Optimal bounds of classical and non-classical means in terms of Q means

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Abstract

We show optimal bounds of the form $Q_\alpha < M < Q_\beta$, where

$$Q_\alpha(x, y) = A(x, y) \frac{A^2(x, y)}{(1 - \alpha)A^2(x, y) + \alpha G^2(x, y)}$$

and M belongs to a broad class of classical homogeneous, symmetric means of two variables.

Keywords Classical means · Seiffert function · Q mean

Mathematics Subject Classification 26D15

1 Introduction, definitions and notations

It is a truism to say that means are important. Means in various forms were used for practical purposes as early as in the antiquity. It is worth mentioning here the Pythagorean means of many variables (harmonic, geometric, arithmetic) or the Heronian mean. The averages have been used in practice in many areas: optics, electricity, mechanics, finance, econometrics, and statistics.

Over time, as it usually happens, the means began to live their own lives and became the subject of research of many scientists (see the list of references). In particular, the means of two variables became an interesting object of investigation. Due to big diversity of means, important information is the mutual relation between them, i.e. information about which of the three cases: $M > N$, $M < N$ or M and N are incomparable, holds. Another useful piece of information is to study the position of the mean relative to some reference family

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of means. The most popular reference family is the power means given by the formula

$$A_r(x, y) = \left(\frac{x^r + y^r}{2} \right)^{1/r} \text{ and } A_0(x, y) = \sqrt{xy}.$$

Here are three examples of such results:

- for the logarithmic means Tung Po Lin [11] proved $A_0 < L < A_{1/3}$,
- for the Heronian mean Alzer and Janous [1] proved optimal inequalities $A_{\log 2 / \log 3} < He < A_{2/3}$,
- the result of Hästö [8] gives optimal bounds for the first Seiffert mean $A_{\log 2 / \log \pi} < P < A_{2/3}$.

In this paper we introduce a family of Q -means defined by

$$Q_\alpha(x, y) = A(x, y) \frac{A^2(x, y)}{(1 - \alpha)A^2(x, y) + \alpha G^2(x, y)}, \quad (1)$$

where $x, y > 0$, $-4 \leq \alpha \leq \frac{1}{2}$, $A(x, y) = \frac{x+y}{2}$ and $G(x, y) = \sqrt{xy}$ are the arithmetic and geometric means. It is easy to see that if $\alpha < \beta$ then $Q_\alpha < Q_\beta$. The Q -means will be used as an alternative reference line for most of the classical means as well as for some recently discovered ones. The reader will find here the optimal bounds for them in terms of Q -means as well as some numerical results.

Based on the shapes of the Seiffert means, one of the authors introduced in [20] the notion of Seiffert function: a function $f : (0, 1) \rightarrow \mathbb{R}$ such that

$$M_f(x, y) = \begin{cases} \frac{|x - y|}{2f\left(\frac{|x-y|}{x+y}\right)} & x \neq y \\ x & x = y \end{cases} \quad (2)$$

is a mean. It was shown that every symmetric and homogeneous mean of positive arguments can be represented in the form (2) and that every function $f : (0, 1) \rightarrow \mathbb{R}$ satisfying

$$\frac{z}{1+z} \leq f(z) \leq \frac{z}{1-z} \quad (3)$$

produces a mean. The correspondence between a mean and its Seiffert function is given by the formula

$$f(z) = \frac{z}{M(1-z, 1+z)}, \quad \text{where } z = \frac{|x-y|}{x+y}. \quad (4)$$

In [20] it was shown also that the functions $\sin$, $\tan$, $\sinh$ and $\tanh$ are also Seiffert functions that produce the means called the sine, tangent, hyperbolic sine and hyperbolic tangent means.

Remark 1 Note that due to identity $HA = G^2$ (where $H(x, y) = \frac{2xy}{x+y}$ is the harmonic mean), (1) can be written as

$$Q_\alpha(x, y) = A(x, y) \frac{A(x, y)}{(1 - \alpha)A(x, y) + \alpha H(x, y)}. \quad (1')$$

Let us use the formula (4) to calculate the Seiffert function of Q_α :

$$\begin{aligned} q_\alpha(z) &= \frac{z}{A(1+z, 1-z) \frac{A(1+z, 1-z)}{(1-\alpha)A(1+z, 1-z) + \alpha H(1+z, 1-z)}} \\ &= z(1 - \alpha + \alpha(1 - z^2)) = z - \alpha z^3. \end{aligned} \quad (5)$$

In this paper we shall establish the optimal bounds of the form $Q_\alpha < M < Q_\beta$, where M is one of the classical (or less classical) means.

Remark 2 For two means M, N the notation $M < N$ means that $M(x, y) < N(x, y)$ holds unless $x = y$.

Remark 3 If m, n are the Seiffert functions of M, N respectively, then the inequality $M < N$ is equivalent to $n < m$ by the formula (2).

Note that the inequalities $Q_\alpha < M < Q_\beta$ in terms of Seiffert functions can be rewritten as $q_\beta(z) < m(z) < q_\alpha(z)$, and, given (5), to find optimal α and β we may use one of the three methods

Method 1 Evaluate

$$\alpha = \inf_{0 < z < 1} \frac{z - m(z)}{z^3},$$

$$\beta = \sup_{0 < z < 1} \frac{z - m(z)}{z^3},$$

or

Method 2 Calculate

$$\alpha = \sup\{\gamma : z - \gamma z^3 - m(z) \geq 0 \text{ holds for all } z \in (0, 1)\},$$

$$\beta = \inf\{\gamma : z - \gamma z^3 - m(z) \leq 0 \text{ holds for all } z \in (0, 1)\},$$

or equivalently

Method 3 Calculate

$$\alpha = \sup\left\{\gamma : 1 - \gamma z^2 - \frac{m(z)}{z} \geq 0 \text{ holds for all } z \in (0, 1)\right\},$$

$$\beta = \inf\left\{\gamma : 1 - \gamma z^2 - \frac{m(z)}{z} \leq 0 \text{ holds for all } z \in (0, 1)\right\}.$$

2 Remarks on Q means

It might be surprising why we consider the Q_α means for $-4 \leq \alpha \leq \frac{1}{2}$ only. The following lemmas give the answer.

Lemma 1 The function $q_\alpha(z) = z - \alpha z^3$ is a Seiffert function if and only if $-4 \leq \alpha \leq \frac{1}{2}$.

Proof The inequalities (3) can be rewritten as

$$-\frac{1}{z(1-z)} \leq \alpha \leq \frac{1}{z(1+z)}$$

and since $\inf_{z \in (0,1)} \frac{1}{z(1+z)} = \frac{1}{2}$ and $\max_{z \in (0,1)} \frac{-1}{z(1-z)} = -4$, the assertion holds. $\square$

Lemma 2 The function $q_\alpha(z) = z - \alpha z^3$ is the Seiffert function of Q_α .

Proof By (4) we have

$$\frac{z}{Q_\alpha(1+z, 1-z)} = \frac{z}{\frac{1}{1-\alpha+\alpha(1-z^2)}} = q_\alpha(z).$$

$\square$

3 Main results

At the beginning we consider means that are lesser than the logarithmic mean:

- harmonic mean $H(x, y) = \frac{2xy}{x+y} = A_{-1}(x, y)$. Its Seiffert function equals $h(z) = \frac{z}{1-z^2}$,
- power mean of order $-1/2$, $A_{-1/2}(x, y) = \left(\frac{x^{-1/2} + y^{-1/2}}{2}\right)^{-2}$ with Seiffert function $a_{-1/2}(z) = \frac{z(1+\sqrt{1-z^2})}{2(1-z^2)}$,
- geometric mean $G(x, y) = \sqrt{xy}$ with $g(z) = \frac{z}{\sqrt{1-z^2}}$,
- logarithmic mean $L(x, y) = \frac{x-y}{\ln x - \ln y} = M_{\text{artanh}}(x, y) = \frac{|x-y|}{2\text{artanh}\frac{|x-y|}{x+y}}$. The last formula shows that $l(z) = \text{artanh}z$.

All the means mentioned above have one common property: $\lim_{x \rightarrow 0^+} M(x, 1) = 0$, which indicates that there is no lower bound for them in the class of Q means, as the limit of Q_α at zero is positive.

Theorem 1 (Bounds for harmonic mean) *The inequality*

$$H(x, y) \leq Q_\beta(x, y)$$

holds if and only if $\beta \geq -1$.

Proof Consider the function from Method 1:

$$u(z) = \frac{z - \frac{z}{1-z^2}}{z^3} = -\frac{1}{1-z^2}.$$

It decreases and assumes values in $(-\infty, -1)$, which completes the proof. $\square$

Theorem 2 (Bounds for power mean of order $-1/2$) *The inequality*

$$A_{-1/2}(x, y) \leq Q_\beta(x, y)$$

holds if and only if $\beta \geq -\frac{3}{4}$. There is no lower bound for the power mean of order $-1/2$ in the family of Q means.

Proof Consider the function in Method 3

$$g_\gamma(z) = 1 - \gamma z^2 - \frac{1}{2} \left(\frac{1}{1-z^2} + \frac{1}{\sqrt{1-z^2}} \right).$$

The function g_γ satisfies $g_\gamma(0) = 0$ and (as every even function) $g'_\gamma(0) = 0$. Moreover

$$g''_\gamma(z) = -2\gamma - \frac{1}{2} \left(\frac{6z^2 + 2}{(1-z^2)^3} + \frac{2z^2 + 1}{(1-z^2)^{5/2}} \right). \quad (6)$$

Both fractions in (6) increase (numerators increase and denominators decrease) thus g''_γ decreases from $-2\gamma - \frac{3}{2}$ to $-\infty$. This implies that if $\gamma \geq -\frac{3}{4}$ the function g_γ is concave and therefore satisfies $g_\gamma(z) < 0$ for $0 < z < 1$. If $\gamma < -\frac{3}{4}$ the function g_γ is convex and thus positive for small z , and cannot preserve sign, because it tends to $-\infty$ at the right end. $\square$

Theorem 3 (Bounds for geometric mean) *The inequality $G < Q_\beta$ holds if and only if $\beta \geq -1/2$. There is no Q_α that bounds the geometric mean from below.*

Proof We use Method 1. The function

$$\frac{z - \frac{z}{\sqrt{1-z^2}}}{z^3} = \frac{\sqrt{1-z^2} - 1}{z^2\sqrt{1-z^2}} = -\frac{1}{\sqrt{1-z^2}(1+\sqrt{1-z^2})}$$

decreases because both functions in denominator decrease. Therefore it assumes values in $(-\infty, -\frac{1}{2})$, which completes the proof. $\square$

Let us recall that the Seiffert function of the logarithmic mean $l(z) = \operatorname{artanh} z$.

Theorem 4 (Bounds for logarithmic mean). *The inequalities $M_{\operatorname{artanh}} < Q_\beta$ hold if and only if $\beta \geq -\frac{1}{3} \approx -0.3333$. There is no Q_α that bounds the logarithmic mean from below.*

Proof Let us consider the function from Method 2: $u_\gamma(z) = z - \gamma z^3 - \operatorname{artanh} z$. It is elementary that $u_\gamma(0) = u'_\gamma(0) = 0$. Let us investigate the sign of its second derivative.

$$u''_\gamma(z) = -6\gamma z - \frac{2z}{(1-z^2)^2} = -2z \left(3\gamma + \frac{1}{(1-z^2)^2} \right).$$

It is easy to see that if $\gamma \geq -\frac{1}{3}$, then u_γ is concave and thus negative for all $z \in (0, 1)$. For $\gamma < -\frac{1}{3}$ our function is convex and positive for small z and negative for z close to 1, which means that it cannot preserve sign, thus α is not a real number. $\square$

There are many interesting means between the logarithmic and the arithmetic means. We shall consider five of them. Here they are:

- power mean of order $\frac{1}{2}$, $A_{1/2}(x, y) = \left(\frac{\sqrt{x} + \sqrt{y}}{2} \right)^2$ with Seiffert function $a_{1/2}(z) = \frac{2z}{1+\sqrt{1-z^2}}$,
- Heronian mean $\operatorname{He}(x, y) = \frac{x+\sqrt{xy}+y}{3}$ used to calculate the volume of a truncated pyramid long before Christ. Its Seiffert function equals $\operatorname{he}(z) = \frac{3z}{2+\sqrt{1-z^2}}$,
- tangent mean $M_{\tan}(x, y) = \frac{|x-y|}{2 \tan \frac{|x-y|}{x+y}}$. Obviously, tangent is its Seiffert function,
- arcsine or first Seiffert mean $P(x, y) = M_{\arcsin}(x, y) = \frac{|x-y|}{2 \arcsin \frac{|x-y|}{x+y}}$ with $p(z) = \arcsin z$,
- hyperbolic sine mean $M_{\sinh}(x, y) = \frac{|x-y|}{2 \sinh \frac{|x-y|}{x+y}}$ with $m_{\sinh}(z) = \sinh z$.

The following inequalities between these means can be found in the literature.

Inequality $L < A_{1/3}$ has been proven in [11]. This combined with monotonicity of power means shows that $L < A_{1/2}$.

Inequalities

$$L < \left\{ \begin{array}{c} P \\ M_{\tan} \end{array} \right\} < M_{\sinh} < A$$

have been proven in [20] (the means in curly brackets are not comparable). The means $A_{1/2}$ and $M_{\tan}$ are not comparable either, and the relations

$$A_{1/2} < P < \operatorname{He}$$

has been proved by Hastö in [8] (left part) and Sándor in [15] (right part).

Finally

$$M_{\tan} < \operatorname{He} < M_{\sinh}$$

follow from Theorems 2.1 and 2.2 in [12]. Summarizing, we have the following chain of inequalities:

$$L < \left\{ \begin{array}{c} A_{1/2} < P \\ M_{\tan} \end{array} \right\} < He < M_{\sinh} < A.$$

Let us begin with the Heronian mean $He = \frac{2A+G}{3}$ and the power mean of order $\frac{1}{2}$ that can be written as $A_{1/2} = \frac{A+G}{2}$. Both are members of the family of means that interpolate between the geometric and the arithmetic means given by $He_p = (1-p)G + pA$, $0 \leq p \leq 1$. Obviously $He = He_{2/3}$. The Seiffert function for He_p is

$$he_p(z) = \frac{z}{(1-p)\sqrt{1-z^2} + p}.$$

Theorem 5 (Bounds for the Heronian family of means). *Let $0 \leq p \leq 1$. The inequalities $He_p < Q_\beta$ hold if and only if $\beta \geq \frac{p-1}{2}$.*

If $p \geq \frac{1}{5}$, then the inequality $Q_\alpha < He_p$ holds if and only if $\alpha \leq \frac{p-1}{p}$.

If $p < \frac{1}{5}$, then there is no lower bound for He_p .

Proof We shall use Method 1 here:

$$\frac{z - he_p(z)}{z^3} = \frac{p-1}{\left[(1-p)\sqrt{1-z^2} + p \right] \left[\sqrt{1-z^2} + 1 \right]},$$

which shows that this function decreases with $z \in (0, 1)$ and assumes values in $\left(\frac{p-1}{p}, \frac{p-1}{2} \right)$. For $p < \frac{1}{5}$, we have $(p-1)/p < -4$, so the corresponding Q is not a mean by Lemma 1. $\square$

Corollary 1 *The optimal inequalities hold:*

$$Q_{-1} < A_{1/2} < Q_{-1/4}.$$

Corollary 2 *The optimal inequalities hold:*

$$Q_{-1/2} < He < Q_{-1/6}.$$

And now it's time for three other means

Theorem 6 (Bounds for tangent mean) *The inequalities $Q_\alpha < M_{\tan} < Q_\beta$ hold if and only if $\alpha \leq 1 - \tan 1 \approx -0.5574$ and $\beta \geq -\frac{1}{3} \approx -0.3333$.*

Proof Consider the function from Method 2:

$$g_\gamma(z) = z - \gamma z^3 - \tan z.$$

It satisfies $g_\gamma(0) = g'_\gamma(0) = 0$ and $g_\gamma(1) = 1 - \gamma - \tan 1$. Moreover,

$$g''_\gamma(z) = -2z \left(3\gamma + \frac{\sin z}{z \cos^3 z} \right). \quad (7)$$

The function $\frac{\sin z}{\cos^3 z} = \frac{1}{\cos z} \cdot \frac{1}{\cos z} \cdot \tan z$ is a product of three positive, increasing and convex functions, so it is convex. By Property 2 its divided difference $\frac{\sin z}{z \cos^3 z}$ increases from $m = 1$ to $M = \frac{\sin 1}{\cos^3 1}$. Thus we see that if $3\gamma + 1 \geq 0$, then g_γ is concave and therefore negative. On the other hand, if $3\gamma + 1 < 0$ then g_γ is convex, and thus positive for small values of z and changes its convexity at most once. Therefore it remains nonnegative in the unit interval if and only if $g_\gamma(1) = 1 - \gamma - \tan 1 \geq 0$. $\square$

Theorem 7 (Bounds for first Seiffert mean). *The inequalities $Q_\alpha < P < Q_\beta$ hold if and only if $\alpha \leq 1 - \pi/2 \approx -0.5708$ and $\beta \geq -\frac{1}{6} \approx -0.1667$.*

Proof We shall use once more Method 2 and investigate

$$g_\gamma(z) = z - \gamma z^3 - \arcsin z.$$

It satisfies $g_\gamma(0) = g'_\gamma(0) = 0$ and $g_\gamma(1) = 1 - \gamma - \arcsin 1$. Moreover,

$$g''_\gamma(z) = -z \left(6\gamma + \frac{1}{(1-z^2)^{3/2}} \right). \quad (8)$$

The function $6\gamma + \frac{1}{(1-z^2)^{3/2}}$ increases from $6\gamma + 1$ to infinity, so if $6\gamma + 1 \geq 0$ the function g_γ is concave, and thus negative, while if $6\gamma + 1 < 0$, the function g_γ is convex and positive for small z and then becomes concave. As a consequence it remains positive on $(0, 1)$ if and only if $g_\gamma(1) \geq 0$. $\square$

Theorem 8 (Bounds for hyperbolic sine mean). *The inequalities $Q_\alpha < M_{\sinh} < Q_\beta$ hold if and only if $\alpha \leq 1 - \sinh 1 \approx -0.1752$ and $\beta \geq -\frac{1}{6} \approx -0.1667$.*

Proof We use again Method 2 and investigate:

$$g_\gamma(z) = z - \gamma z^3 - \sinh z.$$

It satisfies $g_\gamma(0) = g'_\gamma(0) = 0$ and $g_\gamma(1) = 1 - \gamma - \sinh 1$. Moreover,

$$g''_\gamma(z) = -z \left(6\gamma + \frac{\sinh z}{z} \right). \quad (9)$$

The function $\frac{\sinh z}{z}$ increases by Property 2 from 1 to infinity, so if $6\gamma + 1 \geq 0$ the function g_γ is concave, and thus negative, while if $6\gamma + 1 < 0$ the function g_γ is convex and positive for small z and then remains convex or becomes concave. As a consequence it remains positive on $(0, 1)$ if and only if $g_\gamma(1) \geq 0$. $\square$

Next we shall take care of several means that are greater than the arithmetic mean:

- Neuman–Sándor mean called also inverse hyperbolic sine mean $NS(x, y) = \frac{|x-y|}{2 \operatorname{arsinh} \frac{|x-y|}{x+y}}$ with Seiffert function $ns(z) = \operatorname{arsinh} z$,
- sine mean $M_{\sin}(x, y) = \frac{|x-y|}{2 \sin \frac{|x-y|}{x+y}}$,
- second Seiffert mean or arctangent mean $T(x, y) = M_{\arctan}(x, y) = \frac{|x-y|}{2 \arctan \frac{|x-y|}{x+y}}$,
- hyperbolic tangent mean $M_{\tanh}(x, y) = \frac{|x-y|}{2 \tanh \frac{|x-y|}{x+y}}$,
- centroidal mean $Ce(x, y) = \frac{2}{3} \frac{x^2 + xy + y^2}{x+y}$. Its Seiffert mean is $ce(z) = \frac{3z}{3+z^2}$,
- root-mean square or quadratic mean $RMS(x, y) = \sqrt{\frac{x^2 + y^2}{2}}$ with Seiffert function $rms(z) = \frac{z}{\sqrt{1+z^2}}$,
- contraharmonic mean $C(x, y) = \frac{x^2 + y^2}{x+y}$ where $c(z) = \frac{z}{1+z^2}$.

We have

$$A < NS < M_{\sin} < T < M_{\tanh} < Ce < RMS < C.$$

The first four inequalities come from [20], next from [13], the remaining two can be found in many sources.

In this part of our paper Method 1 proves to be very useful.

Theorem 9 (Bounds for Neuman–Sándor mean). *The inequalities $Q_\alpha < M_{\text{arsinh}} < Q_\beta$ hold if and only if $\alpha \leq 1 - \text{arsinh} 1 \approx 0.1186$ and $\beta \geq \frac{1}{6} \approx 0.1667$.*

Proof The function

$$\frac{(z - \text{arsinh} z)'}{(z^3)'} = \frac{1 - \frac{1}{\sqrt{1+z^2}}}{3z^2} = \frac{1}{3\sqrt{1+z^2}(1 + \sqrt{1+z^2})}$$

decreases, so by Lemma 3 $h(z) = \frac{z - \text{arsinh} z}{z^3}$ decreases, assuming values between $\lim_{z \rightarrow 0^+} h(z) = \frac{1}{6}$ and $h(1) = 1 - \text{arsinh} 1$, which completes the proof. $\square$

Theorem 10 (Bounds for the sine mean). *The inequalities $Q_\alpha < M_{\sin} < Q_\beta$ hold if and only if $\alpha \leq 1 - \sin 1 \approx 0.1585$ and $\beta \geq \frac{1}{6} \approx 0.1667$.*

Proof Applying Lemma 3 to the functions $f(z) = z - \sin z$ and $g(z) = z^3$ we see that

$$\frac{f'(z)}{g'(z)} = \frac{1 - \cos z}{3z^2} = \frac{\sin^2(z/2)}{6(z/2)^2}$$

decreases thus $f(z)/g(z)$ does so. Thus $h(z) = \frac{z - \sin z}{z^3}$ from Method 1 assumes values between $\lim_{z \rightarrow 0^+} h(z) = \frac{1}{6}$ and $h(1) = 1 - \sin 1$. $\square$

Theorem 11 (Bounds for the second Seiffert mean). *The inequalities $Q_\alpha < M_{\arctan} < Q_\beta$ hold if and only if $\alpha \leq 1 - \frac{\pi}{4} \approx 0.2146$ and $\beta \geq \frac{1}{3} \approx 0.3333$.*

Proof Applying Lemma 3 to the functions $f(z) = z - \arctan z$ and $g(z) = z^3$ we see that

$$\frac{f'(z)}{g'(z)} = \frac{1 - \frac{1}{1+z^2}}{3z^2} = \frac{1}{3} \frac{1}{1+z^2}$$

decreases and $f(z)/g(z)$ does so. Consequently, $h(z) = \frac{z - \arctan z}{z^3}$ assumes values between $\lim_{z \rightarrow 0^+} h(z) = \frac{1}{3}$ and $h(1) = 1 - \frac{\pi}{4}$. $\square$

Theorem 12 (Bounds for the hyperbolic tangent mean). *The inequalities $Q_\alpha < M_{\tanh} < Q_\beta$ hold if and only if $\alpha \leq 1 - \tanh 1 \approx 0.2384$ and $\beta \geq \frac{1}{3} \approx 0.3333$.*

Proof To show that $h(z) = \frac{z - \tanh z}{z^3}$ decreases we use Lemma 3. We have

$$\frac{(z - \tanh z)'}{(z^3)'} = \frac{1}{3} \left(\frac{\tanh z}{z} \right)^2.$$

The function $\tanh$ is concave and $\lim_{z \rightarrow 0^+} \tanh z = 0$, so by Property 2 the function $\frac{\tanh z}{z}$ decreases and this implies that the function h used in Method 1 decreases. We complete the proof by noting $\lim_{z \rightarrow 0^+} h(z) = \frac{1}{3}$. $\square$

Theorem 13 (Bounds for centroidal mean). *The inequalities $Q_\alpha < C_e < Q_\beta$ hold if and only if $\alpha \leq \frac{1}{4}$ and $\beta \geq \frac{1}{3}$.*

Proof The function

$$h(z) = \frac{z - \frac{3z}{3+z^2}}{z^3} = \frac{1}{3+z^2}$$

decreases, so it assumes values between $h(1) = \frac{1}{4}$ and $h(0) = \frac{1}{3}$. $\square$

Table 1 Absolute errors

Mean	α	$\sup M - Q_\alpha$		$\sup Q_\beta - M$	
		value	at $x =$	β	value at 0^+
Harmonic mean		n/a		-1	$\frac{1}{4} = 0.25$
Power mean of order $-1/2$		n/a		$-\frac{3}{4}$	$\frac{2}{7} \approx 0.2857$
Geometric mean		n/a		$-\frac{1}{2}$	$\frac{1}{3} \approx 0.3333$
Logarithmic mean		n/a		$-\frac{1}{3}$	$\frac{3}{8} = 0.375$
Power mean of order $1/2$	-1	0.1095	0.1642	$-\frac{1}{4}$	$\frac{3}{20} = 0.15$
Heronian mean	$-\frac{1}{2}$	0.0610	0.1327	$-\frac{1}{6}$	$\frac{2}{21} \approx 0.0952$
Tangent mean	$1 - \tan 1$	0.0277	0.2275	$-\frac{1}{3}$	$\frac{3}{8} - \frac{1}{2 \tan 1} \approx 0.0540$
First Seiffert mean	$1 - \frac{\pi}{2}$	0.0729	0.1330	$-\frac{1}{6}$	$\frac{3}{7} - \frac{1}{\pi} \approx 0.1103$
Hyperbolic sine mean	$1 - \sinh 1$	0.0011	0.2218	$-\frac{1}{6}$	$\frac{3}{7} - \frac{1}{2 \sinh 1} \approx 0.0031$
Neuman-Sándor mean	$1 - \operatorname{arsinh} 1$	0.0065	0.2126	$\frac{1}{6}$	$\frac{3}{5} - \frac{1}{2 \operatorname{arsinh} 1} \approx 0.0327$
Sine mean	$1 - \sin 1$	0.0014	0.1778	$\frac{1}{6}$	$\frac{3}{5} - \frac{1}{2 \sin 1} \approx 0.0058$
Second Seiffert mean	$1 - \arctan 1$	0.0170	0.2017	$\frac{1}{3}$	$\frac{3}{4} - \frac{1}{2 \arctan 1} \approx 0.1134$
Hyperbolic tangent mean	$1 - \tanh 1$	0.0153	0.1859	$\frac{1}{3}$	$\frac{3}{4} - \frac{1}{2 \tanh 1} \approx 0.0935$
Centroidal mean	$\frac{1}{4}$	0.0140	0.1802	$\frac{1}{3}$	$\frac{1}{12} \approx 0.8333$
Quadratic mean	$\frac{1}{2+\sqrt{2}}$	0.0313	0.1924	$\frac{1}{2}$	$\frac{\sqrt{2}-1}{\sqrt{2}} \approx 0.2929$

All the maxima in the rightmost column occur at 0^+

Theorem 14 (Bounds for quadratic mean). *The inequalities $Q_\alpha < \text{RMS} < Q_\beta$ hold if and only if $\alpha \leq \frac{1}{2+\sqrt{2}} \approx 0.2929$ and $\beta \geq \frac{1}{2}$.*

Proof Once more we use Method 1. The function

$$h(z) = \frac{z - \frac{z}{\sqrt{1+z^2}}}{z^3} = \frac{\sqrt{1+z^2} - 1}{z^2 \sqrt{1+z^2}} = \frac{1}{\sqrt{1+z^2}(1 + \sqrt{1+z^2})}$$

decreases, so it assumes values between $h(1) = \frac{1}{2+\sqrt{2}}$ and $h(0) = \frac{1}{2}$. $\square$

4 Error rates

Tables 1 on page 9 and Table 2 on page 10 represent absolute and relative errors of the bounds calculated in the preceding section.

5 Remarks about other one-parameter families of means

In this section we consider two more families of one-parameter means given by formulae similar to (1) and (1')

$$V_\alpha = A \frac{A^2}{(1-\alpha)A^2 + \alpha \text{RMS}^2} = A \frac{A}{(1-\alpha)A + \alpha C},$$

Table 2 Relative errors (in percents)

Mean	α	$\sup \frac{M-Q_\alpha}{M}$	at $x =$	$\sup \frac{Q_\beta-M}{M}$	value[%] at 0^+
		value[%]		β	
Power mean of order $1/2$	-1	23.99	0.0893	$-\frac{1}{4}$	60.00
Heronian mean	$-\frac{1}{2}$	12.78	0.0842	$-\frac{1}{6}$	28.57
Tangent mean	$1 - \tan 1$	5.50	0.1670	$-\frac{1}{3}$	16.81
First Seiffert mean	$1 - \frac{\pi}{2}$	15.37	0.0820	$-\frac{1}{6}$	34.64
Hyperbolic sine mean	$1 - \sinh 1$	0.20	0.1801	$-\frac{1}{6}$	0.73
Neuman-Sándor mean	$1 - \operatorname{arsinh} 1$	1.01	0.1899	$\frac{1}{6}$	5.76
Sine mean	$1 - \sin 1$	0.22	0.1627	$\frac{1}{6}$	0.98
Second Seiffert mean	$1 - \arctan 1$	2.51	0.1876	$\frac{1}{3}$	17.81
Hyperbolic tangent mean	$1 - \tanh 1$	2.24	0.1757	$\frac{1}{3}$	14.24
Centroidal mean	$\frac{1}{4}$	2.04	0.1716	$\frac{1}{3}$	12.50
Quadratic mean	$\frac{1}{2+\sqrt{2}}$	4.35	0.1858	$\frac{1}{2}$	41.42

All the maxima in the rightmost column occur at 0^+

where $\operatorname{RMS}(x, y) = \sqrt{\frac{x^2+y^2}{2}}$ is the quadratic (or root-mean square) mean and $\operatorname{C}(x, y) = \frac{x^2+y^2}{x+y}$ is the contraharmonic mean. Using (4) we see that the corresponding Seiffert functions are of the form $z + \alpha z^3$, so by Lemma 1 V_α are means if and only if $-\frac{1}{2} \leq \alpha \leq 4$ and $V_\alpha = Q_{-\alpha}$.

The second family is

$$W_\alpha = A \frac{A}{(1-\alpha)A + \alpha \operatorname{Ce}},$$

where $\operatorname{Ce}(x, y) = \frac{2}{3} \frac{x^2+xy+y^2}{x+y}$ is the centroidal mean. We use (4) to find out that their Seiffert functions are of the form $w_\alpha(z) = z + \frac{\alpha}{3} z^3$. Again by Lemma 1 we see that W_α are means if and only if $-\frac{3}{2} \leq \alpha \leq 12$ and $W_\alpha = Q_{\alpha/3}$.

6 Tools and lemmas

In this section, we place all the technical details needed to prove our main results.

Property 1 A function $f : (a, b) \rightarrow \mathbb{R}$ is convex if, and only if, for every $a < \theta < b$ its divided difference $\frac{f(x)-f(\theta)}{x-\theta}$ increases for $x \neq \theta$.

A simple consequence of Property 1 is

Property 2 If a function $f : (a, b) \rightarrow \mathbb{R}$ is convex and $\lim_{x \rightarrow a} f(x) = \Theta$, then the function $\frac{f(x)-\Theta}{x-a}$ increases.

The next lemma can be found in [2, Theorem 1.25].

Lemma 3 Suppose $f, g : (a, b) \rightarrow \mathbb{R}$ are differentiable with $g'(x) \neq 0$ and such that $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x) = 0$ or $\lim_{x \rightarrow b} f(x) = \lim_{x \rightarrow b} g(x) = 0$. Then

1. if $\frac{f'}{g'}$ is increasing on (a, b) , then $\frac{f}{g}$ is increasing on (a, b) ,
2. if $\frac{f'}{g'}$ is decreasing on (a, b) , then $\frac{f}{g}$ is decreasing on (a, b) .

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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