad \text{as}\, k \to \infty,$$

by assumption.

(5)⇒(2). It follows as in the previous case arguing with the l_∞ norm.

(2)⇒(1). Trivial.

(4)⇒(5). Suppose that M_φ is uniformly mean ergodic. In particular, M_φ is mean ergodic, hence from Theorem 6.5 $\|\varphi\|_\infty \leq 1$. Fix $\overline{v}_n \in \lambda_\infty(V_n)_+$ and $\overline{v} \in \overline{V}$ and set $B := \left\{x \in \omega \mid \left\|\left(\frac{x_i}{\overline{v}_n(i)}\right)_{i\in\mathbb{N}}\right\|_p \leq 1\right\}$. B is bounded in $\lambda_p(V_n)$ from Proposition 4.9 and hence in $l_p(\mathcal{V})$. Let $x \in B$.

Given $j \in \mathbb{N}$ such that $\varphi_j \neq 1$, we set $y_j^j := \overline{v}_n(j)$ and $y_i^j := 0$ if $j \neq i$. Then we put $y^j = (y_i^j)_{i\in\mathbb{N}}$. We have that $y^j \in B$ and

$$p_{\overline{v}}\left((M_\varphi)_{[k]}(y^j)\right) = \frac{\overline{v}_j \overline{v}_n(j)|\varphi_j||1-\varphi_j^k|}{k\,|1-\varphi_j|}.$$

Therefore

$$\sup_{j\in\mathbb{N},\varphi_j\neq 1} \frac{\overline{v}_j \overline{v}_n(j)|\varphi_j||1-\varphi_j^k|}{k\,|1-\varphi_j|} \leq \sup_{(x_i)_i\in B} p_{\overline{v}}\left((M_\varphi)_{[k]}(x)\right) \to 0, \quad \text{as}\, k \to \infty$$

since M_φ is uniformly mean ergodic.

(3)⇒(5). It follows as in the previous case arguing with the l_∞ norm.

(2)⇒(3). Trivial. □

Acknowledgements This paper was planned and prepared during the visit of the author to the Universitat Politècnica de València. He expresses his gratitude to José Bonet and his collaborators for their kind hospitality and their valuable suggestions and comments. The author would also like to express his gratitude to his advisor Angela Albanese for her important remarks.

Funding Open access funding provided by Università del Salento within the CRUI-CARE Agreement.

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