RACSAM (2021) 115:157
https://doi.org/10.1007/s13398-021-01097-1

ORIGINAL PAPER

Convolutors on $\mathcal{S}_\omega(\mathbb{R}^N)$

Angela A. Albanese[1] · Claudio Mele[1]

Received: 30 April 2021 / Accepted: 3 July 2021 / Published online: 15 July 2021

Abstract
In this paper we continue the study of the spaces $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$ and $\mathcal{O}_{C,\omega}(\mathbb{R}^N)$ undertaken in Albanese and Mele (J Pseudo-Differ Oper Appl, 2021). We determine new representations of such spaces and we give some structure theorems for their dual spaces. Furthermore, we show that $\mathcal{O}'_{C,\omega}(\mathbb{R}^N)$ is the space of convolutors of the space $\mathcal{S}_\omega(\mathbb{R}^N)$ of the ω-ultradifferentiable rapidly decreasing functions of Beurling type (in the sense of Braun, Meise and Taylor) and of its dual space $\mathcal{S}'_\omega(\mathbb{R}^N)$. We also establish that the Fourier transform is an isomorphism from $\mathcal{O}'_{C,\omega}(\mathbb{R}^N)$ onto $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$. In particular, we prove that this isomorphism is topological when the former space is endowed with the strong operator lc-topology induced by $\mathcal{L}_b(\mathcal{S}_\omega(\mathbb{R}^N))$ and the last space is endowed with its natural lc-topology.

Keywords Convolutor · Multiplier · Weight function · Ultradifferentiable function space · Fourier transform

Mathematics Subject Classification Primary 46E10 · 46F05; Secondary 47B38

1 Introduction

The study of the space $\mathcal{O}_M(\mathbb{R}^N)$ of multipliers and of the space $\mathcal{O}_C(\mathbb{R}^N)$ of convolutors of the space $\mathcal{O}`_C(\mathbb{R}^N)$ of rapidly decreasing functions was started by Schwartz [34]. Since then, the spaces $\mathcal{O}_M(\mathbb{R}^N)$ and $\mathcal{O}`_C(\mathbb{R}^N)$ attracted the attention of several authors, even recently, (see, f.i., [3,21–24,28–31] and the references therein). Their interest lies in the rich topological structure and in the importance of their application to the study of partial differential equations. In the case of ultradifferentiable classes of rapidly decreasing functions of Beurling or Roumieu type in the sense of Komatsu [25] or in the frame of Gelfand–Shilov spaces, the spaces of multipliers and of convolutors have been also considered and studied in the recent years (see, f.i., [15,17,18,26,27,32,33,36,37]). Inspired by this line of research, in [2] the authors recently introduced and studied the space $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$ of the slowly increasing

✉ Angela A. Albanese
angela.albanese@unisalento.it

Claudio Mele
claudio.mele1@unisalento.it

1 Dipartimento di Matematica e Fisica "E. De Giorgi", Università del Salento, C.P.193, 73100 Lecce, Italy

functions of Beurling type in the setting of ultradifferentiable function space in the sense of [12], showing there that it is the space of multipliers of the space $\mathcal{S}_\omega(\mathbb{R}^N)$ of the ω-ultradifferentiable rapidly decreasing functions of Beurling type, as introduced by Björck [6]. An analogous result for a more general classes of ultradifferentiable rapidly decreasing functions of Beurling type or Roumieu type was recently established also in [14]. We point out that, in general, the ultradifferentiable classes defined in one way cannot be defined in the other way (see [10]).

In this paper we continue the study of the spaces $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$ and $\mathcal{O}_{C,\omega}(\mathbb{R}^N)$ undertaken in [2], where their elements were defined in terms of weighted L^∞-norms. Our main aim is to show that $\mathcal{O}'_{C,\omega}(\mathbb{R}^N)$ is the space of convolutors of the space $\mathcal{S}_\omega(\mathbb{R}^N)$ and of its dual space $\mathcal{S}'_\omega(\mathbb{R}^N)$ (see Sect. 5). To this end, Sects. 3 and 4 are devoted to establish all the necessary results. In particular, in Sect. 3 we first prove that the elements of both spaces $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$ and $\mathcal{O}_{C,\omega}(\mathbb{R}^N)$ can be also defined in terms of weighted L^p-norms and then we give some structure theorems for their dual spaces. The characterization in terms of L^p-norms relies on an appropriate weighted Sobolev embedding theorem (Proposition 3.6). In Sect. 4 we introduce the weighted Fréchet spaces $\mathcal{D}_{L^p_\mu,\omega}(\mathbb{R}^N)$, $1 \leq p \leq \infty$, and study the convolution operators on their duals, i.e., on the weighted spaces of ultradistributions $\mathcal{D}'_{L^p_\mu,\omega}(\mathbb{R}^N)$ (of Beurling type) of $L^{p'}$-growth, p' being the conjugate exponent of p. Finally, in Sect. 6 we establish that the Fourier transform is an isomorphism from $\mathcal{O}'_{C,\omega}(\mathbb{R}^N)$ onto $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$. This isomorphim is topological when the former space is endowed with the strong operator lc-topology induced by $\mathcal{L}_b(\mathcal{S}_\omega(\mathbb{R}^N))$ and the last space is endowed with its natural lc-topology. We point out that the methods of the proofs are different from the ones used in [14,15]. Indeed, in [14,15] the proofs relies on tools from the time-frequency analysis as the short time Fourier transform (STFT).

2 Preliminary

We first give the definition of non-quasianalytic weight function in the sense of Braun-Meise-Taylor [12] suitable for the Beurling case, i.e., we also consider the logarithm as a weight function.

Definition 2.1 A non-quasianalytic weight function is a continuous increasing function $\omega : [0,\infty) \to [0,\infty)$ satisfying the following properties:

(α) there exists $K \geq 1$ such that $\omega(2t) \leq K(1+\omega(t))$ for every $t \geq 0$;
(β) $\int_1^\infty \frac{\omega(t)}{1+t^2}\,dt < \infty$;
(γ) there exist $a \in \mathbb{R}$, $b > 0$ such that $\omega(t) \geq a + b\log(1+t)$, for every $t \geq 0$;
(δ) $\varphi_\omega(t) = \omega \circ \exp(t)$ is a convex function.

We recall some known properties of the weight functions that shall be useful in the following (the proofs can be found in the literature):

(1) Condition (α) implies for every $t_1, t_2 \geq 0$ that

$$\omega(t_1 + t_2) \leq K(1 + \omega(t_1) + \omega(t_2)). \tag{2.1}$$

Observe that this condition is weaker than subadditivity (i.e., $\omega(t_1+t_2) \leq \omega(t_1)+\omega(t_2)$). The weight functions satisfying (α) are not necessarily subadditive in general.

(2) Condition (α) implies that there exists $L \geq 1$ such that for every $t \geq 0$

$$\omega(et) \leq L(1 + \omega(t)). \tag{2.2}$$

(3) By condition (γ) we have for every $\lambda \geq \frac{N+1}{bp}$ that

$$e^{-\lambda\omega(t)} \in L^p(\mathbb{R}^N). \tag{2.3}$$

Given a non-quasianalytic weight function ω, we define the Young conjugate φ^*_ω of φ_ω as the function $\varphi^*_\omega : [0, \infty) \to [0, \infty)$ by

$$\varphi^*_\omega(s) := \sup_{t\geq 0}\{st - \varphi_\omega(t)\}, \quad s \geq 0. \tag{2.4}$$

There is no loss of generality to assume that ω vanishes on $[0, 1]$. Moreover φ^*_ω is convex and increasing, $\varphi^*_\omega(0) = 0$ and $(\varphi^*_\omega)^* = \varphi_\omega$. Further useful properties of φ^*_ω are listed below (see [12]):

(1) $\frac{\varphi^*_\omega(t)}{t}$ is an increasing function in $(0, \infty)$.
(2) For every $s, t \geq 0$ and $\lambda > 0$

$$2\lambda\varphi^*_\omega\left(\frac{s+t}{2\lambda}\right) \leq \lambda\varphi^*_\omega\left(\frac{s}{\lambda}\right) + \lambda\varphi^*_\omega\left(\frac{t}{\lambda}\right) \leq \lambda\varphi^*_\omega\left(\frac{s+t}{\lambda}\right). \tag{2.5}$$

(3) For every $t \geq 0$ and $\lambda > 0$

$$\lambda L\varphi^*_\omega\left(\frac{t}{\lambda L}\right) + t \leq \lambda\varphi^*_\omega\left(\frac{t}{\lambda}\right) + \lambda L, \tag{2.6}$$

where $L \geq 1$ is the constant appearing in formula (2.2).
(4) For every $m, M \in \mathbb{N}$ with $M \geq mL$, where L is the constant appearing in formula (2.2), and for every $t \geq 0$

$$2^t \exp\left(M\varphi^*_\omega\left(\frac{t}{M}\right)\right) \leq C\exp\left(m\varphi^*_\omega\left(\frac{t}{m}\right)\right), \tag{2.7}$$

with $C := e^{mL}$.

We now introduce the ultradifferentiable function space $\mathcal{S}_\omega(\mathbb{R}^N)$ in the sense of Björk [6].

Definition 2.2 Let ω be a non-quasianalytic weight.
We denote by $\mathcal{S}_\omega(\mathbb{R}^N)$ the set of all functions $f \in L^1(\mathbb{R}^N)$ such that $f, \hat{f} \in C^\infty(\mathbb{R}^N)$ and for each $\lambda > 0$ and $\alpha \in \mathbb{N}_0^N$ we have

$$\|\exp(\lambda\omega)\partial^\alpha f\|_\infty < \infty \ \text{ and } \ \|\exp(\lambda\omega)\partial^\alpha \hat{f}\|_\infty < \infty\,, \tag{2.8}$$

where $\hat{f}$ denotes the Fourier transform of f. The elements of $\mathcal{S}_\omega(\mathbb{R}^N)$ are called *ω-ultradifferentiable rapidly decreasing functions of Beurling type*. We denote by $\mathcal{S}'_\omega(\mathbb{R}^N)$ the dual of $\mathcal{S}_\omega(\mathbb{R}^N)$ endowed with its strong topology.

We refer to [12] for the definition and the main properties of ultradifferentiable function spaces $\mathcal{E}_\omega(\Omega)$, $\mathcal{D}_\omega(\Omega)$ and their duals of Beurling type in the sense of Braun, Meise and Taylor. We now recall some properties of $\mathcal{S}_\omega(\mathbb{R}^N)$.

Remark 2.3 Let ω be a non-quasianalytic weight function.
(1) The condition (γ) of Definition 2.1 implies that $\mathcal{S}_\omega(\mathbb{R}^N) \subseteq \mathcal{S}(\mathbb{R}^N)$ with continuous inclusion. Accordingly, we can rewrite the definition of $\mathcal{S}_\omega(\mathbb{R}^N)$ as the set of all the rapidly decreasing functions that satisfy the condition (2.8).
(2) The space $\mathcal{S}_\omega(\mathbb{R}^N)$ is closed under convolution, under point-wise multiplication, translation and modulation, where the translation and modulation operators are defined by

$\tau_y f(x) := f(x-y)$ and $M_t f(x) := e^{itx} f(x)$, respectively, where $t, x, y \in \mathbb{R}^N$ ([6, Propositions 1.8.3 and 1.8.5]).

(3) The Fourier transform $\mathcal{F} : \mathcal{S}_\omega(\mathbb{R}^N) \to \mathcal{S}_\omega(\mathbb{R}^N)$ is a continuous isomorphism, that can be extended in the usual way to $\mathcal{S}'_\omega(\mathbb{R}^N)$ ([6, Proposition 1.8.2]), i.e. $\mathcal{F}(T)(f) := \langle T, \hat{f}\rangle$ for every $f \in \mathcal{S}_\omega(\mathbb{R}^N)$ and $T \in \mathcal{S}'_\omega(\mathbb{R}^N)$. Moreover, for every $f \in \mathcal{S}_\omega(\mathbb{R}^N)$ and $T \in \mathcal{S}'_\omega(\mathbb{R}^N)$ the convolution $(T\star f)(x) := \langle T_y, \tau_x \check{f}\rangle$, for $x \in \mathbb{R}^N$, (where $\check{f}$ is the function $x \mapsto f(-x)$) is a well defined function on $\mathbb{R}^N$ such that $T\star f \in \mathcal{S}'_\omega(\mathbb{R}^N)$, see [6, Theorem 1.8.12], and

$$\mathcal{F}(T\star f) = \hat{f}\mathcal{F}T. \tag{2.9}$$

(4) The space $\mathcal{S}_\omega(\mathbb{R}^N)$ is a nuclear Fréchet space, see, f.i., [9, Theorem 3.3] or [16, Theorem 1.1].

The space $\mathcal{S}_\omega(\mathbb{R}^N)$ is a Fréchet space with different equivalent systems of seminorms. Indeed, the following result holds.

Proposition 2.4 *Let ω be a non-quasianalytic weight function and consider $f \in \mathcal{S}(\mathbb{R}^N)$. Fix $1 \le p \le \infty$. Then $f \in \mathcal{S}_\omega(\mathbb{R}^N)$ if and only if one of the following conditions is satisfied.*

(1) (i) $\forall\lambda > 0,\ \alpha \in \mathbb{N}_0^N\ \exists C_{\alpha,\lambda,p} > 0$ *such that* $\|\exp(\lambda\omega)\partial^\alpha f\|_p \le C_{\alpha,\lambda,p}$, *and*
(ii) $\forall\lambda > 0,\ \alpha \in \mathbb{N}_0^N\ \exists C_{\alpha,\lambda,p} > 0$ *such that* $\|\exp(\lambda\omega)\partial^\alpha \hat{f}\|_p \le C_{\alpha,\lambda,p}$.

(2) (i) $\forall\lambda > 0\ \exists C_{\lambda,p} > 0$ *such that* $\|\exp(\lambda\omega) f\|_p \le C_{\lambda,p}$, *and*
(ii) $\forall\lambda > 0\ \exists C_{\lambda,p} > 0$ *such that* $\|\exp(\lambda\omega) \hat{f}\|_p \le C_{\lambda,p}$.

(3) $\forall\lambda,\ \mu > 0\ \exists C_{\lambda,\mu,p} > 0$ *such that*

$$q_{\lambda,\mu,p}(f) := \sup_{\alpha\in\mathbb{N}_0^N} \|\exp(\mu\omega)\partial^\alpha f\|_p \exp\left(-\lambda\varphi^*_\omega\left(\frac{|\alpha|}{\lambda}\right)\right) \le C_{\lambda,\mu,p}.$$

If $1 \le p < \infty$, then conditions (1)÷(3) are equivalent to

(4) $\forall\lambda,\ \mu > 0\ \exists C_{\lambda,\mu,p} > 0$ *such that*

$$\sigma_{\lambda,\mu,p}(f) := \left(\sum_{\alpha\in\mathbb{N}_0^N} \|\exp(\mu\omega)\partial^\alpha f\|_p^p \exp\left(-p\lambda\varphi^*_\omega\left(\frac{|\alpha|}{\lambda}\right)\right)\right)^{\frac{1}{p}} \le C_{\lambda,\mu,p}.$$

Proof For a proof of (1)⇔(2)⇔(3) we refer to [8, Theorem 4.8] and [7, Theorem 2.6].

(3)⇒(4). Fix $\lambda, \mu > 0$. Then by (2.6)

$$\begin{aligned}(\sigma_{\lambda,\mu,p}(f))^p &= \sum_{\alpha\in\mathbb{N}_0^N} \|\exp(\mu\omega)\partial^\alpha f\|_p^p \exp\left(-p\lambda\varphi^*_\omega\left(\frac{|\alpha|}{\lambda}\right)\right)\\ &\le \sum_{\alpha\in\mathbb{N}_0^N} \|\exp(\mu\omega)\partial^\alpha f\|_p^p \exp\left(-pL\lambda\varphi^*_\omega\left(\frac{|\alpha|}{L\lambda}\right)\right)\exp(-p|\alpha| + pL\lambda)\\ &= \exp(pL\lambda)(q_{L\lambda,\mu,p}(f))^p \sum_{\alpha\in\mathbb{N}_0^N} \exp(-p|\alpha|),\end{aligned}$$

where $\sum_{\alpha\in\mathbb{N}_0^N} \exp(-p|\alpha|) < \infty$. So,

$$\sigma_{\lambda,\mu,p}(f) \le \exp(L\lambda)\left(\sum_{\alpha\in\mathbb{N}_0^N} \exp(-p|\alpha|)\right)^{\frac{1}{p}} q_{L\lambda,\mu,p}(f) < \infty.$$

(4)⇒(3). Fix $\lambda, \mu > 0$. Then for every $\alpha \in \mathbb{N}_0^N$

$$\|\exp(\mu\omega)\partial^\alpha f\|_p \exp\left(-\lambda\varphi_\omega^*\left(\frac{|\alpha|}{\lambda}\right)\right) \le \sigma_{\lambda,\mu,p}(f).$$

Accordingly, $q_{\lambda,\mu,p}(f) \le \sigma_{\lambda,\mu,p}(f) < \infty$. □

In the following, we will often use this system of norms generating the Fréchet topology of $\mathcal{S}_\omega(\mathbb{R}^N)$

$$q_{\lambda,\mu}(f) := q_{\lambda,\mu,\infty}(f),\ \lambda, \mu > 0,\ f \in \mathcal{S}_\omega(\mathbb{R}^N), \tag{2.10}$$

or equivalently, the sequence of norms $\{q_{m,m}\}_{m\in\mathbb{N}}$.

3 The spaces $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$ and $\mathcal{O}_{C,\omega}(\mathbb{R}^N)$ and their duals

The elements of the spaces $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$ and $\mathcal{O}_{C,\omega}(\mathbb{R}^N)$ have been defined in [2] in terms of weighted L^∞-norms. The main aim of this section is to show that the elements of such spaces can be also defined in terms of weighted L^p-norms, with $1 \le p < \infty$. This characterization allows to easily show some structure theorems for their strong dual spaces. In order to do this, we begin by recalling the definition and some basic properties of the spaces $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$ and $\mathcal{O}_{C,\omega}(\mathbb{R}^N)$ given in [2].

Definition 3.1 Let ω be a non-quasianalytic weight function.

(a) For $m \in \mathbb{N}$ and $n \in \mathbb{Z}$ we define the space $\mathcal{O}^m_{n,\omega}(\mathbb{R}^N)$ as the set of all functions $f \in C^\infty(\mathbb{R}^N)$ satisfying the following condition:

$$r_{m,n}(f) := \sup_{\alpha\in\mathbb{N}_0^N} \sup_{x\in\mathbb{R}^N} |\partial^\alpha f(x)| \exp\left(-n\omega(x) - m\varphi_\omega^*\left(\frac{|\alpha|}{m}\right)\right) < \infty. \tag{3.1}$$

The space $(\mathcal{O}^m_{n,\omega}(\mathbb{R}^N), r_{m,n})$ is a Banach space.

(b) We define the space $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$ by

$$\mathcal{O}_{M,\omega}(\mathbb{R}^N) := \bigcap_{m=1}^\infty \bigcup_{n=1}^\infty \mathcal{O}^m_{n,\omega}(\mathbb{R}^N). \tag{3.2}$$

The elements of $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$ are called *slowly increasing functions of Beurling type*. The space $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$ is endowed with its natural lc-topology t, i.e., $\mathcal{O}_{M,\omega}(\mathbb{R}^N) = \text{proj}_{\overleftarrow{m}}\ \text{ind}_{\overrightarrow{n}}\ \mathcal{O}^m_{n,\omega}(\mathbb{R}^N)$ is a projective limit of (LB)-spaces.

(c) We define the space $\mathcal{O}_{C,\omega}(\mathbb{R}^N)$ by

$$\mathcal{O}_{C,\omega}(\mathbb{R}^N) := \bigcup_{n=1}^\infty \bigcap_{m=1}^\infty \mathcal{O}^m_{n,\omega}(\mathbb{R}^N). \tag{3.3}$$

The elements of $\mathcal{O}_{C,\omega}(\mathbb{R}^N)$ are called *very slowly increasing functions of Beurling type*. The space $\mathcal{O}_{C,\omega}(\mathbb{R}^N)$ is endowed with its natural lc-topology, i.e., $\mathcal{O}_{C,\omega}(\mathbb{R}^N) = \text{ind}_{\overrightarrow{n}}\ \text{proj}_{\overleftarrow{m}}\ \mathcal{O}^m_{n,\omega}(\mathbb{R}^N)$ is an (LF)-space.

The space $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$ is the space of multipliers of $\mathcal{S}_\omega(\mathbb{R}^N)$ and of $\mathcal{S}'_\omega(\mathbb{R}^N)$ as proved in [2,14]. In particular, in [14, Theorem 5.3] it is shown that $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$ is an ultrabornological space. On the other hand, by [2, Theorems 4.4, 4.6 and 5.1] the following result holds.

Theorem 3.2 *Let ω be a non-quasianalytic weight function and $f \in C^\infty(\mathbb{R}^N)$. Then the following properties are equivalent.*

(1) $f \in \mathcal{O}_{M,\omega}(\mathbb{R}^N)$.
(2) *For every $g \in \mathcal{S}_\omega(\mathbb{R}^N)$ and $m \in \mathbb{N}$ we have*

$$q_{m,g}(f) := \sup_{\alpha\in\mathbb{N}_0^N} \sup_{x\in\mathbb{R}^N} |g(x)||\partial^\alpha f(x)| \exp\left(-m\varphi_\omega^*\left(\frac{|\alpha|}{m}\right)\right) < \infty. \tag{3.4}$$

(3) *For every $g \in \mathcal{S}_\omega(\mathbb{R}^N)$ we have $fg \in \mathcal{S}_\omega(\mathbb{R}^N)$.*
(4) *For every $T \in \mathcal{S}'_\omega(\mathbb{R}^N)$ we have $fT \in \mathcal{S}'_\omega(\mathbb{R}^N)$.*

Moreover, if $f \in \mathcal{O}_{M,\omega}(\mathbb{R}^N)$, then the linear operators $M_f : \mathcal{S}_\omega(\mathbb{R}^N) \to \mathcal{S}_\omega(\mathbb{R}^N)$ defined by $M_f(g) := fg$, for $g \in \mathcal{S}_\omega(\mathbb{R}^N)$, and $\mathcal{M}_f : \mathcal{S}'_\omega(\mathbb{R}^N) \to \mathcal{S}'_\omega(\mathbb{R}^N)$ defined by $\mathcal{M}_f(T) := fT$, for $T \in \mathcal{S}'_\omega(\mathbb{R}^N)$, are continuous.

The set $\{q_{m,g}\}_{m\in\mathbb{N},g\in\mathcal{S}_\omega(\mathbb{R}^N)}$ defines a complete Hausdorff lc-topology τ on $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$ weaker than t [2, Theorem 5.2(2) and Proposition 5.6]. Actually, by combining [2, Proposition 5.6 and Theorem 5.9] with [14, Theorem 5.2] it follows that $t = \tau$. So, the lc-topology t is described by means of $\{q_{m,g}\}_{m\in\mathbb{N},g\in\mathcal{S}_\omega(\mathbb{R}^N)}$. Moreover, the inclusions $\mathcal{D}_\omega(\mathbb{R}^N) \hookrightarrow \mathcal{S}_\omega(\mathbb{R}^N) \hookrightarrow \mathcal{O}_{C,\omega}(\mathbb{R}^N) \hookrightarrow \mathcal{O}_{M,\omega}(\mathbb{R}^N) \hookrightarrow \mathcal{E}_\omega(\mathbb{R}^N)$ are well-defined, continuous with dense range, see [2, Theorems 3.8, 3.9 and 5.2(1)].

So, denoted by $\mathcal{O}'_{C,\omega}(\mathbb{R}^N)$ ($\mathcal{O}'_{M,\omega}(\mathbb{R}^N)$, resp.) the strong dual space of $\mathcal{O}_{C,\omega}(\mathbb{R}^N)$ ($\mathcal{O}_{M,\omega}(\mathbb{R}^N)$, resp.), the inclusions $\mathcal{E}'_\omega(\mathbb{R}^N) \hookrightarrow \mathcal{O}'_{M,\omega}(\mathbb{R}^N) \hookrightarrow \mathcal{O}'_{C,\omega}(\mathbb{R}^N) \to \mathcal{S}'_\omega(\mathbb{R}^N) \to \mathcal{D}'_\omega(\mathbb{R}^N)$

are also well-defined and continuous. On the other hand, $\mathcal{O}_{M,\omega}(\mathbb{R}^N) \hookrightarrow \mathcal{S}'_\omega(\mathbb{R}^N)$ and $\mathcal{O}_{C,\omega}(\mathbb{R}^N) \hookrightarrow \mathcal{S}'_\omega(\mathbb{R}^N)$ continuously, as it is easy to see.

In order to characterize the elements of the spaces $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$ and $\mathcal{O}_{C,\omega}(\mathbb{R}^N)$ in terms of weighted L^p-norms, we first observe the following fact.

Proposition 3.3 *Let ω be a non-quasianalytic weight function. Then the following properties hold.*

(1) *$f \in \mathcal{O}_{M,\omega}(\mathbb{R}^N)$ if and only if $f \in C^\infty(\mathbb{R}^N)$ and for each $m \in \mathbb{N}$ there exist $C > 0$ and $n \in \mathbb{N}$ such that for every $\alpha \in \mathbb{N}_0^N$ and $x \in \mathbb{R}^N$ we have*

$$|\partial^\alpha f(x)| \leq C \exp\left(n\omega(x) + m\varphi_\omega^*\left(\frac{|\alpha|}{m}\right) - |\alpha|\right). \tag{3.5}$$

(2) *$f \in \mathcal{O}_{C,\omega}(\mathbb{R}^N)$ if and only if $f \in C^\infty(\mathbb{R}^N)$ and there exists $n \in \mathbb{N}$ such that for every $m \in \mathbb{N}$ there exists $C > 0$ so that for every $\alpha \in \mathbb{N}_0^N$ and $x \in \mathbb{R}^N$ we have*

$$|\partial^\alpha f(x)| \leq C \exp\left(n\omega(x) + m\varphi_\omega^*\left(\frac{|\alpha|}{m}\right) - |\alpha|\right). \tag{3.6}$$

Proof It is straightforward. □

Remark 3.4 Suppose that the function $f \in C^\infty(\mathbb{R}^N)$ satisfies the condition

$$|\partial^\alpha f(x)| \leq C \exp\left(n\omega(x) + m\varphi_\omega^*\left(\frac{|\alpha|}{m}\right) - |\alpha|\right),$$

for every $\alpha \in \mathbb{N}_0^N$, $x \in \mathbb{R}^N$ and for some $n, m \in \mathbb{N}$. Then, for a fixed $1 \le p < \infty$, the function f clearly satisfies the condition

$$\exp(-(n+n_0)\omega(x))|\partial^\alpha f(x)| \le C \exp\left(-n_0\omega(x) + m\varphi_\omega^*\left(\frac{|\alpha|}{m}\right) - |\alpha|\right),$$

for every $\alpha \in \mathbb{N}_0^N$ and $x \in \mathbb{R}^N$ and for $n_0 := \left[\frac{N+1}{bp}\right] + 1 > \frac{N+1}{bp}$, where b is the constant appearing in condition (γ). Since $\exp(-n_0\omega) \in L^p(\mathbb{R}^N)$ by (2.3), it follows for every $\alpha \in \mathbb{N}_0^N$ that

$$\| \exp(-(n+n_0)\omega)\partial^\alpha f\|_p \le C\| \exp(-n_0\omega)\|_p \exp\left(m\varphi_\omega^*\left(\frac{|\alpha|}{m}\right) - |\alpha|\right).$$

Accordingly, we obtain that

$$\sum_{\alpha\in\mathbb{N}_0^N} \| \exp(-(n+n_0)\omega)\partial^\alpha f\|_p^p \exp\left(-pm\varphi_\omega^*\left(\frac{|\alpha|}{m}\right)\right) \le (CD)^p \sum_{\alpha\in\mathbb{N}_0^N} \exp(-p|\alpha|) < \infty,$$

where $D := \| \exp(-n_0\omega)\|_p$.

In view of Remark 3.4 above it is natural to introduce the following spaces of C^∞ functions on $\mathbb{R}^N$.

Definition 3.5 Let ω be a non-quasianalytic weight function and $1 \le p < \infty$.

(a) For $m \in \mathbb{N}$ and $n \in \mathbb{Z}$ we define the space $\mathcal{O}^m_{n,\omega,p}(\mathbb{R}^N)$ as the set of all functions $f \in C^\infty(\mathbb{R}^N)$ satisfying the following condition:

$$r^p_{m,n,p}(f) := \sum_{\alpha\in\mathbb{N}_0^N} \| \exp(-n\omega)\partial^\alpha f\|_p^p \exp\left(-mp\varphi_\omega^*\left(\frac{|\alpha|}{m}\right)\right) < \infty. \tag{3.7}$$

The space $(\mathcal{O}^m_{n,\omega,p}(\mathbb{R}^N), r_{m,n,p})$ is a Banach space.

(b) We define the space $\mathcal{O}_{M,\omega,p}(\mathbb{R}^N)$ by

$$\mathcal{O}_{M,\omega,p}(\mathbb{R}^N) := \bigcap_{m=1}^\infty \bigcup_{n=1}^\infty \mathcal{O}^m_{n,\omega,p}(\mathbb{R}^N). \tag{3.8}$$

The space $\mathcal{O}_{M,\omega,p}(\mathbb{R}^N)$ is endowed with its natural lc-topology, i.e., $\mathcal{O}_{M,\omega,p}(\mathbb{R}^N) = \operatorname{proj}_{\leftarrow m} \operatorname{ind}_{\to n} \mathcal{O}^m_{n,\omega,p}(\mathbb{R}^N)$ is a projective limit of (LB)-spaces.

(c) We define the space $\mathcal{O}_{C,\omega,p}(\mathbb{R}^N)$ by

$$\mathcal{O}_{C,\omega,p}(\mathbb{R}^N) := \bigcup_{n=1}^\infty \bigcap_{m=1}^\infty \mathcal{O}^m_{n,\omega,p}(\mathbb{R}^N). \tag{3.9}$$

The space $\mathcal{O}_{C,\omega,p}(\mathbb{R}^N)$ is endowed with its natural lc-topology, i.e., $\mathcal{O}_{C,\omega,p}(\mathbb{R}^N) = \operatorname{ind}_{\to n} \operatorname{proj}_{\leftarrow m} \mathcal{O}^m_{n,\omega,p}(\mathbb{R}^N)$ is an (LF)-space.

The aim is to show that $\mathcal{O}_{M,\omega}(\mathbb{R}^N) = \mathcal{O}_{M,\omega,p}(\mathbb{R}^N)$ and $\mathcal{O}_{C,\omega}(\mathbb{R}^N) = \mathcal{O}_{C,\omega,p}(\mathbb{R}^N)$ algebraically and topologically for every $1 \le p < \infty$. To do this, we introduce the weighted space $W^{k,p}(\mathbb{R}^N, \exp(-n\omega(x))\,dx)$, with $1 \le p \le \infty$, $n \in \mathbb{N}_0$ and $k \in \mathbb{N}_0 \cup \{\infty\}$, defined as the set of all functions $f \in W^{k,p}_{loc}(\mathbb{R}^N)$ such that

$$\|f\|_{k,p,\exp(-n\omega)} := \sum_{|\alpha|\le k} \| \exp(-n\omega)\partial^\alpha f\|_p < \infty.$$

Since the weight $\exp(-n\omega(x)) \in L^\infty(\mathbb{R}^N)$ is a positive function on $\mathbb{R}^N$, it is straithforward to verify that $(W^{k,p}(\mathbb{R}^N, \exp(-n\omega(x))\,dx), \|\cdot\|_{k,p,\exp(-n\omega)})$ is a Banach space and that $C_0^k(\mathbb{R}^N)$ is a dense subspace of $(W^{k,p}(\mathbb{R}^N, \exp(-n\omega(x))\,dx), \|\cdot\|_{k,p,\exp(-n\omega)})$.

We now show that an appropriate embedding theorem is also valid in the setting of the spaces introduced above.

Proposition 3.6 *Let ω be a non-quasianalytic weight function and let $k \in \mathbb{N}$ and $1 \le p < \infty$. If $kp > N$, then for each $n \in \mathbb{N}_0$ there exist $n' \ge n$ with $n' = n'(n,\omega) \in \mathbb{N}_0$ and $C > 0$ with $C = C(n, N, k, p)$ such that for every $f \in W^{k,p}(\mathbb{R}^N, \exp(-n\omega(x))\,dx)$ the following inequality is valid:*

$$\|f \exp(-n'\omega)\|_\infty \le C\|f\|_{k,p,\exp(-n\omega)}. \tag{3.10}$$

Proof We first consider the case $k = 1$ and so $p > N$.

By Morrey's inequality ([19, Theorem 4, p.266]), there exists a constant $C_N > 0$ such that for every $f \in C^1(\mathbb{R}^N)$, $x \in \mathbb{R}^N$ and $r > 0$ we have

$$\frac{1}{|B_r(x)|}\int_{B_r(x)} |f(y) - f(x)|\,dy \le C_N \int_{B_r(x)} \frac{|Df(y)|}{|y-x|^{N-1}}\,dy. \tag{3.11}$$

Now, let $n' \ge Kn$ with $n' \in \mathbb{N}_0$, where $K \ge 1$ is the constant appearing in Definition 2.1(α). Since $\omega(y) = \omega(|(y-x)+x)|) \le \omega(|x-y|+|x|)) \le K(\omega(x-y)+\omega(x)+1) \le K(\omega(1)+\omega(x)+1)$ for $x, y \in \mathbb{R}^N$ with $|x-y| \le 1$, it follows by (3.11) for every $f \in C^1(\mathbb{R}^N)$ and $x \in \mathbb{R}^N$, that

$$\begin{aligned}
|f(x)|\exp(-n'\omega(x)) &\le |f(x)|\exp(-Kn\omega(x)) = \frac{1}{|B_1(x)|}\int_{B_1(x)} |f(x)|\exp(-Kn\omega(x))\,dy\\
&\le \frac{1}{|B_1(x)|}\int_{B_1(x)} |f(x) - f(y) + f(y)|\exp(-Kn\omega(x)\,dy\\
&\le C_N \int_{B_1(x)} \frac{|Df(y)|\exp(-Kn\omega(x))}{|y-x|^{N-1}}\,dy + \frac{1}{|B_1(x)|}\int_{B_1(x)} |f(y)|\exp(-Kn\omega(x))dy\\
&\le C_N \exp(Kn(1+\omega(1)))\int_{B_1(x)} \frac{|Df(y)|\exp(-n\omega(y))}{|y-x|^{N-1}}\,dy\\
&\quad + \frac{\exp(Kn(1+\omega(1)))}{|B_1(x)|}\int_{B_1(x)} |f(y)|\exp(-n\omega(y))dy.
\end{aligned}$$

So, setting $C := \max\left\{C_N \exp(Kn(1+\omega(1))), \frac{\exp(Kn(1+\omega(1)))}{|B_1(x)|}\right\}$ and applying Hölder's inequality, we get for every $f \in C_0^1(\mathbb{R}^N)$ and $x \in \mathbb{R}^N$ that

$$\begin{aligned}
&|f(x)|\exp(-n'\omega(x))\\
&\le C\left(\int_{B_1(x)} |Df(y)|^p \exp(-pn\omega(y))\,dy\right)^{\frac{1}{p}} \left(\int_{B_1(x)} \frac{1}{|y-x|^{(N-1)p'}}\,dy\right)^{\frac{1}{p'}}\\
&\quad + C|B_1(x)|^{1/p'}\left(\int_{B_1(x)} |f(y)|^p \exp(-pn\omega(y))dy\right)^{\frac{1}{p}} \le C'\|f\|_{1,p,\exp(-n\omega)},
\end{aligned}$$

after having observed that $\left(\int_{B_1(x)} \frac{1}{|y-x|^{(N-1)p'}}\,dy\right)^{\frac{1}{p'}} < \infty$ as $p > N$. Therefore, we have for every $f \in C_0^1(\mathbb{R}^N)$ that

$$\|f\exp(-n'\omega)\|_\infty \le C'\|f\|_{1,p,\exp(-n\omega)}.$$

Thus, (3.10) is proved for $k=1$ as $C_0^1(\mathbb{R}^N)$ is a dense subspace of $W^{1,p}(\mathbb{R}^N, \exp(-n\omega(x))dx)$. To conclude the proof in the case $k > 1$, we proceed as follows.

If $k > 1$ but $p > N$, we have by the result proved above that for every $f \in W^{k,p}(\mathbb{R}^N, \exp(-n\omega(x))dx)$ and $x \in \mathbb{R}^N$

$$\begin{aligned}
&|f(x)| \exp(-n'\omega(x)) \\
&\quad \leq C' \left[\left(\int_{B_1(x)} |f(y)|^p \exp(-pn\omega(y))dy \right)^{\frac{1}{p}} + \left(\int_{B_1(x)} |Df(y)|^p \exp(-pn\omega(y))\, dy \right)^{\frac{1}{p}} \right] \\
&\quad \leq C'' \sum_{|\alpha| \leq k} \left(\int_{B_1(x)} |\partial^\alpha f(y)|^p \exp(-pn\omega(y))dy \right)^{\frac{1}{p}} \leq C'' ||f||_{k,p,\exp(-n\omega)},
\end{aligned}$$

with $C''=C''(C', N) > 0$. Accordingly, we obtain for every $f \in W^{k,p}(\mathbb{R}^N, \exp(-n\omega(x))dx)$ that

$$||f \exp(-n'\omega)||_\infty \leq C'' ||f||_{k,p,\exp(-n\omega)}.$$

If $p \leq N < kp$, then there exists $j \in \mathbb{N}$ with $1 \leq j \leq k-1$ such that $jp \leq N < (j+1)p$. If $jp < N$, we set $r := \frac{Np}{N-jp}$. If $jp = N$, we choose $r > \max\{N, p\}$. In both cases, $r > N$ and $r \geq p$. So, by applying again Morrey's inequality, we get for every $f \in C^1(\mathbb{R}^N)$ and $x \in \mathbb{R}^N$, that

$$|f(x)| \leq C_1' \left[\left(\int_{B_1(x)} |f(y)|^r dy \right)^{\frac{1}{r}} + \left(\int_{B_1(x)} |Df(y)|^r \, dy \right)^{\frac{1}{r}} \right],$$

where $C_1' = C_1'(N, r) > 0$. But if either $jp < N$ and $r = \frac{Np}{N-jp}$ or $jp = N$ and $r > \max\{N, p\}$, then $W^{j,p}(B_1(x)) \hookrightarrow L^r(B_1(x))$ for every $x \in \mathbb{R}$, where the constant $C_1'' > 0$ for this imbedding depends only on j, p, N, r (see, f.i., [1, Lemma 5.14, pg. 106]). On the other hand, we have for every $x \in \mathbb{R}$ and $g \in L^p(B_1(x))$ that

$$\begin{aligned}
\int_{B_1(x)} |g(y)|^p \, dy &= \int_{B_1(x)} |g(y)|^p \exp(-pn\omega(y)) \exp(pn\omega(y))dy \\
&\leq \int_{B_1(x)} |g(y)|^p \exp(-pn\omega(y)) \exp(pKn\omega(x-y) + pnK) \exp(pKn\omega(x))\, dy \\
&\leq \exp(pKn\omega(1) + pKn) \exp(pKn\omega(x)) \int_{B_1(x)} |g(y)|^p \exp(-pn\omega(y))\, dy.
\end{aligned}$$

By combining all these facts, we obtain for every $f \in C_0^k(\mathbb{R}^N)$ and $x \in \mathbb{R}$ that

$$\begin{aligned}
|f(x)| &\leq C_1' C_1'' (||f||_{W^{j,p}(B_1(x))} + ||Df||_{W^{j,p}(B_1(x))}) \\
&\leq C_2 \exp(Kn\omega(x)) (||f||_{W^{j,p}(B_1(x),\exp(-n\omega)\,dy)} + ||Df||_{W^{j,p}(B_1(x),\exp(-n\omega(y))\,dy)}) \\
&\leq C_2 \exp(n'\omega(x)) ||f||_{k,p,\exp(-n\omega)}.
\end{aligned}$$

So, also in this case we have for every $f \in W^{k,p}(\mathbb{R}^N, \exp(-n\omega(x))dx)$ that

$$||f \exp(-n'\omega)||_\infty \leq C_2 ||f||_{k,p,\exp(-n\omega)},$$

with $C_2 > 0$ depending on n, N, k, p. So, the proof is complete. □

Remark 3.7 If the weight function ω is sub-additive, i.e., $\omega(s+t) \leq \omega(s)+\omega(t)$ for $s, t \geq 0$, from the proof above it follows that $||f \exp(-n\omega)||_\infty \leq C||f||_{k,p,\exp(-n\omega)}$ whenever $n \in \mathbb{N}_0$ and $f \in W^{k,p}(\mathbb{R}^N, \exp(-n\omega(x))\,dx)$ with $kp > N$.

Thanks to Propositions 3.3 and 3.6 , and Remark 3.4, we are now able to show the main result of this section.

Proposition 3.8 *Let ω be a non-quasianalytic weight function and $1 \leq p < \infty$. Then the following properties are satisfied.*

(1) $\mathcal{O}_{M,\omega}(\mathbb{R}^N) = \mathcal{O}_{M,\omega,p}(\mathbb{R}^N)$ algebraically and topologically.
(2) $\mathcal{O}_{C,\omega}(\mathbb{R}^N) = \mathcal{O}_{C,\omega,p}(\mathbb{R}^N)$ algebraically and topologically.

Proof We first establish that $\mathcal{O}_{M,\omega}(\mathbb{R}^N) \hookrightarrow \mathcal{O}_{M,\omega,p}(\mathbb{R}^N)$ and $\mathcal{O}_{C,\omega}(\mathbb{R}^N) \hookrightarrow \mathcal{O}_{C,\omega,p}(\mathbb{R}^N)$ continuously.

Fix $m, n \in \mathbb{N}$ and set $n_0 := \left[\frac{N+1}{pb}\right] + 1$. Then Proposition 3.3(1) and Remark 3.4 imply for every $f \in \mathcal{O}^m_{n,\omega}(\mathbb{R}^N)$ and $m' \geq Lm$ that

$$r_{m',n+n_0,p}(f) \leq \exp(Lm)Cr_{m,n}(f), \tag{3.12}$$

where $C := ||\exp(-n_0\omega)||_p \left(\sum_{\alpha\in\mathbb{N}_0^N} \exp(-p|\alpha|)\right)^{\frac{1}{p}} < \infty$. Accordingly, the inclusions

$$\mathcal{O}^m_{n,\omega}(\mathbb{R}^N) \hookrightarrow \mathcal{O}^{m'}_{n+n_0,\omega,p}(\mathbb{R}^N) \hookrightarrow \bigcup_{n'=1}^{\infty} \mathcal{O}^{m'}_{n',\omega,p}(\mathbb{R}^N)$$

are continuous for every $m' \geq Lm$. The arbitrarity of $n \in \mathbb{N}$ yields that also the inclusion

$$\bigcup_{n=1}^{\infty} \mathcal{O}^m_{n,\omega}(\mathbb{R}^N) \hookrightarrow \bigcup_{n'=1}^{\infty} \mathcal{O}^{m'}_{n',\omega,p}(\mathbb{R}^N)$$

is continuous for every $m' \geq Lm$. Finally, since $m \in \mathbb{N}$ is arbitrary and the spaces $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$ and $\mathcal{O}_{M,\omega,p}(\mathbb{R}^N)$ are endowed with the projective lc-topology defined by the spectrum $\{\bigcup_{n=1}^{\infty} \mathcal{O}^m_{n,\omega}(\mathbb{R}^N)\}_{m\in\mathbb{N}}$ and $\{\bigcup_{n'=1}^{\infty} \mathcal{O}^{m'}_{n',\omega,p}(\mathbb{R}^N)\}_{m'\in\mathbb{N}}$ respectively, it follows that $\mathcal{O}_{M,\omega}(\mathbb{R}^N) \hookrightarrow \mathcal{O}_{M,\omega,p}(\mathbb{R}^N)$ continuously.

Now, fix $m, n \in \mathbb{N}$ and set $n_0 := \left[\frac{N+1}{pb}\right] + 1$. Then Proposition 3.3(2) and Remark 3.4 clearly imply for every $f \in \mathcal{O}^m_{n,\omega}(\mathbb{R}^N)$, $m' \geq Lm$ and $n' \geq n + n_0$ that

$$r_{m',n',p}(f) \leq \exp(Lm)Cr_{m,n}(f). \tag{3.13}$$

This means that the inclusion

$$\mathcal{O}^m_{n,\omega}(\mathbb{R}^N) \hookrightarrow \mathcal{O}^{m'}_{n',\omega,p}(\mathbb{R}^N)$$

is continuous for every $m' \geq Lm$ and $n' \geq n + n_0$. Since $m \in \mathbb{N}$ is arbitrary and the spaces $\bigcap_{m=1}^{\infty} \mathcal{O}^m_{n,\omega}(\mathbb{R}^N)$ and $\bigcap_{m'=1}^{\infty} \mathcal{O}^{m'}_{n',\omega}(\mathbb{R}^N)$ are endowed with the projective lc-topology defined by the spectrum $\{\mathcal{O}^m_{n,\omega}(\mathbb{R}^N)\}_{m\in\mathbb{N}}$ and $\{\mathcal{O}^{m'}_{n',\omega,p}(\mathbb{R}^N)\}_{m\in\mathbb{N}}$ respectively, it follows that also the inclusion

$$\bigcap_{m=1}^{\infty} \mathcal{O}^m_{n,\omega}(\mathbb{R}^N) \hookrightarrow \bigcap_{m'=1}^{\infty} \mathcal{O}^{m'}_{n',\omega,p}(\mathbb{R}^N)$$

is continuous for every $n' \geq n+n_0$. Finally, taking into account that $\bigcap_{m'=1}^\infty \mathcal{O}_{n',\omega,p}^{m'}(\mathbb{R}^N) \hookrightarrow \mathcal{O}_{C,\omega,p}(\mathbb{R}^N)$ and that $n \in \mathbb{N}$ is arbitrary, it follows that $\mathcal{O}_{C,\omega}(\mathbb{R}^N) \hookrightarrow \mathcal{O}_{C,\omega,p}(\mathbb{R}^N)$ continuously.

It remains to establish that $\mathcal{O}_{M,\omega,p}(\mathbb{R}^N) \hookrightarrow \mathcal{O}_{M,\omega}(\mathbb{R}^N)$ and $\mathcal{O}_{C,\omega,p}(\mathbb{R}^N) \hookrightarrow \mathcal{O}_{C,\omega}(\mathbb{R}^N)$ are also well-defined and continuous inclusions. To do this, we first prove that for every $n \in \mathbb{N}$ there exists $n' \geq n$ such that for every $m \in \mathbb{N}$ the inclusion

$$\mathcal{O}_{n,\omega,p}^{2m}(\mathbb{R}^N) \hookrightarrow \mathcal{O}_{n',\omega}^{m}(\mathbb{R}^N) \tag{3.14}$$

is well-defined and continuous.

So, fix $n \in \mathbb{N}$ and choose $k \in \mathbb{N}$ satisfying $kp > N$. Then by Proposition 3.6 there exist $n' = n'(n,\omega) \geq n$ with $n \in \mathbb{N}$ and $C = C(n,N,k,p) > 0$ such that inequality (3.10) is satisfied. So, for a fixed $m \in \mathbb{N}$, we have for every $f \in \mathcal{O}_{n,\omega,p}^{2m}(\mathbb{R}^N) \subset C^\infty(\mathbb{R}^N)$ and $\alpha \in \mathbb{N}^N$ that

$$\|\partial^\alpha f \exp(-n'\omega)\|_\infty \leq C\|\partial^\alpha f\|_{k,p,\exp(-n\omega)}.$$

Thus, applying the fact that $\varphi_\omega^*(t)/t$ is increasing function in $(0,\infty)$ and inequality (2.5), it follows for every $f \in \mathcal{O}_{n,\omega,p}^{2m}(\mathbb{R}^N)$ and $\alpha \in \mathbb{N}^N$ that

$$\begin{aligned}
\|\partial^\alpha f \exp(-n'\omega)\|_\infty \exp\left(-m\varphi_\omega^*\left(\frac{|\alpha|}{m}\right)\right) &\leq C\|\partial^\alpha f\|_{k,p\exp(-n\omega)} \exp\left(-m\varphi_\omega^*\left(\frac{|\alpha|}{m}\right)\right) \\
&= C\|\partial^\alpha f \exp(-n\omega)\|_p \exp\left(-m\varphi_\omega^*\left(\frac{|\alpha|}{m}\right)\right) \\
&\quad + C\sum_{0<|\beta|\leq k} \|\partial^{\alpha+\beta} f\ \exp(-n\omega)\|_p \exp\left(-m\varphi_\omega^*\left(\frac{|\alpha|}{m}\right)\right) \\
&\leq C\|\partial^\alpha f \exp(-n\omega)\|_p \exp\left(-2m\varphi_\omega^*\left(\frac{|\alpha|}{2m}\right)\right) \\
&\quad + C\sum_{0<|\beta|\leq k} \|\partial^{\alpha+\beta} f\ \exp(-n\omega)\|_p \exp\left(-2m\varphi_\omega^*\left(\frac{|\alpha+\beta|}{2m}\right) + m\varphi_\omega^*\left(\frac{\beta}{m}\right)\right) \\
&\leq C'\left(\sum_{\gamma\in\mathbb{N}^N} \|\partial^\gamma f \exp(-n\omega)\|_p^p \exp\left(-2pm\varphi_\omega^*\left(\frac{|\gamma|}{2m}\right)\right)\right)^{\frac{1}{p}} = C' r_{2m,n,p}(f),
\end{aligned}$$

where $C' := C\left(\sum_{0<|\beta|\leq k} \exp\left(p'm\varphi_\omega^*\left(\frac{|\beta|}{m}\right)\right) + 1\right)^{\frac{1}{p'}}$ whenever $1 < p < \infty$, with $\frac{1}{p} + \frac{1}{p'} = 1$, and $C' := C\sup_{0<|\beta|\leq k} \exp\left(p'm\varphi_\omega^*\left(\frac{|\beta|}{m}\right)\right) + 1$ whenever $p = 1$. Accordingly, for every $f \in \mathcal{O}_{n,\omega,p}^{2m}(\mathbb{R}^N)$ the following inequality holds

$$r_{m,n'}(f) = \sup_{\alpha\in\mathbb{N}_0^N} \|\partial^\alpha f \exp(-n\omega)\|_\infty \exp\left(-m\varphi_\omega^*\left(\frac{|\alpha|}{m}\right)\right) \leq C' r_{2m,n,p}(f).$$

This completes the proof.

Now, the reverse topological inclusions easily follow by applying (3.14) and by arguing in a similar way as above. ◻

Remark 3.9 We observe that if $f \in \mathcal{S}_\omega(\mathbb{R}^N)$, then $\sigma_{m,n,p}(f) = r_{m,-n,p}(f)$ for every $m \in \mathbb{N}$, $n \in \mathbb{N}$ and $p \in [1,\infty)$. Therefore, by Proposition 2.4 it follows that $\mathcal{S}_\omega(\mathbb{R}^N) = \bigcap_{n=1}^\infty \bigcap_{m=1}^\infty \mathcal{O}_{-n,\omega,p}^m(\mathbb{R}^N)$ for every $p \in [1,\infty)$.

Denoting by $\mathcal{O}'_{M,\omega,p}(\mathbb{R}^N)$ ($\mathcal{O}'_{C,\omega,p}(\mathbb{R}^N)$, resp.), for $1 \le p < \infty$, the strong dual of $\mathcal{O}_{M,\omega,p}(\mathbb{R}^N)$ ($\mathcal{O}_{C,\omega,p}(\mathbb{R}^N)$, resp.) Proposition 3.8 above implies the following fact.

Corollary 3.10 *Let ω be a non-quasianalytic weight function and $1 \le p < \infty$. Then the following properties are satisfied.*

(1) $\mathcal{O}'_{M,\omega}(\mathbb{R}^N) = \mathcal{O}'_{M,\omega,p}(\mathbb{R}^N)$ *algebraically and topologically.*
(2) $\mathcal{O}'_{C,\omega}(\mathbb{R}^N) = \mathcal{O}'_{C,\omega,p}(\mathbb{R}^N)$ *algebraically and topologically.*

Thanks to Proposition 3.8 and Corollary 3.10, it is easy to show some structure theorems for the dual spaces $\mathcal{O}'_{M,\omega}(\mathbb{R}^N)$ and $\mathcal{O}'_{C,\omega}(\mathbb{R}^N)$ of $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$ and $\mathcal{O}_{C,\omega}(\mathbb{R}^N)$, respectively. To do this, we introduce the following weighted spaces.

Definition 3.11 Let ω be a non-quasianalytic weight function.

(a) For $1 \le p < \infty$ and for $m, n \in \mathbb{Z}$ we define $\left(\oplus L^p(\mathbb{R}^N, \exp(n\omega(x))\,dx)\right)_{\omega,m,p}$ as the set of all sequences $\{f_\alpha\}_{\alpha\in\mathbb{N}_0^N}$ of Lebesgue measurable functions on $\mathbb{R}^N$ satisfying the following condition:

$$|\{f_\alpha\}_{\alpha\in\mathbb{N}_0^N}|^p_{m,n\omega,p} := \sum_{\alpha\in\mathbb{N}_0^N} \|\exp(n\omega) f_\alpha\|_p^p \exp\left(pm\varphi^*_\omega\left(\frac{|\alpha|}{m}\right)\right) < \infty \tag{3.15}$$

(b) For $m, n \in \mathbb{Z}$ we define the space $\left(\oplus L^\infty(\mathbb{R}^N, \exp(n\omega(x))dx)\right)_{\omega,m,\infty}$ as the set of all the sequences $\{f_\alpha\}_{\alpha\in\mathbb{N}_0^N}$ of Lebesgue measurable functions on $\mathbb{R}^N$ satisfying the following condition:

$$|\{f_\alpha\}_{\alpha\in\mathbb{N}_0^N}|_{m,n\omega,\infty} := \sup_{\alpha\in\mathbb{N}_0^N} \|\exp(n\omega) f_\alpha\|_\infty \exp\left(m\varphi^*_\omega\left(\frac{|\alpha|}{m}\right)\right) < \infty \tag{3.16}$$

In view of Proposition 3.8, Corollary 3.10 and Remark 3.9 we can proceed in a similar way as in [18, Theorem 3.2], [26, Theorem 3.2] and [27, Theorem 2] to show the following representations.

Theorem 3.12 (Structure theorem for $\mathcal{O}'_{M,\omega}(\mathbb{R}^N)$). *Let ω be a non-quasianalytic weight function, $T \in \mathcal{D}'_\omega(\mathbb{R}^N)$ and $1 \le p < \infty$. Then $T \in \mathcal{O}'_{M,\omega}(\mathbb{R}^N)$ if and only if there exists $m \in \mathbb{N}$ such that for every $n \in \mathbb{N}$ there exists $\{f_\alpha\}_{\alpha\in\mathbb{N}_0^N} \in \left(\oplus L^{p'}(\mathbb{R}^N, \exp(n\omega(x))\,dx)\right)_{\omega,m,p'}$, p' being the conjugate exponent of p, such that*

$$T = \sum_{\alpha\in\mathbb{N}_0^N} \partial^\alpha f_\alpha. \tag{3.17}$$

Theorem 3.13 (Structure theorem for $\mathcal{O}'_{C,\omega}(\mathbb{R}^N)$). *Let ω be a non-quasianalytic weight function, $T \in \mathcal{D}'_\omega(\mathbb{R}^N)$ and $1 \le p < \infty$. Then $T \in \mathcal{O}'_{C,\omega}(\mathbb{R}^N)$ if and only if for each $n \in \mathbb{N}$ there exist $m \in \mathbb{N}$ and $\{f_\alpha\}_{\alpha\in\mathbb{N}_0^N} \in \left(\oplus L^{p'}(\mathbb{R}^N, \exp(n\omega(x))\,dx)\right)_{\omega,m,p'}$, p' being the conjugate exponent of p, such that*

$$T = \sum_{\alpha\in\mathbb{N}_0^N} \partial^\alpha f_\alpha. \tag{3.18}$$

Theorem 3.14 (Structure theorem for $\mathcal{S}'_\omega(\mathbb{R}^N)$). *Let ω be a non-quasianalytic weight function, $T \in \mathcal{D}'_\omega(\mathbb{R}^N)$ and $1 \le p < \infty$. Then $T \in \mathcal{S}'_\omega(\mathbb{R}^N)$ if and only if there exist $m \in \mathbb{N}$ and $\{f_\alpha\}_{\alpha\in\mathbb{N}_0^N} \in \left(\oplus L^{p'}(\mathbb{R}^N, \exp(-m\omega(x))\,dx)\right)_{\omega,m,p'}$ such that*

$$T = \sum_{\alpha\in\mathbb{N}_0^N} \partial^\alpha f_\alpha. \tag{3.19}$$

4 The space $\mathcal{D}_{L^p_\mu,\omega}(\mathbb{R}^N)$

In this section we collect further necessary results in order to show that $\mathcal{O}'_{C,\omega}(\mathbb{R}^N)$ is the space of convolutors of $\mathcal{S}_\omega(\mathbb{R}^N)$ and of $\mathcal{S}'_\omega(\mathbb{R}^N)$. To this end we introduce the spaces $\mathcal{D}_{L^p_\mu,\omega}(\mathbb{R}^N)$, where $\mu \in \mathbb{R}$ and $1 \le p \le \infty$.

Definition 4.1 Let ω be a non-quasianalytic weight function and let $\mu \in \mathbb{R}$.

(a) We denote by $\mathcal{D}_{L^p_\mu,\omega}(\mathbb{R}^N)$, for $1 \le p < \infty$, the set of all functions $f \in C^\infty(\mathbb{R}^N)$ such that for every $m \in \mathbb{N}$ the following inequality is satisfied:

$$t^p_{m,\mu,p}(f) := \sum_{\alpha\in\mathbb{N}_0^N} \|\exp(\mu\omega)\partial^\alpha f\|_p^p \exp\left(-pm\varphi^*_\omega\left(\frac{|\alpha|}{m}\right)\right) < \infty. \tag{4.1}$$

(b) We denote by $\mathcal{B}_{L^\infty_\mu,\omega}(\mathbb{R}^N)$ the set of all functions $f \in C^\infty(\mathbb{R}^N)$ such that for every $m \in \mathbb{N}$ the following inequality is satisfied:

$$t_{m,\mu,\infty}(f) := \sup_{\alpha\in\mathbb{N}_0^N} \|\exp(\mu\omega)\partial^\alpha f\|_\infty \exp\left(-m\varphi^*_\omega\left(\frac{|\alpha|}{m}\right)\right) < \infty. \tag{4.2}$$

(c) We denote by $\mathcal{D}_{L^\infty_\mu,\omega}(\mathbb{R}^N)$ the subspace of $\mathcal{B}_{L^\infty_\mu,\omega}(\mathbb{R}^N)$ consisting of all the functions f such that $|\exp(\mu\omega(x))\partial^\alpha f(x)| \to 0$ as $|x| \to \infty$ for all $\alpha \in \mathbb{N}_0^N$.

We denote by $\mathcal{D}'_{L^p_\mu,\omega}(\mathbb{R}^N)$ (by $\mathcal{B}'_{L^\infty_\mu,\omega}(\mathbb{R}^N)$, resp.) the strong dual of $\mathcal{D}_{L^p_\mu,\omega}(\mathbb{R}^N)$ (of $\mathcal{B}_{L^\infty_\mu,\omega}(\mathbb{R}^N)$, resp.).

The spaces $\mathcal{D}_{L^p_\mu,\omega}(\mathbb{R}^N)$, $1 \le p \le \infty$, and $\mathcal{B}_{L^\infty_\mu,\omega}(\mathbb{R}^N)$ are always supposed to be endowed with the lc-topology generated by the sequence of norms $\{t_{m,\mu,p}\}_{m\in\mathbb{N}}$ and $\{t_{m,\mu,\infty}\}_{m\in\mathbb{N}}$, respectively. The elements of the strong dual $\mathcal{D}'_{L^p_\mu,\omega}(\mathbb{R}^N)$ of $\mathcal{D}_{L^p_\mu,\omega}(\mathbb{R}^N)$ are called ultradistributions of $L^{p'}_\mu$-growth, p' being the conjugate exponent of p. In the case $\mu = 0$ such spaces were already considered in [5], [13] and [20] and are extensions of the classical spaces $\mathcal{D}_{L^p}(\mathbb{R}^N)$ and $\mathcal{B}(\mathbb{R}^N)$ as introduced by Schwartz [34] (see also [4]). Moreover, spaces analogous to $\mathcal{D}_{L^p_\mu,\omega}(\mathbb{R}^N)$, $1 \le p \le \infty$, (to $\mathcal{B}_{L^\infty_\mu,\omega}(\mathbb{R}^N)$, resp.) were treated in [27] (in [15], resp.), in the context of ultradifferentiable functions of Beurling type as introduced by Komatsu [25] (of Gelfand-Shilov type spaces, resp.). Moreover, by arguing in a similar way as in [5] (see also [13,20]), we can prove the following properties.

Proposition 4.2 *Let ω be a non-quasianalytic weight function and $1 \le p \le \infty$. Then the following properties are satisfied.*

(1) *For every $\mu \in \mathbb{R}$ the spaces $\mathcal{D}_{L^p_\mu,\omega}(\mathbb{R}^N)$ and $\mathcal{B}_{L^\infty_\mu,\omega}(\mathbb{R}^N)$ are Fréchet spaces.*

(2) *For every* $\mu, \mu' \in \mathbb{R}$ *with* $\mu < \mu'$ *the inclusions* $\mathcal{D}_{L^p_{\mu'},\omega}(\mathbb{R}^N) \hookrightarrow \mathcal{D}_{L^p_{\mu},\omega}(\mathbb{R}^N)$ *and* $\mathcal{B}_{L^\infty_{\mu'},\omega}(\mathbb{R}^N) \hookrightarrow \mathcal{B}_{L^\infty_{\mu},\omega}(\mathbb{R}^N)$ *are continuous.*

(3) *For every* $\mu \in \mathbb{R}$ *the inclusion* $\mathcal{D}_\omega(\mathbb{R}^N) \hookrightarrow \mathcal{D}_{L^p_\mu,\omega}(\mathbb{R}^N)$ *is continuous with dense range.*

(4) *For every* $\mu \in \mathbb{R}$ *the inclusions* $\mathcal{D}_{L^p_\mu,\omega}(\mathbb{R}^N) \hookrightarrow \mathcal{E}_\omega(\mathbb{R}^N)$ *and* $\mathcal{B}_{L^\infty_\mu,\omega}(\mathbb{R}^N) \hookrightarrow \mathcal{E}_\omega(\mathbb{R}^N)$ *are continuous with dense range.*

Remark 4.3 Since $t_{m,\mu,p} = \sigma_{m,\mu,p}$ for $\mu > 0$ and $t_{m,-n,p} = r_{m,n,p}$ for $n \in \mathbb{N}$ ($t_{m,-n,\infty} = r_{m,n}$ for $n \in \mathbb{N}$), the following equalities are valid algebraically and topologically:

$$\mathcal{S}_\omega(\mathbb{R}^N) = \bigcap_{\mu>0} \mathcal{D}_{L^p_\mu,\omega}(\mathbb{R}^N) = \bigcap_{\mu>0} \mathcal{B}_{L^\infty_\mu,\omega}(\mathbb{R}^N) \ \ (1 \le p \le \infty), \tag{4.3}$$

$$\bigcap_{m=1}^{\infty} \mathcal{O}^m_{n,\omega,p}(\mathbb{R}^N) = \mathcal{D}_{L^p_{-n},\omega}(\mathbb{R}^N) \ \ (1 \le p < \infty, \ n \in \mathbb{N}), \tag{4.4}$$

$$\bigcap_{m=1}^{\infty} \mathcal{O}^m_{n,\omega}(\mathbb{R}^N) = \mathcal{B}_{L^\infty_{-n},\omega}(\mathbb{R}^N) \ \ (n \in \mathbb{N}), \tag{4.5}$$

and hence,

$$\mathcal{O}_{C,\omega}(\mathbb{R}^N) = \bigcup_{n=1}^{\infty} \mathcal{D}_{L^p_{-n},\omega}(\mathbb{R}^N) = \bigcup_{n=1}^{\infty} \mathcal{B}_{L^\infty_{-n},\omega}(\mathbb{R}^N) \ \ (1 \le p \le \infty), \tag{4.6}$$

where (4.6) has been established in Proposition 3.8 for $1 \le p < \infty$. For $p = \infty$, (4.6) follows from the fact that if $f \in \mathcal{B}_{L^\infty_{-n},\omega}(\mathbb{R}^N)$ for some $n \in \mathbb{N}$, then $f \in \mathcal{D}_{L^\infty_{-(n+1)},\omega}(\mathbb{R}^N)$, as it is easy to see.

The aim of this section is to study the convolution operators acting on the spaces of ultradistributions of L^p_μ-growth, thereby extending some results in [5] (see also [20]) in the setting of the weigthed spaces $\mathcal{D}_{L^p_\mu,\omega}(\mathbb{R}^N)$. In order to do this, in the following we always assume that the weight ω satisfies the additional condition $\log(1+t) = o(\omega(t))$ as $t \to \infty$, which is stronger than condition (γ).

Let G be an entire function satisfying the condition $\log|G(z)| = O(\omega(z))$ as $|z| \to \infty$. Then the functional T_G defined on $\mathcal{E}_\omega(\mathbb{R}^N)$ by

$$\langle T_G, \phi \rangle := \sum_{\alpha \in \mathbb{N}_0^N} (-i)^\alpha \frac{\partial^\alpha G(0)}{\alpha!} \partial^\alpha \phi(0), \quad \phi \in \mathcal{E}_\omega(\mathbb{R}^N),$$

belongs to $\mathcal{E}'_\omega(\mathbb{R}^N)$. The operator $G(D)$ defined on $\mathcal{D}'_\omega(\mathbb{R}^N)$ through

$$G(D) : \mathcal{D}'_\omega(\mathbb{R}^N) \to \mathcal{D}'_\omega(\mathbb{R}^N), \quad S \mapsto G(D)S := T_G \star S,$$

is called an ultradifferential operator of ω-class. When $G(D)$ is restricted to $\mathcal{E}_\omega(\mathbb{R}^N)$, $G(D)$ is a continuous linear operator from $\mathcal{E}_\omega(\mathbb{R}^N)$ into itself and, for every $\phi \in \mathcal{E}_\omega(\mathbb{R}^N)$,

$$(G(D)\phi)(x) = \sum_{\alpha \in \mathbb{N}_0^N} i^{|\alpha|} \frac{\partial^\alpha G(0)}{\alpha!} \partial^\alpha \phi(x), \quad \forall x \in \mathbb{R}^N.$$

Along the lines in [20] (see also [5,11]), one can show with some straightforward variants that each ultradifferential operator $G(D)$ of ω-class defines a continuous linear operator from $\mathcal{D}_{L^p_\mu,\omega}(\mathbb{R}^N)$ into itself and from $\mathcal{B}_{L^\infty_\mu,\omega}(\mathbb{R}^N)$ into itself, for every $1 \le p \le \infty$ and $\mu \in \mathbb{R}$.

Proposition 4.4 *Let ω be a non-quasianalytic weight function with the additional condition $\log(1+t) = o(\omega(t))$ as $t \to \infty$. Let $\mu \in \mathbb{R}$ and $1 \le p \le \infty$. If $G(D)$ is an ultradifferential operator of ω-class, then*

$$G(D) : \mathcal{D}_{L^p_{\mu,\omega}}(\mathbb{R}^N) \to \mathcal{D}_{L^p_{\mu,\omega}}(\mathbb{R}^N)$$

is a continuous linear operator. Moreover, $G(D)$ is also a continuous linear operator from $\mathcal{D}'_{L^p_{\mu,\omega}}(\mathbb{R}^N)$ into itself. The result is also valid when $G(D)$ acts on $\mathcal{B}_{L^\infty_\mu,\omega}(\mathbb{R}^N)$ and on its strong dual, i.e., $G(D) \in \mathcal{L}(\mathcal{B}_{L^\infty_\mu,\omega}(\mathbb{R}^N))$ and $G(D) \in \mathcal{L}(\mathcal{B}'_{L^\infty_\mu,\omega}(\mathbb{R}^N))$.

From Proposition 4.4 and Remark 4.3, it easily follows the next result.

Proposition 4.5 *Let ω be a non-quasianalytic weight function with the additional condition $\log(1+t) = o(\omega(t))$ as $t \to \infty$ and $G(D)$ be an ultradifferential operator of ω-class. Then the following properties are satisfied.*

(1) *$G(D) : \mathcal{O}_{C,\omega}(\mathbb{R}^N) \to \mathcal{O}_{C,\omega}(\mathbb{R}^N)$ is a continuous linear operator. Moreover, $G(D)$ also acts continuously on $\mathcal{O}'_{C,\omega}(\mathbb{R}^N)$.*
(2) *$G(D) : \mathcal{S}_\omega(\mathbb{R}^N) \to \mathcal{S}_\omega(\mathbb{R}^N)$ is a continuous linear operator. Moreover, $G(D)$ also acts continuously on $\mathcal{S}'_\omega(\mathbb{R}^N)$.*

From now on in this section, we consider only the case $p \in [1, \infty)$. In particular, we give representations of the elements of $\mathcal{D}'_{L^p_{\mu,\omega}}(\mathbb{R}^N)$ similar to the ones in [5,20]. In order to do this, we begin with the following result.

Proposition 4.6 *Let ω be a non-quasianalytic weight function.*
Let $\mu \in \mathbb{R}$ and $1 \le p < \infty$. Then
for every $T \in \mathcal{D}'_{L^p_{\mu,\omega}}(\mathbb{R}^N)$ and $\phi \in \mathcal{D}_\omega(\mathbb{R}^N)$ the function $T \star \phi \in L^{p'}(\mathbb{R}^N,$ $\exp(-\mu'\omega(x))dx)$, p' being the conjugate exponent of p and $\mu' = K\mu$ if $\mu \ge 0$ and $\mu' = \frac{\mu}{K}$ if $\mu < 0$ with K the constant appearing in condition (α).

Proof Since $\mathcal{D}'_{L^p_{\mu,\omega}}(\mathbb{R}^N) \subset \mathcal{D}'_\omega(\mathbb{R}^N)$ by Proposition 4.2(2), the convolution is well posed and $T \star \phi \in \mathcal{E}_\omega(\mathbb{R}^N)$ (see [12, Proposition 6.4]). By assumption there exist $m \in \mathbb{N}$ and $C > 0$ such that for every $\varphi \in \mathcal{D}_{L^p_{\mu,\omega}}(\mathbb{R}^N)$

$$|\langle T, \varphi \rangle| \le C t_{m,\mu,p}(\varphi).$$

Therefore, we get for every $\varphi \in \mathcal{D}_\omega(\mathbb{R}^N)$ that

$$\left| \int_{\mathbb{R}^N} (T \star \phi)(x) \varphi(x)\, dx \right| = |\langle T \star \phi, \varphi \rangle| = |\langle T, \check{\phi} \star \varphi \rangle| \le C t_{m,\mu,p}(\check{\phi} \star \varphi)$$
$$= C \left(\sum_{\alpha \in \mathbb{N}_0^N} \| \exp(\mu\omega) \partial^\alpha (\check{\phi} \star \varphi) \|_p^p \exp\left(-pm\varphi^*_\omega \left(\frac{|\alpha|}{m} \right) \right) \right)^{\frac{1}{p}}.$$

We now suppose that $\mu \geq 0$ and observe that by property (α) of ω, we have for every $\varphi \in \mathcal{D}_\omega(\mathbb{R}^N)$ that

$$\begin{aligned}\|\exp(\mu\omega)\partial^\alpha(\check{\phi}\star\varphi)\|_p^p &= \int_{\mathbb{R}^N} \exp(p\mu\omega(x))|\partial^\alpha(\check{\phi}\star\varphi)(x)|^p\,dx \\ &= \int_{\mathbb{R}^N} \exp(p\mu\omega(x))|(\partial^\alpha\check{\phi}\star\varphi)(x)|^p\,dx \\ &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \exp(p\mu\omega(x-y+y))|\partial^\alpha\phi(x-y)|^p|\varphi(y)|^p\,dxdy \\ &\leq \exp(Kp\mu)\|\exp(\mu K\omega)\varphi\|_p^p\|\exp(\mu K\omega)\partial^\alpha\phi\|_p^p.\end{aligned}$$

Thereby, we obtain for every $\varphi \in \mathcal{D}_\omega(\mathbb{R}^N)$ that

$$\begin{aligned}\left|\int_{\mathbb{R}^N}(T\star\phi)(x)\varphi(x)\,dx\right| &\leq C\exp(\mu K)\|\exp(\mu K\omega)\varphi\|_p t_{m,K\mu,p}(\phi) \\ &= C'\|\exp(\mu K\omega)\varphi\|_p t_{m,K\mu,p}(\phi).\end{aligned}$$

This implies that $T\star\phi \in L^{p'}(\mathbb{R}^N, \exp(-\mu K\omega(x))dx)$ with $||\exp(-\mu K\omega)T\star\phi||_{p'} \leq C't_{m,K\mu,p}(\phi)$. In the case that $\mu < 0$, a similar argument shows that $T\star\phi \in L^{p'}(\mathbb{R}^N,$ $\exp(-\frac{\mu}{K}\omega(x))dx)$ with $||\exp(-\frac{\mu}{K}\omega)T\star\phi||_{p'} \leq C't_{m,-\mu,p}(\phi)$. □

Remark 4.7 In case the weight function ω is sub-additive, i.e., $\omega(s+t) \leq \omega(s)+\omega(t)$ for all $s,t \geq 0$ and hence $K = 1$, we can deduce from the proof above that for every $T \in \mathcal{D}_{L^p_\mu,\omega}(\mathbb{R}^N)$ and $\phi \in \mathcal{D}_\omega(\mathbb{R}^N)$ the function $T\star\phi \in L^{p'}(\mathbb{R}^N, \exp(-\mu\omega(x))dx)$, with p' the conjugate exponent of p.

We recall that an ultradifferential operator $G(D)$ of ω-class is said to be *strongly elliptic* if there exist $M > 0$ and $l > 0$ such that $|G(z)| \geq M\exp(l\omega(z))$, for every $z \in \mathbb{C}^N$ with $|\Im z| < M|\Re z|$.

Proposition 4.8 *Let ω be a non-quasianalytic weight function with the additional condition* $\log(1+t) = o(\omega(t))$ *as* $t \to \infty$. *Let* $T \in \mathcal{D}'_\omega(\mathbb{R}^N)$, $1 < p < \infty$ *and* $\mu \in \mathbb{R}$. *Suppose that* $T\star\phi \in L^p(\mathbb{R}^N, \exp(\mu\omega(x))dx)$ *for every* $\phi \in \mathcal{D}_\omega(\mathbb{R}^N)$. *Then, there exist a strongly elliptic ultradifferential operator* $G(D)$ *of* ω*-class and* $f, g \in L^p(\mathbb{R}^N, \exp(\mu\omega(x))dx)$ *such that* $T = G(D)f + g$.

Proof Let $V_{p'}$ denote the unit ball of $L^{p'}(\mathbb{R}^N, \exp(-\mu\omega(x))dx)$, p' being the conjugate exponent of p. Then, for a fixed $\varphi \in V_{p'} \cap \mathcal{D}_\omega(\mathbb{R}^N)$, we have for every $\phi \in \mathcal{D}_\omega(\mathbb{R}^N)$ that

$$|\langle T\star\check{\varphi}, \check{\phi}\rangle| = |\langle T\star\phi, \varphi\rangle| \leq \|\exp(\mu\omega)(T\star\phi)\|_p\|\exp(-\mu\omega)\varphi\|_{p'} \leq \|\exp(\mu\omega)(T\star\phi)\|_p.$$

This implies that $\{T\star\check{\varphi} : \varphi \in V_{p'} \cap \mathcal{D}_\omega(\mathbb{R}^N)\}$ is a weakly bounded subset, and hence, an equicontinuous subset of $\mathcal{D}'_\omega(\mathbb{R}^N)$. Therefore, if $K_1 := [-2,2]^N$, we can find $m \in \mathbb{N}$ and $C > 0$ such that

$$|\langle T\star\check{\varphi}, \phi\rangle| \leq Cp_{K_1,m}(\phi)$$

for each $\phi \in \mathcal{D}_\omega(K_1)$ and $\varphi \in V_{p'} \cap \mathcal{D}_\omega(\mathbb{R}^N)$. Accordingly, for each $\phi \in \mathcal{D}_\omega(K_1)$ and $\varphi \in \mathcal{D}_\omega(\mathbb{R}^N)$

$$|\langle T\star\check{\varphi}, \phi\rangle| \leq Cp_{K_1,m}(\phi)\|\exp(-\mu\omega)\varphi\|_{p'}. \tag{4.7}$$

We now take $K_2 := [-1,1]^N$ and we show that $T\star\phi \in L^p(\mathbb{R}^N, \exp(\mu\omega(x))dx)$ for every $\phi \in \mathcal{E}_{\omega,2m}(K_2)\cap\mathcal{D}(K_2)$. Let $\eta \in \mathcal{D}_\omega(K_2)$ be such that $\eta \geq 0$, $\int_{K_1} \exp(\mu\omega(x))\eta(x)\,dx = 1$ and consider $\eta_\epsilon(x) := \frac{\eta(\frac{x}{\epsilon})}{\epsilon}$, for $x \in \mathbb{R}^N$ and $\epsilon > 0$. Then, for $\phi \in \mathcal{E}_{\omega,2m}(K_2)\cap\mathcal{D}(K_2)$, $\phi\star\eta_\epsilon \in \mathcal{D}_\omega(K_1)$, $0<\epsilon<1$, and $\phi\star\eta_\epsilon \to \phi$ in $\mathcal{E}_{\omega,m}(K_1)\cap\mathcal{D}(K_1)$, as $\epsilon \to 0^+$. By assumption, $T\star(\phi\star\eta_\epsilon) \in L^p(\mathbb{R}^N, \exp(\mu\omega(x))dx)$, $0<\epsilon<1$. On the other hand, from (4.7) it follows for every $0<\epsilon<1$ that

$$\|\exp(\mu\omega)(T\star(\phi\star\eta_\epsilon))\|_p \leq C p_{K_1,m}(\phi\star\eta_\epsilon) \leq C' p_{K_1,m}(\phi), \tag{4.8}$$

for some $C' > 0$. Thanks to inequality (4.8), we get that $\{T\star(\phi\star\eta_\epsilon)\}_{0<\epsilon<1}$ is a Cauchy net in the space $L^p(\mathbb{R}^N, \exp(\mu\omega(x))dx)$, thereby a convergent net in $L^p(\mathbb{R}^N, \exp(\mu\omega(x))dx)$. Since $T \in \mathcal{D}'_\omega(\mathbb{R}^N)$, there exist $l \in \mathbb{N}$ and $C' > 0$ such that

$$|\langle T, \varphi\rangle| \leq C' p_{K_2,l}(\phi)$$

for each $\phi \in \mathcal{D}_\omega(K_2)$. Then T can be continuously extended to $\mathcal{E}_{\omega,l}(K_2)\cap\mathcal{D}(K_2)$. Hence, if m is large enough, we can conclude for every $x \in \mathbb{R}^N$ that

$$|(T\star(\phi\star\eta_\epsilon) - T\star\phi)(x)| \leq C' p_{K_2,l}(\phi\star\eta_\epsilon - \phi) \leq C' p_{K_1,m}(\phi\star\eta_\epsilon - \phi).$$

So, $T\star(\phi\star\eta_\epsilon) \to T\star\phi$ in $C_b(\mathbb{R}^N)$ as $\epsilon \to 0^+$. From (4.8) we obtain that $T\star\phi \in L^p(\mathbb{R}^N, \exp(\mu\omega(x))dx)$. Now applying [20, Corollary 2.6], we can write $\delta = G(D)\Gamma + \chi$, where $G(D)$ is a strongly elliptic ultradifferential operator of ω-class, $\chi \in \mathcal{D}_\omega(K_2)$ and $\Gamma \in \mathcal{E}_{\omega,2m}(K_2)\cap\mathcal{D}(K_2)$. To get the claim, is sufficient to take $f := T\star\Gamma$ and $g := T\star\chi$. □

The results above lead to the following result about the elements of $\mathcal{D}'_{L^p_\mu,\omega}(\mathbb{R}^N)$.

Theorem 4.9 *Let ω be a non-quasianalytic weight function with the additional condition $\log(1+t) = o(\omega(t))$ as $t\to\infty$ and $T \in \mathcal{D}'_\omega(\mathbb{R}^N)$. Let $1<p<\infty$ and $\mu \in \mathbb{R}$. Set $\mu' = K\mu$ if $\mu \geq 0$ and $\mu' = \frac{\mu}{K}$ if $\mu < 0$ and consider the following properties.*

(1) $T \in \mathcal{D}'_{L^p_\mu,\omega}(\mathbb{R}^N)$.

(2) $T\star\phi \in L^{p'}(\mathbb{R}^N, \exp(-\mu'\omega(x))dx)$ *for every* $\phi \in \mathcal{D}_\omega(\mathbb{R}^N)$, *$p'$ being the conjugate exponent of p.*

(3) *There exists a strongly elliptic ultradifferential operator $G(D)$ of ω-class and $f, g \in L^{p'}(\mathbb{R}^N, \exp(-\mu'\omega(x))dx)$ such that $T = G(D)f + g$, p' being the conjugate exponent of p.*

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3). *If, in addition, the weigth function ω is sub-additive, then all the assertions are equivalent.*

Proof (1) $\Rightarrow$ (2) follows from Proposition 4.6.

(2) $\Rightarrow$ (3) follows from Proposition 4.8.

In case the weight function ω is sub-additive the constant $K = 1$ and hence $\mu' = \mu$.

So, (3) $\Rightarrow$ (1). Indeed, from the facts that $L^{p'}(\mathbb{R}^N, \exp(-\mu\omega(x)dx)) \subset \mathcal{D}'_{L^p_\mu,\omega}(\mathbb{R}^N)$ and $G(D) \in \mathcal{L}(\mathcal{D}'_{L^p_\mu,\omega}(\mathbb{R}^N))$ it follows that $T = G(D)f + g \in \mathcal{D}'_{L^p_\mu,\omega}(\mathbb{R}^N)$. □

Taking into account Remark 4.3 and applying Theorem 4.9 together with Proposition 4.5, we can obtain a second structure theorem for both the spaces $\mathcal{O}'_{C,\omega}(\mathbb{R}^N)$ and $\mathcal{S}'_\omega(\mathbb{R}^N)$.

Theorem 4.10 (Second structure theorem for $\mathcal{O}'_{C,\omega}(\mathbb{R}^N)$). *Let ω be a non-quasianalytic weight function with the additional condition* $\log(1+t) = o(\omega(t))$ *as $t\to\infty$, $T \in \mathcal{D}'_\omega(\mathbb{R}^N)$*

and $1 < p < \infty$. *Then* $T \in \mathcal{O}'_{C,\omega}(\mathbb{R}^N)$ *if and only if for every* $n \in \mathbb{N}$ *there exist* $n'(= [n/K]) \in \mathbb{N}$ *with* $n' \leq n$, *a strongly elliptic differential operator* $G(D)$ *of* ω*-class and* $f, g \in L^{p'}(\mathbb{R}^N, \exp(n'\omega(x))dx)$ *such that* $T = G(D)f + g$.

Theorem 4.11 (Second structure theorem for $\mathcal{S}'_\omega(\mathbb{R}^N)$). *Let* ω *be a non-quasianalytic weight function with the additional condition* $\log(1+t) = o(\omega(t))$ *as* $t \to \infty$, $T \in \mathcal{D}'_\omega(\mathbb{R}^N)$ *and* $1 < p < \infty$. *Then* $T \in \mathcal{S}'_\omega(\mathbb{R}^N)$ *if and only if there exists* $\mu > 0$, *a strongly elliptic differential operator* $G(D)$ *of* ω*-class and* $f, g \in L^{p'}(\mathbb{R}^N, \exp(-\mu\omega(x))dx)$ *such that* $T = G(D)f + g$.

5 $\mathcal{O}'_{C,\omega}(\mathbb{R}^N)$ is the space of convolutors of the spaces $\mathcal{S}_\omega(\mathbb{R}^N)$ and $\mathcal{S}'_\omega(\mathbb{R}^N)$

In this section we show that $\mathcal{O}'_{C,\omega}(\mathbb{R}^N)$ is the space of convolutors of the spaces $\mathcal{S}_\omega(\mathbb{R}^N)$ and $\mathcal{S}'_\omega(\mathbb{R}^N)$. So, we begin by proving that the convolution beetween elements of $\mathcal{S}'_\omega(\mathbb{R}^N)$ with the ones of $\mathcal{S}_\omega(\mathbb{R}^N)$ belongs to $\mathcal{O}_{C,\omega}(\mathbb{R}^N)$. To this end, we observe the following facts.

Lemma 5.1 *Let* ω *be a non-quasianalytic weight function. If* $T \in \mathcal{S}'_\omega(\mathbb{R}^N)$ *and* $f \in \mathcal{S}_\omega(\mathbb{R}^N)$, *then the map*

$$\mathbb{R}^N \ni x \mapsto \langle T_y, \tau_x f\rangle$$

is a C^∞ *function in* $x \in \mathbb{R}^N$. *In particular, for every* $\alpha \in \mathbb{N}_0^N$, *we have*

$$\partial_x^\alpha \langle T_y, \tau_x f\rangle = \langle T_y, \partial_x^\alpha \tau_x f\rangle \tag{5.1}$$

We recall that the notation T_y means that the distribution T acts on a function $\psi(x-y)$, when the latter is regarded as a function of the variable y.

Proof The proof follows by applying [35, Theorem 27.1] to the map $x \mapsto \langle T_y, \tau_x f\rangle$. □

Proposition 5.2 *Let* ω *be a non-quasianalytic weight function. If* $T \in \mathcal{S}'_\omega(\mathbb{R}^N)$ *and* $f \in \mathcal{S}_\omega(\mathbb{R}^N)$, *then* $T\star f \in \mathcal{O}_{C,\omega}(\mathbb{R}^N)$. *Moreover the map* $f \mapsto T\star f$ *is a continuous linear operator from* $\mathcal{S}_\omega(\mathbb{R}^N)$ *into* $\mathcal{O}_{C,\omega}(\mathbb{R}^N)$.

Proof By Theorem 3.14 with $p = 1$, there exist $m \in \mathbb{N}$ and a sequence $\{f_\alpha\}_{\alpha\in\mathbb{N}_0^N} \in \left(\oplus L^\infty(\mathbb{R}^N, \exp(-m\omega(x))\,dx)\right)_{\omega,m,\infty}$ such that $T = \sum_{\alpha\in\mathbb{N}_0^N} \partial^\alpha f_\alpha$. Moreover, by Lemma 5.1 the function $T\star f \in C^\infty(\mathbb{R}^N)$. In particular, we have for every $x \in \mathbb{R}^N$ and $\beta \in \mathbb{N}_0^N$ that

$$\begin{aligned}
|\partial^\beta(T\star f)(x)| &= \left|\sum_{\alpha\in\mathbb{N}_0^N} (\partial^\alpha f_\alpha \star \partial^\beta f)(x)\right| = \left|\sum_{\alpha\in\mathbb{N}_0^N} \int_{\mathbb{R}^N} f_\alpha(y)\partial^{\alpha+\beta} f(x-y)\,dy\right| \\
&\leq \sum_{\alpha\in\mathbb{N}_0^N} \int_{\mathbb{R}^N} |f_\alpha(y)\partial^{\alpha+\beta} f(x-y)|\,dy \\
&\leq \sum_{\alpha\in\mathbb{N}_0^N} \|\exp(-m\omega) f_\alpha\|_\infty \|\exp(m\omega)\partial^{\alpha+\beta}\tau_x \check{f}\|_1.
\end{aligned}$$

On the other hand, we have for every $x \in \mathbb{R}^N$ and $\alpha, \beta \in \mathbb{N}_0^N$ that

$$\begin{aligned}
&\|\exp(m\omega)\partial^{\alpha+\beta}\tau_x \check{f}\|_1 = \int_{\mathbb{R}^N} \exp(m\omega(y))|\partial^{\alpha+\beta} f(x-y)|\,dy \\
&= \int_{\mathbb{R}^N} \exp(m\omega(x-z))|\partial^{\alpha+\beta} f(z)|\,dz \le \int_{\mathbb{R}^N} \exp(mK(1+\omega(x)+\omega(z)))|\partial^{\alpha+\beta} f(z)|\,dz \\
&= \exp(mK(1+\omega(x))) \int_{\mathbb{R}^N} \exp(mK\omega(z))|\partial^{\alpha+\beta} f(z)|\,dz \\
&= \exp(mK(1+\omega(x)))\|\exp(mK\omega)\partial^{\alpha+\beta} f\|_1 \le \exp(n(1+\omega(x)))\|\exp(n\omega)\partial^{\alpha+\beta} f\|_1,
\end{aligned}$$

where $n := [mK]+1$. Accordingly, we get for every $x \in \mathbb{R}^N$ and $\beta \in \mathbb{N}_0^N$ that

$$\begin{aligned}
\exp(-n\omega(x))|\partial^\beta(T\star f)(x)| &\le \exp(n) \sum_{\alpha\in\mathbb{N}_0^N} \|\exp(-m\omega) f_\alpha\|_\infty \|\exp(n\omega)\partial^{\alpha+\beta} f\|_1 \\
&\le \exp(n) \sup_{\alpha\in\mathbb{N}_0^N} \|\exp(-m\omega) f_\alpha\|_\infty \exp\left(m\varphi^*_\omega\left(\frac{|\alpha|}{m}\right)\right) \times \\
&\times \sum_{\alpha\in\mathbb{N}_0^N} \|\exp(n\omega)\partial^{\alpha+\beta} f\|_1 \exp\left(-m\varphi^*_\omega\left(\frac{|\alpha|}{m}\right)\right).
\end{aligned}$$

Let $C := \exp(n) \sup_{\alpha\in\mathbb{N}_0^N} \|\exp(-m\omega) f_\alpha\|_\infty \exp\left(m\varphi^*_\omega\left(\frac{|\alpha|}{m}\right)\right) < \infty$. Then it follows, thanks to (2.5), for every $k \ge m$, $x \in \mathbb{R}^N$ and $\beta \in \mathbb{N}_0^N$ that

$$\begin{aligned}
&\exp(-n\omega(x))|\partial^\beta(T\star f)(x)| \exp\left(-k\varphi^*_\omega\left(\frac{|\beta|}{k}\right)\right) \le \\
&\le C \sum_{\alpha\in\mathbb{N}_0^N} \|\exp(n\omega)\partial^{\alpha+\beta} f\|_1 \exp\left(-m\varphi^*_\omega\left(\frac{|\alpha|}{m}\right)\right) \exp\left(-k\varphi^*_\omega\left(\frac{|\beta|}{k}\right)\right) \\
&\le C \sum_{\alpha\in\mathbb{N}_0^N} \|\exp(n\omega)\partial^{\alpha+\beta} f\|_1 \exp\left(-k\varphi^*_\omega\left(\frac{|\alpha|}{k}\right)\right) \exp\left(-k\varphi^*_\omega\left(\frac{|\beta|}{k}\right)\right) \\
&\le C \sum_{\alpha\in\mathbb{N}_0^N} \|\exp(n\omega)\partial^{\alpha+\beta} f\|_1 \exp\left(-2k\varphi^*_\omega\left(\frac{|\alpha+\beta|}{2k}\right)\right) \le C\sigma_{2k,n,1}(f).
\end{aligned}$$

So,

$$r_{k,n}(T\star f) \le C\sigma_{2k,n,1}(f).$$

If $k \le m$, with a similar argument we obtain for every $x \in \mathbb{R}^N$ and $\beta \in \mathbb{N}_0^N$ that

$$\exp(-n\omega(x))|\partial^\beta(T\star f)(x)| \exp\left(-k\varphi^*_\omega\left(\frac{|\beta|}{k}\right)\right) \le C\sigma_{2m,n,1}(f).$$

This shows that $T\star f \in \bigcap_{k=1}^\infty \mathcal{O}^k_{n,\omega}(\mathbb{R}^N) \subset \mathcal{O}_{C,\omega}(\mathbb{R}^N)$ and that the map $f \mapsto T\star f$ is a continuous linear operator from $\mathcal{S}_\omega(\mathbb{R}^N)$ in $\bigcap_{k=1}^\infty \mathcal{O}^k_{n,\omega}(\mathbb{R}^N)$, and so in $\mathcal{O}_{C,\omega}(\mathbb{R}^N)$. □

We can now prove that $\mathcal{O}'_{C,\omega}(\mathbb{R}^N)$ is the space of convolutors of $\mathcal{S}_\omega(\mathbb{R}^N)$.

Theorem 5.3 *Let ω be a non-quasianalytic weight function and $T \in \mathcal{S}'_\omega(\mathbb{R}^N)$. Consider the following properties.*

(1) $T \in \mathcal{O}'_{C,\omega}(\mathbb{R}^N)$.

(2) *For every $f \in \mathcal{S}_\omega(\mathbb{R}^N)$ we have $T\star f \in \mathcal{S}_\omega(\mathbb{R}^N)$.*

Then (1) $\Rightarrow$ (2). If, in addition, the weight function ω satisfies the additional condition $\log(1+t) = o(\omega(t))$ as $t \to \infty$, then (2) $\Rightarrow$ (1).

Moreover, if $T \in \mathcal{O}'_{C,\omega}(\mathbb{R}^N)$, then the linear operator $C_T : \mathcal{S}_\omega(\mathbb{R}^N) \to \mathcal{S}_\omega(\mathbb{R}^N)$ defined by $C_T(f) := T\star f$, for $f \in \mathcal{S}_\omega(\mathbb{R}^N)$, is continuous.

Proof (1) $\Rightarrow$ (2). Fix $f \in \mathcal{S}_\omega(\mathbb{R}^N)$. Since $\mathcal{O}'_{C,\omega}(\mathbb{R}^N) \subset \mathcal{S}'_\omega(\mathbb{R}^N)$, the function $T\star f \in \mathcal{O}_{C,\omega}(\mathbb{R}^N)$ by Proposition 5.2. We show that $T\star f \in \mathcal{S}_\omega(\mathbb{R}^N)$.

Fix any $p \in (1,\infty)$. By Theorem 3.13, for every $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $T = \sum_{\alpha\in\mathbb{N}_0^N} \partial^\alpha f_\alpha$, with $\{f_\alpha\}_{\alpha\in\mathbb{N}_0^N} \in \left(\oplus L^p(\mathbb{R}^N, \exp(n\omega(x))\,dx)\right)_{\omega,m,p}$. So, for fixed $\beta \in \mathbb{N}_0^N$ and $x \in \mathbb{R}^N$, we have that

$$
\begin{aligned}
|\partial^\beta(T\star f)(x)| &\le \sum_{\alpha\in\mathbb{N}_0^N} \int_{\mathbb{R}^N} |f_\alpha(y)\partial^{\alpha+\beta} f(x-y)|\,dy \\
&\le \sum_{\alpha\in\mathbb{N}_0^N} \|\exp(n\omega) f_\alpha\|_p \|\exp(-n\omega)\partial^{\alpha+\beta}\tau_x \check{f}\|_{p'},
\end{aligned}
$$

where

$$
\begin{aligned}
\|\exp(-n\omega)\partial^{\alpha+\beta}\tau_x \check{f}\|_{p'}^{p'} &= \int_{\mathbb{R}^N} \exp(-np'\omega(y))|\partial^{\alpha+\beta} f(x-y)|^{p'}\,dy \\
&= \int_{\mathbb{R}^N} \exp(-np'\omega(x-z))|\partial^{\alpha+\beta} f(z)|^{p'}\,dz \\
&\le \int_{\mathbb{R}^N} \exp\left(np'\left(1+\omega(z)-\frac{\omega(x)}{K}\right)\right)|\partial^{\alpha+\beta} f(z)|^{p'}\,dz \\
&\le C^{p'} \exp\left(-\frac{np'\omega(x)}{K}\right) \|\exp(n\omega)\partial^{\alpha+\beta} f\|_{p'}^{p'},
\end{aligned}
$$

where $C := \exp(n)$. Therefore, we get for every $x \in \mathbb{R}^N$ and $\beta \in \mathbb{N}_0^N$ that

$$
\begin{aligned}
\exp\left(\frac{n\omega(x)}{K}\right)|\partial^\beta(T\star f)(x)| &\le C \sum_{\alpha\in\mathbb{N}_0^N} \|\exp(n\omega) f_\alpha\|_p \|\exp(n\omega)\partial^{\alpha+\beta} f\|_{p'} \\
&\le C \left(\sum_{\alpha\in\mathbb{N}_0^N} \|\exp(n\omega) f_\alpha\|_p^p \exp\left(pm\varphi^*_\omega\left(\frac{|\alpha|}{m}\right)\right)\right)^{\frac{1}{p}} \times \\
&\quad\times \left(\sum_{\alpha\in\mathbb{N}_0^N} \|\exp(n\omega)\partial^{\alpha+\beta} f\|_{p'}^{p'} \exp\left(-p'm\varphi^*_\omega\left(\frac{|\alpha|}{m}\right)\right)\right)^{\frac{1}{p'}} \\
&= C' \left(\sum_{\alpha\in\mathbb{N}_0^N} \|\exp(n\omega)\partial^{\alpha+\beta} f\|_{p'}^{p'} \exp\left(-p'm\varphi^*_\omega\left(\frac{|\alpha|}{m}\right)\right)\right)^{\frac{1}{p'}},
\end{aligned}
$$

where $C' := C\left(\sum_{\alpha\in\mathbb{N}_0^N}\|\exp(n\omega)f_\alpha\|_p^p\exp\left(pm\varphi_\omega^*\left(\frac{|\alpha|}{m}\right)\right)\right)^{\frac{1}{p}} < \infty$. If $n > m$, it follows for each $x \in \mathbb{R}^N$ and $\beta \in \mathbb{N}_0^N$ that

$$\exp\left(\frac{n\omega(x)}{K}\right)|\partial^\beta(T\star f)(x)| \le C'\left(\sum_{\alpha\in\mathbb{N}_0^N}\|\exp(n\omega)\partial^{\alpha+\beta}f\|_{p'}^{p'}\exp\left(-p'm\varphi_\omega^*\left(\frac{|\alpha|}{m}\right)\right)\right)^{\frac{1}{p'}}$$
$$\le C'\left(\sum_{\alpha\in\mathbb{N}_0^N}\|\exp(n\omega)\partial^{\alpha+\beta}f\|_{p'}^{p'}\exp\left(-p'n\varphi_\omega^*\left(\frac{|\alpha|}{n}\right)\right)\right)^{\frac{1}{p'}}$$

and so

$$q_{\frac{n}{K},n}(T\star f) = \sup_{\beta\in\mathbb{N}_0^N}\sup_{x\in\mathbb{R}^N}\exp\left(-n\varphi_\omega^*\left(\frac{|\beta|}{n}\right)\right)\exp\left(\frac{n\omega(x)}{K}\right)|\partial^\beta(T\star f)(x)|$$
$$\le C'\left(\sum_{\alpha\in\mathbb{N}_0^N}\|\exp(n\omega)\partial^{\alpha+\beta}f\|_{p'}^{p'}\exp\left(-p'n\varphi_\omega^*\left(\frac{|\alpha|}{n}\right)\right)\exp\left(-p'n\varphi_\omega^*\left(\frac{|\beta|}{n}\right)\right)\right)^{\frac{1}{p'}}$$
$$\le C'\left(\sum_{\alpha\in\mathbb{N}_0^N}\|\exp(n\omega)\partial^{\alpha+\beta}f\|_{p'}^{p'}\exp\left(-2np'\varphi_\omega^*\left(\frac{|\alpha+\beta|}{2n}\right)\right)\right)^{\frac{1}{p'}} = \sigma_{2n,n,p'}(f).$$

If $n \le m$, procedding in a similar way we obtain for every $x \in \mathbb{R}^N$ and $\beta \in \mathbb{N}_0^N$ that

$$q_{\frac{n}{K},n}(T\star f) \le \sigma_{2m,n,p'}(f).$$

From the arbitrarity of n, we conclude that $T\star f \in \mathcal{S}_\omega(\mathbb{R}^N)$. This shows also the continuity of the operator C_T.

(2) $\Rightarrow$ (1). The assumption implies that $T\star\phi \in \mathcal{S}_\omega(\mathbb{R}^N)$ for every $\phi \in \mathcal{D}_\omega(\mathbb{R}^N)$. For a fixed $p \in (1,\infty)$, we have by Remark 4.3 that $\mathcal{S}_\omega(\mathbb{R}^N) = \bigcap_{n=1}^\infty \mathcal{D}_{L_n^p,\omega}(\mathbb{R}^N)$ and so, $T\star\phi \in \mathcal{D}_{L_n^p,\omega}(\mathbb{R}^N)$ for every $\phi \in \mathcal{D}_\omega(\mathbb{R}^N)$ and $n \in \mathbb{N}$. This yields that $T\star\phi \in L^p(\mathbb{R}^N, \exp(n\omega(x))dx)$ for every $\phi \in \mathcal{D}_\omega(\mathbb{R}^N)$ and $n \in \mathbb{N}$. Therefore, by Theorem 4.9(2) $\Rightarrow$ (3) and Proposition 4.5, we get for every $n \in \mathbb{N}$ that there exist an elliptic ultradifferentiable operator $G(D)$ of ω-class and $f, g \in L^p(\mathbb{R}^N, \exp(n\omega(x))dx)$ such that $T = G(D)f + g$. So, $T \in \mathcal{D}'_{L_{-n}^{p'},\omega}(\mathbb{R}^N)$ for every $n \in \mathbb{N}$, i.e., $T \in \mathcal{O}'_{C,\omega}(\mathbb{R}^N)$.

This completes the proof. □

Finally, we show that $\mathcal{O}'_{C,\omega}(\mathbb{R}^N)$ is the space of convolutors of $\mathcal{S}'_\omega(\mathbb{R}^N)$. So, we recall that if $T \in \mathcal{O}'_{C,\omega}(\mathbb{R}^N)$ and $S \in \mathcal{S}'_\omega(\mathbb{R}^N)$, we define the convolution $T\star S$ by

$$\langle T\star S, f\rangle := \langle S, \check{T}\star f\rangle \tag{5.2}$$

for every $f \in \mathcal{S}_\omega(\mathbb{R}^N)$ (recall that $\check{T}$ is the distribution defined by $\varphi \mapsto \langle\check{T},\varphi\rangle := \langle T, \check{\varphi}\rangle$).

Proposition 5.4 *Let ω be a non-quasianalytic weight function and $T \in \mathcal{O}'_{C,\omega}(\mathbb{R}^N)$. Then $T\star S \in \mathcal{S}'_\omega(\mathbb{R}^N)$ for every $S \in \mathcal{S}'_\omega(\mathbb{R}^N)$.*

Moreover, the linear operator $\mathcal{C}_T : \mathcal{S}'_\omega(\mathbb{R}^N) \to \mathcal{S}'_\omega(\mathbb{R}^N)$ defined by $\mathcal{C}_T(S) := T\star S$, for $S \in \mathcal{S}'_\omega(\mathbb{R}^N)$, is continuous.

Proof By Theorem 5.3 the convolution $\check{T}\star f$ belongs to $\mathcal{S}_\omega(\mathbb{R}^N)$ whenever $f \in \mathcal{S}_\omega(\mathbb{R}^N)$. Furthermore, the linear operator $f \in \mathcal{S}_\omega(\mathbb{R}^N) \mapsto \check{T}\star f \in \mathcal{S}_\omega(\mathbb{R}^N)$ is continuous. So, as S is a continuous linear form on $\mathcal{S}_\omega(\mathbb{R}^N)$, the map $f \mapsto \langle S, \check{T}\star f\rangle$ is continuous on $\mathcal{S}_\omega(\mathbb{R}^N)$, i.e., $T\star S \in \mathcal{S}'_\omega(\mathbb{R}^N)$. So, the first statement follows is proved.

The operator $\mathcal{C}_T : \mathcal{S}'_\omega(\mathbb{R}^N) \to \mathcal{S}'_\omega(\mathbb{R}^N)$ is continuous as it is the transpose of the continuous linear operator $C_{\check{T}} : \mathcal{S}_\omega(\mathbb{R}^N) \to \mathcal{S}_\omega(\mathbb{R}^N)$. □

6 The action of the Fourier transform in the multiplier and convolutor spaces

In this final section we study the action of the Fourier transform on the spaces $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$ and $\mathcal{O}'_{C,\omega}(\mathbb{R}^N)$ following the approach in [23]. To this end, we recall that the spaces $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$ and $\mathcal{O}'_{C,\omega}(\mathbb{R}^N)$ are both continuously included in $\mathcal{S}'_\omega(\mathbb{R}^N)$. We also recall that the gaussian function $f(x) := \exp\left(-\frac{|x|^2}{2}\right)$, for $x \in \mathbb{R}^N$, belongs to $\mathcal{S}_\omega(\mathbb{R}^N)$.

We point out that by Theorem 3.2 the space $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$ can be identified with the space $\mathcal{M}(\mathcal{S}_\omega(\mathbb{R}^N))$ of all multipliers on $\mathcal{S}_\omega(\mathbb{R}^N)$ via the map $M : \mathcal{O}_{M,\omega}(\mathbb{R}^N) \to \mathcal{M}(\mathcal{S}_\omega(\mathbb{R}^N))$ defined by $M(f) := M_f$ for each $f \in \mathcal{O}_{M,\omega}(\mathbb{R}^N)$. Since $\mathcal{M}(\mathcal{S}_\omega(\mathbb{R}^N))$ is a subspace of $\mathcal{L}(\mathcal{S}_\omega(\mathbb{R}^N))$, the space $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$ (via the map M) can be then endowed with the topology τ_b of uniform convergence on bounded sets of $\mathcal{S}_\omega(\mathbb{R}^N)$, induced by $\mathcal{L}_b(\mathcal{S}_\omega(\mathbb{R}^N))$. In view of [14, Theorem 5.2] we have that $t = \tau_b|_{\mathcal{O}_{M,\omega}(\mathbb{R}^N)}$. On the other hand, by Theorem 5.3 the space $\mathcal{O}'_{C,\omega}(\mathbb{R}^N)$ can be identified with the space $\mathcal{C}(\mathcal{S}_\omega(\mathbb{R}^N))$ of all convolutors on $\mathcal{S}_\omega(\mathbb{R}^N)$ via the map $C : \mathcal{O}'_{C,\omega}(\mathbb{R}^N) \to \mathcal{C}(\mathcal{S}_\omega(\mathbb{R}^N))$ defined by $C(T) := C_T$ for each $T \in \mathcal{O}'_{C,\omega}(\mathbb{R}^N)$. Since $\mathcal{C}(\mathcal{S}_\omega(\mathbb{R}^N))$ is a subspace of $\mathcal{L}(\mathcal{S}_\omega(\mathbb{R}^N))$, the space $\mathcal{O}'_{C,\omega}(\mathbb{R}^N)$ (via the map C) can be then endowed with the topology τ_b induced by $\mathcal{L}_b(\mathcal{S}_\omega(\mathbb{R}^N))$. Now, we can state and prove this result.

Theorem 6.1 *Let ω be a non-quasianalytic weight function with the additional condition $\log(1+t) = o(\omega(t))$ as $t \to \infty$. Then the Fourier transform $\mathcal{F}$ is a topological isomorphism from the space $\mathcal{O}'_{C,\omega}(\mathbb{R}^N)$ onto the space $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$. Furthermore, for $T \in \mathcal{O}'_{C,\omega}(\mathbb{R}^N)$ and $S \in \mathcal{S}'_\omega(\mathbb{R}^N)$, we have*

$$\mathcal{F}(T\star S) = \mathcal{F}(T)\mathcal{F}(S), \tag{6.1}$$

and if $f \in \mathcal{O}_{M,\omega}(\mathbb{R}^N)$ and $T \in \mathcal{S}'_\omega(\mathbb{R}^N)$ we have

$$\mathcal{F}(fT) = (2\pi)^{-N}\hat{f}\star\mathcal{F}(T). \tag{6.2}$$

Proof Fix $T \in \mathcal{O}'_{C,\omega}(\mathbb{R}^N)$. Since $\mathcal{O}'_{C,\omega}(\mathbb{R}^N) \subset \mathcal{S}'_\omega(\mathbb{R}^N)$ and $\mathcal{F}$ is an automorphism into $\mathcal{S}'_\omega(\mathbb{R}^N)$, we have that $\mathcal{F}(T) \in \mathcal{S}'_\omega(\mathbb{R}^N)$. So, we have to prove that $\mathcal{F}(T)$ is an ultradistribution represented by a function belonging to $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$. To do this, we observe that by (2.9) the ultradistribution $\exp\left(-\frac{|x|^2}{2}\right)\mathcal{F}(T) \in \mathcal{S}'_\omega(\mathbb{R}^N)$ as the gaussian function belongs to $\mathcal{S}_\omega(\mathbb{R}^N)$, and it is equal to the ultradistribution $(2\pi)^{\frac{N}{2}}\mathcal{F}\left(T\star\exp\left(-\frac{|x|^2}{2}\right)\right)$. On the other hand, from Theorem 5.3 it follows that $T\star\exp\left(-\frac{|x|^2}{2}\right) \in \mathcal{S}_\omega(\mathbb{R}^N)$. Since $\mathcal{F}$ is an automorphism into $\mathcal{S}_\omega(\mathbb{R}^N)$, then $(2\pi)^{\frac{N}{2}}\mathcal{F}\left(T\star\exp\left(-\frac{|x|^2}{2}\right)\right) \in \mathcal{S}_\omega(\mathbb{R}^N)$.Thus, the ultradistribution $\mathcal{F}(T)$ is represented by the function $\psi := \exp\left(\frac{|x|^2}{2}\right)(2\pi)^{\frac{N}{2}}\mathcal{F}\left(T\star\exp\left(-\frac{|x|^2}{2}\right)\right) \in C^\infty(\mathbb{R}^N)$. But, $\psi \in \mathcal{E}_\omega(\mathbb{R}^N)$ as it is easy to see. So, it remains to prove that the function ψ belongs to

$\mathcal{O}_{M,\omega}(\mathbb{R}^N)$, i.e., thanks to Theorem 3.2 it suffices to show that $f\psi \in \mathcal{S}_\omega(\mathbb{R}^N)$ for every $f \in \mathcal{S}_\omega(\mathbb{R}^N)$. To see this, we fix $f \in \mathcal{S}_\omega(\mathbb{R}^N)$ and notice that by formula (2.9)

$$f\psi = f\mathcal{F}(T) = \mathcal{F}(T\star\mathcal{F}^{-1}(f)) \in \mathcal{S}_\omega(\mathbb{R}^N).$$

This shows that $\mathcal{F}(T) \in \mathcal{O}_{M,\omega}(\mathbb{R}^N)$.

In order to get that $\mathcal{F}$ is an isomorphism between $\mathcal{O}'_{C,\omega}(\mathbb{R}^N)$ and $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$, it remains to prove that if $f \in \mathcal{O}_{M,\omega}(\mathbb{R}^N)$, then $\hat{f} \in \mathcal{O}'_{C,\omega}(\mathbb{R}^N)$. So, fixed $f \in \mathcal{O}_{M,\omega}(\mathbb{R}^N)$, we observe that $\mathcal{F}\hat{f} \in (\mathcal{F}\circ\mathcal{F})(\mathcal{O}_{M,\omega}(\mathbb{R}^N)) = (2\pi)^N(\mathcal{O}_{M,\omega}(\mathbb{R}^N))^\vee = \mathcal{O}_{M,\omega}(\mathbb{R}^N)$. By Theorem 3.2 it follows for every $g \in \mathcal{S}_\omega(\mathbb{R}^N)$ that $(\mathcal{F}\hat{f})\hat{g} \in \mathcal{S}_\omega(\mathbb{R}^N)$. But, taking into account that $\hat{f} \in \mathcal{S}'_\omega(\mathbb{R}^N)$ and that

$$\begin{aligned}(\mathcal{F}\hat{f})\hat{g} \in \mathcal{S}_\omega(\mathbb{R}^N) \quad \forall g \in \mathcal{S}_\omega(\mathbb{R}^N) &\iff \mathcal{F}(\hat{f}\star g) \in \mathcal{S}_\omega(\mathbb{R}^N) \quad \forall g \in \mathcal{S}_\omega(\mathbb{R}^N)\\ &\iff \hat{f}\star g \in \mathcal{S}_\omega(\mathbb{R}^N) \quad \forall g \in \mathcal{S}_\omega(\mathbb{R}^N),\end{aligned}$$

we obtain via Theorem 5.3 that $\hat{f} \in \mathcal{O}'_{C,\omega}(\mathbb{R}^N)$.

We now prove formula (6.1). Fixed $T \in \mathcal{O}'_{C,\omega}(\mathbb{R}^N)$, $S \in \mathcal{S}'_\omega(\mathbb{R}^N)$, $f \in \mathcal{S}_\omega(\mathbb{R}^N)$, we have

$$\langle \mathcal{F}(T\star S), f\rangle = \langle T\star S, \hat{f}\rangle = \langle S, \check{T}\star\hat{f}\rangle,$$

where $\hat{f} \in \mathcal{S}_\omega(\mathbb{R}^N)$ and $\check{T}\star\hat{f} \in \mathcal{S}_\omega(\mathbb{R}^N)$ by Theorem 5.3. Thanks to formula (2.9), we obtain $\mathcal{F}(\mathcal{F}(T)f) = \check{T}\star\hat{f}$. So, we conclude

$$\langle \mathcal{F}(T\star S), f\rangle = \langle S, \mathcal{F}(\mathcal{F}(T)f)\rangle = \langle \mathcal{F}(S), \mathcal{F}(T)f\rangle = \langle \mathcal{F}(T)\mathcal{F}(S), f\rangle,$$

where in the last equality we used the property that $\mathcal{F}(T) \in \mathcal{O}_{M,\omega}(\mathbb{R}^N)$. This shows that (6.1) holds. The proof of formula (6.2) is analogous and so it is omitted.

That $\mathcal{F}$ is a topological isomorphism now follows from the continuity of $\mathcal{F}$, the ultrabornologicity of $\mathcal{O}_{M,\omega}(\mathbb{R}^N)$ (see [14]) and de Wilde's open mapping theorem. □

Funding Open access funding provided by Università del Salento within the CRUI-CARE Agreement.

References

1. Adams, R.A.: Sobolev Spaces. Academic Press, London (1975)
2. Albanese, A.A., Mele, C.: Multipliers on $\mathcal{S}_\omega(\mathbb{R}^N)$. J. Pseudo-Differ. Oper. Appl. **12**, 35 (2021)
3. Bargetz, C., Ortner, N.: Characterization of L. Schwartz' convolutor and multiplier spaces $\mathcal{O}'_C$ and $\mathcal{O}_M$ by the short-time Fourier transform. Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM (2014) **108**, 833–847 (2014)
4. Barros-Neto, J.: An Introduction to the Theory of Distributions. Marcel Dekker Inc, New York (1973)
5. Betancor, J.J., Fernández, C., Galbis, A.: Beurling ultradistributions of L_p-growth. J. Math. Anal. Appl. **279**, 246–265 (2003)
6. Björck, G.: Linear partial differential operators and generalized distributions. Ark. Mat. **6**, 351–407 (1965)
7. Boiti, C., Jornet, D., Oliaro, A.: Real Paley-Wiener theorems in spaces of ultradifferentiable functions. J. Funct. Anal. **278**, 1–45 (2020)

8. Boiti, C., Jornet, D., Oliaro, A.: Regularity of partial differential operators in ultradifferentiable spaces and Wigner type transforms. J. Math. Anal. Appl. **446**, 920–944 (2017)
9. Boiti, C., Jornet, D., Oliaro, A., Schindl, G.: Nuclearity of rapidly decreasing ultradifferentiable functions and time-frequency analysis. Collect. Math. **72**, 423–442 (2021)
10. Bonet, J., Meise, R., Melikhov, S.N.: A comparison of two different ways to define classes of ultradifferentiable functions. Bull. Belg. Math. Soc. Simon Stevin **14**, 425–444 (2007)
11. Braun, R.W.: An extension of Komatsu's second structure theorem for ultradistributions. J. Fac. Sci. Tokyo Sec. IA **40**, 411–417 (1993)
12. Braun, R.W., Meise, R., Taylor, B.A.: Ultradifferentiable functions and Fourier analysis. Result. Math. **17**, 206–237 (1990)
13. Cioranescu, I.: The characterization of the almost periodic ultradistributions of Beurling type. Proc. Am. Math. Soc. **116**, 127–134 (1992)
14. Debrouwere, A., Neyt, L.: Weighted (PLB)-spaces of ultradifferentiable functions and multiplier spaces. arXiv: 2010.02606
15. Debrouwere, A., Vindas, J.: On weighted inductive limits of spaces of ultradifferentiable functions and their duals. Math. Nachr. **292**, 573–602 (2019)
16. Debrouwere, A., Neyt, L., Vindas, J.: Characterization of nuclearity for Beurling-Björk spaces. Proc. Am. Math. Soc. **148**, 5171–5180 (2020)
17. Dimovski, P., Prangoski, B., Velinov, D.: Multipliers and convolutors in the space of tempered ultradistrubutions. Novi Sad J. Math. **44**, 1–18 (2014)
18. Dimovski, P., Pilipović, S., Prangoski, B., Vindas, J.: Convolution of ultradistributions and ultradistribution spaces associated to translation-invariant Banach spaces. Kyoto J. Math. **56**, 401–440 (2016)
19. Evans, L.C.: *Partial differential equations.* Am. Math. Soc. **19** (1998)
20. Gómez-Collado, M.C.: Almost periodic ultradistributions of Beurling and Roumieu type. Proc. Am. Math. Soc. **129**, 2319–2329 (2000)
21. Grothendieck, A.: Produits tensoriels topologiques et espaces nucléaires. In: Graduate Studies in Mathematics , vol. 16. Memoirs of the American Mathematical Society (1998)
22. Horvath, J.: Topological Vector Spaces and Distributions, vol. 1. Addison-Wesley Publishing Company, Boston (1966)
23. Kisinsky, J.: On the exchange between convolution and multiplication via the Fourier transformation. Polska Akademia Nauk. Instytut Matematyczny, Preprint 751 (2017)
24. Kisinsky, J.: Equicontinuity and convergent sequences in the spaces $\mathcal{O}'_C$ and $\mathcal{O}_M$. Bull. Polish Acad. Sci. Math. **59**, 223–235 (2011)
25. Komatsu, H.: Ultradistributions 1. Structure theorems and a characterization. J. Fac. Sci. Tokyo Sec. IA **20**, 25–105 (1973)
26. Kovacević, D.: Some operations on the space $\mathcal{S}'^{(M_p)}$ of tempered ultradistributions. Univ. u Novom Sadu Zb. Rad. Prirod. Mat. Fak. Ser. Mat. **23**, 87–106 (1993)
27. Kovacević, D.: The spaces of weighted and tempered ultradistributions, Part II. Univ. u Novom Sadu, Zb. Rad. Prirod. Mat. Fak. Ser. Mat. **24**, 171–185 (1994)
28. Kucera, J.: Convolution of temperate distributions. Publ. Math. Debrecen **33**, 323–327 (1986)
29. Kucera, J., McKennon, K.: The topology on certain spaces of multipliers of temperate distributions. Rocky Mt. J. Math. **7**, 377–383 (1977)
30. Larcher, J.: Some remarks concerning the spaces of multipliers and convolutors, $\mathcal{O}_M$ *and* $\mathcal{O}'_C$, of Laurent Schwartz. Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM **106**, 407–417 (2012)
31. Larcher, J., Wengenroth, J.: A new proof for the bornologicity of the space of slowly increasing functions. Bull. Belg. Math. Soc. Simon Stevin **21**, 887–894 (2014)
32. Pilipovic, S.: Multipliers, convolutors and hypoelliptic convolutors for tempered ultradistributions. In: Banaras Hindu University, India, Platinum Jubile Year International Symposium on Generalized Functions and Their Applications (December 23–26, 1991). Plenum Press, New York (1991)
33. Pilipovic, S.: Characterizations of bounded sets in spaces of ultradistributions. Proc. Am. Math. Soc. **12**, 1191–1206 (1994)
34. Schwartz, L.: Théorie des Distributions. Hermann, Paris (1966)
35. Treves, F.: Topological Vector Spaces. Distributions and Kernels. Academic Press, New York (1967)
36. Zielezny, Z.: On the space of convolution operators in $\mathcal{K}'_1$. Stud. Math. **31**, 111–124 (1968)
37. Zielezny, Z.: Hypoelliptic and entire elliptic convolution equations in subspaces of the space of distributions (II). Stud. Math. **32**, 47–59 (1969)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.