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ORIGINAL PAPER

Monotonicity properties and solvability of dominated best approximation problem in Orlicz spaces equipped with s-norms

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Abstract
In the paper, Wisła (J Math Anal Appl 483(2):123659, 2020, 10.1016/j.jmaa.2019.123659), it was proved that the classical Orlicz norm, Luxemburg norm and (introduced in 2009) p-Amemiya norm are, in fact, special cases of the s-norms defined by the formula $\|x\|_{\Phi,s} = \inf_{k>0} \frac{1}{k} s\left(\int_T \Phi(kx)d\mu\right)$, where s and Φ are an outer and Orlicz function respectively and x is a measurable real-valued function over a σ-finite measure space (T, Σ, μ). In this paper the strict monotonicity, lower and upper uniform monotonicity and uniform monotonicity of Orlicz spaces equipped with the s-norm are studied. Criteria for these properties are given. In particular, it is proved that all of these monotonicity properties (except strict monotonicity) are equivalent, provided the outer function s is strictly increasing or the measure μ is atomless. Finally, some applications of the obtained results to the best dominated approximation problems are presented.

Keywords Banach lattice · Uniform monotonicity · Strict monotonicity · Lower locally uniform monotonicity · Upper locally uniform monotonicity · Orlicz space, s-Norm

Mathematics Subject Classification 46B20 · 46E30

1 Introduction

Many geometric properties of Banach spaces are directly related to the property of its norm. One such important property is the property of the existence of an element that realizes the distance of a set from a point (the so-called solvability of best approximation problem). More

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precisely, for a given nonempty proper subset A of a Banach space X and $x \notin A$ the question is stated whether there exists $z \in A$ such that

$$\|x - z\| = \inf \{\|x - y\| : y \in A\}.$$

In this paper, we will give results on solvability (even more, on strong solvability) of the dominated best approximation problem, i.e., for the case when A is a closed sublattice of an Orlicz space X and z is an upper or lower boundary of A.

The dominated best approximation problem is strictly connected with the monotonicity properties of the norm. Thus in the first part of the paper we will investigate the strict monotonicity, uniform monotonicity, local lower uniform monotonicity and upper local uniform monotonicity of Orlicz spaces equipped with the so-called s-norms [36].

Orlicz spaces L_Φ, introduced by W. Orlicz in 1932 (see [33]) form a wide class of Banach spaces of measurable functions or sequences. Originally W. Orlicz defined the norm as follows

$$\|x\|_\Phi^o = \sup \left\{ \int_T |x(t)y(t)| d\mu : \int_T \Psi(y(t)) d\mu \leq 1 \right\},$$

where Ψ is the complementary function to Φ in the sense of Young defined by

$$\Psi(u) = \sup \{|u|v - \Phi(v) : v \geq 0\}, \quad u \in \mathcal{R}.$$

In 1955, Luxemburg [25] investigated topologically equivalent norm to the Orlicz one which was defined as follows

$$\|x\|_\Phi = \inf \left\{ \lambda > 0 : I_\Phi\left(\frac{x}{\lambda}\right) \leq 1 \right\},$$

where $I_\Phi(x) = \int_T \Phi(x(t)) d\mu$. In the fifties Ichiro Amemiya (see [30, p. 218]) considered the norm defined by the following formula

$$\|x\|_\Phi^A = \inf_{k>0} \frac{1}{k} \left(1 + I_\Phi(kx)\right).$$

Krasnoselskii and Rutickii [18], Nakano [30], Luxemburg and Zaanen [26] proved, under additional assumptions on the function Φ, that the Orlicz norm can be expressed exactly by the Amemiya formula, i.e. $\|\cdot\|_\Phi^o = \|\cdot\|_\Phi^A$. In the most general case of Orlicz function Φ, the similar result was obtained by Hudzik and Maligranda [12]. Moreover, it is not difficult to verify that the Luxemburg norm can also be expressed by an Amemiya-like formula (see [4,32]), namely

$$\|x\|_\Phi = \inf_{k>0} \frac{1}{k} \max\{1, I_\Phi(kx)\}. \tag{1}$$

In the paper, Hudzik and Maligranda [12] proposed to investigate another class of norms given by the so-called p-Amemiya formula

$$\|x\|_{\Phi,p} = \inf_{k>0} \frac{1}{k}(1 + I_\Phi^p(kx))^{1/p}, \tag{2}$$

where $1 \leq p \leq \infty$ (if $p = \infty$ then we use the formula (1)).

However it is easy to observe that both formulae (1) and (2) can be expressed by the norm (called s-norm) defined by

$$\|x\|_{\Phi,s} = \inf_{k>0} \frac{1}{k} s(I_{\Phi}(kx)), \tag{3}$$

where s is the so-called outer function (see [36]). Let us underline that this concept covers all the previously investigated cases of norms. Moreover, each outer function corresponds to a random nonnegative variable $\mathcal{X}$ (over a probability space $(\Omega, \mathcal{M}, \mathcal{P})$) with expected value $0 \le E(\mathcal{X}) \le 1$ (see the Preliminaries chapter below). So the results presented in this paper may find applications in probability theory or stochastic modelling.

In recent years several papers have been published devoted to the study of the problem of monotonicity of Orlicz space equipped with a certain norm (and often under additional assumptions about the function Φ or the measure μ) see, e.g., [2,6,9,11,19–21,24]. In this paper we consider the most general case of s-norms and Orlicz functions. Moreover, we will investigate the general case of measure space—in particular, there may be atoms among measurable sets.

In the section 3 it was proved that strict monotonicity of the function Φ on $(0, \infty)$ is necessary for $L_{\Phi,s}$ (the Orlicz space L_Φ equipped with the s-norm $\|\cdot\|_{\Phi,s}$) to be strictly monotone, for all possible choices of the outer function s (Lemma 3.7). The strict monotonicity of Φ is also sufficient for $L_{\Phi,s}$ to be strictly monotone as long the outer function s is strictly increasing on $(0, \infty)$ (Theorem 3.9). If the outer function s is constant on some nontrivial interval $(0, \varepsilon)$ and the Orlicz function Φ takes finite values only, then the Orlicz space $L_{\Phi,s}$ is strictly monotone if and only if Φ is strictly increasing on $(0, \infty)$, Φ satisfies the Δ_2-condition (Theorem 3.9).

The uniform monotonicity of $L_{\Phi,s}$ does not depend on the outer function s in the case of atomless measure μ. Namely, if the measure μ is atomless (and for any choice of s) or s is strictly increasing on $(0, \infty)$ (and for any choice of measure μ), the Orlicz space $L_{\Phi,s}$ is uniformly monotone if and only if Φ is strictly increasing on $(0, \infty)$ and satisfies the Δ_2-condition (Theorem 4.10).

Finally, using the results in [19] we proved that, if s is an outer function strictly increasing on $(0, \infty)$ or μ is an atomless measure then, for every closed sublattice A of $L_{\Phi,s}$, the dominated best approximation problem is strongly solvable for any (lower or upper) boundary of A as long as Φ is strictly increasing on $(0, \infty)$ and satisfies the Δ_2-condition (Corollary 5.5).

2 Preliminaries

For any map $\Phi : \mathcal{R} \to [0, \infty]$ define

$$a_\Phi = \sup\{u \ge 0 : \Phi(u) = 0\}, \quad b_\Phi = \sup\{u > 0 : \Phi(u) < \infty\}.$$

If $a_\Phi = 0$ then Φ is positive and strictly increasing on $(0, \infty)$. We will also write $\Phi > 0$ in that case. Analogously, we will write $\Phi < \infty$, provided $b_\Phi = \infty$. A map $\Phi : \mathcal{R} \to [0, \infty]$ is said to be an Orlicz function if $\Phi(0) = 0$, Φ is not identically equal to zero, it is even, convex on the interval $(-b_\Phi, b_\Phi)$ and left-continuous at b_Φ, i.e., $\lim_{u \to b_\Phi^-} \Phi(u) = \Phi(b_\Phi)$.

Throughout the paper we will assume that (T, Σ, μ) is a non-trivial measure space with a σ-finite and complete measure μ. By T_a we will denote the (countable) set of all μ-atoms. We will say that the atomless part of T is not empty, if T contains a measurable set of positive measure that does not contain any atoms. Moreover we will always assume that either the atomless part of T is not empty or T_a does not reduce to a finite set of atoms. Thus the σ-algebra Σ is non-trivial, i.e., it contains a nonempty and proper subset of T of positive and finite measure (indeed, otherwise T would be an atom or μ would not be σ-finite).

By $L^0 = L^0(\mu)$ we will denote the set of all μ-equivalence classes of Σ-measurable real functions defined on T equipped with the topology of convergence in measure on μ-finite sets. For a given Orlicz function Φ, on the space $L^0(\mu)$ we define a convex functional (called a pseudomodular [29]) by

$$I_\Phi(x) = \int_T \Phi(x(t))d\mu.$$

The Orlicz space L_Φ generated by an Orlicz function Φ is a linear lattice of measurable functions defined by the formula

$$L_\Phi = \left\{x \in L^0 : \exists\lambda > 0 \ I_\Phi(\lambda x) < \infty\right\}$$

with the μ-a.e. partial order, i.e., $x \leq y$ if and only if $y(t) - x(t) \geq 0$ for μ-a.e. $t \in T$. By the space E_Φ we will mean a linear subspace of L_Φ defined as follows

$$E_\Phi = \{x \in L_\Phi : \forall\lambda > 0 \ I_\Phi(\lambda x) < \infty\}.$$

In the case of purely atomic measure (i.e., if $T \backslash T_a = \emptyset$), the spaces L_Φ and E_Φ are usually denoted by ℓ_Φ and h_Φ respectively. We will use this notation only if it is more convenient or necessary. Let us note that the space E_Φ may degenerate into one element set $\{0\}$. For instance, this is what happens when the Φ values jump to infinity (i.e., $b_\Phi < \infty$).

We say that an Orlicz function Φ satisfies the condition Δ_2 if

(a) there exists a constant $K > 0$ such that $\Phi(2u) \leq K\Phi(u)$ for every $u \in \mathcal{R}$ (resp., for every $|u| \geq u_0$, where $\Phi(u_0) < \infty$) provided the measure μ is atomless and $\mu(T) = \infty$ (resp. $\mu(T) < \infty$),

(b) there are constants $a > 0$, $K > 0$ and a nonnegative sequence $(c_n) \in \ell^1$ such that

$$\Phi(2u)\mu(e_n) \leq K\Phi(u)\mu(e_n) + c_n$$

for every $u \geq 0$ with $\Phi(u)\mu(e_n) \leq a$ and every $n \in \mathcal{N}$, provided the measure μ is purely atomic and $\{e_n : n \in \mathcal{N}\}$ is the set of atoms of T (this condition is known as the δ_2-condition),

(c) both conditions (a) and (b) are satisfied in the case when T contains an atomless part and the set of atoms is not empty.

It is well known that $L_\Phi = E_\Phi$ if and only if $\Phi \in \Delta_2$ and $\Phi < \infty$ (see, e.g., [3,34] for the atomless measure case and [15] for the purely atomic case). Note that $\Phi \in \Delta_2$ implies that $\Phi < \infty$ provided the measure μ is atomless. But this implication is not true for purely atomic measures.

A function $s : [0, \infty) \to [1, \infty)$ is said to be an outer function, if it is convex and

$$\max\{1, u\} \leq s(u) \leq u + 1 \text{ for all } u \geq 0.$$

To simplify notations, we extend the domain and range of s to the interval $[0, \infty]$ by setting $s(\infty) = \infty$.

Let us note that each outer function corresponds to a random nonnegative variable $\mathcal{X}$ (over a probability space $(\Omega, \mathcal{M}, \mathcal{P})$) with the expected value $0 \leq E(\mathcal{X}) \leq 1$ in the sense that

$$s_\mathcal{X}(u) = 1 + F^{(2)}_\mathcal{X}(u)$$

for $u \geq 0$, where $F_{\mathcal{X}}^{(2)} : \mathcal{R} \to [0, \infty)$ denotes the the second performance function of $\mathcal{X}$, which is defined as the integral of the cumulative distribution function $F_{\mathcal{X}} : \mathcal{R} \to [0, 1]$, $F_{\mathcal{X}}(u) = \mathcal{P}(\mathcal{X} \leq u)$, over the half-line $(-\infty, u)$, i.e.,

$$F_{\mathcal{X}}^{(2)}(u) = \int_{-\infty}^{u} F_{\mathcal{X}}(t)dt, \quad u \in \mathcal{R}$$

(see the O-R diagram in [31]). Indeed, the second performance function $F_{\mathcal{X}}^{(2)}$ is nonnegative, finite, convex on $\mathcal{R}$ and admits an asymptote at infinity with the slope 1 and the y-intercept $-E(\mathcal{X})$ [31]. Moreover, if the random variable $\mathcal{X}$ is nonnegative and $E(\mathcal{X}) \in [0, 1]$ then $F_{\mathcal{X}}^{(2)}(u) = 0$ for all $u \leq 0$, so the function $s_{\mathcal{X}}(u) = 1 + F_{\mathcal{X}}^{(2)}(u)$ restricted to $[0, \infty)$ is an outer function. In particular, if $\mathcal{X} \equiv 0$ then $s_{\mathcal{X}}(u) = 1 + u$ for $u \geq 0$. Analogously, if $\mathcal{X} \equiv 1$ then $s_{\mathcal{X}}(u) = \max\{1, u\}$ for $u \geq 0$.

If s is an outer function and Φ is an Orlicz function then the functional

$$\|x\|_{\Phi,s} = \inf_{k>0} \frac{1}{k} s(I_{\Phi}(kx))$$

is a norm on the Orlicz space L_{Φ} [36]. In the following by $L_{\Phi,s}$ (resp. $E_{\Phi,s}$) we will denote the Orlicz space L_{Φ} (resp. E_{Φ}) equipped with the s-norm $\|\cdot\|_{\Phi,s}$.

Put $s_L(u) = \max\{1, u\}$ and $s_o(u) = 1 + u$. Then the norms $\|\cdot\|_{\Phi,s_L}$ and $\|\cdot\|_{\Phi,s_o}$ are the Luxemburg norm $\|\cdot\|_{\Phi}$ and the Orlicz norm $\|\cdot\|_{\Phi}^{o}$ respectively [25,32]. Since, for any outer function s, $s_L(u) \leq s(u) \leq s_o(u)$ for all $u \geq 0$, we have the following inequality

$$\|x\|_{\Phi} = \|x\|_{\Phi,s_L} \leq \|x\|_{\Phi,s} \leq \|x\|_{\Phi,s_o} = \|x\|_{\Phi}^{o} \leq 2\|x\|_{\Phi}$$

for all $x \in L_{\Phi}$, so all s-norms are (topologically) equivalent to each other. In general, for a fixed Orlicz function Φ two different outer functions $s_1 \neq s_2$ create two different norms $\|\cdot\|_{\Phi,s_1} \not\equiv \|\cdot\|_{\Phi,s_2}$. But in the case of power functions $\Phi(u) = c|u|^p$, $1 \leq p < \infty$, $c > 0$, all Orlicz spaces $L_{\Phi,s}$, no matter how the outer function s is chosen, coincide with the Lebesgue spaces L^p and $\|\cdot\|_{\Phi,s} \equiv d\|\cdot\|_p$ up to a some constant $d > 0$ depending on p and s. Analogously, if $\Phi(u) = 0$ for $|u| \leq 1$ and $\Phi(u) = \infty$ for $|u| > 1$ then, for all outer functions s, the Orlicz space $L_{\Phi,s}$ coincides with the Lebesgue space L^{∞} and $\|\cdot\|_{\Phi,s} = \|\cdot\|_{\infty}$ (see [36]).

Note that, for any outer function s, the s-norm convergence to 0 of a sequence (x_n) of functions of L_{Φ} implies its modular convergence to 0. Moreover, these two types of convergence are equivalent, i.e.,

$$\|x_n\|_{\Phi,s} \to 0 \iff \exists\, \lambda > 0\ I_{\Phi}(\lambda x_n) \to 0$$

as $n \to \infty$ if and only if $\Phi \in \Delta_2$ and, in the case of purely atomic measure, $\Phi > 0$ (see, e.g., [3,15]).

Recall that an element x of a Banach lattice $(X, \|\cdot\|)$ is said to be order continuous if $\|x_n\| \to 0$ for any sequence (x_n) in X such that $0 \leq x_n \leq |x|$ and $x_n \to 0$ μ-a.e. By X_a we will denote the subspace of all order continuous elements in X. If $X = X_a$ then the Banach lattice X is called order continuous (OC).

If μ is purely atomic, then $(L_{\Phi})_a = (\ell_{\Phi})_a \neq \{0\}$ for any Orlicz function Φ (consider finite sequences of numbers). In general $E_{\Phi} \subset (L_{\Phi})_a \subset L_{\Phi}$.

It is well known that if $\Phi < \infty$ then $(L_{\Phi})_a = E_{\Phi}$ [37]. Moreover, $L_{\Phi} = (L_{\Phi})_a$ if and only if $\Phi \in \Delta_2$ (see [3,16,19]).

Another nice characterization of Δ_2-condition is as follows: the modular unit sphere and the Luxemburg-norm unit sphere coincide, i.e.,

$$\|x\|_\Phi = 1 \Leftrightarrow I_\Phi(x) = 1 \tag{4}$$

for all $x \in L_\Phi$ if and only if $\Phi \in \Delta_2$ and, if the measure μ is purely atomic, on each atom $e \in T_a$ the function $u \to \Phi(u)\mu(e)$ achieves the value 1, i.e., $\Phi(b_\Phi)\mu(e) \geq 1$ for all $e \in T_a$ (see [3,6,16]).

More details about Orlicz spaces equipped with the Luxemburg or the Orlicz norm, can be found in [3,18,22,23,27–29,34]. In the paper [12] Hudzik and Maligranda proposed to investigate the family of so-called p-Amemiya norms generated by the outer functions defined by $s_p(u) = (1 + u^p)^{1/p}$. Then several geometric properties of p-Amemiya norms were investigated in a number of papers, e.g., [4,6,7,13,14,35]. Basic properties of Orlicz spaces equipped with the s-norm (and, among others, the theorem on duality) are presented in [36].

3 Strict monotonicity properties of Orlicz spaces $L_{\Phi,s}$

Let X be a Banach lattice with a norm $\|\cdot\|$. By X_+ we will denote the positive cone of X, i.e. $X_+ = \{x \in X, x \geq 0\}$ and by $S(A)$ the interception of the unit sphere of X with the set $A \subset X$, i.e., $S(A) = \{x \in A : \|x\| = 1\}$. A Banach lattice X is said to be strictly monotone (SM for short) if $\|x - y\| < \|x\|$ for all $x \geq y \geq 0$ and $y \neq 0$.

In order to calculate the value of s-norm it is important to know whether the infimum in the formula (3) is achieved at some $k > 0$. Denote

$$K(x) = \left\{0 < k < \infty : \|x\|_{\Phi,s} = \frac{1}{k} s(I_\Phi(kx))\right\}$$

and let $k^*(x) = \inf\{k > 0 : k \in K(x)\}$ and $k^{**}(x) = \sup\{k > 0 : k \in K(x)\}$ with the convention that $k^*(x) = k^{**}(x) = \infty$ as long as $K(x) = \emptyset$. We shall say that the s-norm is k^*-finite (resp. k^{**}-finite) if $k^*(x) < \infty$ (resp. $k^{**}(x) < \infty$) for every $x \in L_\Phi \setminus \{0\}$. If $k^*(x) = k^{**}(x) < \infty$ for all $L_\Phi \setminus \{0\}$ then we say that the s-norm is k-unique. In [36], it was proved, that $K(x) = [k^*(x), k^{**}(x)] \cap (0, \infty)$. Moreover, the criteria for s-norm to be k^*-finite and k^{**}-finite were also established in that paper. We will start with a characterization of the set of those function x for which $k^{**}(x) = \infty$.

Lemma 3.1 *For any outer function s and any Orlicz function Φ, if the space $L_{\Phi,s}$ is not k^{**}-finite then Φ takes only finite values. Moreover*

$$\left\{x \in L_\Phi \setminus \{0\} : k^{**}(x) = \infty\right\} \subset E_\Phi.$$

In consequence, the s-norm $\|\cdot\|_{\Phi,s}$ may be not k^{**}-finite only if the set $E_\Phi \setminus \{0\}$ is not empty.

Proof Let $x \in L_\Phi \setminus \{0\}$ be such that $k^{**}(x) = \infty$ and suppose that $b_\Phi < \infty$. Then $\|x\|_\infty < \infty$, whence $k^{**}(x) \leq b_\Phi \|x\|_\infty^{-1} < \infty$, a contradiction. Thus $\Phi < \infty$. In order to prove the inclusion of sets, take any $x \in L_\Phi \setminus E_\Phi$. Then there exists $k_0 > 0$ such that $I_\Phi(k_0 x) = \infty$. Then

$$\|x\|_{\Phi,s} = \inf_{k>0} \frac{1}{k} s(I_\Phi(kx)) = \inf_{0<k<k_0} \frac{1}{k} s(I_\Phi(kx)),$$

i.e., $k^{**}(x) \leq k_0 < \infty$. □

To make this paper clearer, we present the following lemma here, although some of its statements can also be deduced form the results given in [36].

Lemma 3.2 *Let s be an outer function and Φ an Orlicz function. If the space $L_{\Phi,s}$ is not k^{**}-finite then*

(i) *the limit $c = \lim_{u\to\infty} \frac{\Phi(u)}{u}$ is positive and finite,*
(ii) *Φ admits an oblique asymptote at infinity if and only if $\Psi(c) < \infty$, where Ψ is the Orlicz function complementary to Φ in the sense of Young, in fact, the slope and the y-intercept of the asymptote are equal to $c > 0$ and $-\Psi(c)$ respectively,*
(iii) *the Lebesgue space L^1 is continuously imbedded into the Orlicz space $L_{\Phi,s}$ and $\|y\|_{\Phi,s} \le c\,\|y\|_1$ for every $y \in L^1$,*
(iv) *$\|x\|_{\Phi,s} = c\,\|x\|_1$ for every $x \in L_\Phi\setminus\{0\}$, with $k^{**}(x) = \infty$.*

Proof Let $x \in L_\Phi\setminus\{0\}$ be such that $k^{**}(x) = \infty$. Then, by Lemma 3.1, Φ has to take finite values only and $x \in E_\Phi\setminus\{0\}$. Moreover,

$$\lim_{k\to\infty} \frac{1}{k} I_\Phi(kx) \le \lim_{k\to\infty} \frac{1}{k} s(I_\Phi\,(kx)) = \inf_{k>0} \frac{1}{k} s(I_\Phi\,(kx)) = \|x\|_{\Phi,s} < \infty.$$

(i). Suppose that $\lim_{u\to\infty} \frac{\Phi(u)}{u} = \infty$. Since Φ is convex and $\Phi(0) = 0$, the function $u \to \Phi(u)/u$ is nondecreasing on the half-line $(0,\infty)$. By the Beppo Levy theorem,

$$\lim_{k\to\infty} \frac{1}{k} I_\Phi(kx) = \lim_{k\to\infty} \int_{\mathrm{supp}(x)} \frac{\Phi(kx(t))}{|kx(t)|}\,|x(t)|\,d\mu = \int_{\mathrm{supp}(x)} \lim_{k\to\infty} \frac{\Phi(kx(t))}{|kx(t)|}\,|x(t)|\,d\mu = \infty,$$

a contradiction. Thus $c = \lim_{u\to\infty} \frac{\Phi(u)}{u} < \infty$. Evidently, since $\Phi \not\equiv 0$, $c > 0$ as well.

(ii). If Φ admits an oblique asymptote then, by (i), its slope is equal to $c = \lim_{u\to\infty} \Phi(u)/u$ and $0 < c < \infty$. Since Φ is convex, the slope of each secant of Φ is less or equal to c. Thus $\frac{\Phi(u_2)-\Phi(u_1)}{u_2-u_1} \le c$ for all $0 \le u_1 < u_2$, what implies that the function $u \to cu - \Phi(u)$ is nondecreasing on $(0,\infty)$. Therefore

$$\Psi(c) = \sup_{u\ge 0}(cu - \Phi(u)) = \lim_{u\to\infty} (cu - \Phi(u)) = -\lim_{u\to\infty} (\Phi(u) - cu).$$

Thus Φ admits the oblique asymptote with the slope c if and only if $\Psi(c) < \infty$ and its y-intercept is equal to $-\Psi(c)$.

(iii). By virtue of (ii), for any measurable function $y \in L^0\setminus\{0\}$ we have

$$\|y\|_{\Phi,s} \le \|y\|_\Phi^o = \inf_{k>0} \frac{1}{k}(1 + I_\Phi(k(y)) \le \lim_{k\to\infty} \frac{1}{k}\left(1 + \int_{\{t:y(t)\ne 0\}} \Phi(ky(t))d\mu\right)$$
$$= \lim_{k\to\infty} \int_{\{t:y(t)\ne 0\}} \frac{\Phi(ky(t))}{k\,|y(t)|}\,|y(t)|\,d\mu = c\,\|y\|_1\,.$$

Thus the Lebesgue space $(L^1, \|\cdot\|_1)$ is continuously imbedded into the Orlicz space $(L_\Phi, \|\cdot\|_{\Phi,s})$.

(iv). By Lemma 3.1, $x \in L_\Phi\setminus\{0\}$ and $k^{**}(x) = \infty$ imply that $\Phi < \infty$ and $x \in E_\Phi\setminus\{0\}$. Moreover $I_\Phi(kx) > 0$ for all $k > 0$ large enough and $I_\Phi(kx) \to \infty$ as $k \to \infty$, whence

$$\|x\|_{\Phi,s} = \lim_{k\to\infty} \frac{1}{k} s(I_\Phi(kx)) = \lim_{k\to\infty} \frac{1}{k} \frac{s(I_\Phi(kx))}{I_\Phi(kx)} I_\Phi(kx)$$
$$= \lim_{k\to\infty} \frac{s(I_\Phi(kx))}{I_\Phi(kx)} \int_{\mathrm{supp}(x)} \frac{\Phi(kx(t))}{|kx(t)|}\,|x(t)|\,d\mu = c\,\|x\|_1\,.$$

□

Let us underline that the existence of finite limit of the quotient $\Phi(u)/u$ at infinity is not sufficient for Φ to admit an oblique asymptote. For example, let $\Phi(u) = |u| - \ln(1 + |u|)$. Evidently, $\lim_{u\to\infty} \Phi(u)/u = \lim_{u\to\infty}(1 - \ln(1 + u)/u) = 1$ and $\lim_{u\to\infty}(\Phi(u) - u) = \lim_{u\to\infty}(-\ln(1 + u)) = -\infty$, whence Φ does not attend an asymptote at infinity.

Observe that $\Psi(1) = \infty$ in this case as well, where Ψ is the Orlicz function complementary to Φ in the sense of Young. Indeed, the right-hand derivative p_+ of Φ is equal to $p_+(u) = u/(1 + u)$ for all $u \geq 0$. Hence the inverse function $q_+ = (p_+)^{-1}$, which is the right-hand derivative of Ψ, is given by the formula $q_+(v) = v/(1 - v)$ for all $0 \leq v < 1$. Therefore, for every $|v| < 1$,

$$\Psi(v) = -\int_0^{|v|} \frac{t}{1-t} dt = -\ln(1 - |v|) - |v| .$$

Evidently $\Psi(1) = \lim_{v\to 1^-} \Psi(v) = +\infty$ (in fact, $\Psi(v) = +\infty$ for all $v \geq 1$).

In order to illustrate the relationship between k^{**}-finiteness of the s-norm and the space E_Φ consider the following example.

Example 3.3 Let μ be an atomless measure with $\mu(T) = \infty$ and let $\Phi(u) = \max\{0, |u| - 1\}$ for all $u \in \mathcal{R}$. Then $L_\Phi = L^1 + L^\infty$ and $L^1 \subsetneq E_\Phi$. If, moreover, the space L_Φ is equipped with the Orlicz norm, i.e., $s(u) = 1 + u$, then $\{0\} \subsetneq E_\Phi \backslash K^{**} \subsetneq L^1 \subsetneq E_\Phi$ and $E_\Phi \cap K^{**} \neq \{0\}$, where $K^{**} = \left\{x \in L_{\Phi,s} \backslash \{0\} : k^{**}(x) < \infty\right\}$. Further, for any measurable subset $A \in \Sigma$,

$$\|\chi_A\|_{\Phi,s} = \min\{\mu(A), 1\} .$$

(Note: the set K^{**} depends on Φ and s, but the sets L_Φ, E_Φ depend on Φ only.)

Proof Let $x = y + z$, where $y \in L^1$, $z \in L^\infty$. If $\|z\|_\infty = 0$ then $I_\Phi(x) = I_\Phi(y) \leq \|y\|_1 < \infty$, so $x \in L_\Phi$. If $\|z\|_\infty > 0$ then

$$I_\Phi\left(\frac{y+z}{2\|z\|_\infty}\right) \leq \frac{1}{2} I_\Phi\left(\frac{y}{\|z\|_\infty}\right) + \frac{1}{2} I_\Phi\left(\frac{z}{\|z\|_\infty}\right) \leq \frac{\|y\|_1}{2\|z\|_\infty} + 0 < \infty,$$

i.e., $x \in L_\Phi$.

Conversely, let $x \in L^0$ and $k > 0$ be such that $I_\Phi(kx) < \infty$. Define

$$A_k = \{t \in T : |kx(t)| \leq 1\}, \quad y_k = (x - \frac{sgn\, x}{k})\chi_{T\backslash A_k}, \quad z_k = x\chi_A + \frac{sgn\, x}{k}\chi_{T\backslash A_k}.$$

Evidently $x = y_k + z_k$ and $\|z_k\|_\infty \leq \frac{1}{k}$, so $z_k \in L^\infty$. Moreover, $|y_k| = \left|sgn\, x(|x| - \frac{1}{k})\right| \chi_{T\backslash A_k} = (|x| - \frac{1}{k})\chi_{T\backslash A_k}$, so

$$\|y_k\|_1 = \frac{1}{k}\int_T |ky_k(t)|\, d\mu = \frac{1}{k}\int_{T\backslash A_k} (|kx(t)| - 1) d\mu = \frac{1}{k} I_\Phi(kx) < \infty,$$

whence $y_k \in L^1$. We have proved that $L_\Phi = L^1 + L^\infty$.

If $x \in L^1$ then, since $\Phi(u) \leq u$ for all $u \geq 0$, we have $I_\Phi(kx) \leq \|kx\|_1 < \infty$ for every $k > 0$, so $L^1 \subset E_\Phi$. We will show that $L^1 \subsetneq E_\Phi$. Let (a_n) be any sequence of positive numbers converging to 0. Without loss of generality we can assume that (a_n) decreases to 0. Take a sequence (A_n) of pairwise disjoint measurable sets such that $\mu(A_n) = \frac{1}{a_n}$ for all $n \in \mathcal{N}$ and define $x = \sum_{n=1}^\infty a_n\chi_{A_n}$. Evidently $x \notin L^1$. For any $k > 0$ put $n_k = \min\{n \in \mathcal{N} : ka_n \leq 1\}$. If $n_k = 1$ then $ka_n \leq 1$ for all $n \in \mathcal{N}$, whence $I_\Phi(kx) = 0 < \infty$. So, assume that $n_k > 1$. Then

$$I_\Phi(kx) = \sum_{n=1}^{\infty} \Phi(ka_n)\mu(A_n) = \sum_{n=1}^{n_k-1} (ka_n - 1)\mu(A_n) < \infty.$$

Therefore $x \in E_\Phi$, whence $L^1 \subsetneq E_\Phi$.

Let A be any measurable subset of T (with finite or infinite measure). Take $k = 1$. Since $\Phi(1) = 0$, for any outer function s we have

$$1 = \max\{1, \Phi(1)\mu(A)\} \le s(I_\Phi(\chi_A)) = 1 + \Phi(1)\mu(A) = 1, \tag{5}$$

whence $\|\chi_A\|_{\Phi,s} \le 1$. Further, for every $0 < k < 1$,

$$\frac{1}{k} s(I_\Phi(k\chi_A)) \ge \frac{1}{k} > 1 \ge \|\chi_A\|_{\Phi,s} \tag{6}$$

so the norm $\|\chi_A\|_{\Phi,s}$ cannot be achieved at any $0 < k < 1$.

From now on we will assume that $s(u) = 1 + u$. Finally, for every $k > 1$ put

$$C(k) = \frac{1}{k} s(I_\Phi(k\chi_A)) = \frac{1}{k}(1 + \Phi(k)\mu(A)) = \frac{1}{k} + \left(1 - \frac{1}{k}\right)\mu(A).$$

If $\mu(A) = \infty$ then $C(k) \equiv \infty$, if $\mu(A) = 1$ then $C(k) \equiv 1$ and if $1 \ne \mu(A) < \infty$ then $C(k) > \min\{\mu(A), 1\}$ for all $k > 1$ and $\lim_{k\to\infty} C(k) = \mu(A)$. Therefore $\|\chi_A\|_{\Phi,s} = \min\{\mu(A), 1\}$ and

$$k^*(\chi_A) = \begin{cases} 1 & when \;\; \mu(A) \ge 1, \\ \infty & otherwise. \end{cases}$$

$$k^{**}(\chi_A) = \begin{cases} 1 & when \;\; \mu(A) > 1, \\ \infty & otherwise. \end{cases}$$

Thus

$$K(\chi_A) = \begin{cases} \{1\} & when \;\; \mu(A) > 1, \\ [1, \infty) & when \;\; \mu(A) = 1, \\ \emptyset & otherwise. \end{cases}$$

Evidently, $\chi_A \in E_\Phi$ for all measurable sets A with $\mu(A) < \infty$. Thus $E_\Phi \cap K^{**} \ne \{0\}$ (take χ_A with $1 < \mu(A) < \infty$) and $\{0\} \subsetneq E_\Phi \setminus K^{**}$ (take χ_A with $\mu(A) \le 1$).

The inclusion $E_\Phi \setminus K^{**} \subset L^1$ follows directly from Lemma 3.2. Moreover, $\chi_A \in L^1 \cap K^{**}$ for all $A \in \Sigma$ with $1 < \mu(A) < \infty$, so $E_\Phi \setminus K^{**} \subsetneq L^1$. □

Example 3.4 If the space $L^1 + L^\infty$ is equipped with the Luxemburg norm, i.e., in the Example 3.3 we consider the outer function $s(u) = \max\{1, u\}$, then the Orlicz space $L_{\Phi,s}$ is k^{**}-finite and, for all measurable sets $A \in \Sigma$,

$$\|\chi_A\|_{\Phi,s} = \begin{cases} 1 & when \;\; \mu(A) = \infty, \\ \frac{\mu(A)}{1+\mu(A)} & otherwise. \end{cases}$$

Proof The k^{**}-finitness follows from the definition of the outer function s (see [36]). Let $A \in \Sigma$. For any $0 < \lambda < \infty$ define

$$C(\lambda) = \lambda \max\left\{1, \Phi(\lambda^{-1})\mu(A)\right\} = \max\{\lambda, (1-\lambda)\mu(A)\}.$$

By (5) and (6) we get that $C(1) = 1$ and $C(\lambda) \ge \lambda > 1$ for all $\lambda > 1$. If $\mu(A) = \infty$ then $C(\lambda) = \infty$ for all $0 < \lambda < 1$, whence $\|\chi_A\|_{\Phi,s} = \inf_{\lambda>0} C(\lambda) = C(1) = 1$ and $k^*(\chi_A) = k^{**}(\chi_A) = 1$.

If $\mu(A) < \infty$ then the infimum of values of the function $C(\lambda)$ on the interval $(0, 1)$ is achived at the intersection point of the graphs of the identity function $\lambda \to \lambda$ and the function $\lambda \to (1-\lambda)\mu(A)$, i.e., at the point $\lambda = \frac{\mu(A)}{1+\mu(A)} \in (0, 1)$. Thus

$$\|\chi_A\|_{\Phi,s} = \inf_{\lambda>0} C(\lambda) = C\left(\frac{\mu(A)}{1+\mu(A)}\right) = \frac{\mu(A)}{1+\mu(A)}$$

and $k^*(\chi_A) = k^{**}(\chi_A) = \frac{\mu(A)+1}{\mu(A)}$. □

As it was pointed out in the previous section, the Δ_2-condition plays an importatnt role in Orlicz spaces. In the sequel, we will need the following lemma.

Lemma 3.5 *Let s be an outer function. If $\Phi \notin \Delta_2$ then, for every $0 < \varepsilon < 1$, we can find $x \in L_\Phi \setminus \{0\}$ such that $I_\Phi(x) < \varepsilon < \|x\|_{\Phi,s} = 1$.*

Proof Let $0 < \varepsilon < 1$. If $a_\Phi = b_\Phi$ then, putting $x = a_\Phi \chi_T$, we have $I_\Phi(x) = 0 < \varepsilon < 1 = \|x\|_{\Phi,s}$. Hence, in the following, we will assume that $d = b_\Phi - a_\Phi > 0$.

In the case of Luxemburg norm it is well known that if $\Phi \notin \Delta_2$ then there exists $x \in L_\Phi \setminus \{0\}$ such that $I_\Phi(x) < \|x\|_\Phi = 1$ (see [3,16]). This implies that $I_\Phi(kx) = \infty$ for all $k > 1$ (otherwise, applying the Lebesgue dominated convergence theorem we would get that $\|x\|_\Phi < 1$).

Put $A_0 = \{t \in T : |x(t)| \le a_\Phi\}$. If $I_\Phi\left(kx\chi_{A_0}\right) = \infty$ for every $k > 1$, then

$$I_\Phi\left(x\chi_{A_0}\right) = 0 < \varepsilon < 1 = \inf_{0<k\le 1} \frac{1}{k} s(I_\Phi\left(kx\chi_{A_0}\right)) = \left\|x\chi_{A_0}\right\|_{\Phi,s}$$

and the thesis is proved. Thus we can assume that $I_\Phi\left(k_0 x\chi_{A_0}\right) < \infty$ for some $k_0 > 1$. For each $n \ge 1$ define

$$a_n = \begin{cases} a_\Phi + \frac{1}{n}, & if\ b_\Phi = \infty, \\ a_\Phi + \frac{d}{2n}, & otherwise, \end{cases} \qquad b_n = \begin{cases} a_\Phi + n, & if\ b_\Phi = \infty, \\ b_\Phi - \frac{d}{2n}, & otherwise, \end{cases}$$

and

$$A_n = \{t \in T_n : a_{n+1} < |x(t)| \le a_n\} \cup \{t \in T_n : b_n \le |x(t)| < b_{n+1}\}.$$

Evidently, the sets $A_0, A_1, \ldots$ are pairwise disjoint, $\{t \in T : |x(t)| > a_\Phi\} \subset \bigcup_{n=0}^{\infty} A_n$ and

$$I_\Phi(x) = \sum_{n=0}^{\infty} I_\Phi\left(x\chi_{A_n}\right) < \infty.$$

Thus we can find $m > 1$ such that $I_\Phi(y) < \varepsilon$, where $y = \sum_{n=m}^{\infty} x\chi_{A_n}$.

Suppose that $I_\Phi(ky) < \infty$ for some $k > 1$. Without loss of generality we can assume that $1 < k < k_0$ and, if $b_\Phi < \infty$, that $k < \frac{b_\Phi}{b_m}$. For every $n \ge 1$ we have

$$0 < \Phi(a_{n+1})\mu(A_n) \le I_\Phi\left(x\chi_{A_n}\right) \le I_\Phi(x) < \infty,$$

whence $\mu(A_n) < \infty$. Thus

$$I_\Phi(kx) = \sum_{n=0}^{\infty} I_\Phi\left(kx\chi_{A_n}\right) \le I_\Phi\left(k_0 x\chi_{A_0}\right) + \sum_{n=1}^{m-1} \Phi(kb_{n+1})\mu(A_n) + I_\Phi(ky) < \infty,$$

and this contradicts to the assumption that $I_\Phi(kx) = \infty$ for all $k > 1$. Thus

$$\|y\|_{\Phi,s} = \inf_{0<k\leq 1} \frac{1}{k} s(I_\Phi(ky)) \geq \inf_{0<k\leq 1} \frac{1}{k} = 1,$$

whence, putting $z = y \cdot \|y\|_{\Phi,s}^{-1}$,

$$I_\Phi(z) \leq I_\Phi(y) < \varepsilon < 1 = \|z\|_{\Phi,s}$$

and the proof is completed. □

As the immediate consequence of Lemma 3.5 we get the following corollary.

Corollary 3.6 *If $\Phi \notin \Delta_2$ then, for any outer function s, there exists a sequence (x_n) of functions in the unit sphere $S(L_{\Phi,s})$ such that the modular $I_\Phi(x_n)$ converges to 0, i.e., $\|x_n\|_{\Phi,s} = 1$ for all $n \in \mathcal{N}$ and $I_\Phi(x_n) \to 0$ as $n \to \infty$.*

Now we turn back to the monotonicity properties of Orlicz spaces. We will start with the lemma that will allow us to restrict the number of cases of outer and Orlicz functions that we have to dealt with.

Lemma 3.7 *Let s and Φ be an outer and Orlicz function respectively and denote $a_s = \sup\{u \geq 0 : s(u) = 1\}$. The space $L_{\Phi,s}$ is not strictly monotone if one of the following conditions is satisfied*

(i) $a_\Phi > 0$,
(ii) $a_s > 0$ *and* $\Phi \notin \Delta_2$,
(iii) $a_s > 0$, *the set of atoms T_a of μ is infinite and* $\inf_{e\in T_a} \Phi(b_\Phi)\mu(e) < a_s$.

If $a_\Phi > 0$ and $E_\Phi \neq \{0\}$ then the space $E_{\Phi,s}$ is not strictly monotone.

Proof If T contains a nonempty atomless subset T_0 then take a set $A \subset T_0$ such that $\mu(A) > 0$ and $\mu(T_0 \backslash A) > 0$. In the other case, the set of atoms T_a is infinite, so take $A \subset T_a$ such that the set A is infinite and $T_a \backslash A \neq \emptyset$. Moreover, let $B \subset T_a \backslash A$ be such that $0 < \mu(B) < \infty$.

(i) Assume that $a_\Phi > 0$. Put $z = \chi_B$. Then $z \in L_\Phi \backslash \{0\}$. Further, let $\varepsilon > 0$ be a fixed arbitrary positive number and take $k_z > 0$ such that $\frac{1}{k_z} s(I_\Phi(k_z z)) \leq \|z\|_{\Phi,s} + \varepsilon$. Define $y = a_\Phi k_z^{-1} \chi_A$ and $x = y + z$. Evidently $x, y \in L_\Phi$, $0 \leq y \leq x$ and $y \neq 0$. Moreover,

$$\|x - y\|_{\Phi,s} = \|z\|_{\Phi,s} \geq \frac{1}{k_z} s(I_\Phi(k_z z)) - \varepsilon = \frac{1}{k_z} s(I_\Phi(k_z(z + a_\Phi k_z^{-1}\chi_A))) - \varepsilon \geq \|x\|_{\Phi,s} - \varepsilon,$$

whence, by arbitrariness of $\varepsilon > 0$, $\|x - y\|_{\Phi,s} \geq \|x\|_{\Phi,s}$. Since $y \neq 0$, the space $L_{\Phi,s}$ is not strictly monotone.

(ii) Since $\Phi \notin \Delta_2$, by Lemma 3.5 (with the measure space restricted to $(T \cap A, \Sigma \cap A, \mu|_A)$) we can find $z \in L_\Phi \backslash \{0\}$ such that supp$z \subset A$ and $I_\Phi(z) < a_s \leq \|z\|_{\Phi,s} = 1$. This implies that $I_\Phi(kz) = \infty$ for all $k > 1$.

Take $\varepsilon > 0$ such that $\Phi(\varepsilon)\mu(B) < a_s - I_\Phi(z)$. Put $y = \varepsilon\chi_B$ and $x = z + y$. Then, evidently, $0 \leq y \leq x$ and $y \neq 0$. Since $s(u) = 1$ for all $0 \leq u \leq a_s$,

$$I_\Phi(kx) \leq I_\Phi(x) \leq I_\Phi(z) + \Phi(\varepsilon)\mu(B) < a_s$$

for all $0 < k \leq 1$. Further, $I_\Phi(kx) \geq I_\Phi(kz) = \infty$ for all $k > 1$, so

$$\|x - y\|_{\Phi,s} = \|z\|_{\Phi,s} = 1 = \inf_{0<k\leq 1} \frac{1}{k} s(I_\Phi(kx)) = \|x\|_{\Phi,s}.$$

Since $y \neq 0$, the space $L_{\Phi,s}$ is not strictly monotone.

(iii) If the set of atoms T_a is infinite and $\Phi(b_\Phi)\mu(e) < a_s$ for an atom $e \in T_a$ then, putting $z = b_\Phi \chi_{\{e\}}$, we have $I_\Phi(z) < a_s \leq 1$ and $\|z\|_{\Phi,s} = 1$. Without loss of generality we can assume that $e \in A$. Thus, we can repeat the part (ii) of the proof to conclude that $L_{\Phi,s}$ is not strictly monotone.

Assume that $E_\Phi \neq \{0\}$. This implies that Φ takes only finite values, whence the functions $z = \chi_B$ and $y = a_\Phi k_z^{-1} \chi_A$ in the part (i) of the proof belong to $E_\Phi \setminus \{0\}$. Repeating the arguments of (i) we conclude that $E_{\Phi,s}$ is not strictly monotone. □

In the proof of the theorem on strict monotonicity of the Orlicz space $L_{\Phi,s}$ we will need the following lemma.

Lemma 3.8 *Let s be an outer function, $0 < c < \infty$ and let T_a be the (empty or infinite) set of atoms of T. If $\Phi \in \Delta_2$ and $\inf_{e \in T_a} \Phi(b_\Phi)\mu(e) \geq c$ then, for every $x \in L_\Phi$,*

$$I_\Phi(x) < c \Longrightarrow I_\Phi(kx) < c \text{ for some } k > 1.$$

Proof Let $x \in L_\Phi$ be such that $I_\Phi(x) < c$. We claim that there exists $k > 1$ such that $I_\Phi(kx) < \infty$. If $T \setminus T_a \neq \emptyset$ then, by Δ_2-condition, $I_\Phi\left(2x\chi_{T \setminus T_a}\right) < \infty$.

Now, assume that the set of atoms T_a is infinite. For simplicity, put $T_a = \bigcup_n e_n$, $e_n = \{n\}$ and $a_n = x(n)$ for all $n \in \mathcal{N}$. Let $a, K > 0$ be constants and $(c_n) \in \ell^1$ a sequence that appear in Δ_2-condition. Since

$$\sum_{n=1}^{\infty} \Phi(a_n)\mu(e_n) \leq I_\Phi(x) < c < \infty,$$

there exists $m \in N$ such that $\Phi(a_n)\mu(e_n) \leq a$ for all $n \geq m$. Thus

$$\sum_{n=m}^{\infty} \Phi(2a_n)\mu(e_n) \leq K \sum_{n=m}^{\infty} \Phi(a_n)\mu(e_n) + \sum_{n=m}^{\infty} c_n < \infty.$$

Moreover, for each $n \in \mathcal{N}$,

$$\Phi(a_n)\mu(e_n) \leq I_\Phi(x) < c \leq \Phi(b_\Phi)\mu(e_n).$$

Thus $|a_n| < b_\Phi$ for all $n \in \mathcal{N}$. Hence, for each $1 \leq n < m$ we can find $k_n > 1$ such that $k_n |a_n| < b_\Phi$. Put $k_0 = \min\{k_1, \ldots, k_{m-1}, 2\}$. Then $k_0 > 1$ and

$$\begin{aligned} I_\Phi(k_0 x) &= I_\Phi\left(k_0 x \chi_{T \setminus T_a}\right) + I_\Phi\left(k_0 x \chi_{T_a}\right) \\ &\leq I_\Phi\left(2x\chi_{T \setminus T_a}\right) + \sum_{n=1}^{m-1} \Phi(k_n a_n)\mu(e_n) + \sum_{n=m}^{\infty} \Phi(2a_n)\mu(e_n) < \infty \end{aligned}$$

and the claim is proved.

Therefore $k \to I_\Phi(kx)$ is a nondecreasing continuous function on the interval $(1, k_0)$ and $I_\Phi(x) < c$. Hence we can find $k \in (1, k_0)$ such that $I_\Phi(kx) < c$ as well. □

The next theorem provides criteria for the Orlicz space equipped with the s-norm to be strictly monotone.

Theorem 3.9 *Let s and Φ be an outer and Orlicz function respectively and let $a_s = \sup\{0 \leq u \leq 1 : s(u) = 1\}$.*

(a) *The Orlicz space $L_{\Phi,s}$ is strictly monotone if and only if Φ vanishes only at 0 and one of the following conditions is satisfied:*

(i) *s is strictly increasing on* $[0, \infty)$ *(i.e.,* $a_s = 0$*),*

(ii) $a_s > 0$ *and*

(a) $\Phi \in \Delta_2$,

(b) $\Phi(b_\Phi)\mu(e) \geq a_s$ *for all atoms* $e \in T_a$.

(b) *For any outer function s, the space* $E_{\Phi,s}$ *is strictly monotone if and only if either* Φ *vanishes only at 0 or* $E_\Phi = \{0\}$.

Proof The necessity part of the proof follows directly from Lemma 3.7.

Sufficiency. Assume that Φ vanishes only at 0. Take any $x, y \in L_\Phi$ (respectively, $x, y \in E_\Phi$, provided $E_\Phi \neq \{0\}$) with $0 \leq y \leq x$ and $y \neq 0$. Since Φ vanishes only at 0, we have $0 < I_\Phi(k(x-y)) < I_\Phi(kx)$ for all $k > 0$. Without loss of generality we can also assume that $\|x\|_{\Phi,s} = 1$.

If $K(x) = \emptyset$ then, by Lemma 3.2, Φ admits an asymptote at infinity with the slope $c > 0$ and $x \in L^1$. By strict monotonicity of the L^1-norm, $\|x-y\|_1 < \|x\|_1 < \infty$, so $x - y \in L^1$ and, again by Lemma 3.2,

$$\|x-y\|_{\Phi,s} \leq c\,\|x-y\|_1 < c\,\|x\|_1 = \|x\|_{\Phi,s}\,.$$

Thus, in the following part of the proof we can assume that $K(x) \neq \emptyset$, i.e., $k_x \in K(x)$ for some $1 \leq k_x < \infty$. Put $a_s = \sup\{0 \leq u \leq 1 : s(u) = 1\}$.

If $I_\Phi(k_x(x-y)) \geq a_s$ then $I_\Phi(k_x x) > a_s$ and, applying the fact that s is strictly increasing on (a_s, ∞), we get

$$\|x-y\|_{\Phi,s} \leq \frac{1}{k_x} s(I_\Phi(k_x(x-y))) < \frac{1}{k_x} s(I_\Phi(k_x x)) = \|x\|_{\Phi,s}\,.$$

If $a_s = 0$, the above inequality completes the sufficiency part of the proof.

So, let $0 < a_s \leq 1$ and assume that $I_\Phi(k_x(x-y)) < a_s$. If $x, y \in E_\Phi$ then we can find $k_0 > k_x$ such that $I_\Phi(k_0(x-y)) < a_s$. Therefore

$$\|x-y\|_{\Phi,s} \leq \frac{1}{k_0} s(I_\Phi(k_0(x-y))) = \frac{1}{k_0} < \frac{1}{k_x} \leq \frac{1}{k_x} s(I_\Phi(k_x x)) = \|x\|_{\Phi,s}\,, \tag{7}$$

whence $E_{\Phi,s}$ is strictly monotone and the statement (b) is proved.

If $x, y \in L_\Phi$ then, by (i)(a), (i)(b) and Lemma 3.8, we can also find $k_0 > k_x$ such that $I_\Phi(k_0(x-y)) < a_s$. Thus (7) holds true and the proof is completed. □

4 Uniform, lower local uniform and upper local uniform monotonicity of Orlicz spaces $L_{\Phi,s}$

A Banach lattice $(X, \|\cdot\|)$ is said to be

– uniformly monotone (UM) if for any $\varepsilon \in (0, 1)$ there exists $\delta(\varepsilon) \in (0, 1)$ such that

$$\|y\| \geq \varepsilon \implies \|x-y\| \leq 1 - \delta(\varepsilon) \tag{8}$$

for all $x \in S(X_+)$, $0 \leq y \leq x$. Recall that Birkhoff [1] defined a Banach lattice X to be uniformly monotone if for any $\varepsilon > 0$ there exists $\eta(\varepsilon) > 0$ such that

$$\|y\| \geq \varepsilon \implies \|x+y\| > 1 + \eta(\varepsilon) \tag{9}$$

for all $x, y \in X_+$, $x \in S(X_+)$. Kurc [19] showed that the definitions (8) and (9) are equivalent.

– lower locally uniformly monotone (LLUM) if for every $\varepsilon \in (0, 1]$ and $x \in S(X_+)$ there exists $\delta(x, \varepsilon) > 0$ such that

$$\|y\| \geq \varepsilon \implies \|x - y\| \leq 1 - \delta(x, \varepsilon)$$

for every $0 \leq y \leq x$,

– upper locally uniformly monotone (ULUM) if for every $\varepsilon > 0$ and $x \in S(X_+)$ there exists $\delta(x, \varepsilon) > 0$ such that

$$\|y\| \geq \varepsilon \implies \|x + y\| \geq 1 + \delta(x, \varepsilon)$$

whenever $y \in X_+$.

It is known that a Banach lattice X is LLUM (resp. ULUM) if and only if for every $x \in S(X_+)$ and every sequence (x_n) with $0 \leq x_n \leq x$ (resp. $0 \leq x \leq x_n$) the implication

$$\|x_n\| \to 1 \implies \|x_n - x\| \to 0$$

holds true. Evidently, each of the properties UM, LLUM and ULUM of X implies strict monotonicity (SM) of X.

We will say that an Orlicz function Φ satisfies the Kamińska condition for a constant $0 < d \leq 1$ if $\Phi(b_\Phi)\mu(e) \geq d$ and for each $\varepsilon \in (0, d)$ there exists $\delta > 0$ such that

$$\Phi(u)\mu(e) < d - \varepsilon \implies \Phi((1 + \delta)u)\mu(e) \leq 1$$

for all $u > 0$ and each atom $e \in T_a$ (this condition, for $d = 1$, was introduced by Kamińska in the paper [17] and it was denoted by (*) there).

An important consequence of Kamińska condition is a significant simplification of Δ_2-condition in the purely atomic measure case.

Lemma 4.1 *Let $T_a = \{e_n : n \in \mathcal{N}\}$ and let the Orlicz function Φ satisfies Kamińska condition with a constant $d > 0$. Then Φ satisfies Δ_2-condition on T_a if and only if for each $\varepsilon > 0$ there exist constants $\delta > 0$, $K > 1$ and a sequence $(c_n) \in \ell^1$ such that*

$$\Phi((1 + \delta)u)\mu(e_n) \leq K\Phi(u)\mu(e_n) + c_n \tag{10}$$

for each $n \in \mathcal{N}$ and all $u > 0$ with $\Phi(u)\mu(e_n) < d - \varepsilon$.

Proof By Δ_2-condition, there exist constants $a > 0$, $K_0 > 1$ and a sequence $(c_n) \in \ell^1$ such that

$$\Phi(2u)\mu(e_n) \leq K_0\Phi(u)\mu(e_n) + c_n$$

for all $u > 0$ with $\Phi(u)\mu(e_n) \leq a$ and $n \in \mathcal{N}$. Let $\varepsilon > 0$ and let $d > 0$, $\delta > 0$ be taken from Kamińska condition. Without loss of generality we can assume that $\delta < 1$. If $d - \varepsilon \leq a$ then (10) is evident. So, let $d - \varepsilon > a$. Then, for all $u > 0$ with $a < \Phi(u)\mu(e_n) < d - \varepsilon$ and each $n \in \mathcal{N}$ we have

$$\Phi((1 + \delta)u)\mu(e_n) \leq 1 = \frac{1}{a}a < \frac{1}{a}\Phi(u)\mu(e_n),$$

whence

$$\Phi((1 + \delta)u)\mu(e_n) \leq K\Phi(u)\mu(e_n) + c_n,$$

where $K = \max\left\{K_0, \frac{1}{a}\right\}$.

Conversely, if (10) holds then, using the fact that $2 \leq (1 + \delta)^i$ for some $i \in \mathcal{N}$ and modifying (decreasing) the number $a = d - \varepsilon > 0$ if necessary, we get Δ_2-condition on T_a (for detailed proof see Lemma 4 in [16]). □

Before we prove the criteria for the Orlicz space $L_{\Phi,s}$ to be uniformly monotone, we need three more result on s-norms generated by outer functions that are constant nearby 0.

Lemma 4.2 *Let s be an outer function, $a_s = \sup\{u \geq 0 : s(u) = 1\} > 0$ and let us assume that Φ satisfies Δ_2-condition.*

(i) *For all $x \in L_\Phi$, $\|x\|_{\Phi,s} \geq 1 \implies I_\Phi(x) \geq a_s$,*
(ii) *If $T_a = \emptyset$ then, for any $0 < \varepsilon < a_s$ we can find $\eta > 0$ such that*

$$I_\Phi(x) < a_s - \varepsilon \implies \|x\|_{\Phi,s} < 1 - \eta \tag{11}$$

for all $x \in L_\Phi$.
(iii) *If $T_a \neq \emptyset$ then (11) holds true if and only if Φ also satisfies Kamińska condition with the constant a_s.*

Proof (i) Let $\|x\|_{\Phi,s} \geq 1$ and suppose that $I_\Phi(x) < a_s$. By Lemma 3.8, we can find $k > 1$ such that $I_\Phi(kx) < a_s$, whence

$$1 \leq \|x\|_{\Phi,s} \leq \frac{1}{k} s(I_\Phi(kx)) = \frac{1}{k} < 1,$$

a contradiction.

(ii) (also proof of the sufficiency part of (iii)). Suppose that we can find $\varepsilon \in (0, a_s)$ and a sequence (x_n) of elements of $L_\Phi \setminus \{0\}$ such that $I_\Phi(x_n) \leq a_s - \varepsilon$ and $\|x_n\|_{\Phi,s} \geq (1 - \frac{1}{n+2})$ for all $n \in \mathcal{N}$. By (i) we infer that $\|x_n\|_{\Phi,s} < 1$ for all $n \in \mathcal{N}$.

If $\mu(T \setminus T_a) > 0$ then, by Δ_2-condition, there exist $u_0 \geq 0$ and $K_1 > 0$ such that

$$\begin{aligned} I_\Phi\left(2x_n\chi_{T\setminus T_a}\right) &\leq K_1 I_\Phi\left(x_n\chi_{T\setminus(T_a\cup A_n)}\right) + I_\Phi\left(2x_n\chi_{A_n}\right) \\ &< K_1(a_s - \varepsilon) + \Phi(2u_0)\mu(T\setminus T_a) < \infty \end{aligned}$$

for all $n \in \mathcal{N}$, where $A_n = \{t \in T\setminus T_a : |x_n(t)| < u_0\}$ and $\Phi(u_0)\mu(T\setminus T_a) < \infty$ (with the convention $0 \cdot \infty = \infty$).

Further, if $T_a \neq \emptyset$ then, by Kamińska condition with $d = a_s$ and Lemma 4.1, $\Phi < \infty$ and there exist $\delta > 0$, $K_2 > 0$ and a sequence $(c_n) \in \ell^1$ such that

$$I_\Phi\left((1+\delta)x_n\chi_{T_a}\right) \leq K_2 I_\Phi\left(x_n\chi_{T_a}\right) + \sum_{m=1}^{\infty} c_m \leq K_2(a_s - \varepsilon) + \|(c_m)\|_1 < \infty$$

for every $n \in \mathcal{N}$. Put

$$K = \max\{K_1(a_s - \varepsilon) + \Phi(2u_0)\mu(T\setminus T_a), K_2(a_s - \varepsilon) + \|(c_m)\|_1\}$$

and $\alpha_n = \|x_n\|_{\Phi,s}^{-1} - 1$ for $n \in \mathcal{N}$. Then $0 < \alpha_n < 1$ and $\alpha_n \to 0$, so $\alpha_n < \delta$ for all n large enough. Applying (i) we obtain

$$\begin{aligned} a_s &\leq I_\Phi\left(\frac{x_n}{\|x_n\|_{\Phi,s}}\right) = I_\Phi((\alpha_n + 1)x_n) = I_\Phi\left(\frac{\alpha_n}{\delta}(1+\delta)x_n + (1 - \frac{\alpha_n}{\delta})x_n\right) \\ &\leq \frac{\alpha_n}{\delta} I_\Phi((1+\delta)x_n) + (1 - \frac{\alpha_n}{\delta}) I_\Phi(x_n) \leq \frac{\alpha_n}{\delta} K + (1 - \frac{\alpha_n}{\delta})(a_s - \varepsilon) < a_s \end{aligned}$$

for all n large enough, a contradiction. Thus the condition (11) holds true.

Proof of the necessity part of (iii). Suppose that $\Phi(b_\Phi)\mu(e) < a_s$ for an atom $e \in T_a$. Denote $x = b_\Phi \chi_e$ and $\varepsilon = \frac{1}{2}(a_s - I_\Phi(x))$. Then, evidently, $I_\Phi(x) < a_s - \varepsilon$. Moreover, $s(I_\Phi(x)) = s(a_s - \varepsilon) = 1$, $\frac{1}{k}s(I_\Phi(kx)) \geq \frac{1}{k} > 1$ for all $0 < k < 1$ and $\frac{1}{k}s(I_\Phi(kx)) =$

$s(\infty) = \infty$ for all $k > 1$. Therefore $\|x\|_{\Phi,s} = 1$ and this equality contradicts to the condition (11).

Let $0 < \varepsilon < a_s$. Take any $u > 0$ and $n \in \mathcal{N}$ such that $\Phi(u)\mu(e_n) \leq a_s - \varepsilon$. For $x = u\chi_{e_n}$ we have $I_\Phi(x) \leq a_s - \varepsilon$, whence, by (11), $\|x\|_\Phi \leq \|x\|_{\Phi,s} < 1 - \eta$ for some $0 < \eta < 1$. Thus, putting $\delta = \frac{\eta}{1-\eta}$, and applying the fact, that (for the Luxemburg norm $\|\cdot\|_\Phi$) if $\|z\|_\Phi \leq 1$ then $I_\Phi(z) \leq \|z\|_\Phi \leq 1$, we obtain

$$\Phi((1+\delta)u)\mu(e_n) = \Phi\left(\frac{u}{1-\eta}\right)\mu(e_n) = I_\Phi\left(\frac{x}{1-\eta}\right) \leq \left\|\frac{x}{1-\eta}\right\|_\Phi \leq 1,$$

so Kamińska condition with the constant $d = a_s$ holds true. □

Lemma 4.3 *Let s be an outer function with $a_s > 0$ and let Φ satisfy Δ_2-condition. If the Orlicz space $L_{\Phi,s}$ is uniformly monotone then the condition (11) holds true. If, moreover, the set of atoms T_a is not empty then the Kamińska condition is satisfied with the constant a_s.*

Proof Suppose that (11) does not hold. Then we can find a sequence (z_n) of nonnegative functions in L_Φ such that $I_\Phi(z_n) < a_s - \varepsilon$ and $\|z_n\|_{\Phi,s} > 1 - \frac{1}{n}$. Since the measure μ is σ-finite, without loss of generality we can assume that $T\backslash\operatorname{supp}(z_n) \neq \emptyset$. Note that $\|z_n\|_{\Phi,s} \leq s(a_s - \varepsilon) = 1$.

We claim that for each $n \in N$ we can find a set $A_n \subset T\backslash\operatorname{supp}(z_n)$ of positive and finite measure and a constant $u_n > 0$ such that $I_\Phi(z_n + w_n) = a_s$, where $w_n = u_n\chi_{A_n}$. Indeed, if $A_n \subset T\backslash T_a$ then the atomless part of T is not empty, whence, by Δ_2-condition, $\Phi < \infty$ and the claim follows directly from continuity of the function Φ.

If A_n contains atoms then, without loss of generality, we can assume that A_n consists of one atom only, i.e., $A_n = \{e_n\} \subset T_a$. Since (UM) property implies (SM), by Theorem 3.9, $\Phi(b_\Phi)\mu(e) \geq a_s$ for each atom $e \in T_a$ and the claim is proved.

Since $I_\Phi(z_n) \leq a_s - \varepsilon$ we have $I_\Phi(w_n) \geq \varepsilon$. Hence, by Δ_2-condition, we can find $\eta > 0$ such that $\|w_n\|_{\Phi,s} \geq \eta$ for all $n \in \mathcal{N}$. Define

$$x_n = \frac{z_n + w_n}{\|z_n + w_n\|_{\Phi,s}}, \quad y_n = \frac{w_n}{\|z_n + w_n\|_{\Phi,s}}.$$

Evidently $0 \leq y_n \leq x_n$ and $\|x_n\|_{\Phi,s} = 1$. Moreover

$$\|y_n\|_{\Phi,s} = \frac{\|w_n\|_{\Phi,s}}{\|z_n + w_n\|_{\Phi,s}} \geq \frac{\|w_n\|_{\Phi,s}}{s(I_\Phi(z_n + w_n))} \geq \eta > 0.$$

Finally,

$$\|x_n - y_n\|_{\Phi,s} = \frac{\|z_n\|_{\Phi,s}}{\|z_n + w_n\|_{\Phi,s}} \geq \frac{1 - \frac{1}{n}}{s(I_\Phi(z_n + w_n))} = 1 - \frac{1}{n} \to 1$$

as $n \to \infty$, and this contradicts to the assumption that $L_{\Phi,s}$ is uniformly monotone. The Kamińska condition follows directly from the Lemma 4.2. □

Lemma 4.4 *Let s be an outer function and let $a_s = \sup\{u \geq 0 : s(u) = 1\}$. For every $v > a_s$ and $\varepsilon \in (0, v - a_s)$ we can find $\delta \in (0, 1)$ such that*

$$s(v - u) \leq (1 - \delta)s(v)$$

for every $u \in [\varepsilon, v]$.

Proof Suppose that there are $v > a_s$, $\varepsilon \in (0, v - a_s)$ and a sequence (u_n) of numbers from the interval $[\varepsilon, v]$ such that $s(v) \geq s(v-u_n) > (1-\frac{1}{n})s(v)$ for all $n \in \mathcal{N}$. Since the sequence (u_n) is bounded, passing to a subsequence if necessary, we can assume that $u_n \to u_0 \in [\varepsilon, v]$ as $n \to \infty$. Letting $n \to \infty$ and applying the continuity of s, we get $s(v - u_0) = s(v)$. Since $u_0 > 0$, the last equality can hold true only if $v - u_0, v \in [0, a_s]$, whence $v \leq a_s$, a contradiction. □

Now we will prove the main theorem of this section.

Theorem 4.5 *For any outer function s,*

the Orlicz space $L_{\Phi,s}$ is uniformly monotone if and only if the following conditions are satisfied

(i) *the Orlicz function Φ vanishes only at 0,*
(ii) *Φ satisfies Δ_2-condition,*
(iii) *Φ satisfies Kamińska condition with the constant a_s provided $a_s > 0$ and the set of atoms T_a is not empty.*

Proof *Necessity*. Since UM implies SM, by Theorem 3.9 we infer that $\Phi > 0$. Moreover, UM implies LLUM and this property implies order continuity of the norm (see [8]), so the s-norm norm $\|\cdot\|_{\Phi,s}$ is order continuous. In consequence, the Luxemburg norm $\|\cdot\|_\Phi$ is order continuous as well, whence $\Phi \in \Delta_2$ (see, e.g., [3,5]). Finally, if $a_s > 0$ and $T_a \neq \emptyset$, Kamińska condition with the constant a_s follows from Lemma 4.3.

Sufficiency. Let $\varepsilon > 0$. Take any $x \in S((L_\Phi)_+)$ and let $y \in (L_\Phi)_+$ be such that $0 \leq y \leq x$ and $\|y\|_{\Phi,s} \geq \varepsilon$.

If $K(x) = \emptyset$ then $x \in L^1$. Hence $x - y \in L^1$ as well because $0 \leq x - y \leq x$. Thus applying Lemma 3.2 and the fact that the Lebesgue space L^1 is UM, we can find $0 < \eta_1 < 1$ such that

$$\|x - y\|_{\Phi,s} \leq c\,\|x - y\|_1 < c(1 - \eta_1)\,\|x\|_1 = (1 - \eta_1)\,\|x\|_{\Phi,s}\,,$$

where $c > 0$ is the slope of the oblique asymptote of Φ at infinity. Hence, in the following, we can assume that $K(x) \neq \emptyset$.

Let $k \geq 1$ be such that $k \in K(x)$ and let us assume that $I_\Phi(kx) - I_\Phi(ky) \geq a_s$. Since $\Phi \in \Delta_2$ and $\Phi > 0$, the modular convergence of a sequence of functions to 0 implies its norm convergence to 0. Hence we can find $\delta > 0$ such that $I_\Phi(y) > \delta$ for all $\|y\|_{\Phi,s} \geq \varepsilon$. Thus $I_\Phi(ky) > \delta$ as well, whence $I_\Phi(kx) > a_s$. Moreover, $\delta < I_\Phi(ky) \leq I_\Phi(kx) - a_s$. By superaddtivity of Φ and by Lemma 4.4, we can find $0 < \eta_2 < 1$ such that

$$\begin{aligned}\|x - y\|_{\Phi,s} &\leq \frac{1}{k} s\left(I_\Phi(k(x - y))\right) \leq \frac{1}{k} s\left(I_\Phi(kx) - I_\Phi(ky)\right)\\ &< (1 - \eta_2)\frac{1}{k} s(I_\Phi(kx)) = (1 - \eta_2)\,\|x\|_{\Phi,s}\,.\end{aligned}$$

That completes the proof in the case when the outer function s is strictly increasing (i.e, $a_s = 0$).

Now, assume that $a_s > 0$ and $I_\Phi(kx) - I_\Phi(ky) < a_s$. If $I_\Phi(kx) \leq a_s + \delta/2$ then

$$I_\Phi(k(x - y)) \leq I_\Phi(kx) - I_\Phi(ky) < a_s + \delta/2 - \delta = a_s - \delta/2.$$

By Lemma 4.2, we can find $\eta_3 > 0$ such that

$$\|x - y\|_{\Phi,s} < 1 - \eta_3 = (1 - \eta_3)\,\|x\|_{\Phi,s}\,.$$

Finally, assume that $a_s > 0$, $I_\Phi(kx) - I_\Phi(ky) < a_s$ and $I_\Phi(kx) > a_s + \delta/2$. Since $\|x\|_{\Phi,s} = 1$ and $k \in K(x)$, we have

$$k = s(I_\Phi(kx)) > s(a_s + \delta/2) = \frac{1}{1-\eta_4} > 1,$$

where $\eta_4 = 1 - 1/s(a_s + \delta/2) \in (0, 1)$. Thus

$$\begin{aligned}\|x-y\|_{\Phi,s} &\le \frac{1}{k}s\left(I_\Phi(k(x-y))\right) = (1-\eta_4)s\left(I_\Phi(k(x-y))\right)\\ &\le (1-\eta_4)s\left(I_\Phi(kx) - I_\Phi(ky)\right) = 1-\eta_4 = (1-\eta_4)\|x\|_{\Phi,s}.\end{aligned}$$

Taking $\eta = \min\{\eta_i : 1 \le i \le 4\}$ we have $\|x-y\|_{\Phi,s} < (1-\eta)\|x\|_{\Phi,s}$ for all $x \in S((L_\Phi)_+)$ and all $0 \le y \le x$ with $\|y\|_{\Phi,s} \ge \varepsilon$, i.e., the space $L_{\Phi,s}$ is uniformly monotone. □

We say that a sequence (x_n) of nonnegative elements of a Banach lattice X is nearly order convergent to $x \in X_+$ from below, if $0 \le x_n \le x$ and $\|x_n\| \to \|x\|$.

Lemma 4.6 *If the outer function s is strictly monotone and the Orlicz function Φ vanishes only at 0, then the nearly order convergence from below implies the convergence in measure on the space $L_{\Phi,s}$.*

Proof Let (x_n) be a sequence of nonnegative elements of $(L_\Phi)_+$, $x \in (L_\Phi)_+$, $0 \le x_n \le x$ and $\|x_n\|_{\Phi,s} \to \|x\|_{\Phi,s}$. Without loss of generality we can assume that $\|x\|_{\Phi,s} = 1$.

Suppose that (x_n) does not converge to x in measure μ as $n \to \infty$, i.e., we can find $\varepsilon, \delta > 0$ and a strictly increasing sequence (n_m) of natural numbers such that $\inf_m \mu(A_m) \ge 2\delta$, where

$$A_m = \left\{t \in T : x(t) - x_{n_m}(t) > \varepsilon\right\}.$$

Then $0 \le \varepsilon\chi_{A_m} \le x - x_{n_m} \in L_\Phi$, whence $\varepsilon\chi_{A_m} \in L_\Phi$ and $\mu(A_{n_m}) < \infty$ for all $m \in \mathcal{N}$.

If $K(x) = \emptyset$ then, by Lemma 3.2, $x \in L^1$. Thus $x_n \in L^1$ as well and

$$\left\|x_{n_m}\right\|_{\Phi,s} \le \left\|x_{n_m}\right\|_1 \le \|x\|_1 - \left\|\varepsilon\chi_{A_{n_m}}\right\|_1 = 1 - \varepsilon\mu(A_{n_m}) \le 1-\delta,$$

whence $\left\|x_{n_m}\right\|_{\Phi,s} \not\to 1$, a contradiction.

Now, assume that $K(x) \ne \emptyset$, i.e., $\|x\|_{\Phi,s} = \frac{1}{k_x}s(I_\Phi(k_x x)) = 1$ for some $k_x \ge 1$. Define $\eta = I_\Phi(k_x\varepsilon\delta)$, $v = I_\Phi(k_x x)$ and $u_m = I_\Phi\left(k_x\varepsilon\chi_{A_m}\right)$ for $m \in \mathcal{N}$. Since $\Phi > 0$ we have $0 < \eta < u_m \le v$ for all $m \in \mathcal{N}$. Since $a_s = 0$, by Lemma 4.4 we can find $\gamma > 0$ such that $s(v-u) \le (1-\gamma)s(v)$ for every $u \in (\eta, v]$. Thus, by superadditivity of Φ,

$$\begin{aligned}\left\|x_{n_m}\right\|_{\Phi,s} &\le \frac{1}{k_x}s(I_\Phi\left(k_x x_{n_m}\right)) \le \frac{1}{k_x}s(I_\Phi\left(k_x(x-\varepsilon\chi_{A_m})\right)) \le \frac{1}{k_x}s(v-u_m)\\ &\le (1-\gamma)\frac{1}{k_x}s(v) = (1-\gamma)\frac{1}{k_x}s(I_\Phi(k_x x)) = 1-\gamma\end{aligned}$$

for all $m \in \mathcal{N}$. Therefore $\left\|x_{n_m}\right\|_{\Phi,s} \not\to 1$, a contradiction that ends the proof. □

Theorem 4.7 *If the outer function s is strictly increasing and the Orlicz function Φ vanishes only at 0, then the Orlicz space $E_{\Phi,s}$ is lower locally uniformly monotone.*

Proof Evidently we can assume that $E_\Phi \ne \{0\}$. Let $x \in S_+(E_\Phi)$ and let (x_n) be a sequence of E_Φ such that $0 \le x_n \le x$ and $\|x_n\|_{\Phi,s} \to 1$. Let $\lambda > 0$ be an arbitrary positive number. We have $0 \le \lambda(x - x_n) \le \lambda x$ and, by Lemma 4.6, $\lambda x_n \xrightarrow{\mu} \lambda x$. Since $x \in E_\Phi$, we have $I_\Phi(\lambda x) < \infty$ so, by the Lebesgue dominated convergence theorem, $I_\Phi(\lambda(x - x_n)) \to 0$, whence, by arbitrariness of λ, $\|x - x_n\|_{\Phi,s} \to 0$. Thus the space $E_{\Phi,s}$ is lower locally uniformly monotone. □

Corollary 4.8 *If the outer function s is strictly increasing then the the following conditions are equivalent*

(i) $E_{\Phi,s}$ *is LLUM*
(ii) $E_{\Phi,s}$ *is SM*
(iii) $\Phi > 0$.

Proof Evidently, we can assume that $E_{\Phi,s} \neq \{0\}$. $(i) \Rightarrow (ii)$ is obvious. $(ii) \Rightarrow (iii)$ follows from Theorem 3.9. Finally, by Theorem 4.7 $(iii) \Rightarrow (i)$. □

Theorem 4.9 *Let s and Φ be an outer and Orlicz function respectively. If one of the following conditions is satisfied*

(i) $L_{\Phi,s}$ *is ULUM,*
(ii) $E_{\Phi,s} \neq \{0\}$ *and* $E_{\Phi,s}$ *is ULUM*

then $\Phi > 0$ *and* $\Phi \in \Delta_2$.

Proof Denote by X the space L_Φ in the case when L_Φ is ULUM or the space E_Φ whenever $E_\Phi \neq \{0\}$ and E_Φ is ULUM. Since ULUM property implies SM, by Lemma 3.7 we infer that $\Phi > 0$.

Suppose that $\Phi \notin \Delta_2$. Then, since $X \neq \{0\}$, we can find a sequence (z_n) of elements of X_+ such that (z_n) is modular convergent to 0 but it is not norm convergent to 0 with respect to the norm $\|\cdot\|_{\Phi,s}$. Without loss of generality we can assume that $\mu(T\setminus\bigcup_n \operatorname{supp} z_n) > 0$, $I_\Phi(z_n) \le 2^{-n}$ and $\|z_n\|_{\Phi,s} \ge \eta$ for all $n \in \mathcal{N}$ and some $\eta > 0$.

Take $x \in S_+(X)$ such that $\operatorname{supp}(x) \subset T\setminus\bigcup_n \operatorname{supp} z_n$ and let $\varepsilon > 0$ be an arbitrary number. We can find $k_x \ge 1$ such that $\frac{1}{k_x}s\left(I_\Phi\left(k_x x\right)\right) \le \|x\|_{\Phi,s} + \varepsilon$. Put $y_n = k_x^{-1} z_n$. We have $I_\Phi(k_x y_n) = I_\Phi\left(z_n\right) \le 2^{-n}$ and $\|y_n\|_{\Phi,s} = k_x^{-1}\|z_n\|_{\Phi,s} \ge k_x^{-1}\eta > 0$. Thus

$$\|x + y_n\|_{\Phi,s} \le \frac{1}{k_x}s(I_\Phi(k_x(x+y_n))) = \frac{1}{k_x}s(I_\Phi(k_x x) + I_\Phi(k_x y_n)) \\ \le \frac{1}{k_x}s\left(I_\Phi(k_x x) + \frac{1}{2^n}\right),$$

whence

$$\limsup_{n\to\infty}\|x + y_n\|_{\Phi,s} \le \frac{1}{k_x}s(I_\Phi(k_x x)) \le \|x\|_{\Phi,s} + \varepsilon = 1 + \varepsilon.$$

By arbitrariness of $\varepsilon > 0$ we infer that $\limsup_{n\to\infty}\|x + y_n\|_{\Phi,p} \le 1$, whence the space X is not ULUM, a contradiction. □

The following theorem summarizes the relationship between uniform monotonicity, lower locally uniform monotonicity and upper local uniform monotonicity properties.

Theorem 4.10 *Let s and Φ be an outer and Orlicz function respectively.*

(a) *If the outer function s is strictly increasing on* $(0, \infty)$ *or the measure μ is atomless then the following conditions are equivalent*

  (i) $L_{\Phi,s}$ *is UM,*
  (ii) $L_{\Phi,s}$ *is LLUM,*
  (iii) $L_{\Phi,s}$ *is ULUM,*
  (iv) $\Phi > 0$ *and* $\Phi \in \Delta_2$.

(b) *If all of the assumptions of (a) are satisfied and, moreover,* $\Phi < \infty$ *then in each of the conditions (i), (ii) and (iii) we can put the space* $E_{\Phi,s}$ *instead of* $L_{\Phi,s}$.

Proof (a) Evidently $(i) \Rightarrow (ii)$ and $(i) \Rightarrow (iii)$. $(iii) \Rightarrow (iv)$ follows directly from Theorem 4.9, while $(iv) \Rightarrow (i)$ is a consequence of Theorem 4.5. $(ii) \Rightarrow (iv)$ Since LLUM implies SM, by Theorem 3.9, $\Phi > 0$. Moreover, LLUM implies order continuity of the norm (see [8]), whence (see [5,10]) $\Phi \in \Delta_2$.

(b) It suffices to note that if $\Phi < \infty$ and $\Phi \in \Delta_2$ then $E_{\Phi,s} = L_{\Phi,s}$.

□

5 The dominated best approximation problem

Recall that if $X = (X, \|\cdot\|)$ is a Banach space and $\emptyset \neq A \subset X$, then for any $x \in X$ the number

$$d(x, A) = \inf\{\|x - y\| : y \in A\}$$

is called the distance of x from A and the sequence (y_n) of elements of A is said to be an x-minimizing sequence whenever $\lim_{n\to\infty} \|x - y_n\| = d(x, A)$. It is obvious that for any nonempty set A in X and any $x \in X$ the distance $d(x, A)$ is finite and $d(x, A) = 0$ for any $x \in A$. Further, the function $\mathcal{P}_A(x) = X \to 2^X$ defined by

$$\mathcal{P}_A(x) = \{z \in A : d(x, A) = \|x - z\|\}$$

is called the projection from X onto A and, for any $x \in X$, the set $\mathcal{P}_A(x)$ is called the projection of x onto A.

The best approximation problem deals with the description of the elements of the set $\mathcal{P}_A(x)$. If $\mathcal{P}_A(x) \neq \emptyset$ (resp. card$\mathcal{P}_A(x) = 1$) then we say that the best approximation problem is solvable (resp. uniquely solvable) for $x \in X$. Further, the best approximation problem is said to be stable for $x \in A$, if for every x-minimizing sequence (y_n) in A we have $d(y_n, \mathcal{P}_A(x)) \to 0$ as $n \to \infty$. Finally, the best approximation problem is called strongly solvable if it is uniquely solvable and stable.

A set A is called a sublattice of the Banach lattice X, if $A \subset X$ and for any $x, y \in A$ there exist $x \wedge y \in A$ and $x \vee y \in A$. The best approximation problem restricted to A being a sublattice and x being a boundary (lower or upper) of A is called the dominated best approximation problem. Let us recall theorems that present conditions under which the dominated best approximation problem is solvable (see [9,11,19]).

Theorem 5.1 (See [19, Proposition 3.1] *Let X be a Banach lattice. For every boundary (lower or upper) of the sublattice A of X the dominated best approximation problem is uniquely solvable if and only if it is solvable and X is a strictly monotone space.*

Theorem 5.2 (See [11, Theorem 4.3]) *Let X be a* σ*-Dedekind complete Banach lattice and let A be a closed sublattice of X. If X is lower locally uniformly monotone then the dominated best approximation property is uniquely solvable for any (lower or upper) boundary of A.*

Theorem 5.3 (See [11, Theorem 4.4]) *Let X be a* σ*-Dedekind complete Banach lattice and let A be a closed sublattice of X. If X is order continuous and upper locally uniformly monotone then the dominated best approximation property is strongly solvable for every (lower or upper) boundary of A.*

Now, applying Theorem 3.9 and Theorem 4.10 we get the following corollaries on solvability of the dominated best approximation problem in Orlicz spaces equipped with s-norms.

Corollary 5.4 *Let Φ be strictly increasing Orlicz function on $(0, \infty)$, s an outer function and let $a_s = \sup\{u \geq 0 : s(u) = 1\}$. If the outer function s is strictly increasing on $(0, \infty)$ or Φ satisfies the Δ_2-condition and $\Phi(b_\Phi)\mu(e) \geq a_s > 0$ for all atoms $e \in T$ then, for every closed sublattice A of the Orlicz space $L_{\Phi,s}$, the dominated best approximation problem is uniquely solvable for any (lower or upper) boundary of A if and only if it is solvable.*

Corollary 5.5 *Let Φ be a strictly increasing Orlicz function on $(0, \infty)$ that satisfies the Δ_2-condition. If the outer function s is strictly increasing on $(0, \infty)$ or the measure μ is atomless then for every closed sublattice A of the Orlicz space $L_{\Phi,s}$ the dominated best approximation property is strongly solvable for any boundary (lower or upper) of the sublattice A.*

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