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Multiple solutions for superlinear double phase Neumann problems

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Abstract

We study a double phase Neumann problem with a superlinear reaction which need not satisfy the Ambrosetti-Rabinowitz condition. Using the Nehari manifold method, we show that the problem has at least three nontrivial bounded ground state solutions, all with sign information (positive, negative and nodal).

Keywords Double phase · Musielak-Orlicz-Sobolev · Constant sign and nodal solutions · Nehari manifold · Superlinear reaction

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1 Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a Lipschitz boundary $\partial\Omega$. In this paper we study the following double phase Neumann problem

$$\left.\begin{cases} -\Delta_p^a u(z) - \Delta_q u(z) + \xi(z)|u(z)|^{q-2}u(z) = f(z, u(z)) \text{ in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega, \ 1 < q < p < N. \end{cases}\right\} \tag{1}$$

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Given $a \in L^\infty(\Omega) \setminus \{0\}$ with $a(z) \geq 0$ for a.a. $z \in \Omega$ and $1 < r < +\infty$, we denote by Δ_r^a the weighted r-Laplace differential operator defined by

$$\Delta_r^a u = \text{div} \left(a(z)|Du|^{r-2}Du \right).$$

If $a \equiv 1$, then we have the usual r-Laplacian denoted by Δ_r. The differential operator in problem (1) is the sum of two such operators with different exponents. Hence the differential operator is not homogeneous. There is also a potential term $\xi(z)|u|^{q-2}u$ with $\xi \in L^\infty(\Omega) \setminus \{0\}$, $\xi(z) \geq 0$ for a.a. $z \in \Omega$. The differential operator of problem (1) is related to the so-called "double phase" integral functional defined by

$$\int_\Omega \left(a(z)|Du|^p + |Du|^q \right) dz.$$

The integrand of this functional is

$$\theta(z, x) = a(z)x^p + x^q \text{ for all } z \in \Omega, \text{ all } x \geq 0.$$

We do not assume that the weight $a(\cdot)$ is bounded away from zero (that is, we do not have that $\text{ess inf}_\Omega a > 0$). Therefore the integrand $\theta(z, x)$ exhibits unbalanced growth, namely

$$x^q \leq \theta(z, x) \leq c_0 \left(x^q + x^p \right) \text{ for a.a. } z \in \Omega, \text{ all } x \geq 0, \text{ some } c_0 > 0.$$

Such functionals, wre first considered by Marcellini [14,15] and Zhikov [27,28], in the context of problems of the calculus of variations and of nonlinear elasticity theory. Recently the interest for such problems was revived and some important interior regularity results were obtained for local minimizers of such functionals. We refer to the works of Baroni et al. [1], Mingione and Rădulescu [16], and Ragusa and Tachikawa [24]. However, until now there is no global regularity theory (that is, regularity results up to the boundary) for the solutions of such problems. This fact limits the tools available for the study of such equations and makes double phase problems more difficult to deal with. In fact, the unbalanced growth of $\theta(z, \cdot)$ implies that the usual Sobolev spaces do not suffice for the analysis of the problem and we have to use the general abstract setting of Musielak-Orlicz-Sobolev spaces.

In the reaction of problem (1), we have a Carathéodory function $f(z, x)$ (that is, for all $x \in \mathbb{R}$ the mapping $z \mapsto f(z, x)$ is measurable and for a.a. $z \in \Omega$ the function $x \mapsto f(z, x)$ is continuous). We assume that $f(z, \cdot)$ exhibits $(p - 1)$-superlinear growth as $x \to \pm\infty$ but without satisfying the usual in such cases Ambrosetti-Rabinowitz condition (the AR-condition for short). The absence of a global regularity theory for such problems requires a different approach for double phase equations. In problems with balanced growth ((p, q)-equations), for which a powerful global regularity theory exists (see Lieberman [10]), the main tools are truncation and comparison techniques, critical point theory and Morse theory (critical groups). We refer to the works of Liu and Papageorgiou [12], Papageorgiou and Rădulescu [17,18], Papageorgiou et al. [21], Papageorgiou and Zhang [23]. Here instead, we use the Nehari manifold method as this was developed by Brown and Wu [2], Brown and Zhang [3], Szulkin and Weth [25] and Willem [26].

We show that problem (1) has at least three nontrivial bounded solutions, all with sign information (positive, negative and nodal (sign changing)). Recently there have been some multiplicity results for double phase problems. We refer to the works of Gasiński and Papageorgiou [5], Ge et al. [8], Liu and Dai [11], Papageorgiou et al. [22] (Dirichlet problems) and Gasiński and Winkert [7], Papageorgiou, Rădulescu and Repovš [20] (Neumann and Robin problems). Closer to our work here are the papers of Gasiński and Papageorgiou [5] and Liu and Dai [11], which deal with superlinear Dirichlet problems, employing more restrictive conditions on the reaction and using a different approach.

2 Mathematical background and hypotheses

As we already mentioned in the introduction, in order to accommodate the unbalanced growth of $\theta(z, \cdot)$, we have to use Musielak-Orlicz-Sobolev spaces. A comprehensive presentation of the theory of these spaces can be found in the recent book of Harjulehto and Hästö [9].

We make the following hypotheses on the weight $a(\cdot)$, the potential function $\xi(\cdot)$ and exponents q, p.

H_0: $a \in C^{0,1}(\overline{\Omega})$ (that is, $a : \overline{\Omega} \mapsto \mathbb{R}$ is Lipschitz continuous), $a(z) > 0$ for all $z \in \Omega$, $\xi \in L^{\infty}(\Omega) \setminus \{0\}, \xi(z) \geq 0$ for a.a. $z \in \Omega, 1 < q < p < q^* \left(q^* = \dfrac{Nq}{N-q} \right), p < N$.

Remark 1 In contrast to previous works on Dirichlet double phase problems, we do not require that $\dfrac{p}{q} < 1 + \dfrac{1}{N}$. This condition implies that $p < q^*$, but the converse is not true. For Dirichlet problems this condition guarantees that the Poincaré inequality is valid for the corresponding Musielak-Orlicz-Sobolev space (see Harjulehto & Hästö [9, pp. 100, 138]).

Let $M(\Omega)$ be the space of all Lebesgue measurable functions $u : \Omega \mapsto \mathbb{R}$. As usual we identify two measurable functions which differ only on a Lebesgue-null subset of Ω. Then the Musielak-Orlicz space $L^{\theta}(\Omega)$ is defined by

$$L^{\theta}(\Omega) = \{u \in M(\Omega) : \rho_{\theta}(u) < +\infty\},$$

with $\rho_{\theta}(\cdot)$ being the modular function defined by

$$\rho_{\theta}(u) = \int_{\Omega} \theta(z, u(z)) dz = \int_{\Omega} \left(a(z)|u(z)|^p + |u(z)|^q \right) dz.$$

We equip $L^{\theta}(\Omega)$ with the so-called "Luxemburg norm"

$$\|u\|_{\theta} = \inf \left\{ \lambda > 0 : \rho_{\theta} \left(\frac{u}{\lambda} \right) \leq 1 \right\}.$$

Then $L^{\theta}(\Omega)$ becomes a Banach space which is separable and reflexive (in fact, uniformly convex).

Using $L^{\theta}(\Omega)$ we can define the corresponding Musielak-Orlicz-Sobolev space $W^{1,\theta}(\Omega)$ as follows

$$W^{1,\theta}(\Omega) = \left\{ u \in L^{\theta}(\Omega) : |Du| \in L^{\theta}(\Omega) \right\}.$$

Here Du denotes the weak gradient of $u(\cdot)$. We equip $W^{1,\theta}(\Omega)$ with the following norm

$$\|u\| = \|u\|_{\theta} + \|Du\|_{\theta} \text{ for all } u \in W^{1,\theta}(\Omega),$$

where $\|Du\|_{\theta} = \||Du|\|_{\theta}$. This space is also a Banach space which is separable and reflexive (in fact, uniformly convex).

We have the following embeddings for these spaces (see Gasiński and Winkert [7]).

Proposition 1 *If hypotheses* H_0 *hold, then the following embeddings are true*:

(a) $L^{\theta}(\Omega) \hookrightarrow L^r(\Omega)$ and $W^{1,\theta}(\Omega) \hookrightarrow W^{1,r}(\Omega)$ continuously for all $1 \leq r \leq q$;
(b) $W^{1,\theta}(\Omega) \hookrightarrow L^r(\Omega)$ continuously if $1 \leq r \leq q$ and compactly if $1 \leq r < q$;
(c) $L^p(\Omega) \hookrightarrow L^{\theta}(\Omega)$ continuously.

There is a close relation between the norm $\|\cdot\|_{\theta}$ and the modular function $\rho_{\theta}(\cdot)$.

Proposition 2 *If hypotheses* H_0 *hold, then*

(a) *if* $u \in L^\theta(\Omega)\backslash\{0\}$, *then* $\|u\|_\theta = \mu \Longleftrightarrow \rho_\theta\left(\dfrac{u}{\mu}\right) = 1$;

(b) $\|u\|_\theta < 1$ $(resp. = 1, > 1) \Longleftrightarrow \rho_\theta(u) < 1$ $(resp. = 1, > 1)$;

(c) $\|u\|_\theta < 1 \Longrightarrow \|u\|_\theta^p \le \rho_\theta(u) \le \|u\|_\theta^q$;

(d) $\|u\|_\theta > 1 \Longrightarrow \|u\|_\theta^q \le \rho_\theta(u) \le \|u\|_\theta^p$;

(e) $\|u_n\|_\theta \to 0 \Longleftrightarrow \rho_\theta(u_n) \to 0$;

(f) $\|u_n\|_\theta \to +\infty \Longleftrightarrow \rho_\theta(u_n) \to +\infty$.

Let $\langle \cdot, \cdot \rangle$ denote the duality brackets for the pair $\langle W^{1,\theta}(\Omega)^*, W^{1,\theta}(\Omega)\rangle$ and let $V : W^{1,\theta}(\Omega) \mapsto W^{1,\theta}(\Omega)^*$ be the nonlinear operator defined by

$$\langle V(u), h\rangle = \int_\Omega a(z)|Du|^{p-2}(Du, Dh)_{\mathbb{R}^N} dz + \int_\Omega |Du|^{q-2}(Du, Dh)_{\mathbb{R}^N} dz$$

for all $u, h \in W^{1,\theta}(\Omega)$.

The next proposition summarizes the main properties of this nonlinear operator (see Liu and Dai [11]).

Proposition 3 *If hypotheses* H_0 *hold, then the operator* $V : W^{1,\theta}(\Omega) \mapsto W^{1,\theta}(\Omega)^*$ *is bounded (that is, it maps bounded sets to bounded sets), continuous, monotone and of type* $(S)_+$, *namely it has the following property*:

$$\text{“}u_n \xrightarrow{w} u \text{ in } W^{1,\theta}(\Omega), \ \limsup_{n\to\infty}\langle V(u_n), u_n - u\rangle \le 0$$

imply that

$$u_n \to u \text{ in } W^{1,\theta}(\Omega).\text{”}$$

For every $u \in W^{1,\theta}(\Omega)$, we set

$$|u| = \|Du\|_\theta + \left(\int_\Omega \xi(z)|u|^q dz\right)^{1/q}.$$

Evidently, this is a norm on $W^{1,\theta}(\Omega)$.

Proposition 4 *If hypotheses* H_0 *hold, then* $\|\cdot\|$ *and* $|\cdot|$ *are equivalent norms on* $W^{1,\theta}(\Omega)$.

Proof Since $\xi \in L^\infty(\Omega)$ (see hypotheses H_0), we have

$$\int_\Omega \xi(z)|u|^q dz \le \|\xi\|_\infty \|u\|_q^q \le c_1 \|u\|_q^q$$

for some $c_1 > 0$ (see Proposition 1).

Therefore

$$|u| = \|Du\|_\theta + c_1^{1/q}\|u\|_\theta \le c_2\|u\| \tag{2}$$

for some $c_2 > 0$, all $u \in W^{1,\theta}(\Omega)$.

Claim: There exists $c_3 > 0$ such that $c_3\|u\| \le |u|$ for all $u \in W^{1,\theta}(\Omega)$.

Arguing by contradiction, suppose we can find $\{u_n\}_{n\in\mathbb{N}} \subseteq W^{1,\theta}(\Omega)$ such that

$$|u_n| < \frac{1}{n}\|u_n\| \text{ for all } n \in \mathbb{N}.$$

Let $v_n = \dfrac{u_n}{\|u_n\|}$ $(n \in \mathbb{N})$. Then we have

$$\|v_n\| = 1 \text{ and } \|Dv_n\|_\theta + \left(\int_\Omega \xi(z)|v_n|^q dz\right)^{1/q} < \frac{1}{n} \text{ for all } n \in \mathbb{N}. \tag{3}$$

By passing to a subsequence if necessary, we may assume that

$$v_n \xrightarrow{w} v \text{ in } W^{1,\theta}(\Omega) \text{ and } v_n \to v \text{ in } L^q(\Omega) \text{ (see Proposition 1)}. \tag{4}$$

Then from (3) and (4), we have

$$\|Dv\|_\theta + \left(\int_\Omega \xi(z)|v|^q dz\right)^{1/q} \leq 0,$$

$$\Rightarrow \|Dv\|_\theta + \left(\int_\Omega \xi(z)|v|^q dz\right)^{1/q} = 0,$$

$$\Rightarrow \|Dv\|_\theta = 0 \text{ and so } v = \eta \in \mathbb{R}. \tag{5}$$

From (5) we have

$$|\eta|^q \int_\Omega \xi(z)dz = 0.$$

Since $\int_\Omega \xi(z)dz > 0$ (see hypotheses H_0), we have that $\eta = 0$ and so $v = 0$. From (3) we have

$$\|Dv_n\|_\theta \to 0,$$

$$\Rightarrow v_n \to 0 \text{ in } W^{1,\theta}(\Omega),$$

which contradicts the fact that $\|v_n\| = 1$ for all $n \in \mathbb{N}$ (see (3)). Finally, (2) and the Claim imply that

$$\|\cdot\| \text{ and } |\cdot| \text{ are equivalent norms on } W^{1,\theta}(\Omega).$$

This proof is now complete. □

In a similar fashion we can show also the following result. In what follows, $\|\cdot\|_{1,q}$ denotes the norm of the Sobolev space $W^{1,q}(\Omega)$.

Proposition 5 *If $\xi \in L^\infty(\Omega) \setminus \{0\}$ and $\xi(z) \geq 0$ for a.a. $z \in \Omega$, then $\|\cdot\|_{1,q}$ and $[u] = \|Du\|_q + \left(\int_\Omega \xi(z)|u|^q dz\right)^{1/q}$ are equivalent norms on $W^{1,q}(\Omega)$.*

Given $u \in W^{1,\theta}(\Omega)$, we define

$$u^+(z) = \max\{u(z), 0\}, \ u^-(z) = \max\{-u(z), \ 0\} \text{ for all } z \in \Omega.$$

We know that

$$u^\pm \in W^{1,\theta}(\Omega), \ u = u^+ - u^-, \ |u| = u^+ + u^-.$$

If X is a Banach space and $\varphi \in C^1(X, \mathbb{R})$, then

$$K_\varphi = \left\{u \in X : \varphi'(u) = 0\right\} \text{ (the critical set of } \varphi).$$

Also, if $c \in \mathbb{R}$, then

$$\varphi^c = \{u \in X : \varphi(u) \leq c\}.$$

Now we introduce the hypotheses on the reaction $f(z, x)$.

H_1: $f : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ is a Carathéodory function such that $f(z, 0) = 0$ for a.a. $z \in \Omega$ and

(i) $|f(z, x)| \leq \hat{a}(z) \left(1 + |x|^{r-1}\right)$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$ with $\hat{a} \in L^\infty(\Omega)$ and $p < r < q^*$;

(ii) if $F(z, x) = \int_0^x f(z, s)ds$, then $\lim_{x \to \pm\infty} \dfrac{F(z, x)}{|x|^p} = +\infty$ uniformly for a.a. $z \in \Omega$;

(iii) for a.a. $z \in \Omega$, the quotient function $x \mapsto \dfrac{f(z, x)}{|x|^{p-1}}$ is increasing on $\mathbb{R} \backslash \{0\}$;

(iv) $\lim_{x \to 0} \dfrac{f(z, x)}{|x|^{q-2}x} = 0$ uniformly for a.a. $z \in \Omega$.

Remark 2 For $s \geq 0$ and $x \neq 0$, we have

$$\frac{1 - s^p}{p} f(z, x)x + F(z, sx) - F(z, x)$$

$$= \int_s^1 f(z, x)xt^{p-1}dt - \int_s^1 \frac{d}{dt}F(z, tx)dt$$

$$= \int_s^1 f(z, x)xt^{p-1}dt - \int_s^1 f(z, tx)xdt \text{ (by the chain rule)}$$

$$= \int_s^1 \left(\frac{f(z, x)}{|x|^{p-1}} - \frac{f(z, tx)}{t^{p-1}|x|^{p-1}}\right) |x|^{p-1}xt^{p-1}dt$$

$$\geq 0 \text{ (see hypothesis } H_1 \text{(iii))},$$

$$\Rightarrow F(z, x) - F(z, sx) \leq \frac{1 - s^p}{p} f(z, x)x$$

$$\text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}, \text{ all } s \geq 0. \tag{6}$$

If in (6) we choose $s = 0$, then

$$pF(z, x) \leq f(z, x)x \text{ for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}, \tag{7}$$

$$\Rightarrow \lim_{x \to \pm\infty} \frac{f(z, x)}{|x|^{p-2}x} = +\infty \text{ uniformly for a.a. } z \in \Omega \text{ (see hypothesis } H_1 \text{(ii))}.$$

So, we see that $f(z, \cdot)$ is $(p - 1)$-superlinear as $x \to \pm\infty$. However this superlinearity property is not formulated using the AR-condition which is common in superlinear problems, see Liu & Dai [11], hypothesis (f_4). Also, in hypothesis H_1(iii) we do not require that the quotient function $x \mapsto \dfrac{f(z, x)}{|x|^{p-1}}$ is strictly increasing on $\mathbb{R} \backslash \{0\}$ as it is common in the literature, see Liu & Dai [11], hypothesis (f_6). Compared to the other superlinear Dirichlet double phase work of Gasiński & Papageorgiou [5], we do not require $f(z, \cdot)$ to be locally Lipschitz and replace hypothesis $H(f)$(iii) of [5] by hypothesis H_1(iii). In case $f(z, \cdot)$ is strictly differentiable, then hypothesis $H(f)$(iii) of [5] has the form $(p - 1)f(z, x)x \leq f_x'(z, x)x^2$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$ (see Clarkc [4, p. 33]), which of course implies the monotonicity of the quotient function. Note that hypothesis H_1(iii) is equivalent to saying that for a.a. $z \in \Omega$, the mapping $x \mapsto \dfrac{f(z, x)x}{|x|^p}$ is increasing in $|x| \neq 0$.

First we will study problem (1) under the following stronger set of hypotheses.

H_1': $f : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ is a Carathéodory function such that $f(z, 0) = 0$ for a.a. $z \in \Omega$, hypotheses H_1' (i), (ii), (iv) are the same as the corresponding hypotheses H_1 (i), (ii), (iv) and

(iii) for a.a. $z \in \Omega$, the quotient function $x \mapsto \dfrac{f(z, x)}{|x|^{p-1}}$ is strictly increasing on $\mathbb{R} \backslash \{0\}$.

Example 1 The following function satisfies hypotheses H_1 but not H_1'. For the sake of simplicity we drop the z-dependence.

$$f(x) = \begin{cases} |x|^{p-2}x & \text{if } |x| \leq 1, \\ |x|^{p-2}x \, (\ln|x| + 1) & \text{if } 1 < |x|. \end{cases}$$

This function does not satisfy the AR-condition and so it does not fit in the framework of Liu & Dai [11]. Also it does not fit the framework of Gasiński & Papageorgiou [5] since it does not satisfy hypothesis $H(f)$(iii) of [5].

In what follows, $\rho_a(\cdot)$ denotes the modular function defined by

$$\rho_a(Du) = \int_\Omega a(z)|Du|^p dz \text{ for all } u \in W^{1,\theta}(\Omega).$$

Let $\varphi : W^{1,\theta}(\Omega) \mapsto \mathbb{R}$ be the energy (Euler) functional for problem (1) defined by

$$\varphi(u) = \frac{1}{p}\rho_a(Du) + \frac{1}{q}\|Du\|_q^q + \frac{1}{q}\int_\Omega \xi(z)|u|^q dz - \int_\Omega F(z, u)dz$$

for all $u \in W^{1,\theta}(\Omega)$.

We have that $\varphi \in C^1(W^{1,\theta}(\Omega))$ and the Nehari manifold N of $\varphi(\cdot)$, is defined by

$$\mathrm{N} = \left\{ u \in W^{1,\theta}(\Omega) : \langle \varphi'(u), u \rangle = 0, \ u \neq 0 \right\}.$$

Evidently, every weak solution of problem (1) is in the Nehari manifold. Since we want to produce constant sign solution for problem (1), we introduce the C^1-functionals $\varphi_\pm$: $W^{1,\theta}(\Omega) \mapsto \mathbb{R}$ defined by

$$\varphi_\pm(u) = \frac{1}{p}\rho_a(Du) + \frac{1}{q}\|Du\|_q^q + \frac{1}{q}\int_\Omega \xi(z)|u|^q dz - \int_\Omega F(z, \pm u^\pm)dz.$$

We associate with these functionals, the following submanifolds of N

$$\mathrm{N}_+ = \left\{ u \in W^{1,\theta}(\Omega) : \langle \varphi_+'(u), u \rangle = 0, \ u \geq 0, \ u \neq 0 \right\},$$
$$\mathrm{N}_- = \left\{ u \in W^{1,\theta}(\Omega) : \langle \varphi_-'(u), u \rangle = 0, \ u \leq 0, \ u \neq 0 \right\}.$$

Finally, in order to produce a nodal (sign changing) solution of problem (1), we introduce the set

$$\mathrm{N}_0 = \left\{ u \in W^{1,\theta}(\Omega) : \langle \varphi'(u), u^+ \rangle = \langle \varphi'(u), u^- \rangle = 0, \ u^\pm \neq 0 \right\}.$$

Note that

$$\mathrm{N}_+, \ \mathrm{N}_-, \ \mathrm{N}_0 \subseteq \mathrm{N}. \tag{8}$$

For the study of N and of its submanifolds, we introduce the fibering function. So, let $u \in W^{1,\theta}(\Omega) \backslash \{0\}$ and consider the function

$$k_u(t) = \varphi(tu) \text{ for all } t \geq 0.$$

This is the fibering function for u. Note that

$$tu \in \mathrm{N} \text{ if and only if } k_u'(t) = 0.$$

3 Constant sign solutions under strict monotonicity

In this section, working with the functionals $\varphi_{\pm}$ and the submanifolds $N_{\pm}$, we will produce two constant sign (positive and negative) solutions for problem (1) under H_1'.

The Nehari manifold N is much smaller than the space $W^{1,\theta}(\Omega)$. So, the energy functional $\varphi(\cdot)$ restricted on N may exhibit properties which fail globally. In our case note that on account of hypothesis H_1(ii), $\varphi(\cdot)$ is unbounded below. However, as we show in the next proposition, $\varphi|_N$ is coercive, thus bounded below.

Proposition 6 *If hypotheses H_0, H_1 hold, then $\varphi|_N$ is coercive.*

Proof Arguing by contradiction, suppose we can find $\{u_n\}_{n\in\mathbb{N}} \subseteq W^{1,\theta}(\Omega)$ such that

$$u_n \in N \text{ for all } n \in \mathbb{N}, \ \|u_n\| \to +\infty, \ \varphi(u_n) \le c_4 \text{ for some } c_4 > 0, \text{ all } n \in \mathbb{N}. \quad (9)$$

Since $u_n \in N$, we have

$$\rho_a(Du_n) + \|Du_n\|_q^q + \int_\Omega \xi(z)|u|^q dz = \int_\Omega f(z, u_n)u_n dz,$$

$$\Rightarrow \frac{1}{p}\rho_a(Du_n) + \frac{1}{p}\|Du_n\|_q^q + \frac{1}{p}\int_\Omega \xi(z)|u_n|^q dz = \frac{1}{p}\int_\Omega f(z, u_n)u_n dz$$

$$\ge \int_\Omega F(z, u_n)dz$$

$$\text{for all } n \in \mathbb{N} \text{ (see (7))}. \quad (10)$$

From (9) and (10), we obtain

$$\left(\frac{1}{q} - \frac{1}{p}\right)\left(\|Du_n\|_q^q + \int_\Omega \xi(z)|u_n|^q dz\right) \le c_4 \text{ for all } n \in \mathbb{N}, \quad (11)$$

$$\Rightarrow \{u_n\}_{n\in\mathbb{N}} \subseteq W^{1,q}(\Omega) \text{ is bounded (see Proposition 5)},$$

$$\Rightarrow \{u_n\}_{n\in\mathbb{N}} \subseteq L^r(\Omega) \text{ is relatively compact (see Proposition 1)}. \quad (12)$$

From (9), (11) and (12) it follows that

$$\frac{1}{p}\rho_a(Du_n) \le c_5 \text{ for some } c_5 > 0, \text{ all } n \in \mathbb{N}, \quad (13)$$

$$\Rightarrow \rho_\theta(Du_n) + \int_\Omega \xi(z)|u_n|^q dz \le c_6 \text{ for some } c_6 > 0, \text{ all } n \in \mathbb{N} \text{ (see (11), (13))},$$

$$\Rightarrow \{u_n\}_{n\in\mathbb{N}} \subseteq W^{1,\theta}(\Omega) \text{ is bounded (see Proposition 4)}. \quad (14)$$

Comparing (9) and (14), we have a contradiction. This proves that $\varphi|_N$ is coercive. □

Clearly we have

$$\varphi_+|_{N_+} = \varphi|_{N_+}, \ \varphi_-|_{N_-} = \varphi|_{N_-}.$$

Since N_+, $N_- \subseteq N$ (see (8)), from Proposition 6 we infer the following corollary.

Corollary 7 *If hypotheses H_0, H_1 hold, then $\varphi_+|_{N_+}$ and $\varphi_-|_{N_-}$ are both coercive.*

Next we show that $N \ne \emptyset$. Now we will start using the stronger hypotheses H_1'.

Proposition 8 *If hypotheses* H_0, H_1' *hold and* $u \in W^{1,\theta}(\Omega) \backslash \{0\}$, *then there exists a unique* $t_u > 0$ *such that*

$$t_u u \in N.$$

Proof We introduce the function $\beta_u : \mathbb{R}_+ \mapsto \mathbb{R}$ defined by

$$\beta_u(t) = t^p \rho_a(Du) + t^q \left(\|Du\|_q^q + \int_\Omega \xi(z)|u|^q dz \right) - \int_\Omega f(z, tu)(tu) dz \quad (15)$$

for all $t \geq 0$.

We see that

$$\beta_u(t) = t k_u'(t) \text{ for all } t \geq 0. \quad (16)$$

Hypothesis H_1'(iii) implies that if $x \neq 0$ and $t \in (0, 1)$, then

$$\frac{f(z, tx)(tx)}{t^p |x|^p} < \frac{f(z, x)x}{|x|^p} \text{ for a.a. } z \in \Omega,$$
$$\Rightarrow f(z, tx)(tx) < t^p f(z, x)x \text{ for a.a. } z \in \Omega, \text{ all } x \neq 0, \text{ all } t \in (0, 1). \quad (17)$$

We use (17) in (15) and obtain

$$\beta_u(t) \geq t^p \rho_a(Du) + t^q \left(\|Du\|_q^q + \int_\Omega \xi(z)|u|^q dz \right) - t^p \int_\Omega f(z, u) u \, dz$$
$$\geq t^q \frac{1}{2^{q-1}} [u]^q - t^p \int_\Omega f(z, u) u \, dz.$$

Since $q < p$, we see that

$$\beta_u(t) > 0 \text{ for all } t \in (0, 1) \text{ small.} \quad (18)$$

Also we have

$$\frac{\beta_u(t)}{t^p} = \rho_a(Du) + \frac{1}{t^{p-q}} \left(\|Du\|_q^q + \int_\Omega \xi(z)|u|^q dz \right) - \int_\Omega \frac{f(z, tu)(tu)}{t^p} dz$$
$$\leq \rho_a(Du) + \frac{2}{t^{p-q}} [u]^q - \int_\Omega \frac{pF(z, tu)}{t^p} dz \text{ (see (7)),}$$
$$\Rightarrow \lim_{t \to +\infty} \frac{\beta_u(t)}{t^p} = -\infty \text{ (see hypothesis } H_1'(\text{ii})),$$
$$\Rightarrow \beta_u(t) < 0 \text{ for } t > 0 \text{ large.} \quad (19)$$

Then relations (18), (19) and Bolzano's theorem, imply that there exists $t_u > 0$ such that

$$\beta_u(t_u) = 0,$$
$$\Rightarrow t_u k_u'(t_u) = 0 \text{ (see (16)),}$$
$$\Rightarrow k_u'(t_u) = 0,$$
$$\Rightarrow t_u u \in N.$$

Finally, note that

$$\beta_u(t) = 0 \Longleftrightarrow \rho_u(Du) = \int_\Omega \frac{f(z, tu)(tu)}{t^p} dz - \frac{1}{t^{p-q}} \left(\|Du\|_q^q + \int_\Omega \xi(z)|u|^q dz \right).$$

In the last equality the right-hand side is strictly increasing in $t > 0$ (see hypothesis H_1'(ii)) and so it follows that $t_u > 0$ is unique. $\qquad \square$

In a similar fashion, working with φ_+ and φ_-, we show the nonemptiness of the submanifolds N_+ and N_-, respectively. Recall that $N_\pm \subseteq N$ (see (8)).

Proposition 9 *If hypotheses* H_0, H_1' *hold and* $u \in W^{1,\theta}(\Omega) \setminus \{0\}$ *with* $u \geq 0$ *(resp.* $u \leq 0$), *then there exists a unique* $t_u^+ > 0$ *(resp.* $t_u^- > 0$) *such that*

$$t_u^+ u \in N_+ \ (resp. \ t_u^- u \in N_-).$$

The next proposition presents a useful connection between the fibering function and the elements of the Nehari manifold.

Proposition 10 *If hypotheses* H_0, H_1' *hold and* $u \in N$, *then* $\varphi(tu) \leq \varphi(u)$ *for all* $t \geq 0$.

Proof From (16) we have

$$\beta_u(1) = k_u'(1),$$
$$\Rightarrow \beta_u(1) = 0 \text{ (since } u \in N).$$

Note that

$$\frac{k_u(t)}{t^p} = \frac{1}{p}\rho_a(Du) + \frac{1}{qt^{p-q}}\left(\|Du\|_q^q + \int_\Omega \xi(z)|u|^q dz\right) - \int_\Omega \frac{F(z, tu)}{t^p}dz,$$
$$\Rightarrow \frac{k_u(t)}{t^p} \to -\infty \text{ as } t \to +\infty \text{ (see hypothesis } H_1'(ii)),$$
$$\Rightarrow k_u(t) < 0 \text{ for all } t > 0 \text{ large.} \tag{20}$$

On the other hand, on account of hypotheses H_1'(i), (iv) given $\varepsilon > 0$, we can find $c_7 = c_7(\varepsilon) > 0$ such that

$$F(z, x) \leq \frac{\varepsilon}{q}|x|^q + c_7|x|^r \text{ for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}. \tag{21}$$

Then using (21) we have

$$k_u(t) \geq \frac{t^p}{p}\rho_a(Du) + \frac{t^q}{q}\left(\frac{1}{2^{q-1}} - \varepsilon c_8\right)[u]^q - t^r c_7\|u\|_r^r$$

for some $c_8 > 0$ (see Proposition 5).

Choosing $\varepsilon \in \left(0, \dfrac{1}{2^{q-1}c_8}\right)$, we obtain

$$k_u(t) \geq c_9 t^p - c_{10} t^r \text{ for some } c_9, \ c_{10} > 0, \text{ all } t \in (0, 1),$$
$$\Rightarrow k_u(t) > 0 \text{ for all } t \in (0, 1) \text{ small (recall that } p < r). \tag{22}$$

From (21), (22) it follows that there exists $\hat{t}_u > 0$ local maximizer of the fibering function. Then

$$k_u'(\hat{t}_u) = 0,$$
$$\Rightarrow \hat{t}_u u \in N,$$
$$\Rightarrow \hat{t}_u = 1 \text{ and it is a global maximizer of } k_u(\cdot)$$
$$\text{(recall that } u \in N \text{ and see Proposition 8),}$$
$$\Rightarrow \varphi(tu) \leq \varphi(u) \text{ for all } t \geq 0.$$

This proof is now complete. □

The same result is also true for $\varphi_{\pm}$.

Proposition 11 *If hypotheses* H_0, H_1' *hold and* $u \in N_+$ *(resp.* $u \in N_-$*), then* $\varphi_+(tu) \leq \varphi_+(u)$ *(resp.* $\varphi_-(tu) \leq \varphi_-(u)$*) for all* $t \geq 0$.

In the next proposition we show that the elements of the Nehari manifold are in norm bounded away from zero.

Proposition 12 *If hypotheses* H_0, H_1' *hold, then there exists* $\mu_0 > 0$ *such that*

$$0 < \mu_0 \leq \|u\|_r, \ \|u\| \text{ for all } u \in N.$$

Proof From hypotheses H_1'(i), (iv), we see that given $\varepsilon > 0$, we can find $c_{11} = c_{11}(\varepsilon) > 0$ such that

$$f(z, x)x \leq \varepsilon|x|^q + c_{11}|x|^r \text{ for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}. \tag{23}$$

Let $u \in N$ and we can always assume that $\|u\|, |u| \leq 1$. Then

$$\rho_a(Du) + \left(\|Du\|_q^q + \int_\Omega \xi(z)|u|^q dz \right)$$

$$\leq \varepsilon c_{12}[u]^q + c_{11}\|u\|_r^r \text{ for some } c_{12} > 0,$$

$$\Rightarrow \rho_a(Du) + \left(\frac{1}{2^{q-1}} - \varepsilon c_{12} \right) [u]^q \leq c_{11}\|u\|_r^r.$$

Choosing $\varepsilon \in \left(0, \dfrac{1}{2^{q-1}c_{12}} \right)$, we obtain

$$\rho_\theta(Du) + \int_\Omega \xi(z)|u|^q dz \leq c_{13}\|u\|_r^r \text{ for some } c_{13} > 0,$$

$$\Rightarrow \rho_\theta(Du) + \left(\int_\Omega \xi(z)|u|^q dz \right)^{p/q} \leq c_{14}\|u\|_r^r$$

for some $c_{14} > 0$ (recall that $|u| \leq 1$ and that $q < p$),

$$\Rightarrow \|Du\|_\theta^p + \left(\int_\Omega \xi(z) |u|^q dz \right)^{p/q} \leq c_{14}\|u\|_r^r \text{ (see Proposition 2),}$$

$$\Rightarrow |u|^p \leq c_{15}\|u\|_r^r \text{ for some } c_{15} > 0,$$

$$\Rightarrow \|u\|^p \leq c_{16}\|u\|_r^r \text{ for some } c_{16} > 0.$$

Therefore we conclude that we can find $\mu_0 > 0$ such that

$$0 < \mu_0 \leq \|u\|_r, \ \|u\| \text{ for all } u \in N.$$

This proof is now complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

From (8) it follows the following auxiliary property.

Corollary 13 *If hypotheses* H_0, H_1' *hold, then*

$$0 < \mu_0 \leq \|u\|_r, \ \|u\| \text{ for all } u \in N_+.$$

Now let

$$m_+ = \inf_{N_+} \varphi_+ \text{ and } m_- = \inf_{N_-} \varphi_-.$$

We show that both infima are attained.

Proposition 14 *If hypotheses* H_0, H_1' *hold, then there exist* $u_0 \in N_+$ *and* $v_0 \in N_-$ *such that*

$$\varphi_+(u_0) = m_+ \text{ and } \varphi_-(v_0) = m_-.$$

Proof We consider a sequence $\{u_n\}_{n \in \mathbb{N}} \subseteq W^{1,\theta}(\Omega)$ such that

$$u_n \in N_+ \text{ for all } n \in \mathbb{N} \text{ and } \varphi_+(u_n) \downarrow m_+. \tag{24}$$

From Corollary 7, we know that $\{u_n\}_{n \in \mathbb{N}} \subseteq W^{1,\theta}(\Omega)$ is bounded and so by passing to a subsequence, we may assume that

$$u_n \xrightarrow{w} u_0 \text{ in } W^{1,\theta}(\Omega) \text{ and } u_n \to u_0 \text{ in } L^r(\Omega). \tag{25}$$

Since $N_+ \subseteq N$ (see (8)), we have

$$\rho_a(Du_n) + \|Du_n\|_q^q + \int_\Omega \xi(z)|u_n|^q dz = \int_\Omega f(z, u_n)u_n dz, \ u_n \geq 0 \text{ for all } n \in \mathbb{N}. \tag{26}$$

Then $u_0 \geq 0$ (see (25)) and from Corollary 13 and (25), we see that $0 < \mu_0 \leq \|u_0\|_r$, hence $u_0 \neq 0$.

From (25) and (26), we have

$$\rho_a(Du_0) + \|Du_0\|_q^q + \int_\Omega \xi(z)|u_0|^q dz \leq \int_\Omega f(z, u_0)u_0 dz.$$

Suppose that the above inequality is strict, that is,

$$\rho_a(Du_0) + \|Du_0\|_q^q + \int_\Omega \xi(z)|u_0|^q dz < \int_\Omega f(z, u_0)u_0 dz. \tag{27}$$

We use the function $\beta_{u_0}(\cdot)$ from the proof of Proposition 8. We know that

$$\beta_{u_0}(t) > 0 \text{ for } t \in (0, 1) \text{ small (see (18))}, \ \beta_{u_0}(1) < 0 \text{ (see (15), (27))}.$$

So, we can find $t_0 \in (0, 1)$ such that

$$\beta_{u_0}(t_0) = 0,$$
$$\Rightarrow t_0 k_{u_0}'(t_0) = 0 \text{ since } u_0 \geq 0 \text{ and so } \varphi_+(tu_0) = \varphi(tu_0), \ t \geq 0,$$
$$\Rightarrow t_0 u_0 \in N_+.$$

We have

$$m_+ \leq \varphi_+(t_0 u_0)$$
$$= \frac{1}{p}\rho_a(D(t_0 u_o)) + \frac{1}{q}\left(\|D(t_0 u_0)\|_q^q + \int_\Omega \xi(z)|t_0 u_0|^q dz\right) - \int_\Omega F(z, t_0 u_0) dz$$
$$= \frac{1}{p}\int_\Omega \left(f(z, t_0 u_0)(t_0 u_0) - pF(z, t_0 u_0)\right) dz$$
$$+ \left(\frac{1}{q} - \frac{1}{p}\right)\left(\|D(t_0 u_0)\|_q^q + \int_\Omega \xi(z)|t_0 u_0|^q dz\right) \text{ (since } t_0 u_0 \in N_+). \tag{28}$$

On account of (6), we have

$$f(z, sx)(sx) - pF(z, sx) \leq f(z, x)x - pF(z, x) \tag{29}$$

for a.a. $z \in \Omega$, all $x \in \mathbb{R}$, all $s \in (0, 1)$.

Recall that $0 < t_0 < 1$. So, using (27) and (29) in (28), we obtain

$$
\begin{aligned}
m_+ &< \frac{1}{p} \int_\Omega (f(z, u_0)u_0 - pF(z, u_0))dz \\
&\quad + \left(\frac{1}{q} - \frac{1}{p}\right) \left(\|Du_0\|_q^q + \int_\Omega \xi(z)|u_0|^q dz\right) \\
&\leq \liminf_{n\to\infty} \left(\frac{1}{p} \int_\Omega (f(z, u_n)u_n - pF(z, u_n))\, dz \right. \\
&\quad \left. + \left(\frac{1}{q} - \frac{1}{p}\right) \left(\|Du_n\|_q^q + \int_\Omega \xi(z)|u_n|^q dz\right)\right) \text{ (see (25))} \\
&= \liminf_{n\to\infty} \left(\frac{1}{p}\rho_a(Du_n) + \frac{1}{q}\left(\|Du_n\|_q^q + \int_\Omega \xi(z)|u_n|^q dz\right) - \int_\Omega F(z, u_n)dz\right) \\
&= \liminf_{n\to\infty} \varphi_+(u_n) = m_+,
\end{aligned}
$$

a contradiction.

So, we have

$$
\rho_a(Du_0) + \|Du_0\|_q^q + \int_\Omega \xi(z)|u_0|^q dz = \int_\Omega f(z, u_0)u_0 dz,
$$

$$
\Rightarrow u_0 \in N_+.
$$

Then it follows that

$$
\rho_\theta(Du_n) \to \rho_\theta(Du_0) \text{ (see (25))}.
$$

The uniform convexity of $\theta(z, \cdot)$ implies that

$$
\rho_\theta(Du_n - Du_0) \to 0 \text{ (see Harjuletho \& Hästö [9, p. 65])},
$$

$$
\Rightarrow Du_n \to Du_0 \text{ in } L^\theta\left(\Omega, \mathbb{R}^N\right) \text{ (see Proposition 2)},
$$

$$
\Rightarrow u_n \to u_0 \text{ in } W^{1,\theta}(\Omega) \text{ (see(25) and Proposition 4)}.
$$

So, finally we can say that

$$
\varphi_+(u_0) = m_+, \ u_0 \in N_+.
$$

Similarly, working with $\varphi_-(\cdot)$ and N_-, we show that there exists $v_0 \in W^{1,\theta}(\Omega)$ such that

$$
\varphi_-(v_0) = m_-, \ v_0 \in N_-.
$$

This proof is now complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Next, we show that the two minimizers produced in Proposition 14 are in fact critical points of the functionals φ_+ and φ_- respectively. This means that N_+ and N_- are natural constraints for φ_+ and φ_-, respectively (see Papapgeorgiou et al. [19, p. 422]). Our proof is inspired by the works of Willem [26, p. 74] and Szulkin and Weth [25, p. 611].

Proposition 15 *If hypothesis* H_0, H_1' *hold and* $u_0 \in N_+$ *and* $v_0 \in N_-$ *are the two minimizers from Proposition 14, then* $u_0 \in K_{\varphi_+}$ *and* $v_0 \in K_{\varphi_-}$.

Proof Let $\tau \in \mathring{\mathbb{R}}_+ \setminus \{1\}$ ($\mathring{\mathbb{R}}_+ = (0, +\infty)$). Since $u_0 \in N_+$ from Proposition 11 we have

$$
\varphi_+(\tau u_0) \leq \varphi_+(u_0) = m_+. \tag{30}
$$

We argue indirectly. So, suppose that $u_0 \notin K_{\varphi_+}$. Then

$$\varphi'_+(u_0) \neq 0.$$

We can find $\delta > 0$ and $d > 0$ such that

$$\|u - u_0\| \leq 3\delta \Rightarrow \|\varphi'_+(u)\|_* \geq d.$$

We consider the interval $I = \left(\dfrac{1}{2}, \dfrac{3}{2} \right)$. From (30) we have

$$\hat{m}_+ = \max_{\tau \in \partial I} \varphi_+(\tau u_0) < m_+. \tag{31}$$

We can always choose $\delta > 0$ even smaller if necessary in order to have $\dfrac{d\delta}{8} \leq \dfrac{m_+ - \hat{m}_+}{4}$. Then we apply the quantitative deformation theorem of Willem [26, p. 38] with $\varepsilon = \dfrac{d\delta}{8}$ and $S = \overline{B}_\delta(u_0) = \left\{ u \in W^{1,\theta}(\Omega) : \|u - u_0\| \leq \delta \right\}$. So, we produce a deformation $h : [0, 1] \times W^{1,\theta}(\Omega) \mapsto W^{1,\theta}(\Omega)$ such that

- $h(1, u) = u$ if $\varphi_+(u) \notin [m_+ - 2\varepsilon, m_+ + 2\varepsilon] \cap \overline{B}_{2\delta}(u_0)$;
- $h\left(1, \varphi_+^{m_+ + \varepsilon} \cap \overline{B}_\delta(u_0) \right) \subseteq \varphi_+^{m_+ - \varepsilon}$;
- $\varphi_+(h(1, u)) \leq \varphi_+(u)$ for all $u \in W^{1,\theta}(\Omega)$.

From these properties of the deformation and (31), we deduce that

$$\max_{\tau \in \overline{I}} \varphi_+(h(1, \tau u_0)) < m_+. \tag{32}$$

Claim: $h(1, Iu_0) \cap N_+ \neq \emptyset$.

Let $\psi(\tau) = h(1, \tau u_0)$ and consider the following two functions

$$\sigma_0(\tau) = \langle \varphi'_+(\tau u_0), u_0 \rangle,$$

$$\sigma_1(\tau) = \frac{1}{\tau} \langle \varphi'_+(\psi(\tau)), \psi(\tau) \rangle, \ \tau > 0.$$

Recall that $u_0 \in N_+$. So, we have

$$\langle \varphi'_+(\tau u_0), u_0 \rangle > 0 \text{ if } 0 < \tau < 1, \tag{33}$$

$$\langle \varphi'_+(\tau u_0), u_0 \rangle < 0 \text{ if } 1 < \tau$$

$$\text{(see Propositions 11 and 9).} \tag{34}$$

By $d_B(\cdot, \cdot, \cdot)$ we denote the Brouwer degree. Then from (33), (34) and Lloyd [13, see Illustration 1, pp. 19, 20], we have

$$d_B(\sigma_0, I, 0) = -1. \tag{35}$$

From the properties of the deformation, we have

$$\sigma_0|_{\partial I} = \sigma_1|_{\partial I},$$

$$\Rightarrow d_B(\sigma_0, I, 0) = d_B(\sigma_1, I, 0),$$

$$\Rightarrow d_B(\sigma_1, I, 0) \neq 0 \text{ (see (35))},$$

$$\Rightarrow \sigma_1(\tau) = 0 \text{ for some } \tau \in I \text{ (see [9], [19])},$$

$$\Rightarrow \psi(\tau) = h(1, \tau u_0) \in N_+.$$

This proves the Claim.

But then on account of the Claim, we have

$$m_+ \leq \max_{\tau \in \overline{I}} \varphi_+(h(1, \tau u_0)).$$

Comparing with (32) we have a contradiction. Therefore $u_0 \in K_{\varphi_+}$.

Similarly we show that $v_0 \in K_{\varphi_-}$. □

Now we are ready to produce two constant sign solutions for problem (1) (positive and negative solutions), under the strict monotonicity condition H_1'(iii).

Proposition 16 *If hypotheses* H_0, H_1' *hold, then problem* (1) *has at least two nontrivial constant sign solutions*

$$u_0 \in N_+ \cap L^\infty(\Omega), \ u_0(z) > 0 \ for \ a.a. \ z \in \Omega,$$
$$v_0 \in N_- \cap L^\infty(\Omega), \ v_0(z) < 0 \ for \ a.a. \ z \in \Omega.$$

Proof From Proposition 15 we know that

$$u_0 \in N_+ \text{ and } v_0 \in N_-$$

are nontrivial constant sign solutions of problem (1). As in Gasiński and Winkert [6, Theorem 3.1], we have that

$$u_0 \in N_+ \cap L^\infty(\Omega), \ v_0 \in N_- \cap L^\infty(\Omega).$$

Finally using Proposition 2.4 of Papageorgiou, Vetro and Vetro [22], we have

$$v_0(z) < 0 < u_0(z) \text{ for a.a. } z \in \Omega.$$

This proof is now complete. □

4 Nodal solutions under strict monotonicity

In this section we prove the existence of a nodal (that is, sign changing) solution for problem (1), under the strict monotonicity hypothesis H_1'(iii). For this purpose, we bring in the picture the submanifold N_0.

We define

$$m_0 = \inf_{N_0} \varphi.$$

Proposition 17 *If hypotheses* H_0, H_1' *hold, then there exists* $y_0 \in N_0$ *such that*

$$m_0 = \varphi(y_0).$$

Proof We consider a sequence $\{y_n\}_{n \in \mathbb{N}} \subseteq W^{1,\theta}(\Omega)$ such that

$$y_n \in N_0 \text{ for all } n \in \mathbb{N} \text{ and } \varphi(y_n) \downarrow m_0.$$

From Proposition 6 we know that $\{y_n\}_{n \in \mathbb{N}} \subseteq W^{1,\theta}(\Omega)$ is bounded. Then $\{y_n^+\}_{n \in \mathbb{N}} \subseteq W^{1,\theta}(\Omega)$ and $\{y_n^-\}_{n \in \mathbb{N}} \subseteq W^{1,\theta}(\Omega)$ are bounded, too. So, we may assume that

$$y_n^+ \overset{w}{\to} \hat{y}_1 \text{ and } y_n^- \overset{w}{\to} \hat{y}_2 \text{ in } W^{1,\theta}(\Omega). \tag{36}$$

Suppose that $\hat{y}_1 = 0$. Since $y_n \in N$, we have that $y_n^+ \in N_+$ for all $n \in \mathbb{N}$ and so

$$\rho_a(Dy_n^+) + \|Dy_n^+\|_q^q + \int_\Omega \xi(z)(y_n^+)^q dz = \int_\Omega f(z, y_n^+)y_n^+ dz \text{ for all } n \in \mathbb{N},$$

$$\Rightarrow \rho_\theta(Dy_n^+) + \int_\Omega \xi(z)(y_n^+)^q dz \to 0$$

(see (36) and recall that $\hat{y}_1 = 0$),

$$\Rightarrow |y_n^+| \to 0,$$

$$\Rightarrow y_n^+ \to 0 \text{ in } W^{1,\theta}(\Omega) \text{ (see Proposition 4).} \quad (37)$$

But from Corollary 13 we know that

$$0 < \mu_0 \leq \|y_n^+\| \text{ for all } n \in \mathbb{N}. \quad (38)$$

Comparing (37) and (38), we have a contradiction.
Therefore $\hat{y}_1 \neq 0$. Similarly we show that $\hat{y}_2 \neq 0$.
According to Proposition 9, there exist a unique $\hat{t}_1, \hat{t}_2 > 0$ such that

$$\hat{t}_1\hat{y}_1 \in N_+ \text{ and } \hat{t}_2\hat{y}_2 \in N_-.$$

We set

$$y_0 = \hat{t}_1\hat{y}_1 - \hat{t}_2\hat{y}_2.$$

Evidently we have

$$y_0^+ = \hat{t}_1\hat{y}_1, \ y_0^- = \hat{t}_2\hat{y}_2 \text{ and so } y_0 \in N_0.$$

We have

$$\begin{aligned}
m_0 &= \lim_{n\to\infty} \varphi(y_n) \\
&= \lim_{n\to\infty} \left(\varphi_+(y_n^+) + \varphi_-(-y_n^-)\right) \\
&\geq \liminf_{n\to\infty} \left(\varphi_+(\hat{t}_1 y_n^+) + \varphi_-(-\hat{t}_2 y_n^-)\right) \text{ (see Proposition 11)} \\
&\geq \varphi(y_0) \\
&\geq m_0 \text{ (since } y_0 \in N_0), \\
\Rightarrow m_0 &= \varphi(y_0) \text{ with } y_0 \in N_0.
\end{aligned}$$

This proof is now complete.   □

Reasoning as in the proof of Proposition 15, we show that N_0 is a natural constraint for $\varphi(\cdot)$, that is, $y_0 \in K_\varphi$. In this case we apply the quantitative deformation theorem of Willem [26], using the following data: the set

$$D = \left(\frac{1}{2}, \frac{3}{2}\right) \times \left(\frac{1}{2}, \frac{3}{2}\right)$$

and the functions

$$\psi(\tau, t) = h(1, \tau\hat{y}_1 - t\hat{y}_2), \ \tau, \ t \geq 0,$$

$$\sigma_0(\tau, t) = \left(\langle\varphi'(\tau\hat{y}_1), \hat{y}_1\rangle, \langle\varphi'(-t\hat{y}_2), -\hat{y}_2\rangle\right)$$

$$\sigma_1(\tau, t) = \left(\frac{1}{\tau}\langle\varphi'(\psi(\tau, t)^+), \psi(\tau, t)^+\rangle, \frac{1}{t}\langle\varphi'(-\psi(\tau, t)^-), -\psi(\tau, t)^-\rangle\right).$$

Using the quantitative deformation theorem of Willem [26, p. 38], with the above data, we have the following result.

Proposition 18 *If hypotheses* H_0, H_1' *hold and* $y_0 \in N_0$ *is the minimizer from Proposition 17, then* $y_0 \in K_\varphi$.

So, under the stronger strict monotonicity condition, we can state the following multiplicity theorem for problem (1).

Proposition 19 *If hypotheses* H_0, H_1' *hold, then problem* (1) *has at least three nontrivial solutions*

$$u_0 \in N_+ \cap L^\infty(\Omega), \ u_0(z) > 0 \ for \ a.a. \ z \in \Omega, \ m_+ = \varphi_+(u_0),$$
$$v_0 \in N_- \cap L^\infty(\Omega), \ v_0(z) < 0 \ for \ a.a. \ z \in \Omega, \ m_- = \varphi_-(v_0),$$
$$y_0 \in N_0 \cap L^\infty(\Omega) \ nodal, \ m_0 = \varphi(y_0).$$

5 Multiple solutions under relaxed monotonicity

In this section we drop the strict monotonicity condition H_1'(iii) and prove a multiplicity result (three solutions theorem) with sign information for all the solutions, under the weaker set of hypotheses H_1.

For this purpose, we introduce the following perturbation of the reaction

$$f_\ell(z, x) = f(z, x) + \ell|x|^{r-2}x, \ \ell > 0.$$

This perturbation of $f(\cdot, \cdot)$ satisfies hypotheses H_1'. We set $F_\ell(z, x) = \int_0^x f_\ell(z, s)ds$ and we consider the corresponding energy functionals $\varphi_\ell, \varphi_\ell^\pm : W^{1,\theta}(\Omega) \mapsto \mathbb{R}$ defined as $\varphi, \varphi_\pm$ only now $F(z, x)$ is replaced by $F_\ell(z, x)$. Also, to these functionals we associate the corresponding Nehari manifolds $N^\ell, N_\pm^\ell, N_0^\ell$.

Proposition 20 *If hypotheses* H_0, H_1 *hold, then there exists* $\hat\mu > 0$ *such that*

$$0 < \hat\mu \le \varphi_\ell(u) \ for \ all \ u \in N^\ell, \ all \ \ell \in (0, 1).$$

Proof On account of hypotheses H_1(i), (iv), given $\varepsilon > 0$, we can find $c_{17} = c_{17}(\varepsilon) > 0$ such that

$$F(z, x) \le \frac{\varepsilon}{q}|x|^q + c_{17}|x|^r \ \text{for a.a.} \ z \in \Omega, \ \text{all} \ x \in \mathbb{R}. \tag{39}$$

Let $u \in N^\ell$ ($\ell \in (0, 1)$). From Proposition 10, we have

$$\varphi_\ell(u) = \max_{t \ge 0} \varphi_\ell(tu)$$
$$\ge \max_{t \ge 0} \left(\frac{t^p}{p} \rho_a(Du) + \frac{t^q}{q} \left(\|Du\|_q^q + \int_\Omega \xi(z)|u|^q dz \right) \right.$$
$$\left. - \int_\Omega F(z, tu)dz - t^r \|u\|_r^r \right) \ (\text{since } 0 < \ell < 1).$$

Using (39), Proposition 5 and choosing $\varepsilon > 0$ small, we obtain

$$\varphi_\ell(u) \ge \max_{t \ge 0} \left(c_{18}t^p - c_{19}t^r \right) \ \text{for some } c_{18}, \ c_{19} > 0, \ \text{all} \ t \ge 0,$$
$$\Rightarrow \varphi_\ell(u) \ge \hat\mu > 0 \ (\text{recall that } p < r).$$

This proof is now complete.   □

In a similar fashion we show similar results for the functionals $\varphi_\ell^\pm(\cdot)$.

Proposition 21 *If hypotheses* H_0, H_1 *hold, then there exist* $\hat{\mu}_\pm > 0$ *such that*

$$0 < \hat{\mu}_+ \leq \varphi_\ell^+(u) \text{ for all } u \in N_+^\ell, \text{ all } \ell \in (0, 1),$$
$$0 < \hat{\mu}_- \leq \varphi_\ell^-(u) \text{ for all } u \in N_-^\ell, \text{ all } \ell \in (0, 1).$$

Now letting $\ell \to 0^+$ we will produce constant sign and nodal solutions for problem (1) under the relaxed monotonicity condition H_1 (iii).

Theorem 22 *If hypotheses* H_0, H_1 *hold, then problem* (1) *has at least three nontrivial solutions*

$$u_0 \in N_+ \cap L^\infty(\Omega), \ u_0(z) > 0 \text{ for a.a. } z \in \Omega, \ \varphi_+(u_0) = m_+,$$
$$v_0 \in N_- \cap L^\infty(\Omega), \ v_0(z) < 0 \text{ for a.a. } z \in \Omega, \ \varphi_-(v_0) = m_-,$$
$$y_0 \in N_0 \cap L^\infty(\Omega) \text{ nodal, } \varphi(y_0) = m_0.$$

Proof Let $\ell_n \to 0^+$. From Proposition 19 we know that we can find $y_n \in W^{1,\theta}(\Omega)$ such that

$$y_n \in N^{\ell_n} \ (n \in \mathbb{N}), \ \varphi_{\ell_n}(y_n) = m_0^{\ell_n} \to \hat{m}_0 \geq 0 \text{ as } n \to \infty, \ \varphi_{\ell_n}'(y_n) = 0 \ (n \in \mathbb{N})$$

(note that $\left\{ m_0^{\ell_n} \right\}_{n \in \mathbb{N}} \subseteq \mathring{\mathbb{R}}_+$ is decreasing).   (40)

We will show that $\{y_n\}_{n \in \mathbb{N}} \subseteq W^{1,\theta}(\Omega)$ is bounded. Arguing by contradiction, suppose that at least for a subsequence, we have

$$\|y_n\| \to +\infty.$$ (41)

We set $\hat{w}_n = \dfrac{y_n}{\|y_n\|}$ $(n \in \mathbb{N})$. Then $\|\hat{w}_n\| = 1$ for all $n \in \mathbb{N}$ and so we may assume that

$$\hat{w}_n \xrightarrow{w} \hat{w} \text{ in } W^{1,\theta}(\Omega) \text{ and } \hat{w}_n \to \hat{w} \text{ in } L^r(\Omega).$$ (42)

Suppose that $\hat{w} \equiv 0$ and let $k > 1$. Then on account of Proposition 11, we have

$$m_0^{\ell_n} = \varphi_{\ell_n}(y_n)$$
$$\geq \varphi_{\ell_n}\left(\frac{k}{\|y_n\|} y_n \right)$$
$$= \varphi_{\ell_n}(k\hat{w}_n)$$
$$= \frac{k^p}{p} \rho_a(D\hat{w}_n) + \frac{k^q}{q} \left(\|D\hat{w}_n\|_q^q + \int_\Omega \xi(z)|\hat{w}_n|^q dz \right)$$
$$- \int_\Omega F(z, k\hat{w}_n)dz - \ell k^r \|\hat{w}_n\|_r^r$$
$$\geq \frac{k^q}{p} c_{20} - \int_\Omega F(z, k\hat{w}_n)dz - \ell k^r \|\hat{w}_n\|_r^r$$

for some $c_{20} > 0$ (since $k > 1$, $q < p$, $\|\hat{w}_n\| = 1$).

Passing to the limit as $n \to \infty$ and recalling that $\hat{w} = 0$, we obtain

$$\hat{m}_0 \geq \frac{k^q}{p} c_{20} \text{ (see (40), (42)).}$$

Since $k > 1$ is arbitrary, we let $k \to +\infty$ and have a contradiction.

Now assume that $\hat{w} \neq 0$ and let $\hat{\Omega} = \{z \in \Omega : \hat{w}(z) \neq 0\}$.

Then $|\hat{\Omega}|_N > 0$ (by $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$). We have

$$|y_n(z)| \to +\infty \text{ for a.a. } z \in \hat{\Omega}.$$

From (40) we have

$$
\begin{aligned}
0 &\leq \frac{m_0^{\ell_n}}{\|y_n\|^p} \\
&= \frac{\varphi_{\ell_n}(y_n)}{\|y_n\|^p} \\
&\leq \frac{1}{p}\rho_a(D\hat{w}_n) + \frac{1}{q\|y_n\|^{p-q}}\left(\|D\hat{w}_n\|_q^q + \int_\Omega \xi(z)|\hat{w}_n|^q dz\right) - \int_\Omega \frac{F(z, y_n)}{\|y_n\|^p}dz \\
&\leq \frac{1}{q}\left(\rho_\theta(D\hat{w}_n) + \int_\Omega \xi(z)|\hat{w}_n|^q dz\right) - \int_\Omega \frac{F(z, y_n)}{\|y_n\|^p}dz \text{ for all } n \geq n_1 \\
&\leq \frac{1}{q}\left(\|D\hat{w}_n\|_\theta^q + \int_\Omega \xi(z)|\hat{w}_n|^q dz\right) - \int_\Omega \frac{F(z, y_n)}{\|y_n\|^p}dz \\
&\quad \text{(since } \|D\hat{w}_n\|_\theta \leq \|\hat{w}_n\| = 1, \text{ see Proposition 2)} \\
&\leq \frac{1}{q}c_{21} - \int_\Omega \frac{F(z, y_n)}{\|y_n\|^p}dz \\
&\quad \text{for some } c_{21} > 0, \text{ all } n \geq n_1 \text{ (see Proposition 4).} \quad\quad (43)
\end{aligned}
$$

On account of hypothesis H_1(ii), we have

$$
\begin{aligned}
&\int_{\hat{\Omega}} \frac{F(z, y_n)}{\|y_n\|^p}dz \to +\infty, \\
\Rightarrow &\int_\Omega \frac{F(z, y_n)}{\|y_n\|^p}dz \to +\infty \\
&\quad \text{(see } H_1\text{(ii) and use Fatou's lemma).} \quad\quad (44)
\end{aligned}
$$

If in (43) we pass to the limit as $n \to \infty$ and use (44) we have a contradiction.

Therefore $\{y_n\}_{n\in\mathbb{N}} \subseteq W^{1,\theta}(\Omega)$ is bounded and so we may assume that

$$y_n \xrightarrow{w} y_0 \text{ in } W^{1,\theta}(\Omega) \text{ and } y_n \to y_0 \text{ in } L^r(\Omega). \quad\quad (45)$$

From (40) we have

$$\langle V(y_n), h\rangle + \int_\Omega \xi(z)|y_n|^{q-2}y_n h dz = \int_\Omega f(z, y_n)h dz \text{ for all } h \in W^{1,\theta}(\Omega).$$

We choose $h = y_n - y_0 \in W^{1,\theta}(\Omega)$, pass to the limits as $n \to \infty$ and use (45). We obtain

$$
\begin{aligned}
&\lim_{n\to\infty} \langle V(y_n), y_n - y_0\rangle = 0, \\
\Rightarrow &y_n \to y_0 \text{ in } W^{1,\theta}(\Omega) \text{ (see Proposition 3).} \quad\quad (46)
\end{aligned}
$$

From (46) and Proposition 20, we have

$$
\begin{aligned}
&0 < \hat{\mu} \leq \varphi(y_0), \\
\Rightarrow &y_0 \neq 0 \text{ and so } y_0 \in \mathbb{N}.
\end{aligned}
$$

Reasoning as in the proof Proposition 17, using Proposition 21, we show that $y_0^+ \neq 0$, $y_0^- \neq 0$, hence $y_0 \in N_0$. Therefore we have

$$y_0 \in N_0, \ \varphi'(y_0) = 0, \ \varphi(y_0) = m_0,$$
$$\Rightarrow \ y_0 \in N \cap L^\infty(\Omega) \text{ is a ground state nodal solution of problem (1).}$$

Similarly, working with $\varphi_{\ell_n}^+$ and $N_+^{\ell_n}$ and passing to the limit as $n \to \infty$, we obtain $u_0 \in W^{1,\theta}(\Omega)$ such that

$$u_0 \in N_+ \cap L^\infty(\Omega), \ \varphi'(u_0) = 0, \ u_0(z) > 0 \text{ for a.a. } z \in \Omega, \ \varphi(u_0) = m_+$$

(positive ground state solution) and $v_0 \in W^{1,\theta}(\Omega)$ such that

$$v_0 \in N_- \cap L^\infty(\Omega), \ \varphi'(v_0) = 0, \ v_0(z) < 0 \text{ for a.a. } z \in \Omega, \ \varphi(v_0) = m_-$$

(negative ground state solution). □

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