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ORIGINAL PAPER

A quantitative approach to disjointly non-singular operators

Manuel González[1] · Antonio Martinón[2]

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Abstract
We introduce and study some operational quantities which characterize the disjointly non-singular operators from a Banach lattice E to a Banach space Y when E is order continuous, and some other quantities which characterize the disjointly strictly singular operators for arbitrary E.

Keywords Disjointly non-singular operator · Disjointly strictly singular operator · Order continuous Banach lattice · Operational quantity · L_p space

Mathematics Subject Classification Primary: 47B60 · 47A55 · 46B42

1 Introduction

The disjointly strictly singular operators (DSS operators) were introduced in [12] as those operators $T : E \to Y$ from a Banach lattice E into a Banach space Y such that T is not an isomorphism in any subspace of E generated by a disjoint sequence of non-zero vectors. These operators have been useful in the study of the structure of Banach lattices (see [2], [3] and references therein). More recently, the disjointly non-singular operators (DN-S operators) where introduced in [6] (see also [1]) as those operators $T : E \to Y$ that are not strictly singular in any subspace of E generated by a disjoint sequence of non-zero vectors. Note that the properties in the definition of these two classes are opposite.

In this paper we study the classes of operators DSS and DN-S from a quantitative point of view by introducing four operational quantities $\Gamma_d(T)$, $\Delta_d(T)$, $\tau_d(T)$ and $\kappa_d(T)$. When E is order continuous, $T \in$ DN-S(E, Y) is equivalent to $\Gamma_d(T) > 0$, or $\kappa_d(T) > 0$; and for E

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✉ Antonio Martinón
anmarce@ull.es

Manuel González
manuel.gonzalez@unican.es

1 Departamento de Matemáticas, Facultad de Ciencias, Universidad de Cantabria, Santander 39071, Spain

2 Departamento de Análisis Matemático, Facultad de Ciencias, Universidad de La Laguna, La Laguna (Tenerife) 38271, Spain

arbitrary, $T \in \mathrm{DSS}(E, Y)$ is equivalent to $\Delta_d(T) = 0$, or $\tau_d(T) = 0$. These four quantities are inspired by some others introduced by Schechter [19] in his study of Fredholm theory.

In [6], the quantity $\beta(T) = \inf_{(x_n)} \liminf_{n\to\infty} \|Tx_n\|$, where the infimum is taken over the normalized disjoint sequences (x_n) in E, was defined. We show that $T \in \text{DN-S}(E, Y)$ if and only if $\beta(T) > 0$ when E is order continuous. This result was proved in [1][Theorem 5.3] using different techniques. We also prove that $\beta(T) \leq \Gamma_d(T)$, but there is no $C > 0$ such that $\Gamma_d(T) \leq C\beta(T)$ for each $T \in \mathrm{L}(\ell_2, Y)$; hence Γ_d and β are not equivalent. Moreover, $\tau_d(T) \leq \Delta_d(T)$, but the quantities τ_d and Δ_d are not equivalent.

We also prove some inequalities for these operational quantities; e.g., for $T, S \in \mathrm{L}(E, Y)$, we have $\Gamma_d(T + S) \leq \Gamma_d(T) + \Delta_d(S)$. When E is order continuous, this inequality allows us to improve the stability result for DN-S operators under DSS perturbations obtained in [6].

Notation

Throughout the paper X and Y are Banach spaces, and E is a Banach lattice. The unit sphere of X is $S_X = \{x \in X : \|x\| = 1\}$, and for a sequence (x_n) in X, $[x_n]$ denotes the closed subspace generated by (x_n).

All the operators are linear and bounded, and $\mathrm{L}(X, Y)$ denotes the set of all the operators from X into Y. Given $T \in \mathrm{L}(X, Y)$, its *injection modulus* is $j(T) := \inf_{\|x\|=1} \|Tx\|$. Recall that $j(T) > 0$ if and only if T is an isomorphism from X onto TX. We denote by T_M the restriction of $T \in \mathrm{L}(X, Y)$ to a closed subspace M of X.

If (Ω, Σ, μ) is a measure space, the domain of a measurable function $f : \Omega \to \mathbb{R}$ is the set $D(f) = \{t \in \Omega : f(t) \neq 0\}$, and 1_A denotes the characteristic function of $A \in \Sigma$. We write L_p for $L_p[0, 1]$, $1 \leq p \leq \infty$.

2 Preliminaries

An operator $T \in \mathrm{L}(X, Y)$ is *strictly singular* if there is no closed infinite dimensional subspace M of X such that the restriction T_M is an isomorphism, and T is *upper semi-Fredholm* if its kernel is finite dimensional and its range is closed.

An operator $T \in \mathrm{L}(E, Y)$ is *disjointly strictly singular* if there is no disjoint sequence of non-zero vectors (x_n) in E such that $T_{[x_n]}$ is an isomorphism. We denote by $\mathrm{DSS}(E, Y)$ the set of all $T \in \mathrm{L}(E, Y)$ which are disjointly strictly singular. The class DSS was introduced by Hernández and Rodríguez-Salinas in [12]. More information on this class can be found in [11].

An operator $T \in \mathrm{L}(E, Y)$ is *disjointly non-singular* if there is no disjoint sequence of non-zero vectors (x_n) in E such that $T_{[x_n]}$ is strictly singular. We denote $\text{DN-S}(E, Y)$ the set of all $T \in \mathrm{L}(E, Y)$ which are disjointly non-singular. These operators were recently introduced in [6], and have been studied by Bilokopytov in [1]. They are related to the tauberian operators, defined by Kalton and Wilansky [13]; in fact, they coincide when $E = L_1$ (see [4] and [6]). We refer to [9] and [5] for additional information on tauberian operators.

The disjointly non-singular operators can be characterized as follows.

Theorem 2.1 [6][Theorem 2.8] *For $T \in L(E, Y)$, the following assertions are equivalent:*

(1) *T is disjointly non-singular.*
(2) *There is no disjoint sequence of non-zero vectors (x_n) in E such that the restriction $T_{[x_n]}$ is a compact operator.*
(3) *For every disjoint sequence of non-zero vectors (x_n) in E, the restriction $T_{[x_n]}$ is an upper semi-Fredholm operator.*
(4) *For every normalized disjoint sequence (x_n) in E, $\liminf_{n\to\infty} \|Tx_n\| > 0$.*

It was proved in [4][Proposition 14] and [6] [Theorem 3.15] that, for $1 \le p < \infty$, $\text{DSS}(L_p, Y)$ is the perturbation class of $\text{DN-S}(L_p, Y)$.

Representation of Banach lattices

It is well-known (see [16][Theorem 1.b.14]) that every order continuous Banach lattice with a weak unit E admits a representation as a Köthe function space, in the sense that there exists a probability space (Ω, Σ, μ) so that

- $L_\infty(\mu) \subset E \subset L_1(\mu)$ with E dense in $L_1(\mu)$ and $L_\infty(\mu)$ dense in E,
- $\|f\|_1 \le \|f\|_E \le 2\|f\|_\infty$ when $f \in L_\infty(\mu)$,

and the order in E is the order induced by $L_1(\mu)$.

The following fact will allow us to state some of our results omitting the existence of a weak unit in the Banach lattice.

Lemma 2.2 *Let E be a Banach lattice. Then each sequence in E is contained in a closed ideal of E with a weak unit.*

Proof If (f_n) is a bounded sequence in E, then $e = \sum_{n=1}^{\infty} |f_n|/2^{-n}$ is a weak unit in the closed ideal generated by (f_n). □

We also will need the following result.

Lemma 2.3 *Let E be an order continuous Banach lattice with a weak unit, and let $f \in E$. If (A_k) is a disjoint sequence in the σ-algebra Σ associated to the representation of E, then* $\lim_{k\to\infty} \|f 1_{A_k}\|_E = 0$.

Proof Let $B_k = \cup_{i=k}^{\infty} A_i$. Since the norm on E is order continuous, (B_k) is decreasing and $\lim_{k\to\infty} \mu(B_k) = 0$ we have $\lim_{k\to\infty} \|f 1_{B_k}\|_E = 0$, hence $\lim_{k\to\infty} \|f 1_{A_k}\|_E = 0$. □

3 Operational quantities

An *operational quantity* is a map $a : \text{L}(X, Y) \to [0, \infty)$ satisfying certain conditions. Given two operational quantities a and b, we write $a \le b$ when $a(T) \le b(T)$ for each $T \in \text{L}(X, Y)$. Moreover, the quantities a and b are *equivalent* if there exist positive constants $c_1 < c_2$ such that $c_1 a \le b \le c_2 a$.

We are interested in some classical operational quantities and some new ones that we introduce here. To describe the classical ones, let $S(X)$ be set of all closed infinite dimensional subspaces of X. Then, given an operational quantity $a : \text{L}(X, Y) \to [0, \infty)$, we define two derived quantities $i\,a$ and $s\,a$ as follows:

$$i\,a(T) := \inf_{M \in S(X)} a(T_M) \quad \text{and} \quad s\,a(T) := \sup_{M \in S(X)} a(T_M), \tag{1}$$

where $T \in \mathrm{L}(X, Y)$.

Note that $a \leq b$ implies $ia \leq ib$ and $sa \leq sb$. Taking the operator norm as a in (1), for $T \in \mathrm{L}(X, Y)$ we obtain

- $\Gamma(T) := i\,\|T\| = \inf_{M\in S(X)} \|T_M\|$ and
- $\Delta(T) := s\,\Gamma(T) = \sup_{M\in S(X)} \Gamma(T_M) = \sup_{M\in S(X)} \inf_{N\in S(M)} \|T_N\|$.

The quantities $\Gamma = i\,\|\cdot\|$ and $\Delta = i\,\Gamma$ were introduced by Gramsch and Schechter (see [19,20]), who proved that $\Gamma(T) > 0$ if and only if T is upper semi-Fredholm, and $\Delta(T) = 0$ if and only if T is strictly singular.

To introduce the new quantities, we denote by $\mathrm{d}(E)$ the set of all sequences of disjoint non-zero vectors of E. Now, given an operational quantity $a : \mathrm{L}(F, Y) \to [0, \infty)$ defined for $F = E$ and $F \in \mathrm{d}(E)$, for each $T \in \mathrm{L}(E, Y)$ we define two derived quantities $i_d\, a$ and $s_d\, a$ as follows:

$$i_d\, a(T) := \inf_{(x_n)\in \mathrm{d}(E)} a(T_{[x_n]}) \quad \text{and} \quad s_d\, a(T) := \sup_{(x_n)\in \mathrm{d}(E)} a(T_{[x_n]})\,. \tag{2}$$

Again, $a \leq b$ implies $i_d a \leq i_d b$ and $s_d a \leq s_d b$. We are interested in two operational quantities derived from the norm, whose notation is inspired by that of Schechter:

- $\Gamma_d(T) := i_d\,\|T\| = \inf_{(x_n)\in \mathrm{d}(E)} \|T_{[x_n]}\|$ and
- $\Delta_d(T) := s_d\,\Gamma_d(T) = \sup_{(x_n)\in \mathrm{d}(E)} \Gamma_d(T_{[x_n]}) = \sup_{(x_n)\in \mathrm{d}(E)} \inf_{(y_n)\in \mathrm{d}([x_n])} \|T_{[y_n]}\|$,

that will allow us to characterize the operators in DN-S and DSS.

In a similar way, for $T \in \mathrm{L}(X, Y)$ we consider two classical operational quantities derived from the injection modulus j:

- $\tau(T) := s\, j(T) = \sup_{M\in S(X)} j(T_M)$ and
- $\kappa(T) := i\,\tau(T) = \inf_{M\in S(X)} \tau(T_M) = \inf_{M\in S(X)} \sup_{N\in S(M)} j(T_N)$,

and derive two new quantities for $T \in \mathrm{L}(E, Y)$:

- $\tau_d(T) := s_d\, j(T) = \sup_{(x_n)\in \mathrm{d}(E)} j(T_{[x_n]})$ and
- $\kappa_d(T) := i_d\,\tau_d(T) = \inf_{(x_n)\in \mathrm{d}(E)} \tau_d(T_{[x_n]}) = \inf_{(x_n)\in \mathrm{d}(E)} \sup_{(y_n)\in \mathrm{d}([x_n])} j(T_{[y_n]})$,

The operational quantities $\tau = s\, j$ and $\kappa = i\,\tau$ were introduced in [19] and [7], where it was proved that $\tau(T) = 0$ if and only if T is strictly singular, and $\kappa(T) > 0$ if and only if T is upper semi-Fredholm. We will show that the quantities τ_d and κ_d characterize the operators in DSS and DN-S, respectively.

The proof of the next lemma shows that for each closed infinite dimensional subspace of a Banach space with a monotone basis (x_n), in particular with a 1-unconditional basis, there is a block basis (y_k) such that $[y_k]$ is 'arbitrarily close' (in the sense of the gap between subspaces; see [14] [Section IV.2]) to a subspace N of M; so the action of an operator on $[y_k]$ is also close to its action on N. This idea will appear several times in our arguments.

Lemma 3.1 *Let X be a Banach space with a monotone basis (x_n), let $M \in S(X)$ and $0 < \varepsilon < 1$. Then there exist a normalized block basis (y_k) of (x_n) and a subspace $N \in S(M)$ such that for every operator $T \in L(X, Y)$,*

$$\big|\|T_{[y_k]}\| - \|T_N\|\big| \leq \varepsilon\|T\| \quad \textit{and} \quad \big|j(T_{[y_k]}) - j(T_N)\big| \leq \varepsilon\|T\|.$$

Proof We will choose (y_k) and N so that the distance between the unit spheres of N and $[y_k]$ is smaller than ε; hence for each $n \in S_N$ there is $y \in S_{[y_k]}$ with $\|n - y\| < \varepsilon$, and for each $z \in S_{[y_k]}$ there is $m \in S_N$ with $\|z - m\| < \varepsilon$. Clearly this fact implies our result.

Let $r = \varepsilon/8$. Inductively, we will find integers $1 = j_1 \le l_1 < j_2 \le l_2 \le \cdots$ and a sequence (a_i) of scalars so that $y_k = \sum_{i=j_k}^{l_k} a_i x_i$ satisfies $\|y_k\| = 1$ and $\text{dist}(y_k, M) < r/2^{k+1}$.

Clearly, y_1 exists; so assume that y_k has been found for $k \le k_0$. Let (x_i^*) be the sequence in X^* such that $x_i^*(x_j) = \delta_{i,j}$. Since $M \cap \left(\cap_{i=1}^{l_{k_0}} N(x_i^*)\right)$ is infinite dimensional, y_{k_0+1} exists.

Since (y_k) is a monotone basic sequence (comment after [15][Definition 1.a.10]), there exists a sequence (y_k^*) in X^* with $\|y_k^*\| \le 2$ and $y_k^*(y_j) = \delta_{k,j}$.

For each $k \in \mathbb{N}$ we choose $m_k \in M$ with $\|y_k - m_k\| < r/2^{k+1}$, and define $K \in L(X)$ by

$$Kx := \sum_{k=1}^{\infty} y_k^*(x)(y_k - m_k).$$

Then K is bounded with $\|K\| \le \sum_{k=1}^{\infty} \|y_k^*\| \cdot \|y_k - m_k\| < r$; hence $I - K$ is bijective. Moreover $(I - K)y_k = m_k$ for each $k \in \mathbb{N}$. We take $N = [m_k] = (I - K)([y_k])$. Note that

$$(I - K)^{-1} = \sum_{l=0}^{\infty} K^l = I - L \text{ with } \|L\| \le \sum_{l=1}^{\infty} r^l = r/(1-r) < 2r.$$

For $n \in S_N$ we take $y = \|(I-L)n\|^{-1}(I-L)n \in S_{[y_k]}$. Then $1-2r < \|(I-L)n\| < 1+2r$ and

$$\|n - y\| = \frac{\big\|(\|(I-L)n\| - 1)n + Ln\big\|}{\|(I-L)n\|} \le \frac{4r}{1-2r} < 8r = \varepsilon.$$

Similarly, for each $z \in S_{[y_k]}$, we have $m = \|(I-K)z\|^{-1}(I-K)z \in S_N$ and $\|z - m\| < \varepsilon$. □

A Banach lattice is called *atomic* if its order is induced by a 1-unconditional basis.

Proposition 3.2 *Let E be an atomic Banach lattice. For an operator $T \in L(E, Y)$,*

$$\Gamma_d(T) = \Gamma(T)\,,\ \Delta_d(T) = \Delta(T)\,,\ \tau_d(T) = \tau(T) \text{ and } \kappa_d(T) = \kappa(T)\,.$$

Proof The inequality $\Gamma_d(T) \ge \Gamma(T)$ is valid in general. The converse inequality is obtained by applying Lemma 3.1. Suppose without loss generality that $\|T\| = 1$. Given $0 < \varepsilon < 1$ and a subspace M of E, there is a block basis (y_k) of the unconditional basis of E such that $[y_k]$ is arbitrarily close to some subspace N of M, and consequently

$$\Big|\|T_{[y_k]}\| - \|T_N\|\Big| \le \varepsilon\,.$$

Hence $\Gamma_d(T) \le \|T_{[y_k]}\| \le \|T_N\| + \varepsilon \le \|T_M\| + \varepsilon$. Therefore $\Gamma_d(T) \le \Gamma(T)$.

The other equalities can be proved in a similar way. □

Corollary 3.3 *We have $s_d\,\Gamma_d = s_d\,\Gamma$ and $i_d\,\tau_d = i_d\,\tau$. Moreover $\Gamma_d = i_d\,\Gamma_d = i_d\,\Gamma$ and $\tau_d = s_d\,\tau_d = s_d\,\tau$.*

Proof For each $(x_n) \in \text{d}(E)$, (x_n) is a 1-unconditional basis; hence $[x_n]$ is an atomic Banach lattice. Therefore

$$s_d\,\Gamma_d(T) = \sup_{(x_n)\in \text{d}(E)} \Gamma_d(T_{[x_n]}) = \sup_{(x_n)\in \text{d}(E)} \Gamma(T_{[x_n]}) = s_d\,\Gamma(T).$$

The proof of $i_d\,\tau_d = i_d\,\tau$, $i_d\,\Gamma_d = i_d\,\Gamma$ and $s_d\,\tau_d = s_d\,\tau$ is identical, and for the remaining equalities, note that $i_d\,i_d\,a = i_d\,a$ and $s_d\,s_d\,a = s_d\,a$ for any quantity a. □

4 Operational quantities derived from the norm

Our first result gives some alternative expressions for $\Gamma_d(T)$ in terms of the classical quantities.

Proposition 4.1 *For $T \in L(E, Y)$, we have $\Gamma_d(T) = i_d\Gamma(T) = i_d\Delta(T)$.*

Proof Note that $\Gamma_d = i_d \|\cdot\|$. Applying i_d to the inequalities $\Gamma \leq \Delta \leq \|\cdot\|$, we obtain $i_d\,\Gamma \leq i_d\,\Delta \leq i_d\,\|\cdot\|$, and Corollary 3.3 completes the proof. □

It was proved in [6] that $T \in \mathrm{L}(E, Y)$ is disjointly non-singular if and only if for every $(f_n) \in \mathrm{d}(E)$, the restriction $T_{[f_n]}$ is upper semi-Fredholm. Next we give a quantitative version of this result when E is an order continuous Banach lattice. Since $\Gamma_d(T) = i_d\Gamma(T)$ by Proposition 4.1, our result says that if $T \in \text{DN-S}(E, Y)$ then the restrictions $T_{[x_n]}$ are “uniformly" upper semi-Fredholm, in the sense that $\inf_{(x_n)\in \mathrm{d}(E)} \Gamma(T_{[x_n]}) > 0$.

Theorem 4.2 *Let E be an order continuous Banach lattice, and let $T \in L(E, Y)$. Then $T \in$ DN-S if and only if $\Gamma_d(T) > 0$.*

Proof Suppose that $\Gamma_d(T) > 0$. For every $(f_n) \in \mathrm{d}(E)$ we have that $\Gamma(T_{[f_n]}) > 0$, hence $T_{[f_n]}$ is upper semi-Fredholm. Consequently, T is disjointly non-singular (Theorem 2.1).

Conversely, we assume that $\Gamma_d(T) = 0$. By Theorem 2.1, it is enough to construct a normalized sequence $(h_n) \in \mathrm{d}(E)$ such that $\lim_{n\to\infty} \|Th_n\| = 0$.

For each $n \in \mathbb{N}$ there exists a normalized sequence $(f_{n,k})_k \in \mathrm{d}(E)$ such that $\|T_{[(f_{n,k})_k]}\| < 1/n$, and by Lemma 2.2 we can assume that the functions $f_{n,k}$ $(n, k \in \mathbb{N})$ are contained in a closed ideal of E which has a representation as a Köthe space.

Let $g_1 = f_{1,1}$. As $\lim_{k\to\infty} \mu(D(f_{2,k})) = 0$, by Lemma 2.3 we have $\lim_{k\to\infty} \|g_1 1_{D(f_{2,k})}\|_E = 0$. So we can find $k_2 > 1$ such that

$$\|g_1\| = 1\,,\ \|Tg_1\| < 1 \text{ and } \|g_1 1_{D(f_{2,k_2})}\|_E < \frac{1}{2^2}.$$

Then, taking $g_2 = f_{2,k_2}$, a similar argument using Lemma 2.3 shows that there exists $k_3 > k_2$ such that

$$\|g_2\| = 1\,,\ \|Tg_2\| < \frac{1}{2} \text{ and } \|g_i 1_{D(f_{3,k_3})}\|_E < \frac{1}{2^3} \quad \text{for} 1 \leq i < 3.$$

In this way we find a sequence $k_1 = 1 < k_2 < k_3 < \cdots$ such that, taking $g_l = f_{l,k_l}$ for each $l \in \mathbb{N}$, we have

$$\|g_l\| = 1\,,\ \|Tg_l\| < \frac{1}{l} \text{ and } \|g_i 1_{D(f_{l,k_{l+1}})}\| < \frac{1}{2^{l+1}} \ (1 \leq i < l+1).$$

Let $A_k = \cup_{j=k+1}^{\infty} D(g_j)$ and $\tilde{h}_k := g_k - g_k 1_{A_k}$. For $k < l$ we have $D(\tilde{h}_k) \cap D(g_l) = \emptyset$ and $D(\tilde{h}_l) \subset D(g_l)$, hence $D(\tilde{h}_k) \cap D(\tilde{h}_l) = \emptyset$. Thus the sequence $(\tilde{h}_k)$ is disjoint. Since $\|g_n\| = 1$,

$$\begin{aligned}|1 - \|\tilde{h}_n\|| &\leq \|g_n - \tilde{h}_n\| = \|g_n 1_{A_n}\| \\ &\leq \left\| \sum_{i=n+1}^{\infty} g_n 1_{D(g_i)} \right\| \leq \sum_{i=n+1}^{\infty} \|g_n 1_{D(g_i)}\| \\ &\leq \sum_{i=n+1}^{\infty} \frac{1}{2^i} = \frac{1}{2^n}.\end{aligned}$$

Taking $h_n = \|\tilde{h}_n\|^{-1}\tilde{h}_n$, we obtain $(h_n) \in \mathrm{d}(E)$ is normalized and

$$\begin{aligned}\|h_n - g_n\| &\leq \left\| \frac{\tilde{h}_n}{\|\tilde{h}_n\|} - \frac{g_n}{\|\tilde{h}_n\|} \right\| + \left\| \frac{g_n}{\|\tilde{h}_n\|} - g_n \right\| \\ &= \frac{\|\tilde{h}_n - g_n\|}{\|\tilde{h}_n\|} + \frac{|1 - \|\tilde{h}_n\|| \, \|g_n\|}{\|\tilde{h}_n\|} \\ &\leq \frac{2\|\tilde{h}_n - g_n\|}{\|\tilde{h}_n\|} \leq \frac{1}{2^{n-1}\|\tilde{h}_n\|}.\end{aligned}$$

Consequently $\lim_{n\to\infty} \|h_n - g_n\| = 0$, and $\|Th_n\| \leq \|T(h_n - g_n)\| + \|Tg_n\|$ and $\|Tg_n\| < 1/n$; hence $\lim_{n\to\infty} \|Th_n\| = 0$. □

Next we give some alternative expressions for $\Delta_d(T)$.

Proposition 4.3 *For $T \in L(E, Y)$, we have $\Delta_d(T) = s_d\Delta(T) = s_d\Gamma(T)$.*

Proof Note that $\Delta_d(T) = s_d\Gamma_d(T)$ and, by Corollary 3.3, $s_d\,\Gamma(T) = s_d\,\Gamma_d(T)$. So it is enough to observe that $s_d\,a(T) = s_d\,s_d\,a$ for any quantity a. □

Proposition 4.4 *$T \in L(E, Y)$ is disjointly strictly singular if and only if $\Delta_d(T) = 0$.*

Proof As $\Delta_d(T) = s_d\Delta(T)$, we have that $\Delta_d(T) = 0$ means that for every $(x_n) \in \mathrm{d}(E)$ we have that $\Delta(T_{[x_n]}) = 0$; that is, all the restrictions $T_{[x_n]}$ are strictly singular. By [6][Proposition 2.6], that is equivalent to T being disjointly strictly singular. □

Obviously, given $T \in \mathrm{L}(E, Y)$ and a scalar λ, $\Gamma_d(\lambda T) = |\lambda|\Gamma_d(T)$ and $\Delta_d(\lambda S) = |\lambda|\Delta_d(S)$. The following result complements these facts.

Proposition 4.5 *For operators $T, S \in L(E, Y)$, we have the following inequalities:*

(1) $\Gamma_d(T + S) \leq \Gamma_d(T) + \Delta_d(S)$ *and*
(2) $\Delta_d(T + S) \leq \Delta_d(T) + \Delta_d(S)$.

Proof Let $(x_n) \in \mathrm{d}(E)$. Then $\|(T + S)_{[x_n]}\| \leq \|T\| + \|S_{[x_n]}\|$, and taking the infimum over $(x_n) \in \mathrm{d}(E)$ we obtain $\Gamma_d(T + S) \leq \|T\| + \Gamma_d(S)$. Therefore

$$\Gamma_d(T + S) \leq \Gamma_d\left((T + S)_{[x_n]}\right) \leq \|T_{[x_n]}\| + \Gamma_d(S_{[x_n]}) \leq \|T_{[x_n]}\| + \Delta_d(S),$$

and taking again the infimum over $(x_n) \in \mathrm{d}(E)$ we get (1).

Let $(x_n) \in \mathrm{d}(E)$. From (1) we derive

$$\Gamma_d((T + S)_{[x_n]}) \leq \Gamma_d(T_{[x_n]}) + \Delta_d(S_{[x_n]}) \leq \Gamma_d(T_{[x_n]}) + \Delta_d(S),$$

and taking the supremum over (x_n) we get $\Delta_d(T + S) \leq \Delta_d(T) + \Delta_d(S)$. □

Since $\Delta_d(T) \leq \|T\|$, Theorem 4.2 and part (1) of Proposition 4.5 improve the results proved in [6] that, under some conditions, DN-S(E, Y) is stable under perturbation by small norm operators and DSS operators.

Corollary 4.6 *Let E be an order continuous Banach lattice. Then*

(1) *DSS(E, Y) is a closed subspace of $L(E, Y)$;*
(2) *DN-S(E, Y) is an open subset of $L(E, Y)$;*

(3) *If* $S \in DSS(E, Y)$*, then* $\Gamma_d(T + S) = \Gamma_d(T)$*, for all* $T \in L(E, Y)$*;* *in particular,* $T \in DN\text{-}S(E, Y)$ *implies* $T + S \in DN\text{-}S(E, Y)$*.*

Proof (1) If $T, S \in \mathrm{DSS}(E, Y)$, then $\Delta_d(T + S) \le \Delta_d(T) + \Delta_d(S) = 0$, so $T + S \in \mathrm{DSS}(E, Y)$; and $\Delta_d(\lambda T) = |\lambda|\Delta_d(T)$ implies $\lambda T \in \mathrm{DSS}(E, Y)$.

(2) If $T \in \mathrm{DN\text{-}S}(E, Y)$ and $S \in \mathrm{L}(E, Y)$ with $\|S\| < \Gamma_d(T)$, then $\Gamma_d(T + S) \ge \Gamma_d(T) - \Delta_d(S) \ge \Gamma_d(T) - \|S\| > 0$. Hence $T + S \in \mathrm{DN\text{-}S}(E, Y)$.

(3) Let $S \in \mathrm{DSS}(E, Y)$, so $\Delta_d(S) = 0$. For all $T \in \mathrm{L}(E, Y)$,

$$\Gamma_d(T + S) \le \Gamma_d(T) + \Delta_d(S) = \Gamma_d(T),$$

and similarly $\Gamma_d(T) = \Gamma_d(T + S - S) \le \Gamma_d(T + S)$. □

Part (2) of Corollary 4.6 was proved by Bilokopytov [1] using different techniques.

A closed subspace M of E is said to be *dispersed* if there is no sequence $(x_n) \in \mathrm{d}(E)$ such that $\lim_{n\to\infty} \mathrm{dist}(x_n, M) = 0$ (see [6][Definition 2.1]).

Remark 4.7 Let M be a non-dispersed closed subspace of E. Denoting by $\mathrm{ND}(M)$ the set of all closed subspaces of M which are non-dispersed in E, it readily follows from Lemma 3.1 that, for $T \in \mathrm{L}(E, Y)$,

$$\Gamma_d(T) = \inf_{M\in \mathrm{ND}(E)} \|T_M\| \quad \text{and} \quad \Delta_d(T) = \sup_{M_1\in \mathrm{ND}(E)} \inf_{M_2\in \mathrm{ND}(M_1)} \|T_{M_2}\|.$$

5 Operational quantities derived from the injection modulus

Next result gives other expressions for the quantity τ_d.

Proposition 5.1 *For* $T \in L(E, Y)$*, we have* $\tau_d(T) = s_d\kappa(T) = s_d\tau(T)$*.*

Proof As $j \le \kappa \le \tau$, we have $\tau_d = s_d j \le s_d\kappa \le s_d\tau$. Moreover, $s_d\,\tau = s_d\,\tau_d$ by Corollary 3.3. Hence

$$s_d\,\tau(T) = s_d\,\tau_d(T) = s_d\,s_d\,j(T) = s_d\,j(T) = \tau_d(T),$$

because $s_d\,s_d\,a = s_d\,a$ for every quantity a. □

Proposition 5.2 *Let* $T \in L(E, Y)$*. Then* $T \in DSS$ *if and only if* $\tau_d(T) = 0$*.*

Proof We have that $\tau_d(T) = 0$ is equivalent to $j(T_{[x_n]}) = 0$, for every sequence $(x_n) \in \mathrm{d}(E)$. This means that T is not an isomorphism on any subspace $[x_n]$ generated by a disjoint sequence. That is, T is disjointly strictly singular. □

Proposition 5.3 *For an operator* $T \in L(E, Y)$*, we have* $\kappa_d(T) = i_d\kappa(T) = i_d\tau(T)$*.*

Proof By Proposition 5.1, $\kappa \le \tau_d \le \tau$, hence $i_d\kappa \le i_d\tau_d = \kappa_d \le i_d\tau$. Moreover, arguing as in the proof of Corollary 3.3 we get $i_d\kappa = i_d\,\kappa_d = i_d\,i_d\,\tau_d = i_d\,\tau_d = i_d\,\tau$, and the result is proved. □

Like Theorem 4.2, by Proposition 5.3 the following result says that $T \in \mathrm{DN\text{-}S}(E, Y)$ if and only if the restrictions $T_{[x_n]}$ with $(x_n) \in \mathrm{d}(E)$ are “uniformly” upper semi-Fredholm, in the sense that $\inf_{(x_n)\in \mathrm{d}(E)} \kappa(T_{[x_n]}) > 0$.

Theorem 5.4 *Let E be an order continuous Banach lattice and let $T \in L(E, Y)$. Then $T \in DN\text{-}S$ if and only if $\kappa_d(T) > 0$.*

Proof By Proposition 5.3, $\kappa_d(T) = i_d\tau(T)$. Then if $\kappa_d(T) > 0$ and $(f_n) \in \mathrm{d}(E)$, $\tau(T_{[f_n]}) > 0$. Hence $T_{[f_n]}$ is not strictly singular, and T is disjointly non-singular by Theorem 2.1.

Conversely, suppose that $\kappa_d(T) = 0$. By Theorem 2.1, in order to show that T is not disjointly non-singular, it is enough to find a normalized $(h_n) \in \mathrm{d}(E)$ such that $\lim_{n\to\infty} Th_n = 0$.

For each $n \in \mathbb{N}$ there exists a normalized sequence $(f_{n,k})_k \in \mathrm{d}(E)$ such that

$$\tau_d(T_{[f_{n,k}]_k}) < \frac{1}{n},$$

and by Lemma 2.2 we can assume that the vectors $f_{n,k}$ are contained in a closed ideal that admits a representation as a Köthe space.

As $j(T_{[f_{1,k}]_k}) < 1$, there exists $g_1 \in [(f_{1,k})_k]$ with $\|Tg_1\| < 1$. From $\lim_{k\to\infty} \mu(D(f_{2,k})) = 0$, by Lemma 2.3 we have $\lim_{k\to\infty} \|g_1 1_{D(f_{2,k})}\|_E = 0$. So we can to take $k_2 > 1$ such that

$$\|g_1\| = 1\,,\ \|Tg_1\| < 1 \ \text{ and } \ \|g_1 1_{D(f_{2,k_2})}\|_E < \frac{1}{2^2}.$$

Moreover, from

$$j(T_{[(f_{2,k})_{k\geq k_2}]}) \leq \tau_d(T_{[(f_{2,k})_k]}) < \frac{1}{2}\,,$$

we obtain that there is $g_2 \in [(f_{2,k})_{k\geq k_2}]$ with $\|Tg_2\| < 1/2$. As $\lim_{k\to\infty} \mu(D(f_{3,k})) = 0$, by Lemma 2.3 we get $\lim_{k\to\infty} \|g_i 1_{D(f_{3,k})}\|_E = 0$, so we can take $k_3 > k_2$ such that

$$\|g_2\| = 1\,,\ \|Tg_2\| < \frac{1}{2} \ \text{ and } \ \|g_i 1_{D(f_{3,k_3})}\|_E < \frac{1}{2^3}\ (i \leq i < 3)\,.$$

Now, proceeding as in the proof of Theorem 4.2, we take $A_n = \cup_{j=n+1}^{\infty} D(g_j)$ and obtain a normalized sequence $h_n := \|g_n - g_n 1_{A_n}\|^{-1}(g_n - g_n 1_{A_n})$ in $\mathrm{d}(E)$. Since $\lim_{n\to\infty} \|Th_n\| = 0$, we conclude that $T \notin \mathrm{DN\text{-}S}(E, Y)$. □

To compare Theorem 5.4 with Theorem 4.2, observe that $\kappa_d \leq \Gamma_d$.

Proposition 5.5 *For operators $T, S \in L(E, Y)$, we have the following inequalities:*

(1) $\tau_d(T + S) \leq \tau_d(T) + \Delta_d(S)$ *and*
(2) $\kappa_d(T + S) \leq \kappa_d(T) + \Delta_d(S)$.

Proof Since $j(T + S) \leq j(T) + \|S\|$, for each $(x_n) \in \mathrm{d}(E)$ we get

$$j(T + S) \leq j((T + S)_{[x_n]}) \leq j(T_{[x_n]}) + \|S_{[x_n]}\| \leq \tau_d(T) + \|S_{[x_n]}\|,$$

and taking the infimum over (x_n) we obtain $j(T + S) \leq \tau_d(T) + \Gamma_d(S)$.

(1) For $(x_n) \in \mathrm{d}(E)$, we have $j((T + S)_{[x_n]}) \leq \tau_d(T_{[x_n]}) + \Gamma_d(S_{[x_n]}) \leq \tau_d(T) + \Gamma_d(S_{[x_n]})$, and taking the supremum over (x_n) we get $\tau_d(T + S) \leq \tau_d(T) + \Delta_d(S)$.
(2) Applying (1), $\tau_d((T + S)_{[x_n]}) \leq \tau_d(T_{[x_n]}) + \Delta_d(S_{[x_n]}) \leq \tau_d(T_{[x_n]}) + \Delta_d(S)$ for each $(x_n) \in \mathrm{d}(E)$. So taking the infimum over (x_n), we obtain $\kappa_d(T + S) \leq \kappa_d(T) + \Delta_d(S)$.

□

From Proposition 5.5, we could derive an alternative proof of Corollary 4.6.

Remark 5.6 As in Remark 4.7, we can give expressions for $\kappa_d(T)$ and $\tau_d(T)$ in terms of the restrictions of T to non-dispersed subspaces. For $T \in \mathrm{L}(E, Y)$,
$\tau_d(T) = \sup_{M \in \mathrm{ND}(E)} j(T_M)$ and $\kappa_d(T) = \inf_{M_1 \in \mathrm{ND}(E)} \sup_{M_2 \in \mathrm{ND}(M_1)} j(T_{M_2})$.

6 The quantity β

For an operator $T \in \mathrm{L}(E, Y)$, the following quantity was defined in [6]:

$$\beta(T) := \inf \left\{ \liminf_{n \to \infty} \|Tx_n\| : (x_n) \text{normalized disjoint in } E \right\}.$$

We have shown in Theorem 4.2 that the quantity Γ_d characterizes DN-S(E, Y) for E an order continuous Banach lattice. Moreover, it is related with β as follows:

Proposition 6.1 *Every operator $T \in L(E, Y)$ satisfies $\beta(T) \leq \Gamma_d(T)$.*

Proof Note that
$\beta(T) = \inf_{(x_n) \in \mathrm{d}(E)} \liminf_{n \to \infty} \left\| T \frac{x_n}{\|x_n\|} \right\| \leq \inf_{(x_n) \in \mathrm{d}(E)} \|T_{[x_n]}\| = \Gamma_d(T)$. □

It was proved in [6][Proposition 3.1] (see [4] for $p = 1$) that, for $1 \leq p < \infty$, an operator $T \in \mathrm{L}(L_p, Y)$ is disjointly non-singular if and only if $\beta(T) > 0$. Now we extend this result.

Proposition 6.2 *Let E be an order continuous Banach lattice. Then an operator $T \in L(E, Y)$ is disjointly non-singular if and only if $\beta(T) > 0$.*

Proof If $\beta(T) > 0$, then condition (4) in Theorem 2.1 is satisfied, hence $T \in$ DN-S(E, Y).

Suppose that $\beta(T) = 0$. Then for every $n \in \mathbb{N}$ we can find a normalized disjoint sequence $(f_{n,k})_{k \in \mathbb{N}}$ with $\|Tf_{n,k}\| < 1/n$ for every $k \in \mathbb{N}$, and proceeding as in the proof of Theorem 4.2, for each n we select k_n so that taking $g_n = f_{n,k_n}$ we have $\|g_i 1_{D(g_n)}\| < 2^{-n}$ for $1 \leq i < n$. The sequence (g_n) is almost disjoint (there exists a normalized disjoint sequence (h_n) in E such that $\lim_{n \to \infty} \|g_n - h_n\|_E = 0$). Then $\lim_{n \to \infty} \|Th_n\| = 0$, hence $T \notin$ DN-S(E, Y). □

By Proposition 6.1, $\beta \leq \Gamma_d$. In some cases, these two quantities coincide; for example, if $1 \leq p < 2$ and M is a dispersed subspace of L_p, then the quotient map $Q_M : L_p \to L_p/M$ satisfies $\beta(Q_M) = 1$ (see [6]), hence $\Gamma_d(Q_M) = \|Q_M\| = 1$. However, using the fact proved by Odell and Schlumprecht in [18] that the Banach space ℓ_2 is arbitrarily distortable, we show that these two quantities are not equivalent:

Example 6.3 For every $\lambda > 1$ and $\varepsilon > 0$, there exists a Banach space Y_λ isomorphic to ℓ_2 and an operator $T_\lambda \in \mathrm{L}(\ell_2, Y_\lambda)$ such that $0 < \lambda \cdot \beta(T_\lambda) \leq \Gamma_d(T_\lambda) + \varepsilon$. Thus there is no $C > 0$ such that $\Gamma_d \leq C \cdot \beta$.

Proof Since ℓ_2 is arbitrarily distortable [18], for every $\lambda > 1$ there is a norm $|\cdot|_\lambda$ on ℓ_2 equivalent to the usual one $\|\cdot\|_2$ such that, for each closed infinite dimensional subspace M of ℓ_2,

$$\sup \left\{ \frac{|x|_\lambda}{|y|_\lambda} : x, y \in M, \|x\|_2 = \|y\|_2 = 1 \right\} > \lambda. \tag{3}$$

We denote $Y_\lambda = (\ell_2, |\cdot|_\lambda)$ and T_λ the identity operator from ℓ_2 onto Y_λ.

Note that the operator T_λ is bounded below, and passing to a closed infinite dimensional subspace of ℓ_2 (that we can identify with ℓ_2, with the lattice structure determined by any orthonormal basis) we can assume that $\|T_\lambda\| < \Gamma_d(T_\lambda) + \varepsilon$.

By inequality (3), $\lambda j(T_\lambda) \leq \|T_\lambda\|$ and there exists g_1 with $\|g_1\|_2 = 1$ and $\lambda \cdot |g_1|_\lambda < \Gamma_d(T_\lambda) + \varepsilon$. Moreover, by the denseness of the span of the basis (e_n) of ℓ_2, we can choose $g_1 \in [e_1, \ldots, e_{m_1}]$ for some $m_1 \in \mathbb{N}$. Similarly, there exists $g_2 \in [e_i : i > m_1]$ with $\|g_2\|_2 = 1$ and $\lambda \cdot |g_2|_\lambda < \Gamma_d(T_\lambda) + \varepsilon$, and again we can choose $g_2 \in [e_{m_1+1}, \ldots, e_{m_2}]$ for some $m_2 > m_1$ in $\mathbb{N}$.

In this way we get a sequence $(g_n) \in \mathrm{d}(\ell_2)$ such that $\lambda \cdot |g_n|_\lambda = \lambda \cdot |T_\lambda g_n|_\lambda \leq \Gamma_d(T_\lambda) + \varepsilon$, which implies $\lambda \cdot \beta(T_\lambda) \leq \Gamma_d(T_\lambda) + \varepsilon$. □

7 Order between operational quantities

The order between the operational quantities derived from the norm and the injection modulus j is showed in the following diagram, where " $\rightarrow''$ means " $\leq''$:

$$\begin{array}{ccccccccc} & & \Gamma & \longrightarrow & \Gamma_d & \longrightarrow & \Delta_d & \longrightarrow & \Delta & \longrightarrow & \|\cdot\| \\ & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\ j & \longrightarrow & \kappa & \longrightarrow & \kappa_d & \longrightarrow & \tau_d & \longrightarrow & \tau & & \end{array}$$

The vertical arrows in the above diagram connect quantities that characterize the same classes of operators: upper semi-Fredholm, DN-S, DSS and strictly singular. We observe that none of these pairs are equivalent quantities.

Indeed, the quantities κ and Γ are not equivalent because ℓ_2 is arbitrarily distortable. Hence, by [8] [Theorem 3.4 and Corollary 3.5], there exist spaces $Y_n \simeq \ell_2$ and operators $T_n \in \mathrm{L}(\ell_2, Y_n)$ $(n \in \mathbb{N})$ such that $n \cdot \kappa(T_n) \leq \Gamma(T_n)$. Since ℓ_2 is an atomic Banach lattice, $\kappa_d(T_n) = \kappa(T_n)$ and $\Gamma_d(T_n) = \Gamma(T_n)$; hence κ_d and Γ_d are not equivalent.

Similarly, by [17][Proposition 1], the operators $T_n \in \mathrm{L}(\ell_2, Y_n)$ in the previous paragraph satisfy $n \cdot \tau(T_n) \leq \Delta(T_n)$, showing that τ and Δ are not equivalent, and also that τ_d and Δ_d are not equivalent.

7.1 Open questions

We finish the paper stating some open questions.

Question 1 Is $\kappa_d \leq D \cdot \beta$ for some constant $D > 0$?

If E is an order continuous Banach lattice then E is an ideal in E^{**} [16][Theorem 1.b.16], hence the quotient E^{**}/E is a Banach lattice [16][Sect. 1.a]. Moreover, every operator $T \in \mathrm{L}(E, Y)$ induces a *residuum operator* $T^{co} \in \mathrm{L}(E^{**}/E, Y^{**}/Y)$ defined by $T^{co}(x^{**} + E) = T^{**}x^{**} + Y$.

Question 2 Suppose that E is order continuous and $T \in$ DN-S(E, Y). Is $T^{co} \in$ DN-S?

It was proved in [4] that the answer is positive in the case $E = L_1$. We refer to [10] for information on the residuum operator T^{co}.

In [6][Theorem 3.16] it is shown that for $1 \leq p < \infty$, DSS(L_p, Y) is the perturbation class of DN-S(L_p, Y) in the sense that when DN-S$(L_p, Y) \neq \emptyset$, $K \in \mathrm{L}(L_p, Y)$ is DSS if and only if $T + K \in$ DN-S for each $T \in$ DN-S(L_p, Y).

Question 3 Suppose that E is an order continuous Banach lattice and DN-S$(E, Y) \neq \emptyset$. Is DSS(E, Y) the perturbation class of DN-S(E, Y)?

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