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ORIGINAL PAPER

Some q-supercongruences modulo the square and cube of a cyclotomic polynomial

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Abstract

Two q-supercongruences of truncated basic hypergeometric series containing two free parameters are established by employing specific identities for basic hypergeometric series. The results partly extend two q-supercongruences that were earlier conjectured by the same authors and involve q-supercongruences modulo the square and the cube of a cyclotomic polynomial. One of the newly proved q-supercongruences is even conjectured to hold modulo the fourth power of a cyclotomic polynomial.

Keywords Basic hypergeometric series · Supercongruences · q-congruences · Cyclotomic polynomial · Andrews' transformation · Gasper's summation

Mathematics Subject Classification Primary 33D15 · Secondary 11A07 · 11B65

1 Introduction

In 1914, Ramanujan [25] listed a number of representations of $1/\pi$, including

$$\sum_{k=0}^{\infty}(6k+1)\frac{\left(\frac{1}{2}\right)_k^3}{k!^3 4^k}=\frac{4}{\pi}, \tag{1.1}$$

where $(a)_n = a(a+1)\cdots(a+n-1)$ denotes the Pochhammer symbol. Ramanujan's formulas gained unprecedented popularity in the 1980's when they were discovered to provide

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fast algorithms for calculating decimal digits of π. See, for instance, the monograph [2] by the Borwein brothers.

In 1997, Van Hamme [29] conjectured 13 intriguing p-adic analogues of Ramanujan-type formulas, such as

$$\sum_{k=0}^{(p-1)/2}(6k+1)\frac{(\frac{1}{2})_k^3}{k!^3 4^k}\equiv p(-1)^{(p-1)/2} \pmod{p^4}, \tag{1.2}$$

where $p>3$ is a prime. Van Hamme himself supplied proofs for three of them. Supercongruences like (1.2) are called Ramanujan-type supercongruences (see [33]). The proof of the supercongruence (1.2) was first given by Long [22]. As of today, all of Van Hamme's 13 supercongruences have been confirmed by various techniques (see [24,28]).

In recent years, q-congruences and q-supercongruences have been established by different authors (see, for example, [5–13,15–21,23,27,30–32,34]). In particular, the present authors [9] proved that, for any odd integer $d\ge 5$,

$$\sum_{k=0}^{n-1}[2dk+1]\frac{(q;q^d)_k^d}{(q^d;q^d)_k^d}q^{d(d-3)k/2}\equiv\begin{cases}0 \pmod{\Phi_n(q)^2}, & \text{if } n\equiv -1 \pmod{d},\\ 0 \pmod{\Phi_n(q)^3}, & \text{if } n\equiv -1/2 \pmod{d}.\end{cases} \tag{1.3}$$

Here and in what follows, we adopt the standard q-notation: $[n]=1+q+\cdots+q^{n-1}$ is the q-*integer*; $(a;q)_n=(1-a)(1-aq)\cdots(1-aq^{n-1})$ is the q-*shifted factorial*, with the compact notation $(a_1,a_2,\ldots,a_m;q)_n=(a_1;q)_n(a_2;q)_n\cdots(a_m;q)_n$ used for their products; and $\Phi_n(q)$ denotes the n-th *cyclotomic polynomial* in q, which may be defined as

$$\Phi_n(q)=\prod_{\substack{1\le k\le n\\ \gcd(k,n)=1}}(q-\zeta^k),$$

where ζ is an n-th primitive root of unity.

We should point out that the q-congruence (1.3) does not hold for $d=3$. The present authors [9] also established the following companion of (1.3): for any odd integer $d\ge 3$ and integer $n>1$,

$$\sum_{k=0}^{n-1}[2dk-1]\frac{(q^{-1};q^d)_k^d}{(q^d;q^d)_k^d}q^{d(d-1)k/2}\equiv\begin{cases}0 \pmod{\Phi_n(q)^2}, & \text{if } n\equiv 1 \pmod{d},\\ 0 \pmod{\Phi_n(q)^3}, & \text{if } n\equiv 1/2 \pmod{d}.\end{cases} \tag{1.4}$$

They also proposed the following conjectures [9, Conjectures 1 and 2], which are generalizations of (1.3) and (1.4).

Conjecture 1 *Let $d\ge 5$ be an odd integer. Then*

$$\sum_{k=0}^{n-1}[2dk+1]\frac{(q;q^d)_k^d}{(q^d;q^d)_k^d}q^{d(d-3)k/2}\equiv\begin{cases}0 \pmod{\Phi_n(q)^3}, & \text{if } n\equiv -1 \pmod{d},\\ 0 \pmod{\Phi_n(q)^4}, & \text{if } n\equiv -1/2 \pmod{d}.\end{cases}$$

Conjecture 2 *Let $d\ge 5$ be an odd integer and let $n>1$. Then*

$$\sum_{k=0}^{n-1}[2dk-1]\frac{(q^{-1};q^d)_k^d}{(q^d;q^d)_k^d}q^{d(d-1)k/2}\equiv\begin{cases}0 \pmod{\Phi_n(q)^3}, & \text{if } n\equiv 1 \pmod{d},\\ 0 \pmod{\Phi_n(q)^4}, & \text{if } n\equiv 1/2 \pmod{d}.\end{cases}$$

q-Supercongruences such as those above (modulo a third and even fourth power of a cyclotomic polynomial) are rather special. In fact, concrete results for truncated basic hypergeometric sums being congruent to 0 modulo a high power of a cyclotomic polynomial are very rare. See [8,10–12,14,18] for recent papers featuring such results. The main goal of this paper is to add two complete two-parameter families of q-supercongruences to the list of such q-supercongruences (see Theorems 1 and 2).

We shall prove that the respective first cases of Conjectures 1 and 2 are true by establishing the following more general result.

Theorem 1 *Let d and r be odd integers satisfying $d \geq 3$, $r \leq d-4$ (in particular, r may be negative) and $\gcd(d, r)=1$. Let n be an integer such that $n \geq d-r$ and $n \equiv -r \pmod{d}$. Then*

$$\sum_{k=0}^{M}[2dk+r]\frac{(q^r;q^d)_k^d}{(q^d;q^d)_k^d}q^{d(d-r-2)k/2} \equiv 0 \pmod{[n]\Phi_n(q)^2}, \tag{1.5}$$

where $M=(dn-n-r)/d$ or $n-1$.

We shall also prove the following q-supercongruences.

Theorem 2 *Let d and r be odd integers satisfying $d \geq 3$, $r \leq d-4$ (in particular, r may be negative) and $\gcd(d, r)=1$. Let n be an integer such that $n \geq (d-r)/2$ and $n \equiv -r/2 \pmod{d}$. Then*

$$\sum_{k=0}^{M}[2dk+r]\frac{(q^r;q^d)_k^d}{(q^d;q^d)_k^d}q^{d(d-r-2)k/2} \equiv 0 \pmod{[n]\Phi_n(q)}, \tag{1.6}$$

where $M=(dn-2n-r)/d$ or $n-1$.

The following generalization of the respective second cases of Conjectures 1 and 2 should be true.

Conjecture 3 *The q-supercongruence (1.6) holds modulo $[n]\Phi_n(q)^3$ for $d \geq 5$.*

We shall prove Theorems 1 and 2 in Sections 2 and 3, respectively, by making use of Andrews' multiseries extension (2.2) of the Watson transformation [1, Theorem 4], along with Gasper's very-well-poised Karlsson–Minton type summation [3, Eq. (5.13)]. It should be pointed out that Andrews' transformation plays an important part in combinatorics and number theory (see [7] and the introduction of [12] for more such examples).

2 Proof of Theorem 1

We need a simple q-congruence modulo $\Phi_n(q)^2$, which was already used in [10,12].

Lemma 1 *Let α, r be integers and n a positive integer. Then*

$$(q^{r-\alpha n}, q^{r+\alpha n}; q^d)_k \equiv (q^r;q^d)_k^2 \pmod{\Phi_n(q)^2}. \tag{2.1}$$

We will further utilize a powerful transformation formula due to Andrews [1, Theorem 4], which may be stated as follows:

$$\begin{aligned}&\sum_{k\ge 0}\frac{(a,q\sqrt{a},-q\sqrt{a},b_1,c_1,\dots,b_m,c_m,q^{-N};q)_k}{(q,\sqrt{a},-\sqrt{a},aq/b_1,aq/c_1,\dots,aq/b_m,aq/c_m,aq^{N+1};q)_k}\left(\frac{a^mq^{m+N}}{b_1c_1\cdots b_mc_m}\right)^k\\&=\frac{(aq,aq/b_mc_m;q)_N}{(aq/b_m,aq/c_m;q)_N}\sum_{j_1,\dots,j_{m-1}\ge 0}\frac{(aq/b_1c_1;q)_{j_1}\cdots(aq/b_{m-1}c_{m-1};q)_{j_{m-1}}}{(q;q)_{j_1}\cdots(q;q)_{j_{m-1}}}\\&\quad\times\frac{(b_2,c_2;q)_{j_1}\dots(b_m,c_m;q)_{j_1+\dots+j_{m-1}}}{(aq/b_1,aq/c_1;q)_{j_1}\dots(aq/b_{m-1},aq/c_{m-1};q)_{j_1+\dots+j_{m-1}}}\\&\quad\times\frac{(q^{-N};q)_{j_1+\dots+j_{m-1}}}{(b_mc_mq^{-N}/a;q)_{j_1+\dots+j_{m-1}}}\frac{(aq)^{j_{m-2}+\dots+(m-2)j_1}q^{j_1+\dots+j_{m-1}}}{(b_2c_2)^{j_1}\cdots(b_{m-1}c_{m-1})^{j_1+\dots+j_{m-2}}}.\end{aligned}\tag{2.2}$$

This transformation is a multiseries generalization of Watson's ${}_8\phi_7$ transformation formula (listed in [4, Appendix (III.18)]; cf. [4, Chapter 1] for the notation of a basic hypergeometric ${}_r\phi_s$ series we are using),

$$\begin{aligned}&{}_8\phi_7\left[\begin{matrix}a,\ qa^{\frac12},\ -qa^{\frac12},\ b,\ c,\ d,\ e,\ q^{-n}\\ a^{\frac12},\ -a^{\frac12},\ aq/b,\ aq/c,\ aq/d,\ aq/e,\ aq^{n+1}\end{matrix};q,\ \frac{a^2q^{n+2}}{bcde}\right]\\&\quad=\frac{(aq,aq/de;q)_n}{(aq/d,aq/e;q)_n}\,{}_4\phi_3\left[\begin{matrix}aq/bc,\ d,\ e,\ q^{-n}\\ aq/b,\ aq/c,\ deq^{-n}/a\end{matrix};q,\ q\right],\end{aligned}\tag{2.3}$$

to which it reduces for $m=2$.

Next, we require a very-well-poised Karlsson–Minton type summation due to Gasper [3, Eq. (5.13)] (see also [4, Ex. 2.33 (i)]):

$$\begin{aligned}&\sum_{k=0}^{\infty}\frac{(a,q\sqrt{a},-q\sqrt{a},b,a/b,d,e_1,aq^{n_1+1}/e_1,\dots,e_m,aq^{n_m+1}/e_m;q)_k}{(q,\sqrt{a},-\sqrt{a},aq/b,bq,aq/d,aq/e_1,e_1q^{-n_1},\dots,aq/e_m,e_mq^{-n_m};q)_k}\left(\frac{q^{1-\nu}}{d}\right)^k\\&\quad=\frac{(q,aq,aq/bd,bq/d;q)_\infty}{(bq,aq/b,aq/d,q/d;q)_\infty}\prod_{j=1}^{m}\frac{(aq/be_j,bq/e_j;q)_{n_j}}{(aq/e_j,q/e_j;q)_{n_j}},\end{aligned}\tag{2.4}$$

where $n_1,\dots,n_m$ are non-negative integers, $\nu=n_1+\dots+n_m$, and the convergence condition $|q^{1-\nu}/d|<1$ is required if the series does not terminate. We point out that an elliptic extension of the terminating $d=q^{-\nu}$ case of (2.4) can be found in [26, Eq. (1.7)].

In particular, we note that for $d=bq$ the right-hand side of (2.4) vanishes. Putting in addition $b=q^{-N}$ we get the following terminating summation formula:

$$\sum_{k=0}^{N}\frac{(a,q\sqrt{a},-q\sqrt{a},e_1,aq^{n_1+1}/e_1,\dots,e_m,aq^{n_m+1}/e_m,q^{-N};q)_k}{(q,\sqrt{a},-\sqrt{a},aq/e_1,e_1q^{-n_1},\dots,aq/e_m,e_mq^{-n_m},aq^{N+1};q)_k}q^{(N-\nu)k}=0,\tag{2.5}$$

which is valid for $N>\nu=n_1+\dots+n_m$.

A suitable combination of (2.2) and (2.5) yields the following multi-series summation formula, derived in [12, Lemma 2] (whose proof we nevertheless give here, to make the paper self-contained):

Lemma 2 *Let $m \ge 2$. Let q, a and $e_1, \ldots, e_{m+1}$ be arbitrary parameters with $e_{m+1} = e_1$, and let $n_1, \ldots, n_m$ and N be non-negative integers such that $N > n_1 + \cdots + n_m$. Then*

$$\begin{aligned}0 = \sum_{j_1,\ldots,j_{m-1}\ge 0} &\frac{(e_1q^{-n_1}/e_2;q)_{j_1}\cdots(e_{m-1}q^{-n_{m-1}}/e_m;q)_{j_{m-1}}}{(q;q)_{j_1}\cdots(q;q)_{j_{m-1}}}\\ &\times\frac{(aq^{n_2+1}/e_2,e_3;q)_{j_1}\ldots(aq^{n_m+1}/e_m,e_{m+1};q)_{j_1+\cdots+j_{m-1}}}{(e_1q^{-n_1},aq/e_2;q)_{j_1}\ldots(e_{m-1}q^{-n_{m-1}},aq/e_m;q)_{j_1+\cdots+j_{m-1}}}\\ &\times\frac{(q^{-N};q)_{j_1+\cdots+j_{m-1}}}{(e_1q^{n_m-N+1}/e_m;q)_{j_1+\cdots+j_{m-1}}}\frac{(aq)^{j_{m-2}+\cdots+(m-2)j_1}q^{j_1+\cdots+j_{m-1}}}{(aq^{n_2+1}e_3/e_2)^{j_1}\cdots(aq^{n_{m-1}+1}e_m/e_{m-1})^{j_1+\cdots+j_{m-2}}}.\end{aligned} \tag{2.6}$$

Proof By specializing the parameters in the multi-sum transformation (2.2) by $b_i \mapsto aq^{n_i+1}/e_i$, $c_i \mapsto e_{i+1}$, for $1 \le i \le m$ (where $e_{m+1} = e_1$), and dividing both sides of the identity by the prefactor of the multi-sum, we obtain that the series on the right-hand side of (2.6) equals

$$\frac{(e_mq^{-n_m},aq/e_1;q)_N}{(aq,e_mq^{-n_m}/e_1;q)_N}\times\sum_{k=0}^{N}\frac{(a,q\sqrt{a},-q\sqrt{a},e_1,aq^{n_1+1}/e_1,\ldots,e_m,aq^{n_m+1}/e_m,q^{-N};q)_k}{(q,\sqrt{a},-\sqrt{a},aq/e_1,e_1q^{-n_1},\ldots,aq/e_m,e_mq^{-n_m},aq^{N+1};q)_k}q^{(N-\nu)k},$$

with $\nu = n_1+\cdots+n_m$. Now the last sum vanishes by the special case of Gasper's summation stated in (2.5). □

Using [11, Lemma 2.1], we can prove the following result which is similar to [11, Lemma 2.2].

Lemma 3 *Let d, n be positive integers with $\gcd(d,n) = 1$. Let r be an integer. Then*

$$\sum_{k=0}^{m}[2dk+r]\frac{(q^r;q^d)_k^d}{(q^d;q^d)_k^d}q^{d(d-r-2)k/2} \equiv 0 \pmod{[n]},$$
$$\sum_{k=0}^{n-1}[2dk+r]\frac{(q^r;q^d)_k^d}{(q^d;q^d)_k^d}q^{d(d-r-2)k/2} \equiv 0 \pmod{[n]},$$

where $0 \le m \le n-1$ and $dm \equiv -r \pmod{n}$.

We have collected enough ingredients which enables us to prove Theorem 1.

Proof of Theorem 1 The q-congruence (1.5) modulo $[n]$ follows from Lemma 3 immediately. In what follows, we shall prove the modulus $\Phi_n(q)^3$ case of (1.5).

For $M = (dn-n-r)/d$, the left-hand side of (1.5) can be written as the following multiple of a terminating ${}_{d+5}\phi_{d+4}$ series:

$$[r]\sum_{k=0}^{(dn-n-r)/d}\frac{(q^r,q^{d+r/2},-q^{d+r/2},q^r,\ldots,q^r,q^{(d+r)/2},q^{d+(d-1)n},q^{r-(d-1)n};q^d)_k}{(q^d,q^{r/2},-q^{r/2},q^d,\ldots,q^d,q^{(d+r)/2},q^{r-(d-1)n},q^{d+(d-1)n};q^d)_k}q^{d(d-r-2)k/2}.$$

Here, the $q^r,\ldots,q^r$ in the numerator means $d-1$ instances of q^r, and similarly, the $q^d,\ldots,q^d$ in the denominator means $d-1$ instances of q^d. By Andrews' transformation

(2.2), we may rewrite the above expression as

$$
\begin{aligned}
&[r]\frac{(q^{d+r},q^{(r-d)/2-(d-1)n};q^d)_{(dn-n-r)/d}}{(q^{(d+r)/2},q^{r-(d-1)n};q^d)_{(dn-n-r)/d}}\sum_{j_1,\dots,j_{m-1}\ge 0}\frac{(q^{d-r};q^d)_{j_1}\cdots(q^{d-r};q^d)_{j_{m-1}}}{(q^d;q^d)_{j_1}\cdots(q^d;q^d)_{j_{m-1}}}\\
&\quad\times\frac{(q^r,q^r;q^d)_{j_1}\dots(q^r,q^r;q^d)_{j_1+\dots+j_{m-2}}(q^{(d+r)/2},q^{d+(d-1)n};q^d)_{j_1+\dots+j_{m-1}}}{(q^d,q^d;q^d)_{j_1}\dots(q^d,q^d;q^d)_{j_1+\dots+j_{m-1}}}\\
&\quad\times\frac{(q^{r-(d-1)n};q^d)_{j_1+\dots+j_{m-1}}}{(q^{(3d+r)/2};q^d)_{j_1+\dots+j_{m-1}}}q^{(d-r)(j_{m-2}+\dots+(m-2)j_1)+d(j_1+\dots+j_{m-1})},
\end{aligned}
\tag{2.7}
$$

where $m=(d+1)/2$.

It is easy to see that the q-shifted factorial $(q^{d+r};q^d)_{(dn-n-r)/d}$ contains the factor $1-q^{(d-1)n}$ which is a multiple of $1-q^n$. Moreover, since none of $(r-d)/2$, $(d+r)/2$ and $(d+r)/2+dn-n-r-d$ are multiples of n, the q-shifted factorials

$$
(q^{(r-d)/2-(d-1)n};q^d)_{(dn-n-r)/d}\quad\text{and}\quad(q^{(d+r)/2};q^d)_{(dn-n-r)/d}
$$

have the same number (0 or 1) of factors of the form $1-q^{\alpha n}$ ($\alpha\in\mathbb{Z}$). Besides, the q-shifted factorial $(q^{r-(d-1)n};q^d)_{(dn-n-r)/d}$ is relatively prime to $\Phi_n(q)$. Thus we conclude that the fraction before the multi-sum in (2.7) is congruent to 0 modulo $\Phi_n(q)$.

Note that the non-zero terms in the multi-summation in (2.7) are those indexed by $(j_1,\dots,j_{m-1})$ that satisfy the inequality $j_1+\dots+j_{m-1}\le(dn-n-r)/d$ because the factor $(q^{r-(d-1)n};q^d)_{j_1+\dots+j_{m-1}}$ appears in the numerator. None of the factors appearing in the denominator of the multi-sum of (2.7) contain a factor of the form $1-q^{\alpha n}$ (and are therefore relatively prime to $\Phi_n(q)$), except for $(q^{(3d+r)/2};q^d)_{j_1+\dots+j_{m-1}}$ when

$$
(dn-d-n-r)/(2d)\le j_1+\dots+j_{m-1}\le(dn-n-r)/d.
$$

Since

$$
\frac{(q^{(d+r)/2};q^d)_{j_1+\dots+j_{m-1}}}{(q^{(3d+r)/2};q^d)_{j_1+\dots+j_{m-1}}}=\frac{1-q^{(d+r)/2}}{1-q^{(d+r)/2+(j_1+\dots+j_{m-1})d}},
$$

the denominator of the above fraction contains a factor of the form $1-q^{\alpha n}$ if and only if $j_1+\dots+j_{m-1}=(dn-d-n-r)/(2d)$ (in this case, the denominator contains the factor $1-q^{(d-1)n/2}$). Writing $n=ad-r$ (with $a\ge 1$), we have $j_1+\dots+j_{m-1}=a(d-1)/2-(r+1)/2$. Noticing that $m-1=(d-1)/2$ and $r\le d-4$, there must exist an i such that $j_i\ge a$. Then $(q^{d-r};q^d)_{j_i}$ has the factor $1-q^{d-r+d(a-1)}=1-q^n$ which is divisible by $\Phi_n(q)$. Hence the denominator of the reduced form of the multi-sum in (2.7) is relatively prime to $\Phi_n(q)$. It remains to show that the multi-sum in (2.7), without the previous fraction, is congruent to 0 modulo $\Phi_n(q)^2$.

By repeated applications of Lemma 1, the multi-sum in (2.7) (without the previous fraction), modulo $\Phi_n(q)^2$, is congruent to

$$
\begin{aligned}
&\sum_{j_1,\dots,j_{m-1}\ge 0}q^{(d-r)(j_{m-2}+\dots+(m-2)j_1)+d(j_1+\dots+j_{m-1})}\frac{(q^{d-r};q^d)_{j_1}\cdots(q^{d-r};q^d)_{j_{m-1}}}{(q^d;q^d)_{j_1}\cdots(q^d;q^d)_{j_{m-1}}}\\
&\qquad\times\frac{(q^{r+(m+1)n},q^{r-(m+1)n};q^d)_{j_1}\dots(q^{r+(2m-2)n},q^{r-(2m-2)n};q^d)_{j_1+\dots+j_{m-2}}}{(q^{d-mn},q^{d+mn};q^d)_{j_1}\dots(q^{d-(2m-3)n},q^{d+(2m-3)n};q^d)_{j_1+\dots+j_{m-2}}}\\
&\qquad\times\frac{(q^{d+(d-1)n},q^{(d+r)/2};q^d)_{j_1+\dots+j_{m-1}}(q^{r-(d-1)n};q^d)_{j_1+\dots+j_{m-1}}}{(q^{d-(2m-2)n},q^{d+(2m-2)n};q^d)_{j_1+\dots+j_{m-1}}(q^{(3d+r)/2};q^d)_{j_1+\dots+j_{m-1}}},
\end{aligned}
$$

where $m = (d+1)/2$. However, this sum vanishes in light of the $m = (d+1)/2$, $q \mapsto q^d$, $a = q^r$, $e_1 = q^{(d+r)/2}$, $e_m = q^{r-(2m-2)n}$, $e_i = q^{r-(m+i-2)n}$, $n_1 = (dn-d+n+r)/(2d)$, $n_m = 0$, $n_i = (n+r-d)/d$, $2 \le i \le m-1$, $N = (dn-n-r)/d$ case of Lemma 2. (It is easy to verify that $N - n_1 - \cdots - n_m = d(d-r-2)/2 > 0$.) This proves that (1.5) holds modulo $\Phi_n(q)^3$ for $M = (dn-n-r)/d$.

Since $(q^r; q^d)_k/(q^d; q^d)_k$ is congruent to 0 modulo $\Phi_n(q)$ for $(dn-n-r)/d < k \le n-1$, we conclude that (1.5) also holds modulo $\Phi_n(q)^3$ for $M = n-1$. □

3 Proof of Theorem 2

We first give a simple lemma on a property of certain arithmetic progressions.

Lemma 4 *Let d and r be odd integers satisfying $d \ge 3$, $r \le d-4$ and $\gcd(d,r) = 1$. Let n be an integer such that $n \ge (d-r)/2$ and $n \equiv -r/2 \pmod d$. Then there are no multiples of n in the arithmetic progression*

$$\frac{d+r}{2},\ \frac{d+r}{2}+d, \ldots, \frac{d+r}{2}+dn-2n-r-d. \tag{3.1}$$

Proof By the condition $\gcd(d,r) = 1$, we have $\gcd((d+r)/2, (d-r)/2) = 1$. Suppose that

$$(d+r)/2 + ad = bn \tag{3.2}$$

for some integers a and b with $a \ge 0$. Then $(d+r)/2 + ad > (r-d)/2 \ge -n$ and so $b \ge 0$. Since $n \equiv (d-r)/2 \pmod d$, we deduce from (3.2) that $b \equiv -1 \pmod d$ and thereby $b \ge d-1$. But we have

$$\frac{d+r}{2} + dn - 2n - r - d = dn - 2n + \frac{d-r}{2} - d \le (d-1)n - d,$$

thus implying that no number in the arithmetic progression (3.1) is a multiple of n. □

Proof of Theorem 2 As before, the q-congruence (1.6) modulo $[n]$ can be deduced from Lemma 3. It remains to prove the modulus $\Phi_n(q)^2$ case of (1.6).

For $M = (dn-2n-r)/d$, the left-hand side of (1.6) can be written as the following multiple of a terminating ${}_{d+5}\phi_{d+4}$ series (this time we changed the position of $q^{(d+r)/2}$):

$$[r] \sum_{k=0}^{(dn-2n-r)/d} \frac{(q^r, q^{d+r/2}, -q^{d+r/2}, q^{(d+r)/2}, q^r, \ldots, q^r, q^{d+(d-2)n}, q^{r-(d-2)n}; q^d)_k}{(q^d, q^{r/2}, -q^{r/2}, q^{(d+r)/2}, q^d, \ldots, q^d, q^{r-(d-2)n}, q^{d+(d-2)n}; q^d)_k} \times q^{d(d-r-2)k/2}.$$

Here, the $q^r, \ldots, q^r$ in the numerator stands for $d-1$ instances of q^r, and similarly, the $q^d, \ldots, q^d$ in the denominator stands for $d-1$ instances of q^d. By Andrews' transformation (2.2), we may rewrite the above expression as

$$\begin{aligned} &[r]\frac{(q^{d+r}, q^{-(d-2)n}; q^d)_{(dn-2n-r)/d}}{(q^d, q^{r-(d-2)n}; q^d)_{(dn-2n-r)/d}} \sum_{j_1,\ldots,j_{m-1}\ge 0} \frac{(q^{(d-r)/2}; q^d)_{j_1} (q^{d-r}; q^d)_{j_2} \cdots (q^{d-r}; q^d)_{j_{m-1}}}{(q^d; q^d)_{j_1} (q^d; q^d)_{j_2} \cdots (q^d; q^d)_{j_{m-1}}} \\ &\quad \times \frac{(q^r, q^r; q^d)_{j_1} \ldots (q^r, q^r; q^d)_{j_1+\cdots+j_{m-2}} (q^r, q^{d+(d-2)n}; q^d)_{j_1+\cdots+j_{m-1}}}{(q^{(d+r)/2}, q^d; q^d)_{j_1} (q^d, q^d; q^d)_{j_1+j_2} \cdots (q^d, q^d; q^d)_{j_1+\cdots+j_{m-1}}} \\ &\quad \times \frac{(q^{r-(d-2)n}; q^d)_{j_1+\cdots+j_{m-1}}}{(q^{d+r}; q^d)_{j_1+\cdots+j_{m-1}}} q^{(d-r)(j_{m-2}+\cdots+(m-2)j_1)+d(j_1+\cdots+j_{m-1})}, \end{aligned} \tag{3.3}$$

where $m = (d+1)/2$.

It is easily seen that the q-shifted factorial $(q^{d+r}; q^d)_{(dn-2n-r)/d}$ has the factor $1-q^{(d-2)n}$ which is a multiple of $1-q^n$. Clearly, the q-shifted factorial $(q^{-(d-2)n}; q^d)_{(dn-2n-r)/d}$ has the factor $1-q^{-(d-1)n}$ (again being a multiple of $1-q^n$) since $(dn-2n-r)/d \geq 1$ holds according to the conditions $d \geq 3$, $r \leq d-4$, and $n \geq (d-r)/2$. This indicates that the q-factorial $(q^{d+r}, q^{-(d-2)n}; q^d)_{(dn-2n-r)/d}$ in the numerator of the fraction before the multi-sum in (3.3) is divisible by $\Phi_n(q)^2$. Further, it is not difficult to see that the q-factorial $(q^d, q^{r-(d-2)n}; q^d)_{(dn-2n-r)/d}$ in the denominator is relatively prime to $\Phi_n(q)$.

Like the proof of Theorem 1, the non-zero terms in the multi-sum in (3.3) are those indexed by $(j_1, \ldots, j_{m-1})$ satisfying the inequality $j_1 + \cdots + j_{m-1} \leq (dn-2n-r)/d$ because of the appearance of the factor $(q^{r-(d-2)n}; q^d)_{j_1+\cdots+j_{m-1}}$ in the numerator. By Lemma 4, the q-shifted factorial $(q^{(d+r)/2}, q^d)_{j_1}$ in the denominator does not contain a factor of the form $1-q^{\alpha n}$ for $j_1 \leq (dn-2n-r)/d$ (and are therefore relatively prime to $\Phi_n(q)$). In addition, none of the other factors appearing in the denominator of the multi-sum of (3.3) contain a factor of the form $1-q^{\alpha n}$, except for $(q^{d+r}; q^d)_{j_1+\cdots+j_{m-1}}$ when $j_1 + \cdots + j_{m-1} = (dn-2n-r)/d$ (in this case the denominator contains the factor $1-q^{(d-2)n}$).

Letting $n = ad + (d-r)/2$ (with $a \geq 0$), we get $j_1 + \cdots + j_{m-1} = a(d-2) + (d-r)/2 - 1$. If $j_1 \geq a+1$, then $(q^{(d-r)/2}; q^d)_{j_1}$ contains the factor $1-q^{(d-r)/2+ad} = 1-q^n$. If $j_1 \leq a$, then $j_2 + \cdots + j_{m-1} \geq a(d-3) + (d-r)/2 - 1$. Since $m-2 = (d-3)/2$, $d \geq 3$, and $r \leq d-4$, there must be an i with $2 \leq i \leq m-1$ and $j_i \geq 2a+1$. Then $(q^{d-r}; q^d)_{j_i}$ contains the factor $1-q^{d-r+2ad} = 1-q^{2n}$ which is a multiple of $\Phi_n(q)$. Therefore, the denominator of the reduced form of the multi-sum in (3.3) is relatively prime to $\Phi_n(q)$. This proves that (3.3) is congruent to 0 modulo $\Phi_n(q)^2$.

For $M = n-1$, since $(q^r; q^d)_k/(q^d; q^d)_k$ is congruent to 0 modulo $\Phi_n(q)$ for $(dn-2n-r)/d < k \leq n-1$, we conclude that (1.6) is also true modulo $\Phi_n(q)^2$ in this case. $\square$

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