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Approximation properties of mixed sampling-Kantorovich operators

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Abstract
In the present paper we study the pointwise and uniform convergence properties of a family of multidimensional sampling Kantorovich type operators. Moreover, besides convergence, quantitative estimates and a Voronovskaja type theorem have been established.

Keywords Sampling-Kantorovich operators · Pointwise convergence · Uniform convergence · Order of approximation · Asymptotic expansions · Voronovskaja formulae

Mathematics Subject Classification 41A30 · 41A05

1 Introduction

In the last years there was an increasing interest in approximation by means of families of discrete operators in several function spaces both in the one-dimensional case and in multidimensional setting, also thanks to the applicative outcome of such results (see, e.g., [18,27,28,36,37,41]). For example, the generalized sampling series defined as

$$(S_w f)(\mathtt{t}) = \sum_{\mathtt{k}\in\mathbb{Z}^N} f\left(\frac{\mathtt{k}}{w}\right)\chi(w\mathtt{t}-\mathtt{k}), \qquad (\star)$$

$\mathtt{t} \in \mathbb{R}^N$, $w > 0$, where χ is a kernel, have been widely studied with respect to several notions of convergence, such as pointwise, uniform, L^p, modular convergence [23–25] and also, recently [5,7], convergence in variation (for other approximation results in BV-spaces see, e.g., [9–13]). The interest of such operators is also due to their deep connections with problems of Signal and Image Processing: indeed they furnish an approximate version of

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the Shannon Sampling Theorem (see, e.g., [46,47]) that allows to reconstruct not necessarily band-limited signals (images).

In order to obtain convergence in variation for the operators ($\star$), in [7] a new family of discrete operators has been introduced:

$$(K_{w,j} f)(\mathtt{t}) := \sum_{\mathtt{k} \in \mathbb{Z}^N} \left[w \int_{\frac{k_j}{w}}^{\frac{k_j+1}{w}} f\left(\frac{k_1}{w}, \ldots, u \ldots, \frac{k_N}{w}\right) du \right] \chi(w\mathtt{t} - \mathtt{k}), \qquad (*)$$

for every $\mathtt{t} \in \mathbb{R}^N$, $w > 0$ and $j = 1, \ldots, N$. The operators ($*$), that here will be called *mixed sampling-Kantorovich operators*, are essentially a Kantorovich version of the generalized sampling series where the integral mean is computed on just one section of the function involved, while the usual Kantorovich version replaces the whole value $f\left(\frac{\mathtt{k}}{w}\right)$ with an integral mean on a multidimensional interval around the sampling node $\frac{\mathtt{k}}{w}$, i.e.,

$$(K_w f)(\mathtt{t}) := \sum_{\mathtt{k} \in \mathbb{Z}^N} \left[w^N \int_{\prod_{j=1}^N [\frac{k_j}{w}, \frac{k_j+1}{w}]} f(\mathtt{u})\, d\mathtt{u} \right] \chi(w\mathtt{t} - \mathtt{k}), \qquad (**)$$

for every $\mathtt{t} \in \mathbb{R}^N$, $w > 0$.

The introduction of the operators ($*$) was naturally motivated by the fact that such operators allow to obtain a multidimensional generalization of the classical relation proved by Lorentz among the derivative of the Bernstein polynomials and the Kantorovich polynomials acting on the derivative of the function, in the one-dimensional case: in particular, in [7] it is proved that the j-th partial derivative of the generalized sampling series $\frac{\partial S_w f}{\partial x_j}$ coincides with a combination of shifted mixed sampling-Kantorovich operators $\left(K_{w,j} \frac{\partial f}{\partial x_j}\right)$ acting on the j-th partial derivative of f in case of kernels of averaged type.

Nevertheless, the family of operators ($*$) appear as an interesting intermediate case between the generalized sampling series ($\star$) and the classical Kantorovich operators ($**$) and therefore the approximation results that we will present cannot be derived by the analogous results that have been previously obtained in the literature for the operators ($\star$) or ($**$).

We recall that the study of the approximation properties of the Kantorovich version of families of operators is a widely investigated topic in the literature. Just to mention some examples, in [3,35] approximation results of the Stancu–Kantorovich operators based on Polya–Eggenberger distribution are presented; in [40] it is established an inverse result for bivariate Kantorovich type sampling series and for their generalized Boolean sum (see also, e.g., [42,43,45] for some modifications and generalizations), while in [4] the rate of convergence of perturbed Kantorovich–Choquet univariate and multivariate normalized neural network operators with respect to the uniform norm is obtained. About the Kantorovich version of the generalized sampling series ($**$), results about pointwise and uniform convergence, L^p-convergence, modular convergence, rate of approximation and inverse results have been obtained (see, e.g, [6,15,26,28–33].

It is well-known that approximation by means of the Kantorovich version of the sampling operators presents several advantages with respect to the class $\{S_w f\}_{w>0}$, also from the point of view of the applications to Signal Processing (see, e.g., [15,39,44]), due to the presence of the integral mean. For instance, this allows to reduce the so-called time-jitter error that occurs when, in the practice, the sampled values are not computed exactly over the sample nodes. Nevertheless, since the mixed operators ($*$) appear to be an intermediate class among the sampling series and the Kantorovich operators, they can be applied in situations in which, in the approximation process, it is more suitable to keep the value of the function on the sample nodes except just one direction, where the integral mean is computed. For example, this can be useful in situations in which the interest is to reduce time-jitter error in just one direction to guarantee a good reconstruction saving computational operations with respect to the usual sampling-Kantorovich operators.

In [7] convergence in $L^p(\mathbb{R}^N)$ for the family of operators $(*)$ is proved. Here we give results about pointwise and uniform convergence. Moreover, we establish a quantitative estimate in terms of the modulus of continuity of the function that, as a consequence, gives a result about the order of approximation if the function belongs to a Lipschitz class. Finally, we prove an asymptotic expansion for $(K_{w,j} f)$ that allows us to obtain a Voronovskaja type theorem.

2 Notations and preliminaries

We will denote by $C(\mathbb{R}^N)$ the space of all the bounded and uniformly continuous functions $f : \mathbb{R}^N \to \mathbb{R}$ endowed with the supremum norm $\|f\|_\infty$.

We will study the family of mixed sampling-Kantorovich operators recently introduced in [7], defined as

$$(K_{w,j} f)(\mathtt{t}) := \sum_{\mathtt{k}\in\mathbb{Z}^N} \left[w \int_{\frac{k_j}{w}}^{\frac{k_j+1}{w}} f\left(\frac{k_1}{w}, \ldots, u \ldots, \frac{k_N}{w}\right) du \right] \chi(w\mathtt{t} - \mathtt{k}), \qquad (*)$$

for every $\mathtt{t} \in \mathbb{R}^N$, $w > 0$ and $j = 1, \ldots, N$.

Here χ is a kernel, that is, a function $\chi : \mathbb{R}^N \to \mathbb{R}$ that satisfies the following assumptions:

(χ_1) χ is continuous and such that $\sum_{\mathtt{k}\in\mathbb{Z}^N} \chi(\mathtt{t} - \mathtt{k}) = 1$, for every $\mathtt{t} \in \mathbb{R}^N$;

(χ_2) $M_0(\chi) := \sup_{\mathtt{u}\in\mathbb{R}^N} \sum_{\mathtt{k}\in\mathbb{Z}^N} |\chi(\mathtt{u} - \mathtt{k})| < +\infty$, where the convergence of the series $\sum_{\mathtt{k}\in\mathbb{Z}^N} |\chi(\mathtt{u} - \mathtt{k})|$ is uniform on the compact subsets of $\mathbb{R}^N$.

We point out that above assumptions are standard working with discrete families of operators (see, e.g., [7,8,17,20,21]) and it is easy to provide examples of kernels that fulfill them. Among them, for example we can mention the multivariate version of the Jackson, Fejér, and central B-spline (product) kernels, or the well-known Bochner–Riesz (radial) kernels. For more details, see, e.g., [16,19,22,30,34,38,48].

It is easy to see that mixed sampling-Kantorovich operators $(*)$ are well-defined if, for example, f is bounded. Indeed, if $|f(\mathtt{t})| \le L$, for every $\mathtt{t} \in \mathbb{R}^N$,

$$\begin{aligned} |(K_{w,j} f)(\mathtt{t})| &\le \sum_{\mathtt{k}\in\mathbb{Z}^N} \left[w \int_{\frac{k_j}{w}}^{\frac{k_j+1}{w}} \left| f\left(\frac{k_1}{w}, \ldots, u \ldots, \frac{k_N}{w}\right) du \right| \right] |\chi(w\mathtt{t} - \mathtt{k})| \\ &\le L \sum_{\mathtt{k}\in\mathbb{Z}^N} |\chi(w\mathtt{t} - \mathtt{k})| \le L\, M_0(\chi) < +\infty, \end{aligned} \qquad (1)$$

by (χ_2), for every $\mathtt{t} \in \mathbb{R}^N$, $w > 0$, $j = 1, \ldots, N$.

We finally recall the following multidimensional notations that we will use in the paper: for a function $f : \mathbb{R}^N \to \mathbb{R}$ and $\mathtt{x} = (x_1, \ldots, x_N) \in \mathbb{R}^N$, we will write

$$\mathtt{x}'_j = (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_N) \in \mathbb{R}^{N-1}, \quad \mathtt{x} = (\mathtt{x}'_j, x_j), \quad f(\mathtt{x}) = f(\mathtt{x}'_j, x_j),$$

to emphasize the j-th coordinate of $\mathtt{x}$, $j = 1, \ldots, N$ and, for $\mathtt{x} \in \mathbb{R}^N$ and $\alpha \in \mathbb{R}$, we will use the usual notation for products and quotients, i.e., $\alpha\mathtt{x} = (\alpha x_1, \ldots, \alpha x_N)$ and, for $\alpha \ne 0$,

$\frac{\mathtt{x}}{\alpha} = \left(\frac{x_1}{\alpha}, \ldots, \frac{x_N}{\alpha}\right)$. Finally, $[\mathtt{x}] = ([x_1], \ldots, [x_N])$ will denote the integer part of $\mathtt{x} \in \mathbb{R}^N$, $\|\mathtt{x}\|$ will denote the usual Euclidean norm of $\mathbb{R}^N$, and $\mathtt{x}^{\mathtt{y}} = \prod_{i=1}^N x_i^{y_i}$, $\mathtt{x}, \mathtt{y} \in \mathbb{R}^N$, when the power is well-defined.

3 Pointwise and uniform convergence

We will first prove pointwise convergence for the mixed sampling-Kantorovich operators $\{K_{w,j} f\}_{w>0}$, $j = 1, \ldots, N$.

Theorem 1 *If $f : \mathbb{R}^N \to \mathbb{R}$ is bounded, then, for every $j = 1, \ldots, N$,*

$$\lim_{w\to+\infty} (K_{w,j} f)(\mathtt{t}_0) = f(\mathtt{t}_0),$$

at each point $\mathtt{t}_0 \in \mathbb{R}^N$ where f is continuous.

Proof Let $\mathtt{t}_0 \in \mathbb{R}^N$ be a point of continuity for f and let us fix $j = 1, \ldots, N$ and $\epsilon > 0$.
Let $\delta > 0$ be such that, for every $\mathtt{t} \in \mathbb{R}^N$ for which $\|\mathtt{t} - \mathtt{t}_0\| < \delta$,

$$|f(\mathtt{t}) - f(\mathtt{t}_0)| < \frac{\epsilon}{2M_0(\chi)}, \tag{2}$$

where $M_0(\chi)$ is the absolute moment of order 0 of χ of assumption (χ_2). By (χ_1) there holds, for $w > 0$,

$$\begin{aligned} &|(K_{w,j} f)(\mathtt{t}_0) - f(\mathtt{t}_0)| \leq \\ &\leq \left\{ \sum_{\|\mathtt{k}-w\mathtt{t}_0\| \leq \frac{w\delta}{2}} + \sum_{\|\mathtt{k}-w\mathtt{t}_0\| > \frac{w\delta}{2}} \right\} \left\{ w \int_{\frac{k_j}{w}}^{\frac{k_j+1}{w}} \left| f\left(\frac{k_1}{w}, \ldots, u \ldots, \frac{k_N}{w}\right) - f(\mathtt{t}_0) \right| du \right\} |\chi(w\mathtt{t}_0 - \mathtt{k})| \\ &:= \Sigma_1 + \Sigma_2. \end{aligned}$$

About the first sum, for $\mathtt{k} \in \mathbb{Z}^N$ such that $\|\mathtt{k} - w\mathtt{t}_0\| \leq \frac{w\delta}{2}$, there holds

$$\left\| \left(\frac{\mathtt{k}'_j}{w}, u \right) - \mathtt{t}_0 \right\| \leq \left\| \left(\frac{\mathtt{k}'_j}{w}, u \right) - \frac{\mathtt{k}}{w} \right\| + \left\| \frac{\mathtt{k}}{w} - \mathtt{t}_0 \right\| < \delta,$$

for sufficiently large $w > 0$. Therefore, by (2) and (χ_2), $\Sigma_1 < \frac{\epsilon}{2}$.
About Σ_2 notice that (see Remark 3.3 of [28]), by (χ_2), $\lim_{R\to+\infty} \sum_{\|\mathtt{u}-\mathtt{k}\|>R} |\chi(\mathtt{u} - \mathtt{k})| = 0$ uniformly on $\mathtt{u} \in \mathbb{R}^N$, and hence, for sufficiently large $w > 0$, $\sum_{\|w\mathtt{t}-\mathtt{k}\|>\frac{w\delta}{2}} |\chi(\mathtt{u} - \mathtt{k})| < \frac{\epsilon}{4\|f\|_\infty}$ (without any loss of generality, $\|f\|_\infty > 0$). This implies that $\Sigma_2 < \frac{\epsilon}{2}$ and the proof is complete. □

We now prove the uniform convergence result for the mixed sampling-Kantorovich operators $\{K_{w,j} f\}_{w>0}$, for every $j = 1, \ldots, N$.

Theorem 2 *Let χ be a kernel such that $\chi \in C(\mathbb{R}^N)$. If $f \in C(\mathbb{R}^N)$ then, for every $j = 1, \ldots, N$, $K_{w,j} f \in C(\mathbb{R}^N)$ and*

$$\lim_{w\to+\infty} \|K_{w,j} f - f\|_\infty = 0. \tag{3}$$

Proof Let us fix $\epsilon > 0$. By the uniform convergence of the series $\sum_{\mathtt{k}\in\mathbb{Z}^N} |\chi(\mathtt{u}-\mathtt{k})|$ on $[-2,2]^N$, for example, there exists $k_0 \in \mathbb{Z}$ such that, for every $\mathtt{u} \in [-2,2]^N$,

$$\sum_{\|\mathtt{k}\|>k_0} |\chi(\mathtt{u}-\mathtt{k})| < \frac{\epsilon}{3\|f\|_\infty}. \tag{4}$$

Since $\chi \in C(\mathbb{R}^N)$ by assumption, there exists $\delta = \delta(\epsilon) > 0$ such that, for every $\mathtt{t}, \mathtt{t}^0 \in \mathbb{R}^N$ with $\|\mathtt{t}-\mathtt{t}^0\| < \delta$, then

$$|\chi(\mathtt{t}) - \chi(\mathtt{t}^0)| < \frac{\epsilon}{3(2k_0+1)^N\|f\|_\infty}. \tag{5}$$

Now (similarly to [21]) define $\bar{\delta} := \min\left\{\frac{\delta}{w}, \frac{1}{w}\right\}$ and consider $\mathtt{t}, \mathtt{t}^0 \in \mathbb{R}^N$ with $\|\mathtt{t}-\mathtt{t}^0\| < \bar{\delta}$: then $w\mathtt{t}^0 - [w\mathtt{t}^0] \in [-2,2]^N$ and $w\mathtt{t} - [w\mathtt{t}^0] = w(\mathtt{t}-\mathtt{t}^0) + w\mathtt{t}^0 - [w\mathtt{t}^0] \in [-2,2]^N$, and therefore, putting $\mathtt{n} := \mathtt{k} - [w\mathtt{t}^0] \in \mathbb{Z}^N$,

$$\begin{aligned}
&\sum_{\mathtt{k}\in\mathbb{Z}^N} |\chi(w\mathtt{t}-\mathtt{k}) - \chi(w\mathtt{t}^0-\mathtt{k})| \\
&\leq \left(\sum_{\|\mathtt{n}\|\leq k_0} + \sum_{\|\mathtt{n}\|>k_0}\right) |\chi(w\mathtt{t}-[w\mathtt{t}^0]-\mathtt{n}) - \chi(w\mathtt{t}_0-[w\mathtt{t}^0]-\mathtt{n})| \\
&\leq \sum_{\|\mathtt{n}\|\leq k_0} \frac{\epsilon}{3(2k_0+1)^N\|f\|_\infty} + 2\frac{\epsilon}{3\|f\|_\infty} \leq \frac{\epsilon}{\|f\|_\infty},
\end{aligned}$$

by (4) and (5). Therefore we conclude that

$$|(K_{w,j}f)(\mathtt{t}) - (K_{w,j}f)(\mathtt{t}_0)| \leq \|f\|_\infty \sum_{\mathtt{k}\in\mathbb{Z}^N} |\chi(w\mathtt{t}-\mathtt{k}) - \chi(w\mathtt{t}_0-\mathtt{k})| < \epsilon$$

for every $\mathtt{t}, \mathtt{t}^0 \in \mathbb{R}^N$ with $\|\mathtt{t}-\mathtt{t}^0\| < \bar{\delta}$, that is, $K_{w,j}f \in C(\mathbb{R}^N)$, for every $w > 0$, $j = 1, \ldots, N$, taking into account that $(K_{w,j}f)$ is bounded (see (1)).

Now, to prove that $\lim_{w\to+\infty} \|K_{w,j}f - f\|_\infty = 0$ it is sufficient to follow the proof of Theorem 1 taking into account of the uniform continuity of f. □

4 Estimates and order of approximation

In this section we first establish a quantitative estimate for the above operators. We recall that

$$\omega(f,\delta) := \sup\left\{|f(\mathtt{x}) - f(\mathtt{y})| : \|\mathtt{x}-\mathtt{y}\| \leq \delta,\ \mathtt{x}, \mathtt{y} \in \mathbb{R}^N\right\}, \qquad \delta > 0,$$

denotes the usual modulus of continuity of the function f. It is well-known that, if $f \in C(\mathbb{R}^N)$ then $\omega(f,\delta) \to 0$ as $\delta \to 0^+$, and moreover, the following estimate there holds

$$\omega(f,\lambda\delta) \leq (1+\lambda)\,\omega(f,\delta), \qquad \lambda > 0. \tag{6}$$

Now we are able to prove the following result.

Theorem 3 *Let χ be a kernel that satisfies the following assumption:*

$$M_1(\chi) := \sup_{\mathtt{t}\in\mathbb{R}^N} \sum_{\mathtt{k}\in\mathbb{Z}^N} |\chi(\mathtt{t}-\mathtt{k})| \cdot \|\mathtt{t}-\mathtt{k}\| < +\infty.$$

Then, for any $f \in C(\mathbb{R}^N)$ *there holds*

$$|(K_{w,j} f)(\mathtt{t}) - f(\mathtt{t})| \ \le\ \omega(f, 1/w) \left\{ \frac{3}{2} M_0(\chi) \ +\ M_1(\chi) \right\},$$

$w > 0$, $1 \le j \le N$.

Proof Proceeding as in the proof of Theorem 1, for every $\mathtt{t} \in \mathbb{R}^N$ we can write

$$|(K_{w,j} f)(\mathtt{t}) - f(\mathtt{t})| \le \sum_{\mathtt{k}\in\mathbb{Z}^N} \left\{ w \int_{\frac{k_j}{w}}^{\frac{k_j+1}{w}} \left| f\left(\frac{k_1}{w}, \ldots, u \ldots, \frac{k_N}{w}\right) - f(\mathtt{t}) \right| du \right\} |\chi(w\mathtt{t} - \mathtt{k})|,$$

$w > 0$. By using inequality (6) it is easy to see that

$$\begin{aligned} |(K_{w,j} f)(\mathtt{t}) - f(\mathtt{t})| &\le \sum_{\mathtt{k}\in\mathbb{Z}^N} \left\{ w \int_{\frac{k_j}{w}}^{\frac{k_j+1}{w}} \omega\left(f, \left\| \left(\frac{\mathtt{k}'_j}{w}, u\right) - \mathtt{t} \right\| \right) du \right\} |\chi(w\mathtt{t} - \mathtt{k})|, \\ &\le \omega(f, 1/w) \sum_{\mathtt{k}\in\mathbb{Z}^N} \left\{ w \int_{\frac{k_j}{w}}^{\frac{k_j+1}{w}} \left(1 + w \left\| \left(\frac{\mathtt{k}'_j}{w}, u\right) - \mathtt{t} \right\| \right) du \right\} |\chi(w\mathtt{t} - \mathtt{k})| \\ &\le \omega(f, 1/w) \sum_{\mathtt{k}\in\mathbb{Z}^N} |\chi(w\mathtt{t} - \mathtt{k})| \left\{ 1 + w^2 \int_{\frac{k_j}{w}}^{\frac{k_j+1}{w}} \left\| \left(\frac{\mathtt{k}'_j}{w}, u\right) - \frac{\mathtt{k}}{w} \right\| du + \|\mathtt{k} - w\,\mathtt{t}\| \right\} \\ &= \omega(f, 1/w) \sum_{\mathtt{k}\in\mathbb{Z}^N} |\chi(w\mathtt{t} - \mathtt{k})| \left\{ 1 + w^2 \int_{\frac{k_j}{w}}^{\frac{k_j+1}{w}} |u - k_j/w|\, du + \|\mathtt{k} - w\,\mathtt{t}\| \right\} \\ &\le \omega(f, 1/w) \left\{ \frac{3}{2} M_0(\chi) \ +\ M_1(\chi) \right\}, \end{aligned}$$

$w > 0$. This completes the proof. □

Note that the requirement of Theorem 3 about the finiteness of the discrete absolute moment $M_1(\chi)$ is quite standard and not restrictive. Indeed in general, if we define

$$M_j(\chi) \ :=\ \sup_{\mathtt{t}\in\mathbb{R}^N} \sum_{\mathtt{k}\in\mathbb{Z}^N} |\chi(\mathtt{t} - \mathtt{k})|\, \|\mathtt{t} - \mathtt{k}\|^j, \qquad j > 0,$$

and χ is a kernel such that $\chi(\mathtt{t}) = \mathcal{O}(\|\mathtt{t}\|^{-r-1-\varepsilon})$, as $\|\mathtt{t}\| \to +\infty$, for some $r > 0$, $\varepsilon > 0$, it turns out that

$$M_j(\chi) \ <\ +\infty, \qquad 0 \le j \le r$$

(see [30]).

Now, recalling the definition of the spaces

$$\mathrm{Lip}\,(\alpha) \ :=\ \left\{ f \in C(\mathbb{R}^N) : \ \omega(f, \delta) = \mathcal{O}(\delta^\alpha), \ \text{as}\ \delta \to 0^+ \right\},$$

for $0 < \alpha \le 1$, as a consequence of Theorem 3, we can immediately obtain the following corollary.

Corollary 1 *Under the assumptions of Theorem 3 and for every* $f \in \mathrm{Lip}\,(\alpha)$, $0 < \alpha \le 1$, *it turns out that:*

$$\|K_{w,j} f - f\|_\infty \; = \; \mathcal{O}(w^{-\alpha}), \qquad as \qquad w \to +\infty,$$

$j = 1, \ldots, N$.

Now, we aim to prove an asymptotic expansion for the above operators. First of all we recall that, for a function $f : \mathbb{R}^N \to \mathbb{R}$, we denote by

$$D^{\mathrm{h}} f \; := \; \frac{\partial^{|\mathrm{h}|}}{\partial \mathrm{x}^{\mathrm{h}}} f \; = \; \frac{\partial^{|\mathrm{h}|}}{\partial x_1^{h_1} \ldots \partial x_1^{h_N}} f \qquad (|\mathrm{h}| = h_1 + \cdots + h_N = r)$$

the r-th order derivatives of f, $\mathrm{h} \in \mathbb{N}_0^N$.

In order to reach the above aim, we need the following multivariate version of the Taylor formula for $f \in C^r(\mathbb{R}^N)$, $r \in \mathbb{N}$:

$$f(\mathrm{u}) \; = \; f(\mathrm{t}) \; + \sum_{\nu=1}^{r} \sum_{|\mathrm{h}|=\nu} \frac{D^{\mathrm{h}} f(\mathrm{t})}{\mathrm{h}!} (\mathrm{u} - \mathrm{t})^{\mathrm{h}} \; + \; R_r(\mathrm{u}; \mathrm{t}), \quad \mathrm{u}, \mathrm{t} \in \mathbb{R}^N, \tag{7}$$

where the term $R_r(\mathrm{u}, \mathrm{t})$ denotes a suitable remainder, and

$$\mathrm{h}! \; := \; h_1! \, h_2! \ldots h_N!.$$

Now, for any given kernel χ, the multivariate algebraic moments, for $\nu \in \mathbb{N}$ and $\mathrm{h} = (h_1, \ldots, h_N)$ with $|\mathrm{h}| = \nu$, are defined as

$$m_{\mathrm{h}}^{\nu}(\chi, \mathrm{u}) \; := \sum_{\mathrm{k} \in \mathbb{Z}^N} \chi(\mathrm{u} - \mathrm{k}) \, (\mathrm{k} - \mathrm{u})^{\mathrm{h}}, \qquad \mathrm{u} \in \mathbb{R}^N.$$

Now, the following asymptotic expansion can be proved.

Theorem 4 *Let χ be a kernel such that, for every $\gamma > 0$,*

$$\lim_{w \to +\infty} \sum_{\|w\mathrm{t} - \mathrm{k}\| > \gamma\, w} |\chi(w\mathrm{t} - \mathrm{k})| \cdot \|w\mathrm{t} - \mathrm{k}\|^r \; = \; 0, \tag{8}$$

uniformly with respect to $\mathrm{t} \in \mathbb{R}^N$, for a certain $r \in \mathbb{N}$. Moreover, we also assume that $M_r(\chi) < +\infty$. Then, for any $f \in C^r(\mathbb{R}^N)$ there holds:

$$(K_{w,j} f)(\mathrm{t}) \; = \; f(\mathrm{t}) \; + \sum_{\nu=1}^{r} \sum_{|\mathrm{h}|=\nu} \frac{D^{\mathrm{h}} f(\mathrm{t})}{\mathrm{h}!} \left[\prod_{i=1}^{N} w^{-h_i} \right] \sum_{\ell=0}^{h_j} \binom{h_j}{\ell} \frac{m_{(\mathrm{h}'_j, \ell)}^{\nu - h_j + \ell}(\chi, w\mathrm{t})}{(h_j - \ell + 1)} \; + \; o(w^{-r}),$$

as $w \to +\infty$, $\mathrm{t} \in \mathbb{R}^N$, $j = 1, \ldots, N$.

Proof Expanding $f\left(\frac{\mathrm{k}'_j}{w}, u\right)$ by the Taylor formula (7) with remainder of the form

$$R_r\left(\left(\frac{\mathrm{k}'_j}{w}, u\right); \mathrm{t}\right) \; = \; \lambda\left(\left(\frac{\mathrm{k}'_j}{w}, u\right) - \mathrm{t}\right) \left\| \left(\left(\frac{\mathrm{k}'_j}{w}, u\right) - \mathrm{t}\right) \right\|^r,$$

where λ is a bounded function such that $\lim_{\mathrm{v}\to 0} \lambda(\mathrm{v}) = 0$, we can write what follows:

$$(K_{w,j} f)(\mathrm{t}) = f(\mathrm{t}) + \sum_{\nu=1}^{r} \sum_{|\mathrm{h}|=\nu} \frac{D^{\mathrm{h}} f(\mathrm{t})}{\mathrm{h}!} \sum_{\mathrm{k}\in\mathbb{Z}^N}$$
$$\times \left[\prod_{i=1, i\neq j}^{N} \left(\frac{k_i}{w} - t_i \right)^{h_i} w \int_{k_j/w}^{(k_j+1)/w} (u - t_j)^{h_j} du \right] \chi(w\mathrm{t} - \mathrm{k})$$
$$+ \sum_{\mathrm{k}\in\mathbb{Z}^N} \chi(w\mathrm{t} - \mathrm{k}) \left[w \int_{k_j/w}^{(k_j+1)/w} \lambda\left(\left(\frac{\mathrm{k}'_j}{w}, u \right) - \mathrm{t} \right) \left\| \left(\left(\frac{\mathrm{k}'_j}{w}, u \right) - \mathrm{t} \right) \right\|^r du \right] =: f(\mathrm{t}) + I_1 + I_2.$$

For $0 \le h_j \le r$ we have:

$$w \int_{k_j/w}^{(k_j+1)/w} (u - t_j)^{h_j} du = w \sum_{\ell=0}^{h_j} \binom{h_j}{\ell} \left(\frac{k_j}{w} - t_j \right)^{\ell} \int_{k_j/w}^{(k_j+1)/w} \left(u - \frac{k_j}{w} \right)^{h_j - \ell} du$$
$$= \sum_{\ell=0}^{h_j} \binom{h_j}{\ell} \left(\frac{k_j}{w} - t_j \right)^{\ell} \frac{w^{-h_j+\ell}}{(h_j - \ell + 1)} = w^{-h_j} \sum_{\ell=0}^{h_j} \binom{h_j}{\ell} \frac{(k_j - wt_j)^{\ell}}{(h_j - \ell + 1)}.$$

Hence for I_1 we get

$$I_1 = \sum_{\nu=1}^{r} \sum_{|\mathrm{h}|=\nu} \frac{D^{\mathrm{h}} f(\mathrm{t})}{\mathrm{h}!} w^{-h_j} \sum_{\ell=0}^{h_j} \binom{h_j}{\ell} \sum_{\mathrm{k}\in\mathbb{Z}^N} \left[\frac{(k_j - wt_j)^{\ell} \prod_{i=1, i\neq j}^{N} \left(\frac{k_i}{w} - t_i \right)^{h_i}}{(h_j - \ell + 1)} \right] \chi(w\mathrm{t} - \mathrm{k})$$
$$= \sum_{\nu=1}^{r} \sum_{|\mathrm{h}|=\nu} \frac{D^{\mathrm{h}} f(\mathrm{t})}{\mathrm{h}!} \left[\prod_{i=1}^{N} w^{-h_i} \right]$$
$$\times \sum_{\ell=0}^{h_j} \binom{h_j}{\ell} \frac{1}{(h_j - \ell + 1)} \sum_{\mathrm{k}\in\mathbb{Z}^N} \left\{ (k_j - wt_j)^{\ell} \prod_{i=1, i\neq j}^{N} (k_i - wt_i)^{h_i} \right\} \chi(w\mathrm{t} - \mathrm{k}),$$

and finally

$$I_1 = \sum_{\nu=1}^{r} \sum_{|\mathrm{h}|=\nu} \frac{D^{\mathrm{h}} f(\mathrm{t})}{\mathrm{h}!} \left[\prod_{i=1}^{N} w^{-h_i} \right] \sum_{\ell=0}^{h_j} \binom{h_j}{\ell} \frac{m^{\nu-h_j+\ell}_{(\mathrm{h}'_j, \ell)}(\chi, w\mathrm{t})}{(h_j - \ell + 1)}.$$

Now we can estimate the remainder term I_2. Since $\lim_{\mathrm{v}\to 0} \lambda(\mathrm{v}) = 0$ for a fixed $\varepsilon > 0$ there exists $\gamma > 0$ such that, for $\|\mathrm{v}\| \le \gamma$ there holds $|\lambda(\mathrm{v})| < \varepsilon$. Thus:

$$I_2 = \left\{ \sum_{\|w\mathrm{t}-\mathrm{k}\| \le w\gamma/2} + \sum_{\|w\mathrm{t}-\mathrm{k}\| > w\gamma/2} \right\} \chi(w\mathrm{t} - \mathrm{k})$$
$$\times \left[w \int_{k_j/w}^{(k_j+1)/w} \lambda\left(\left(\frac{\mathrm{k}'_j}{w}, u \right) - \mathrm{t} \right) \left\| \left(\left(\frac{\mathrm{k}'_j}{w}, u \right) - \mathrm{t} \right) \right\|^r du \right]$$
$$=: I_{2,1} + I_{2,2}.$$

Concerning $I_{2,1}$, for every $\mathrm{k} \in \mathbb{Z}^N$ such that $\|w\mathrm{t} - \mathrm{k}\| \le w\gamma/2$ and $u \in [k_j/w, (k_j+1)/w]$:

$$\left\| \left(\left(\frac{\mathrm{k}'_j}{w}, u \right) - \mathrm{t} \right) \right\| \le \left\| \left(\left(\frac{\mathrm{k}'_j}{w}, u \right) - \frac{\mathrm{k}}{w} \right) \right\| + \|\mathrm{k}/w - \mathrm{t}\| \le \left| u - \frac{k_j}{w} \right| + \frac{\gamma}{2} < \gamma,$$

for $w > 0$ sufficiently large, hence:

$$
\begin{aligned}
\left|I_{2,1}\right| &< \varepsilon \sum_{\|w\mathtt{t}-\mathtt{k}\| \le w\gamma/2} |\chi(w\mathtt{t}-\mathtt{k})| \left[w \int_{k_j/w}^{(k_j+1)/w} \left\| \left(\left(\frac{\mathtt{k}'_j}{w}, u \right) - \mathtt{t} \right) \right\|^r du \right] \\
&\le \varepsilon\, 2^{r-1} \sum_{\|w\mathtt{t}-\mathtt{k}\| \le w\gamma/2} |\chi(w\mathtt{t}-\mathtt{k})| \left[w \int_{k_j/w}^{(k_j+1)/w} \left\| \left(\frac{\mathtt{k}'_j}{w}, u \right) - \frac{\mathtt{k}}{\mathtt{w}} \right\|^r du + \left\| \frac{\mathtt{k}}{\mathtt{w}} - \mathtt{t} \right\|^r \right] \\
&\le \varepsilon\, 2^{r-1} \sum_{\|w\mathtt{t}-\mathtt{k}\| \le w\gamma/2} |\chi(w\mathtt{t}-\mathtt{k})| \left[\frac{w^{-r}}{r+1} + w^{-r} \|w\mathtt{t}-\mathtt{k}\|^r \right] \\
&\le \varepsilon\, w^{-r}\, 2^{r-1} \left[\frac{M_0(\chi)}{r+1} + M_r(\chi) \right] < +\infty,
\end{aligned}
$$

for $w > 0$ sufficiently large. Furthermore, by exploiting the above computations, and by using assumption (8) we have:

$$
\begin{aligned}
|I_{2,2}| &\le \|\lambda\|_\infty \sum_{\|w\mathtt{t}-\mathtt{k}\| > w\gamma/2} |\chi(w\mathtt{t}-\mathtt{k})| \left[w \int_{k_j/w}^{(k_j+1)/w} \left\| \left(\left(\frac{\mathtt{k}'_j}{w}, u \right) - \mathtt{t} \right) \right\|^r du \right] \\
&\le \|\lambda\|_\infty\, 2^{r-1}\, w^{-r} \sum_{\|w\mathtt{t}-\mathtt{k}\| > w\gamma/2} |\chi(w\mathtt{t}-\mathtt{k})| \left[\frac{1}{r+1} + \|w\mathtt{t}-\mathtt{k}\|^r \right] \\
&\le \|\lambda\|_\infty\, 2^{r-1}\, w^{-r} \left[\left(\frac{2}{\gamma\, w} \right)^r \frac{1}{r+1} + 1 \right] \sum_{\|w\mathtt{t}-\mathtt{k}\| > w\gamma/2} |\chi(w\mathtt{t}-\mathtt{k})| \cdot \|w\mathtt{t}-\mathtt{k}\|^r \\
&< \|\lambda\|_\infty\, 2^r\, w^{-r} \varepsilon,
\end{aligned}
$$

for $w > 0$ sufficiently large. This completes the proof. □

Now, from Theorem 4 it is easy to establish the following Voronovskaja type theorem.

Theorem 5 *Under the assumption of Theorem 4 with $r = 1$, if we assume in addition that the following algebraic moments:*

$$
m^1_{e_i}(\chi, \mathtt{u}) = A^1_{e_i} \in \mathbb{R}, \qquad \mathtt{u} \in \mathbb{R}^N,
$$

are constants, where $e_i := (0, \dots, 0, 1, 0, \dots 0)$, $i = 1, \dots, N$, we have:

$$
\lim_{w \to +\infty} w \left\{ (K_{w,j} f)(\mathtt{t}) - f(\mathtt{t}) \right\} = \sum_{i=1, i \ne j}^{N} \frac{\partial^1}{\partial x_i} f(\mathtt{t}) \cdot A^1_{e_i} + \frac{\partial^1}{\partial x_j} f(\mathtt{t}) \left[\frac{1}{2} + A^1_{e_j} \right],
$$

$\mathtt{t} \in \mathbb{R}^N$, $1 \le j \le N$.

Proof By Theorem 4 in the case of $r = 1$ we know:

$$
w \left\{ (K_{w,j} f)(\mathtt{t}) - f(\mathtt{t}) \right\} = \sum_{i=1, i \ne j}^{N} \frac{\partial^1}{\partial x_i} f(\mathtt{t}) \cdot A^1_{e_i} + \frac{\partial^1}{\partial x_j} f(\mathtt{t}) \left[\frac{1}{2} + A^1_{e_j} \right] + w\, o(w^{-1}),
$$

then the proof follows immediately by passing to the limit for $w \to +\infty$. □

It can be useful to observe that the moment-type assumptions required in Theorems 4 and Theorem 5 are quite standard and are satisfied by several examples of kernels, such as those mentioned in Sect. 2. A wide list of them can be found, e.g., in [2,14,16,22].

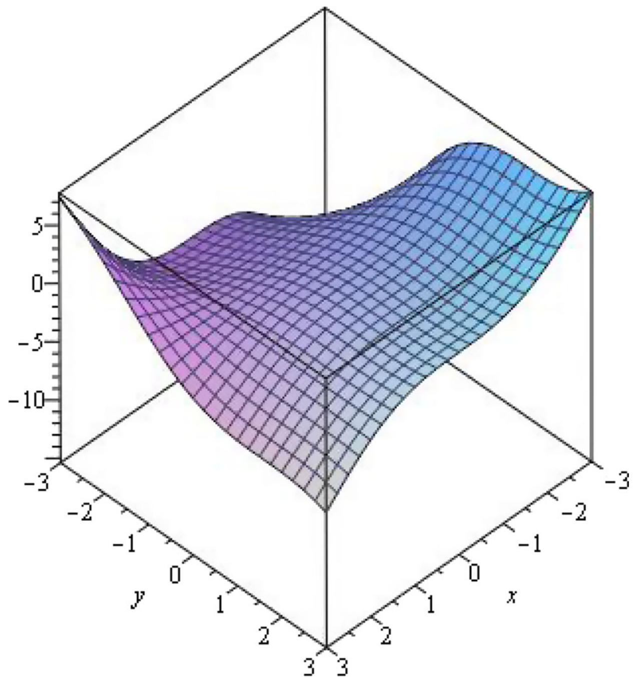

Fig. 1 The plot of the bi-dimensional function f

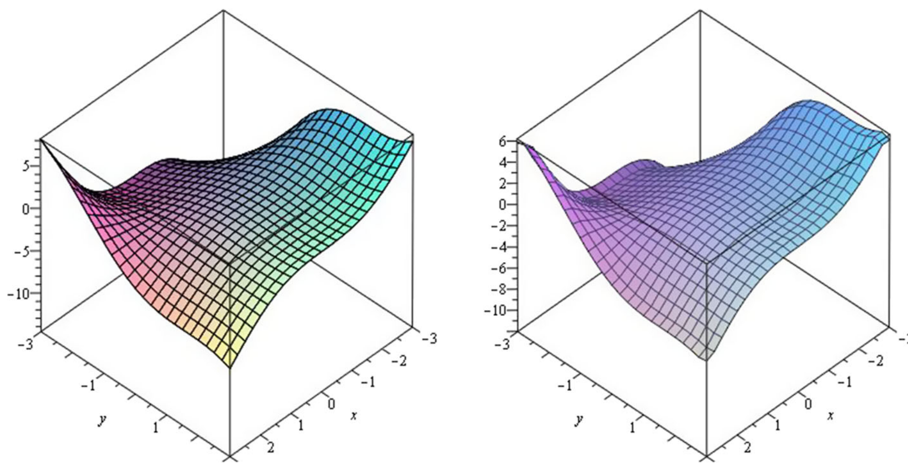

Fig. 2 On the left, we have the operator $K_{w,1} f$, with $w = 5$, while on the right we have the operator $K_{w,1} f$, with $w = 10$, both based upon the bi-dimensional central B-spline of order 3

Remark 1 Note that a quantitative version of Theorems 4 and 5 can be easily established by repeating the above proof using the Lagrange remainder in the Taylor expansion (7) and by using some well-known inequalities; for more details see, e.g., [1,16].

Further, we can also observe that Theorem 5 can be also generalized for higher orders (i.e., $r > 1$). In order to get such generalization, we must require that the algebraic moments

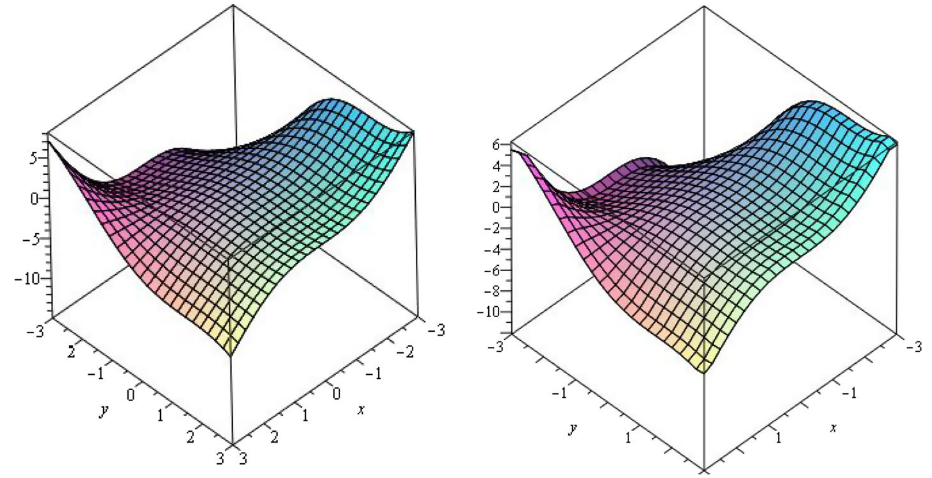

Fig. 3 On the left, we have the operator $K_{w,2}f$, with $w = 5$, while on the right we have the operator $K_{w,2}f$, with $w = 10$, both based upon the bi-dimensional central B-spline of order 3

of the kernel χ:

$$m^{\nu}_{\mathtt{h}}(\chi, \mathtt{u}) \;=\; A^{\nu}_{\mathtt{h}} \in \mathbb{R}, \qquad \mathtt{u} \in \mathbb{R}^N,$$

are constants, for every $\nu = 1, \ldots, r$, and $\mathtt{h} \in \mathbb{N}^N$ with $|\mathtt{h}| = \nu$, and that

$$\sum_{\ell=0}^{h_j} \binom{h_j}{\ell} \frac{A^{\nu-h_j+\ell}_{(\mathtt{h}'_j,\ell)}}{(h_j - \ell + 1)} \;=\; 0,$$

for all $\nu = 1, \ldots, r-1$, and $\mathtt{h} = (h_1, \ldots, h_N) \in \mathbb{N}^N$ with $|\mathtt{h}| = \nu$, $j = 1, \ldots, N$. Examples of kernels satisfying the above conditions can be generated using product kernels, in which the one-dimensional factors are given by suitable finite linear combination of well-known univariate kernels. The procedure for the construction of such multivariate functions is analogous to that one used in [16].

Finally, we give a numerical example showing the approximations that can be achieved by the above theory. Consider the function $f(x_1, x_2) := (\sin x_1 - x_1 + 1)(x_2 + \cos x_2)$, $(x_1, x_2) \in \mathbb{R}^2$ (see Fig. 1). The approximations obtained by the mixed sampling-Kantorovich operators based upon the bi-dimensional central B-spline of order 3 are depicted in Figs. 2 and 3.

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