Rate of convergence of exponential type operators related to $p(x) = 2x^{3/2}$ for functions of bounded variation

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Abstract

The present article deals with the approximation of certain exponential type operators defined by Ismail and May. We estimate the rate of convergence of these operators for functions of bounded variation.

Keywords Ismail–May operators · Rate of convergence · Bounded variation · Exponential type operators

Mathematics Subject Classification 41A25 · 41A30

1 Introduction

Ismail and May [9] studied exponential type operators L_n . These operators have the form

$$(L_n f)(x) = \int_0^\infty k_n(x, t) f(t) dt \quad (1)$$

with kernels $k_n(x, t)$ satisfying the partial differential equation

$$\frac{\partial}{\partial x} k_n(x, t) = \frac{n}{p(x)} k_n(x, t) (t - x).$$

The special case $p(x) = 2x^{3/2}$ leads to the kernel

$$k_n(x, t) = e^{-n\sqrt{x}} \left(n e^{-nt/\sqrt{x}} t^{-1/2} I_1(2n\sqrt{t}) + \delta(t) \right), \quad (2)$$

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where $\delta(\cdot)$ denotes the Dirac delta function and I_1 the modified Bessel function of first kind given by

$$I_1(z) = \sum_{k=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{1+2k}}{k! \Gamma(k+2)}.$$

The corresponding operator [9, (3.16)] has the explicit representation

$$(T_n f)(x) = e^{-n\sqrt{x}} \left\{ n \int_0^{\infty} e^{-nt/\sqrt{x}} t^{-1/2} I_1(2n\sqrt{t}) f(t) dt + f(0) \right\}, \quad (3)$$

where $x \in (0, \infty)$.

Note that the operators (3) can alternatively be written in the form

$$(T_n f)(x) = \frac{n}{\sqrt{x}} \sum_{k=1}^{\infty} s_k(n\sqrt{x}) \int_0^{\infty} s_{k-1}\left(\frac{nt}{\sqrt{x}}\right) f(t) dt + e^{-n\sqrt{x}} f(0), \quad (4)$$

where $x \in (0, \infty)$ and

$$s_k(x) = e^{-x} \frac{x^k}{k!}.$$

We observe that the operators T_n are closely related to the well-known Phillips operators [10] (see [11, Eq. (1.1), case $c=0$]) given by

$$(P_n f)(x) = n \sum_{k=0}^{\infty} s_k(nx) \int_0^{\infty} s_{k-1}(nt) f(t) dt + e^{-nx} f(0).$$

If we substitute n in the Phillips operators by the real parameter $n/\sqrt{x}$, we immediately get the operators (3) by Ismail and May in the form (4). More precisely, we have $(P_{n/\sqrt{x}} f)(x) = (T_n f)(x)$. However, unlike the operators T_n , the Phillips operators P_n are not exponential type operators. Many other integral type operators (see, e.g., [1, 2, 6, 7]) including the Phillips operators are not of exponential type.

Recently Gupta [5] gave some direct results including an estimate in terms of the second order modulus of continuity and a Voronovskaja-type result for these operators. The recent paper [8] contains a quantitative asymptotic formula in terms of the modulus of continuity with exponential growth, a Korovkin-type result for exponential functions and also a Voronovskaja-type asymptotic formula in the simultaneous approximation.

The present paper deals with the rate of convergence of the operators T_n for functions of bounded variation.

2 Some Lemmas

Let e_r ($r = 0, 1, 2, \dots$) denote the monomials $e_r(x) = x^r$. For real A , put $h_A(t) = e^{At}$. The moments $T_n e_r$ of the operators (3) are encoded in the moment generating function (m.g.f.) defined by $(T_n h_A)(x)$. Gupta [5, Remark 1] derived the following formula.

Lemma 1 *The moment generating function of the operators (3) is given by*

$$(T_n h_A)(x) = \exp\left(\frac{nAx}{n - A\sqrt{x}}\right) \quad (n > |A|\sqrt{x}).$$

Expanding in powers of A , we obtain, for $|A| < n/\sqrt{x}$,

$$(T_n h_A)(x) = 1 + xA + \left(\frac{2x^{3/2}}{n} + x^2\right) \frac{A^2}{2!} + \frac{(6x^2 + 6nx^{5/2} + n^2x^3)}{n^2} \frac{A^3}{3!} \\ + \frac{(24x^{5/2} + 36nx^3 + 12n^2x^{7/2} + n^3x^4)}{n^3} \frac{A^4}{4!} + \dots$$

Obviously, the coefficients of $A^r/r!$ provide the r -th order moments $(T_n e_r)(x)$. Moreover, we have

$$(T_n e_r)(x) = x^r + \frac{r(r-1)}{n} x^{(2r-1)/2} + \frac{r(r-1)^2(r-2)}{2n^2} x^{r-1} + O(n^{-3})$$

as $n \rightarrow \infty$.

For $x \in \mathbb{R}$, define $\psi_x(t) = t - x$. The central moments $\mu_{n,r}(x) := (T_n \psi_x^r)(x)$ are crucial for the approximation properties of the operators (3).

Remark 1 The central moments $\mu_{n,r}(x) = (T_n \psi_x^r)(x)$ of the operators (3) are the coefficients of $A^r/r!$ in the expansion

$$\exp\left(\frac{nAx}{n - A\sqrt{x}} - Ax\right) \\ = 1 + \frac{2x^{3/2}}{n} \frac{A^2}{2!} + \frac{6x^2}{n^2} \frac{A^3}{3!} + \left(\frac{24x^{5/2}}{n^3} + \frac{12x^3}{n^2}\right) \frac{A^4}{4!} + \dots$$

In particular, we have $\mu_{n,0}(x) = 1$, $\mu_{n,1}(x) = 0$, $\mu_{n,2}(x) = 2x^{3/2}/n$.

Lemma 2 Let $x \in (0, \infty)$ and let the kernel $k_n(x, t)$ be as defined in (2). Then we have

$$\eta_n(x, y) = \int_0^y k_n(x, t) dt \leq \frac{2x^{3/2}}{n(x-y)^2}, \quad 0 < y < x$$

and

$$1 - \eta_n(x, z) = \int_z^\infty k_n(x, t) dt \leq \frac{2x^{3/2}}{n(z-x)^2}, \quad x < z < \infty.$$

Proof For $0 < y < x$ in view of Remark 1, we have

$$\int_0^y k_n(x, t) dt \leq \int_0^y k_n(x, t) \frac{(x-t)^2}{(x-y)^2} dt \\ \leq (x-y)^{-2} \mu_{n,2}(x) \\ = \frac{2x^{3/2}}{n(x-y)^2}, \quad 0 < y < x.$$

The other estimate follows analogously. $\square$

For the proof of the main result we need an estimation of $(T_n \operatorname{sgn} \psi_x)(x)$. It is easy to see that

$$(T_n \operatorname{sgn} \psi_x)(x) = 1 - 2(T_n \chi_{(0,x)})(x),$$

where $\chi_{(0,x)}$ denotes the characteristic function of the interval $(0, x)$. Therefore, we consider

$$(T_n \chi_{(0,x)})(x) = \int_0^x k_n(x, t) dt.$$

The next result describes the asymptotic behaviour of the latter expression.

Lemma 3 For each $x \in (0, \infty)$, we have

$$\int_0^x k_n(x, t) dt = \frac{1}{2} + \frac{1}{4\sqrt{x}\sqrt{\pi n}} + \frac{1}{64\sqrt[4]{x^3}\sqrt{\pi n}^{3/2}} + O(n^{-2}) \quad (n \rightarrow \infty).$$

Proof From the definition (3) we obtain

$$(T_n \chi_{(0,x)})(x) = ne^{-n\sqrt{x}} \int_0^x e^{-nt/\sqrt{x}} t^{-1/2} I_1(2n\sqrt{t}) dt.$$

The change of variable replacing t with $(\sqrt{x} - t)^2$ leads to the representation

$$(T_n \chi_{(0,x)})(x) = 2ne^{-n\sqrt{x}} \int_0^{\sqrt{x}} e^{-n(\sqrt{x}-t)^2/\sqrt{x}} I_1(2n(\sqrt{x}-t)) dt.$$

Taking advantage of the integral representation [3, (9.6.19)]

$$I_1(t) = \frac{1}{\pi} \int_0^\pi e^{z \cos \vartheta} \cos \vartheta d\vartheta$$

we obtain

$$(T_n \chi_{(0,x)})(x) = \frac{2n}{\pi} \int_0^{\sqrt{x}} \int_0^\pi e^{-nh(t,\vartheta)} \cos \vartheta d\vartheta dt,$$

where

$$\begin{aligned} h(t, \vartheta) &= \sqrt{x} + \frac{(\sqrt{x} - t)^2}{\sqrt{x}} - 2(\sqrt{x} - t) \cos \vartheta \\ &= \frac{t^2}{\sqrt{x}} + 2(\sqrt{x} - t)(1 - \cos \vartheta). \end{aligned}$$

Note that $h(0, 0) = 0$. For a sufficiently small positive real δ , there exists $\varepsilon \in (0, \sqrt{x})$, such that $h(t, \vartheta) > \delta$ if $t \in [\varepsilon, \sqrt{x}]$ and $\vartheta \in [0, \pi]$. This implies that, for arbitrary small $a > 0$, there is a constant $c > 0$ such that

$$(T_n \chi_{(0,x)})(x) = \frac{2n}{\pi} \int_0^a \int_0^a e^{-nh(t,\vartheta)} \cos \vartheta d\vartheta dt + O(e^{-cn}) \quad (n \rightarrow \infty).$$

We study

$$J_n(x) := \frac{2n}{\pi} \int_0^a \int_0^a e^{-nh(t,\vartheta)} \cos \vartheta d\vartheta dt.$$

This equation can be rewritten in the form

$$J_n(x) = \frac{2n}{\pi} \int_0^a \int_0^a \exp\left(-n\left(\frac{t^2}{\sqrt{x}} + \sqrt{x}\vartheta^2\right)\right) g_n(t, \vartheta) d\vartheta dt,$$

where

$$g_n(t, \vartheta) = e^{-n(2(\sqrt{x}-t)(1-\cos \vartheta) - \sqrt{x}\vartheta^2)} \cos \vartheta.$$

A further change of variable replacing t with $\sqrt{x/nt}$ and replacing ϑ with $\vartheta/\sqrt{n}$ yields

$$J_n(x) = \frac{2\sqrt{x}}{\pi} \int_0^{a\sqrt{n}} \int_0^{a\sqrt{n}} \exp(-\sqrt{x}(t^2 + \vartheta^2)) g_n\left(\sqrt{\frac{x}{n}}t, \frac{\vartheta}{\sqrt{n}}\right) d\vartheta dt.$$

Noting that $-2(1 - \cos \vartheta) + \vartheta^2 =: \vartheta^4 F(\vartheta)$ and $2(1 - \cos \vartheta) =: \vartheta^2 G(\vartheta)$, where F and G are entire functions, we see that

$$\begin{aligned} g_n(\sqrt{x/nt}, \vartheta/\sqrt{n}) &= \exp\left(\frac{\sqrt{x}}{n} \vartheta^4 F(\vartheta) + \frac{t}{\sqrt{n}} \vartheta^2 G\left(\frac{\vartheta}{\sqrt{n}}\right)\right) \cos \frac{\vartheta}{\sqrt{n}} \\ &= \sum_{i_1, i_2, i_3 \geq 0} a(i_1, i_2, i_3) \left[\frac{\sqrt{x}}{n} \vartheta^4 F\left(\frac{\vartheta}{\sqrt{n}}\right)\right]^{i_1} \left[\frac{t}{\sqrt{n}} \vartheta^2 G\left(\frac{\vartheta}{\sqrt{n}}\right)\right]^{i_2} \left[\frac{\vartheta}{\sqrt{n}}\right]^{i_3}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} g_n(\sqrt{x/nt}, \vartheta/\sqrt{n}) &= \sum_{\substack{i_1, i_2, i_3 \geq 0, \\ 2i_1 + i_2 + i_3 \leq 3}} n^{-(2i_1 + i_2 + i_3)/2} a(i_1, i_2, i_3) \left[\sqrt{x} F\left(\frac{\vartheta}{\sqrt{n}}\right)\right]^{i_1} \\ &\quad \left[G\left(\frac{\vartheta}{\sqrt{n}}\right)\right]^{i_2} t^{i_2} \vartheta^{4i_1 + 2i_2 + i_3} + O(n^{-2}) \end{aligned}$$

as $n \rightarrow \infty$. After some calculations we arrive at

$$J_n(x) = \frac{1}{2} + \frac{1}{4\sqrt[4]{x}\sqrt{\pi n}} + \frac{1}{64\sqrt[4]{x^3}\sqrt{\pi n^{3/2}}} + O(n^{-2})$$

as $n \rightarrow \infty$. □

3 Rate of convergence

Theorem 1 Let f be a function of bounded variation on each finite subinterval of $(0, \infty)$. Suppose that f satisfies the growth condition $|f(t)| \leq Ce^{\gamma t}$, for $t > 0$. Then, it exists a sequence $(\varepsilon_n(x))$ with $\varepsilon_n(x) \rightarrow 0$ as $n \rightarrow \infty$, such that, for $n > 2\gamma\sqrt{x}$,

$$\begin{aligned} &\left| (T_n f)(x) - \frac{1}{2}(f(x+) + f(x-)) \right| \\ &\leq \left(\frac{1}{2\sqrt[4]{x}\sqrt{\pi n}} + \frac{1}{32\sqrt[4]{x^3}\sqrt{\pi n^{3/2}}} \right) |f(x+) - f(x-)| + \frac{2 + \sqrt{x}}{n\sqrt{x}} \sum_{k=1}^n V_{x-\sqrt{k}/\sqrt{k}}^{x+\sqrt{k}/\sqrt{k}}(f_x) \\ &\quad + \frac{2Ce^{\gamma x}}{n\sqrt{x}} + \frac{\sqrt{2}Cx^{3/4}}{\sqrt{n}} \exp\left(\frac{n\gamma x}{n - 2\gamma\sqrt{x}}\right) + \frac{\varepsilon_n(x)}{n^2}, \end{aligned}$$

where

$$f_x(t) = \begin{cases} f(t) - f(x-) & (0 < t < x), \\ 0 & (t = x), \\ f(t) - f(x+) & (x < t < \infty). \end{cases}$$

Proof Let $x \in (0, \infty)$. Our starting point is the inequality

$$\begin{aligned} &\left| (T_n f)(x) - \frac{1}{2}(f(x+) + f(x-)) \right| \\ &\leq \frac{1}{2} |f(x+) - f(x-)| \cdot |(T_n \operatorname{sgn} \psi_x)(x)| + |(T_n f_x)(x)|. \end{aligned}$$

Firstly, by Lemma 3, we obtain

$$\begin{aligned}(T_n \operatorname{sgn} \psi_x)(x) &= 1 - 2(T_n \chi_{(0,x)})(x) = 1 - 2\left(\frac{1}{2} + \frac{1}{4\sqrt{x}\sqrt{\pi n}} + \frac{1}{64\sqrt{x^3}\sqrt{\pi n^{3/2}}} + O(n^{-2})\right) \\ &= \frac{-1}{2\sqrt{x}\sqrt{\pi n}} + \frac{-1}{32\sqrt{x^3}\sqrt{\pi n^{3/2}}} + O(n^{-2})\end{aligned}$$

as $n \rightarrow \infty$. Next we estimate $(T_n f_x)(x)$ as follows:

$$\begin{aligned}(T_n f_x)(x) &= \int_0^\infty k_n(x, t) f_x(t) dt \\ &= \left(\int_0^{x-x/\sqrt{n}} + \int_{x-x/\sqrt{n}}^{x+x/\sqrt{n}} + \int_{x+x/\sqrt{n}}^\infty \right) k_n(x, t) f_x(t) dt \\ &=: E_1 + E_2 + E_3.\end{aligned}$$

Integrating the first term by parts we obtain

$$\begin{aligned}E_1 &= \int_0^{x-x/\sqrt{n}} f_x(t) d_t(\eta_n(x, t)) \\ &= f_x\left(x - \frac{x}{\sqrt{n}}\right) \eta_n\left(x, x - \frac{x}{\sqrt{n}}\right) - \int_0^{x-x/\sqrt{n}} \eta_n(x, t) d_t(f_x(t)).\end{aligned}$$

Since $|f_x(y)| \leq V_y^x(f_x)$, we have

$$|E_1| \leq V_{x-x/\sqrt{n}}^x(f_x) \cdot \eta_n\left(x, x - \frac{x}{\sqrt{n}}\right) + \int_0^{x-x/\sqrt{n}} \eta_n(x, t) d_t(-V_t^x(f_x)).$$

Applying Lemma 2, and in the next step integrating by parts, we get

$$\begin{aligned}|E_1| &\leq \frac{2x^{3/2}}{n[x - (x - x/\sqrt{n})]^2} V_{x-x/\sqrt{n}}^x(f_x) + \frac{2x^{3/2}}{n} \int_0^{x-x/\sqrt{n}} \frac{1}{(x-t)^2} d_t(-V_t^x(f_x)) \\ &= \frac{2x^{3/2}}{n} \left[\frac{1}{x^2} V_0^x(f_x) + 2 \int_0^{x-x/\sqrt{n}} \frac{1}{(x-t)^3} V_t^x(f_x) dt \right] \\ &\leq \frac{2}{n\sqrt{x}} \sum_{k=1}^n V_{x-x/\sqrt{k}}^x(f_x).\end{aligned}$$

Next for $t \in [x - x/\sqrt{n}, x + x/\sqrt{n}]$ and by the fact $\int_{x-x/\sqrt{n}}^{x+x/\sqrt{n}} d_t(\eta_n(x, t)) \leq 1$, we conclude that

$$|E_2| \leq \frac{1}{n} \sum_{k=1}^n V_{x-x/\sqrt{k}}^{x+x/\sqrt{k}}(f_x).$$

Finally,

$$\begin{aligned}E_3 &= \int_{x+x/\sqrt{n}}^\infty k_n(x, t) f_x(t) dt \\ &= \left(\int_{x+x/\sqrt{n}}^{2x} + \int_{2x}^\infty \right) k_n(x, t) f_x(t) dt \\ &=: E_{31} + E_{32}.\end{aligned}$$

Arguing analogously as in estimate of E_1 , we have

$$|E_{31}| \leq \frac{2}{n\sqrt{x}} \sum_{k=1}^n V_x^{x+x/\sqrt{k}}(f_x).$$

By assumption, we have $|f_x(t)| \leq |f(x+)| + |f(t)| \leq C(e^{\gamma x} + e^{\gamma t})$, such that

$$|E_{32}| \leq \int_{2x}^{\infty} k_n(x, t) |f_x(t)| dt \leq C \int_{2x}^{\infty} k_n(x, t) (e^{\gamma x} + e^{\gamma t}) dt.$$

Since $|t - x|/x \geq 1$, for $t \geq 2x$, we conclude that

$$\begin{aligned} |E_{32}| &\leq \frac{C e^{\gamma x}}{x^2} \int_0^{\infty} k_n(x, t) (t - x)^2 dt + \frac{C}{x} \int_0^{\infty} k_n(x, t) e^{\gamma t} |t - x| dt \\ &= \frac{C e^{\gamma x}}{x^2} \mu_{n,2}(x) + \frac{C}{x} (\mu_{n,2}(x) \cdot (T_n e^{2\gamma \cdot})(x))^{1/2} \\ &= \frac{2C e^{\gamma x}}{n\sqrt{x}} + C \left(\frac{2x^{3/2}}{n} e^{\frac{2n\gamma x}{n-2\gamma\sqrt{x}}} \right)^{1/2}. \end{aligned}$$

Collecting the estimates of E_1, E_2, E_3 , we get the desired result. $\square$

Remark 2 One may extend the present result to generalized bounded variations of integral type (see [4]) and for $\bigwedge BV^{(p)}$ and $\varphi \bigwedge BV$ discussed in [12]. This may be treated as further research in this direction.

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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