Measures of noncompactness and superposition operator in the space of regulated functions on an unbounded interval

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Abstract

In this paper, we formulate necessary and sufficient conditions for relative compactness in the space $BG(\mathbb{R}_+, E)$ of regulated and bounded functions defined on $\mathbb{R}_+$ with values in the Banach space E. Moreover, we construct four new measures of noncompactness in the space $BG(\mathbb{R}_+, E)$. We investigate their properties and we describe relations between these measures. We provide necessary and sufficient conditions so that the superposition operator (Niemytskii) maps $BG(\mathbb{R}_+, E)$ into $BG(\mathbb{R}_+, E)$ and, additionally, be compact.

1 Introduction

The measures of noncompactness and fixed point theorems are often chosen to investigate solvability of the nonlinear equations. Using a suitable function space together with convenient measures of noncompactness we can obtain elegant existence theorems. In recent years, there have appeared a lot of papers concerning the space $G(J, E)$ of regular functions defined on a bounded interval J and with values in the Banach space E [2,4–11,14]. On the other hand, there have not been any papers focused on the space of regular functions on an unbounded interval $\mathbb{R}_+$. Throughout this paper we are going to fill this gap. We will investigate the space $BG(\mathbb{R}_+, E)$ of regular and bounded functions defined on $\mathbb{R}_+$ and with values in the Banach space E.

In Sect. 3 we formulate necessary and sufficient relative compactness conditions in the space $BG(\mathbb{R}_+, E)$. In the sequel we construct four new and convenient measures of noncompactness in $BG(\mathbb{R}_+, E)$ and investigate their properties.

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In Sect. 4 we present necessary and sufficient conditions for superposition operator so that it maps $BG(\mathbb{R}_+, E)$ into $BG(\mathbb{R}_+, E)$. Moreover, we formulate necessary and sufficient conditions so that the superposition operator $F_f : BG(\mathbb{R}_+, E) \to BG(\mathbb{R}_+, E)$ is compact.

Finally, Sect. 5 shows the applicability one of the mentioned measures of noncompactness to the existence result for some nonlinear integral equation.

2 Notation, definitions and auxiliary facts

This section is devoted to recalling some facts which will be used in our further investigations. Assume that E is a real Banach space with the norm $\| \cdot \|$ and the zero element θ. Denote by $B_E(x, r)$ the closed ball centered at x and with radius r. The symbol $B_E(r)$ stands for the ball $B_E(\theta, r)$. We write $\overline{X}, \mathrm{conv}X, \mathrm{Conv}X$ to denote the closure, convex hull and the convex closure of a set X, respectively. Further, let $\mathfrak{M}_E$ denote the family of all nonempty and bounded subsets of E and $\mathfrak{N}_E$ its subfamily consisting of all relatively compact sets. The characteristic function of the set A is denoted by $\mathbf{1}_A$.

We accept the following definition of the notion of a measure of noncompactness [3].

Definition 2.1 A mapping $\mu : \mathfrak{M}_E \to \mathbb{R}_+ := [0, \infty)$ is said to be a *measure of noncompactness* in a Banach space E if it satisfies the following conditions:

1° The family $\ker\mu := \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ is nonempty and $\ker\mu \subset \mathfrak{N}_E$.
2° $X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$.
3° $\mu(\mathrm{Conv}X) = \mu(X)$.
4° $\mu(\lambda X + (1 - \lambda)Y) \leq \lambda\mu(X) + (1 - \lambda)\mu(Y)$ for $\lambda \in [0, 1]$.
5° If $\{X_n\}$ is a sequence of closed sets from $\mathfrak{M}_E$ such that $X_{n+1} \subset X_n$ for $n = 1, 2, \ldots$ and if $\lim_{n \to \infty} \mu(X_n) = 0$, then the intersection $X_\infty := \bigcap_{n=1}^{\infty} X_n$ is nonempty.

In the sequel we will use measures of noncompactness having some additional properties. Namely, a measure μ is said to be *sublinear* if it satisfies the following two conditions:

6° $\mu(\lambda X) = |\lambda|\mu(X), \quad \lambda \in \mathbb{R}$.
7° $\mu(X + Y) \leq \mu(X) + \mu(Y)$.

A sublinear measure of noncompactness μ satisfying the condition (maximum property)

8° $\mu(X \cup Y) = \max\{\mu(X), \mu(Y)\}$

and such that $\ker\mu = \mathfrak{N}_E$ is said to be *regular*.

For a given nonempty bounded subset X of E we denote by $\beta_E(X)$ the so-called Hausdorff measure of noncompactness of X. This quantity is defined by the formula

$$\beta_E(X) := \inf\{r > 0 : X \text{ has a finite } r \text{ -net in } E\}.$$

The function β_E is an example of the regular measure of noncompactness in E.

Let J denote the interval, bounded $J = [0, T]$ or unbounded $J = \mathbb{R}_+ := [0, \infty)$. We use the following notation which will be needed further on: if $J = [0, T]$, then we put $J^+ := [0, T)$ and $J^- := (0, T]$, but if we take $J = \mathbb{R}_+$, then $J^+ := [0, \infty)$ and $J^- := (0, \infty)$.

Now we recall some facts concerning regulated functions.

Definition 2.2 A function $x : J \to E$, where E is a topological vector space, is said to be a *regulated function* if for every $t \in J^+$ the right-sided limit $x(t^+) := \lim_{s \to t^+} x(s)$ exists and for every $t \in J^-$ the left-sided limit $x(t^-) := \lim_{s \to t^-} x(s)$ exists.

From now on, E denotes a real Banach space.

Denote by $G(J, E)$ the space consisting of all regulated functions defined on the interval J with values in a Banach space E. Obviously, $G(J, E)$ is a linear space.

In the case of $J = [0, T]$, every function $x \in G([0, T], E)$ is bounded. Hence, $G([0, T], E)$ can be equipped with the classical supremum norm $\|x\|_T := \sup\{\|x(t)\| : t \in J\}$. It is easy to show that $G([0, T], E)$ is a Banach space.

Compactness criteria and measures of noncompactness in the space $G([0, T], E)$ were investigated in several research papers (see [4–6,11]). First, we recall the concept of a equiregulated subset of the space $G([0, T], E)$.

Definition 2.3 We will say that the set $X \subset G([0, T], E)$ is *equiregulated* on the interval $J = [0, T]$ if the following two conditions are satisfied:

$$\forall_{t\in(0,T]} \ \forall_{\varepsilon>0} \ \exists_{\delta>0} \ \forall_{x\in X} \ \forall_{t_1,t_2\in(t-\delta,t)\cap[0,T]} \ \|x(t_2) - x(t_1)\| \le \varepsilon,$$
$$\forall_{t\in[0,T)} \ \forall_{\varepsilon>0} \ \exists_{\delta>0} \ \forall_{x\in X} \ \forall_{t_1,t_2\in(t,t+\delta)\cap[0,T]} \ \|x(t_2) - x(t_1)\| \le \varepsilon.$$

Let us recall the compactness criterion in $G([0, T], E)$. This result was formulated by Fraňková [11], (see also [4–6]).

Theorem 2.4 *A nonempty subset $X \subset G([0, T], E)$ is relatively compact in $G([0, T], E)$ if and only if X is equiregulated on the interval $[0, T]$ and the sets $X(t)$ are relatively compact in E for $t \in [0, T]$.*

Now we are going to recall the construction of regular measure of noncompactness in the space $G([0, T], E)$. To this end, let us take a set $X \in \mathfrak{M}_{G([0,T],E)}$. For $x \in X$ and $\varepsilon > 0$ let us denote the following quantities:

$$\omega_T^-(x, t, \varepsilon) := \sup\{\|x(t_2) - x(t_1)\| : t_1, t_2 \in (t - \varepsilon, t) \cap [0, T]\}, \quad t \in (0, T],$$
$$\omega_T^+(x, t, \varepsilon) := \sup\{\|x(t_2) - x(t_1)\| : t_1, t_2 \in (t, t + \varepsilon) \cap [0, T]\}, \quad t \in [0, T).$$

The quantities $\omega^-(x, t, \varepsilon)$ and $\omega^+(x, t, \varepsilon)$ can be interpreted as left-hand and right-hand sided moduli of convergence of the function x at the point t. Further, let us put:

$$\omega_T^-(X, t, \varepsilon) := \sup\{\omega_T^-(x, t, \varepsilon) : x \in X\}, \quad t \in (0, T],$$
$$\omega_T^+(X, t, \varepsilon) := \sup\{\omega_T^+(x, t, \varepsilon) : x \in X\}, \quad t \in [0, T),$$
$$\omega_T^-(X, t) := \lim_{\varepsilon\to0^+} \omega_T^-(X, t, \varepsilon), \quad t \in (0, T],$$
$$\omega_T^+(X, t) := \lim_{\varepsilon\to0^+} \omega_T^+(X, t, \varepsilon), \quad t \in [0, T),$$
$$\omega_T^-(X) := \sup_{t\in(0,T]} \omega_T^-(X, t),$$
$$\omega_T^+(X) := \sup_{t\in[0,T)} \omega_T^+(X, t),$$
$$\overline{\beta}_T(X) := \sup_{t\in[0,T]} \beta_E(X(t)).$$

Finally, let us define quantity

$$\mu_T(X) := \max\{\omega_T^-(X), \omega_T^+(X)\} + \overline{\beta}_T(X). \tag{1}$$

Theorem 2.5 [14] *The function μ_T given by formula* (1) *satisfies conditions $1°–7°$ in the space $G([0, T], E)$.*

Further, we define the space mentioned in the title of this paper. Denote by $BG(\mathbb{R}_+, E)$ the space consisting of all bounded and regulated functions defined on the interval $\mathbb{R}_+$ with values in a Banach space E, equipped with the norm $\|x\|_\infty := \sup\{\|x(t)\| : t \in \mathbb{R}_+\}$. It is easy to show that $BG(\mathbb{R}_+, E)$ is a Banach space. In the next section we are going to introduce a simple criterion of compactness in $BG(\mathbb{R}_+, E)$ and a few measures of noncompactness in this space.

3 Measures of noncompactness in $BG(\mathbb{R}_+, E)$

For fixed $x \in BG(\mathbb{R}_+, E)$ and $T > 0$ we put

$$\|x\|_T := \sup\{\|x(t)\| : t \in [0, T]\}.$$

Further, we give the following compactness criterion based on ideas from [16].

Theorem 3.1 *Let E be a real Banach space. Then a nonempty subset $X \subset BG(\mathbb{R}_+, E)$ is relatively compact iff the following three conditions are satisfied:*

(a) *The set $X(t)$ is relatively compact in E for each $t \in \mathbb{R}_+$.*
(b) *For each $T > 0$ the set $X|_{[0,T]}$ is equiregulated.*
(c) *For each $\varepsilon > 0$ there are $\delta > 0$ and $T > 0$ such that, for any $x_1, x_2 \in X$, if $\|x_1 - x_2\|_T < \delta$ then $\|x_1 - x_2\|_\infty < \varepsilon$.*

Proof ($\Rightarrow$) In view of the fact that X is a totally bounded set, we obtain that the sets $X(t)$ for arbitrary $t \in \mathbb{R}_+$ and $X|_{[0,T]}$ for arbitrary $T > 0$ have this property as well. Therefore, we obtain (a) and, in virtue of Theorem, 2.4 we have (b). Let us assume that (c) is not fulfilled. Then, there exist $\varepsilon_0 > 0$ and two sequences $\{x_n\}, \{y_n\} \subset X$ such that

$$\|x_n - y_n\|_\infty \geq \varepsilon_0 \tag{2}$$

and

$$\|x_n - y_n\|_n \leq \frac{1}{n}, \quad n \in \mathbb{N}. \tag{3}$$

Relative compactness of the set X implies the existence of convergent subsequences $\{x_{k_n}\}, \{y_{k_n}\}$ of the sequences $\{x_n\}$ and $\{y_n\}$, i.e. $x_{k_n} \to x \in BG(\mathbb{R}_+, E)$ and $y_{k_n} \to y \in BG(\mathbb{R}_+, E)$. However, we obtain a contradiction. Namely, taking $n \to \infty$ we obtain from (2) inequality $\|x - y\|_\infty \geq \varepsilon_0$. On the other hand, in view of (3) we get $\|x - y\|_\infty = 0$.

($\Leftarrow$) Let us consider an arbitrary sequence $\{x_n\} \subset X$. Conditions (a) and (b) connected with Theorem 2.4 imply that the sequence $\{x_n\}$ has a subsequence $\{x_{1,n}\}$ converging on $[0, 1]$ with respect to the pseudonorm $\|\cdot\|_1$ to some regulated function defined on $[0, 1]$. Similarly, the sequence $\{x_{1,n}\}$ has a subsequence $\{x_{2,n}\}$ converging on $[0, 2]$ with respect to the pseudonorm $\|\cdot\|_2$ to some regulated function defined on $[0, 2]$. Therefore, we obtain the sequence of subsequences $\{x_{i,n}\}$, $i = 1, 2, \ldots$ converging on the interval $[0, i]$ with respect to the pseudonorm $\|\cdot\|_i$ to some regulated function defined on $\mathbb{R}_+$. Next, putting $z_n := x_{n,n}$, $n = 1, 2, \ldots$ we obtain the sequence $\{z_n\}$, uniformly convergent on bounded intervals to some regulated function $z : \mathbb{R}_+ \to E$. Let us fix $\varepsilon > 0$. Further, if $T > 0$ and $\delta > 0$ are like in (c), then for sufficiently big n we have $\|z - z_n\|_T \leq \delta$ and in virtue of (c) we obtain $\|z - z_n\|_\infty \leq \varepsilon$ for sufficiently big n. This yields that $z \in BG(\mathbb{R}_+, E)$ and $z_n \to z$ with respect to the norm $\|\cdot\|_\infty$ which implies relative compactness of the set X in $BG(\mathbb{R}_+, E)$.   □

Using this theorem we introduce the first measure of noncompactness in the space $BG(\mathbb{R}_+, E)$. For this purpose we define several symbols. For a given set $X \in \mathfrak{M}_{BG(\mathbb{R}_+,E)}$ and $\delta > 0$ let us denote

$$\omega_\infty^-(X) := \lim_{T \to \infty} \omega_T^-(X), \quad \omega_\infty^+(X) := \lim_{T \to \infty} \omega_T^+(X),$$

$$\overline{\beta}_\infty(X) := \lim_{T \to \infty} \overline{\beta}_T(X),$$

$$\varphi^T(X, \delta) := \sup\{\|x - y\|_\infty : x, y \in \mathrm{conv}X, \|x - y\|_T < \delta\},$$

$$\varphi_0^T(X) := \lim_{\delta \to 0^+} \varphi^T(X, \delta),$$

$$\varphi_0^\infty(X) := \lim_{T \to \infty} \varphi_0^T(X). \tag{4}$$

Using the above quantities we define the mapping $\mu : \mathfrak{M}_{BG(\mathbb{R}_+,E)} \to \mathbb{R}_+$ given by the formula

$$\mu(X) := \max\{\omega_\infty^-(X), \omega_\infty^+(X)\} + \overline{\beta}_\infty(X) + \varphi_0^\infty(X). \tag{5}$$

Remark 3.2 In the definition of the quantity $\varphi^T(X, \delta)$ in the formula (4) the convex hull $\mathrm{conv}X$ appears. It is caused by the fact that without it the function μ would not fulfil the condition 3° of Definition 2.1. This condition, in turn, is necessary to use many fixed point theorems.

The main properties of the function μ are contained in the below given theorem.

Theorem 3.3 *The function* $\mu : \mathfrak{M}_{BG(\mathbb{R}_+,E)} \to \mathbb{R}_+$ *given by the formula* (5) *satisfies the conditions* 2°, 3°, 5°, 6° *and the equality* $\ker\mu = \mathfrak{N}_{BG(\mathbb{R}_+,E)}$ *holds.*

Proof The equality $\ker\mu = \mathfrak{N}_{BG(\mathbb{R}_+,E)}$ is a simple consequence of Theorem 3.1. We gather several facts concerning the function φ_0^∞. First, let us notice that the properties

$$\varphi_0^\infty(X) \le \varphi_0^\infty(Y) \quad \text{for } X \subset Y, \tag{6}$$

$$\varphi_0^\infty(\mathrm{conv}X) = \varphi_0^\infty(X), \tag{7}$$

$$\varphi_0^\infty(\lambda X) = |\lambda|\varphi_0^\infty(X) \quad \text{for } \lambda \in \mathbb{R}, \tag{8}$$

are obvious.

Now we will show that

$$\varphi_0^\infty(\overline{X}) = \varphi_0^\infty(X). \tag{9}$$

Let us fix $X \in \mathfrak{M}_{BG(\mathbb{R}_+,E)}$, $T > 0$, $\delta > 0$ and take $x, y \in \mathrm{conv}\overline{X}$ such that $\|x - y\|_T < \delta$. Then there exist $\{x_n\}, \{y_n\} \subset \mathrm{conv}X$ such that $x_n \to x$, $y_n \to y$. This means that for arbitrary $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for each $n \ge n_0$ we have

$$\|x - x_n\|_\infty \le \frac{\varepsilon}{2}, \quad \|y_n - y\|_\infty \le \frac{\varepsilon}{2}.$$

Hence

$$\|x - y\|_\infty \le \|x_n - y_n\|_\infty + \varepsilon. \tag{10}$$

Since $x_n - y_n \to x - y$, then we may choose sufficiently big $n \in \mathbb{N}$ such that $\|x_n - y_n\|_T < \delta$. Therefore, in virtue of inequality (10) we obtain

$$\|x - y\|_\infty \le \|x_n - y_n\|_\infty + \varepsilon \le \varphi^T(X, \delta) + \varepsilon$$

and hence

$$\varphi^T(\overline{X}, \delta) \le \varphi^T(X, \delta) + \varepsilon.$$

Letting $\delta \to 0$ and $T \to \infty$ we obtain (in view of arbitrariness of ε)

$$\varphi_0^\infty(\overline{X}) \le \varphi_0^\infty(X),$$

which together with (6) confirms (9). Further, the property (7) in connection with (9) give us

$$\varphi_0^\infty(\mathrm{Conv}X) = \varphi_0^\infty(X). \tag{11}$$

Keeping in mind Theorem 2.5 we have that the mapping μ_T given by the formula (1) satisfies conditions $1°$–$7°$. Hence, to obtain the properties $2°$, $6°$, $3°$ for the measure μ, it is sufficient to join (6), (8), (11) to the conditions $2°$, $6°$, $3°$ for the measure μ_T and to take limit $T \to \infty$.

Further, the property $5°$ will be proved. Let us assume that $\{X_n\}$ is a decreasing sequence of closed and bounded subsets of the space $BG(\mathbb{R}_+, E)$ and $\lim_{n\to\infty} \mu(X_n) = 0$. Let us take arbitrarily $x_n \in X_n$. In particular, it follows that $\lim_{n\to\infty} \mu_T(\{x_i : i \ge n\}) = 0$ for each $T > 0$. Since the measure μ_T satisfies the condition $5°$ in the space $G([0, T], E)$, putting $T = 1, 2, \ldots$ and applying the diagonal method we ensure the existence of subsequence $\{u_n\}$ of the sequence $\{x_n\}$, which is uniformly convergent on bounded intervals to the function $u \in BG(\mathbb{R}_+, E)$. In view of $\lim_{n\to\infty}\varphi_0^\infty(\{u_i : i \ge n\}) = 0$ we conclude that $\{u_n\}$ is also convergent in the norm $\|\cdot\|_\infty$ to u. This means that

$$u \in \bigcap_{n=1}^\infty \overline{\{u_i : i \ge n\}} \subset \bigcap_{n=1}^\infty X_n \ne \emptyset. \qquad \square$$

In what follows we provide two examples illustrating that the mapping μ does not fulfil the conditions $4°$, $7°$ and $8°$.

Example 3.4 Let us fix arbitrarily $e \in E$, $\|e\| = 1$ and put $\lambda := \frac{1}{2}$,

$$X := \left\{ x_n := \mathbf{1}_{(\frac{n-1}{n}, \frac{n}{n+1})}(t)e + \mathbf{1}_{(n, n+1)}(t)e : n \in \mathbb{N} \right\},$$

$$Y := \left\{ y_n := -\mathbf{1}_{(\frac{n-1}{n}, \frac{n}{n+1})}(t)e + \mathbf{1}_{(n, n+1)}(t)e : n \in \mathbb{N} \right\}.$$

Then we get

$$\sup_{t\in\mathbb{R}_+} \beta_E(X(t)) = 0, \quad \sup_{t\in\mathbb{R}_+} \beta_E(Y(t)) = 0, \tag{12}$$

$$\omega_\infty^+(X) = 0, \ \omega_\infty^-(X) = 1, \ \omega_\infty^+(Y) = 0, \ \omega_\infty^-(Y) = 1. \tag{13}$$

Let us notice that if $T > 1, \alpha_i \ge 0, i = 1, \ldots, N, \gamma_j \ge 0, j = 1, \ldots, M, \sum_{i=1}^N \alpha_i = 1$ and $\sum_{j=1}^M \gamma_j = 1$, then

$$\left\| \sum_{i=1}^N \alpha_i x_{n_i} - \sum_{j=1}^M \gamma_j x_{m_j} \right\|_T = \left\| \sum_{i=1}^N \alpha_i x_{n_i} - \sum_{j=1}^M \gamma_j x_{m_j} \right\|_\infty.$$

Thus, if $\|x - y\|_T < \delta$ for some $x, y \in \text{conv}X$, then $\|x - y\|_\infty < \delta$ and finally $\varphi_0^\infty(X) = 0$. Reasoning similarly, we infer that $\varphi_0^\infty(Y) = 0$. Now, combining it with (12) and (13) we obtain

$$\mu(X) = 1, \quad \mu(Y) = 1. \tag{14}$$

Further, we have

$$\sup_{t \in \mathbb{R}_+} \beta_E \left(\frac{1}{2} X(t) + \frac{1}{2} Y(t) \right) = 0,$$

$$\omega_\infty^+ \left(\frac{1}{2} X + \frac{1}{2} Y \right) = 0, \quad \omega_\infty^- \left(\frac{1}{2} X + \frac{1}{2} Y \right) = 1.$$

Moreover,

$$1 = \varphi_0^\infty \left(\left\{ \frac{1}{2}(x_n + y_n) : n \in \mathbb{N} \right\} \right) \leq \varphi_0^\infty \left(\frac{1}{2} X + \frac{1}{2} Y \right).$$

Hence, we obtain $\mu(\frac{1}{2} X + \frac{1}{2} Y) \geq 2$ which together with (14) proves that the property $4°$ is not fulfilled for $\lambda = \frac{1}{2}$.

The above mentioned sets X and Y show that the property $7°$ does not hold either.

Now, let us consider the set $X \cup Y$. We have

$$\sup_{t \in \mathbb{R}_+} \beta_E((X \cup Y)(t)) = 0,$$

$$\omega_\infty^+(X \cup Y) = 0, \quad \omega_\infty^-(X \cup Y) = 1,$$

$$1 = \varphi_0^\infty \left(\left\{ \frac{1}{2}(x_n + y_n) : n \in \mathbb{N} \right\} \right) \leq \varphi_0^\infty(X \cup Y).$$

The above properties yields $\mu(X \cup Y) \geq 2$ which together with (14) gives a contradiction to $8°$.

Open question *It is not known if the measure μ satisfies the weak maximum property. In other words, whether the condition*

$$\mu(X \cup \{x\}) = \mu(X)$$

holds for $X \in \mathfrak{M}_{BG(\mathbb{R}_+, E)}, \ x \in E$?

Remark 3.5 Although the mapping μ given by the formula (5) does not fulfil some conditions from Definition 2.1, the conditions which are satisfied are sufficient to apply basic fixed point theorems (i.e. Darbo, Sadovskii). Moreover, the fact $\ker \mu = \mathfrak{N}_{BG(\mathbb{R}_+, E)}$ is undoubtedly an advantage of the function μ. Indeed, this means that μ "catches" every relative compact subset of $BG(\mathbb{R}_+, E)$.

The below given result presents a relationship between the function μ and the Hausdorff measure of noncompactness in $BG(\mathbb{R}_+, E)$. As it turns out, in spite of having identical kernels, they are not equivalent.

Theorem 3.6 *For each nonempty and bounded set $X \subset BG(\mathbb{R}_+, E)$ we have the inequality*

$$\mu(X) \leq 7\beta_{BG(\mathbb{R}_+, E)}(X).$$

The above estimation is optimal (see Example 3.7).
However, there does not exist a constant $k > 0$ such that

$$k\beta_{BG(\mathbb{R}_+, E)}(X) \leq \mu(X)$$

for bounded $X \subset BG(\mathbb{R}_+, E)$.

Proof Let us denote $R := \beta_{BG(\mathbb{R}_+, E)}(X)$ and fix $\varepsilon > 0$. Then there exists a cover $X \subset \cup_{i=1}^{n} B_{BG(\mathbb{R}_+, E)}(a_i, R + \varepsilon)$, where $a_i \in BG(\mathbb{R}_+, E)$.
Since $X(t) \subset \cup_{i=1}^{n} B_{BG(\mathbb{R}_+, E)}(a_i(t), R + \varepsilon)$, we obtain

$$\sup_{t \in \mathbb{R}_+} \beta_E(X(t)) \leq R + \varepsilon. \tag{15}$$

Further, we will show that

$$\omega_\infty^+(X) \leq 2R + 3\varepsilon. \tag{16}$$

Let us fix $t \in \mathbb{R}_+$. Since $\{a_i : i = 1, \ldots, n\}$ is a compact set, Theorem 2.4 yields that there exists $\delta > 0$ such that

$$\|a_i(t_1) - a_i(t_2)\| \leq \varepsilon, \quad t_1, t_2 \in (t, t + \delta), \ i = 1, \ldots, n.$$

Let us take $x \in X$. There exists i such that $x \in B_{BG(\mathbb{R}_+, E)}(a_i, R + \varepsilon)$. Using these considerations we have

$$\|x(t_1) - x(t_2)\| \leq \|x(t_1) - a_i(t_1)\| + \|a_i(t_1) - a_i(t_2)\| + \|a_i(t_2) - x(t_2)\| \leq 2R + 3\varepsilon,$$

which confirms (16). Similarly, we can show that $\omega_\infty^-(X) \leq 2R + 3\varepsilon$, hence we obtain

$$\max\{\omega_\infty^-(X), \omega_\infty^+(X)\} \leq 2R + 3\varepsilon. \tag{17}$$

In the sequel we show that

$$\lim_{T \to \infty} \lim_{\delta \to 0} \sup\{\|x - y\|_\infty : x, y \in \text{conv}X, \|x - y\|_T < \delta\} \leq 4R + 5\varepsilon. \tag{18}$$

Since $\beta_{BG(\mathbb{R}_+, E)}(\text{conv}X) = \beta_{BG(\mathbb{R}_+, E)}(X)$ there exist vectors $b_1, \ldots, b_m \in BG(\mathbb{R}_+, E)$ such that

$$\text{conv}X \subset \cup_{i=1}^{m} B_{BG(\mathbb{R}_+, E)}(b_i, R + \varepsilon).$$

Let us denote the left side of the inequality (18) by g. Hence, there exist sequences $\{x_n\}, \{y_n\} \subset \text{conv}X$ such that

$$\|x_n - y_n\|_\infty \to g \tag{19}$$

and

$$\|x_n - y_n\|_n \leq \frac{1}{n}, \quad n = 1, 2, \ldots. \tag{20}$$

We may further choose an increasing subsequence of natural numbers $\{k_n\}$ such that there exist indexes i and j which give us

$$\{x_{k_n}\} \subset B_{BG(\mathbb{R}_+, E)}(b_i, R + \varepsilon), \ \{y_{k_n}\} \subset B_{BG(\mathbb{R}_+, E)}(b_j, R + \varepsilon), n = 1, 2, \ldots. \tag{21}$$

We will show that

$$\|b_i - b_j\|_\infty \leq 2R + 3\varepsilon. \tag{22}$$

Suppose, on the contrary, that $\|b_i - b_j\|_\infty > 2R + 3\varepsilon$. Then, for some $t \in \mathbb{R}_+$ we would have

$$\|b_i(t) - b_j(t)\| > 2R + 3\varepsilon.$$

Thus

$$\|x_{k_n}(t) - y_{k_n}(t)\| \geq \|b_i(t) - b_j(t)\| - \|b_i(t) - x_{k_n}(t)\| - \|b_j(t) - y_{k_n}(t)\|$$
$$> 2R + 3\varepsilon - (R + \varepsilon) - (R + \varepsilon) = \varepsilon.$$

In connection with (20) we have (for $n \in \mathbb{N}$ such that $k_n \geq t$)

$$\frac{1}{k_n} \geq \|x_{k_n}(t) - y_{k_n}(t)\| > \varepsilon$$

which under $n \to \infty$ is a contradiction and confirms (22). Combining (21) and (22) we conclude that

$$\|x_{k_n} - y_{k_n}\|_\infty \leq \|x_{k_n} - b_i\|_\infty + \|b_i - b_j\|_\infty + \|b_j - y_{k_n}\|_\infty \leq 4R + 5\varepsilon.$$

Letting $n \to \infty$ and applying (19) we have (18).

Summing the inequalities (15), (17) and (18), in view of arbitrariness of $\varepsilon > 0$ we obtain the first part of thesis.

For further purposes let us assume that there exists $k > 0$ such that

$$k\beta_{BG(\mathbb{R}_+, E)}(X) \leq \mu(X) \text{ for } X \in \mathfrak{M}_{BG(\mathbb{R}_+, E)}. \tag{23}$$

Without loss of generality we may assume that $k \leq 1$. Let us fix arbitrarily $e \in E$, $\|e\| = 1$ and put

$$X := \left\{ x_n := \frac{k}{3} \mathbf{1}_{(\frac{n-1}{n}, \frac{n}{n+1})}(t)e + \mathbf{1}_{(n, n+1)}(t)e : n \in \mathbb{N} \right\}.$$

Hence

$$\beta_{BG(\mathbb{R}_+, E)}(X) = \frac{1}{2}, \quad \sup_{t \in \mathbb{R}_+} \beta_E(X(t)) = 0, \quad \omega_\infty^+(X) = 0, \quad \omega_\infty^-(X) = \frac{k}{3}. \tag{24}$$

Let us notice that if $T > 1$, $\alpha_i \geq 0$, $i = 1, \ldots, N$, $\gamma_j \geq 0$, $j = 1, \ldots, M$, $\sum_{i=1}^N \alpha_i = 1$ and $\sum_{j=1}^M \gamma_j = 1$, then

$$\left\| \sum_{i=1}^N \alpha_i x_{n_i} - \sum_{j=1}^M \gamma_j x_{m_j} \right\|_T \geq \frac{k}{3} \left\| \sum_{i=1}^N \alpha_i x_{n_i} - \sum_{j=1}^M \gamma_j x_{m_j} \right\|_\infty.$$

Thus, if $\|x - y\|_T < \delta$ for $x, y \in \text{conv} X$, then $\|x - y\|_\infty < \frac{3}{k}\delta$ and this implies $\varphi_0^\infty(X) = 0$. Combining this fact with (24) we get $\mu(X) = \frac{k}{3}$. Hence, in virtue of (23) and (24) we have $\frac{k}{2} \leq \frac{k}{3}$, which yields a contradiction. $\square$

Now, we show that upper estimation from Theorem 3.6 is attained.

Example 3.7 Let $e \in E$ is such that $\|e\| = 1$ and let us denote by $\widehat{e}$ the function $\widehat{e} : \mathbb{R}_+ \to E$ given by the formula $\widehat{e}(t) := e$, $t \in \mathbb{R}_+$. Next, let us put

$$X := B_{BG(\mathbb{R}_+, E)}(\widehat{\theta}, 1) \cup B_{BG(\mathbb{R}_+, E)}(2\widehat{e}, 1).$$

Then we get $\beta_{BG(\mathbb{R}_+, E)}(X) = 1$, $\max\{\omega_\infty^-(X), \omega_\infty^+(X)\} = 2$, $\sup_{t \in \mathbb{R}_+} \beta_E(X(t)) = 1$.

Further, for arbitrarily fixed $T > 0$ we define two functions of a real variable $t \in \mathbb{R}_+$, i.e.

$$x(t) := \begin{cases} e & \text{for } t \in [0, T] \\ -e & \text{for } t \in (T, \infty) \end{cases},$$

$$y(t) := \begin{cases} e & \text{for } t \in [0, T] \\ 3e & \text{for } t \in (T, \infty) \end{cases}.$$

Since $x, y \in X$, $\|x - y\|_T = 0$, $\|x - y\|_\infty = 4$, we get

$$\varphi_0^\infty(X) = \lim_{T \to \infty} \lim_{\delta \to 0} \sup\{\|x - y\|_\infty : x, y \in \text{conv}X, \|x - y\|_T < \delta\} = 4$$

and then $\mu(X) = 7$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Inspired by some measures in the space $BC(\mathbb{R}_+, E)$ of bounded and continuous functions (see [3]), we will introduce several new measures of noncompactness in $BG(\mathbb{R}_+, E)$. For this purpose we will define a few new set functions. Let us take arbitrarily $X \in \mathfrak{M}_{BG(\mathbb{R}_+,E)}$. So, for a fixed $T > 0$ let us define

$$a_T(X) := \sup_{x \in X} \left\{ \sup\{\|x(t)\| : t \geq T\}\right\}. \tag{25}$$

Next, notice that there exists the limit

$$a_\infty(X) := \lim_{T \to \infty} a_T(X). \tag{26}$$

Moreover we put

$$b_\infty(X) := \lim_{T \to \infty} b_T(X), \tag{27}$$

where

$$b_T(X) := \sup_{x \in X} \left\{ \sup\{\|x(t) - x(s)\| : t, s \geq T\}\right\}, \tag{28}$$

and we can also define

$$c(X) := \limsup_{t \to \infty} \text{diam } X(t), \tag{29}$$

where diam $X(t)$ is understood as

$$\text{diam } X(t) := \sup\{\|x(t) - y(t)\| : x, y \in X\}. \tag{30}$$

And now let us consider the functions $\gamma_a, \gamma_b, \gamma_c$ defined on the family $\mathfrak{M}_{BG(\mathbb{R}_+,E)}$ as follows

$$\gamma_a(X) := \max\{\omega_\infty^-(X), \omega_\infty^+(X)\} + \overline{\beta}_\infty(X) + a_\infty(X), \tag{31}$$

$$\gamma_b(X) := \max\{\omega_\infty^-(X), \omega_\infty^+(X)\} + \overline{\beta}_\infty(X) + b_\infty(X), \tag{32}$$

$$\gamma_c(X) := \max\{\omega_\infty^-(X), \omega_\infty^+(X)\} + \overline{\beta}_\infty(X) + c(X). \tag{33}$$

The main properties of the above mentioned mappings are contained in the below given theorem.

Theorem 3.8 *The mappings $\gamma_a, \gamma_b, \gamma_c$ satisfy axioms $1° - 7°$ of Definition* 2.1.

Proof First, let us assume that $\gamma(X) = 0$, where $X \in \mathfrak{M}_{BG(\mathbb{R}_+,E)}$ and γ is one of the measures $\gamma_a, \gamma_b, \gamma_c$. Since equalities $\max\{\omega_\infty^-(X), \omega_\infty^+(X)\} = 0$ and $\overline{\beta}_\infty(X) = 0$ are equivalent to conditions (a) and (b) from Theorem 3.1 respectively and each of the conditions $a_\infty(X) = 0$, $b_\infty(X) = 0$, $c(X) = 0$ implies (c) from Theorem 3.1, we have

$\ker \gamma \subset \mathfrak{N}_{BG(\mathbb{R}_+, E)}$. Moreover, let us notice that each of the functions a_∞, b_∞ and c satisfies axioms $2° - 7°$. The proof of this fact is simple and will be omitted. Since, in view of Theorem 2.5 the component $\mu_T = \max\{\omega_T^-, \omega_T^+\} + \overline{\beta}_T$ satisfies axioms $2° - 7°$, it is sufficient to add the appropriate conditions for components a_∞, b_∞, c and to take limit $T \to \infty$.

Now, we prove $5°$. Let us assume that $\{X_n\}$ is a decreasing sequence of closed and bounded subsets of the space $BG(\mathbb{R}_+, E)$ and $\lim_{n \to \infty} \gamma(X_n) = 0$, where γ is one of the measures γ_a, γ_b, γ_c. Let us take arbitrarily $x_n \in X_n$. Reasoning similarly as in the proof of Theorem 3.3, we can state that the sequence $\{x_n\}$ contains subsequence which is convergent in the norm $\| \cdot \|_\infty$ to some element $u \in BG(\mathbb{R}_+, E)$, and it implies that $u \in \bigcap_{n=1}^\infty X_n \neq \emptyset$. $\quad\square$

Remark 3.9 The above mentioned measures γ_a, γ_b, γ_c satisfy several additional conditions (for example $4°$, $7°$) that are not fulfilled by the measure μ. However, the measure μ "catches" relative compact subsets in a better way than γ_a, γ_b and γ_c, namely $\ker \gamma_a$, $\ker \gamma_b$, $\ker \gamma_c \subsetneq \mathfrak{N}_{BG(\mathbb{R}_+, E)}$. We omit the simple examples showing that there is no equality.

The mutual relations between these five measures $\beta_{BG(\mathbb{R}_+, E)}$, μ, γ_a, γ_b and γ_c are presented in the following theorem.

Theorem 3.10 *For arbitrary $X \in \mathfrak{M}_{BG(\mathbb{R}_+, E)}$ the following inequalities are satisfied*

$$\mu(X) \leq 2\gamma_a(X), \ \mu(X) \leq 2\gamma_b(X), \ \mu(X) \leq \gamma_c(X), \tag{34}$$

$$\gamma_b(X) \leq 2\gamma_a(X), \ \gamma_c(X) \leq 2\gamma_a(X), \tag{35}$$

$$\beta_{BG(\mathbb{R}_+, E)}(X) \leq \gamma_a(X), \ \beta_{BG(\mathbb{R}_+, E)}(X) \leq \gamma_b(X), \ \beta_{BG(\mathbb{R}_+, E)}(X) \leq \gamma_c(X). \tag{36}$$

Proof To obtain the inequalities (34), it is sufficient to show that $\varphi_0^\infty(X) \leq 2a_\infty(X)$, $\varphi_0^\infty(X) \leq 2b_\infty(X)$ and $\varphi_0^\infty(X) \leq c(X)$. As an example we will prove only the second inequality. Namely, let us fix arbitrarily $\varepsilon > 0$ and let $T > 0$ be such that $b_T(X) \leq b_\infty(X) + \varepsilon$. Next, let $x, y \in \text{conv}X$ and $\|x - y\|_T \leq \varepsilon$. Then we obtain

$$\|x - y\|_\infty \leq \varepsilon + \sup_{t \geq T} \|x(t) - y(t)\|$$

$$\leq \varepsilon + \sup_{t \geq T}(\|x(t) - x(T)\| + \|x(T) - y(T)\| + \|y(T) - y(t)\|)$$

$$\leq 2\varepsilon + \sup_{t \geq T}(\|x(t) - x(T)\| + \|y(T) - y(t)\|)$$

$$\leq 2\varepsilon + 2b_T(\text{conv}X) \leq 4\varepsilon + 2b_\infty(X)$$

since $b_T(\text{conv}X) = b_T(X)$. Hence $\varphi^T(X, \varepsilon) \leq 4\varepsilon + 2b_\infty(X)$. Letting $\varepsilon \to 0$ and $T \to \infty$ we derive $\varphi_0^\infty(X) \leq 2b_\infty(X)$.

It is sufficient to show that $b_\infty(X) \leq 2a_\infty(X)$, $c(X) \leq 2a_\infty(X)$ to obtain the inequalities (35). We omit the simple calculations. $\quad\square$

Now, let us fix $\varepsilon > 0$ and choose $T > 0$ such that

$$a_T(X) \leq a_\infty(X) + \varepsilon, \ b_T(X) \leq b_\infty(X) + \varepsilon, \ c(X) \leq \sup_{t \geq T} \text{diam}\, X(t) + \varepsilon.$$

It has been shown (see [14]) that $\beta_{G([0,T], E)}(X|_{[0,T]}) \leq \mu_T(X|_{[0,T]})$. Thus there exist functions $u_1, \ldots, u_n \in G([0, T], E)$ such that

$$X|_{[0,T]} \subset \bigcup_{i=1}^n B(u_i, \mu_T(X|_{[0,T]}) + \varepsilon). \tag{37}$$

Now, let u_i^a be a function belonging to the space $BG(\mathbb{R}_+, E)$ formed by the function u_i by extension on the interval (T, ∞) using vector θ. Then the functions $\{u_i^a : i = 1, \ldots, n\}$ form $\mu_T(X) + a_\infty(X) + 2\varepsilon$-net of the set X in the space $BG(\mathbb{R}_+, E)$. Indeed, let us fix arbitrarily $x \in X$ and let i be such that

$$\|x - u_i\|_T \leq \mu_T(X|_{[0,T]}) + \varepsilon. \tag{38}$$

Then, for $t > T$ we get

$$\|x(t) - u_i^a(t)\| \leq \|x(t)\| \leq a_T(X) \leq a_\infty(X) + \varepsilon.$$

Combining this fact with (38) we obtain $\|x - u_i^a\|_\infty \leq \mu_T(X|_{[0,T]}) + a_\infty(X) + 2\varepsilon$, and hence $\beta_{BG(\mathbb{R}_+, E)}(X) \leq \mu_T(X|_{[0,T]}) + a_\infty(X) + 2\varepsilon$. Letting $T \to \infty$ and $\varepsilon \to 0$ we get the first of inequalities (36).

Now, we prove the second of the inequalities (36). Let us extend each function u_i on the interval (T, ∞) using vector $u_i(T)$. We denote such a function by u_i^b, $i = 1, \ldots, n$. We show that $\{u_i^b : i = 1, \ldots, n\}$ forms $\mu_T(X|_{[0,T]}) + b_\infty(X) + 2\varepsilon$-net of the set X in the space $BG(\mathbb{R}_+, E)$. To do this let us take an arbitrary $x \in X$ and let i be such that (38) holds. Since $u_i^b(t) = u_i(T)$ for $t > T$, we get

$$\|x(t) - u_i^b(t)\| \leq \|x(t) - x(T)\| + \|x(T) - u_i(T)\| \leq b_T(X) + \mu_T(X|_{[0,T]}) + \varepsilon$$
$$\leq \mu_T(X|_{[0,T]}) + b_\infty(X) + 2\varepsilon$$

for $t > T$. Hence $\beta_{BG(\mathbb{R}_+, E)}(X) \leq \mu_T(X|_{[0,T]}) + b_\infty(X) + 2\varepsilon$ and letting $T \to \infty$ and $\varepsilon \to 0$ we obtain $\beta_{BG(\mathbb{R}_+, E)}(X) \leq \gamma_b(X)$. We omit similar proof of the last inequality from (36). □

Remark 3.11 Below is given an example showing that the estimations from Theorem 3.10 are optimal. Moreover, let us notice that it is not possible to majorize any of the measures $\gamma_a, \gamma_b, \gamma_c$ by $\beta_{BG(\mathbb{R}_+, E)}$ or μ. The lack of such an estimation is a consequence of the sharp inclusions $\ker \gamma_a, \ker \gamma_b, \ker \gamma_c \subsetneq \mathfrak{N}_{BG(\mathbb{R}_+, E)}$.

Example 3.12 Let us put $E := \mathbb{R}$. We define two sets $X, Y \subset BG(\mathbb{R}_+, \mathbb{R})$ as follows

$$X := \{1 - 2 \cdot \mathbf{1}_{[0,n]}(t) : n \in \mathbb{N}\},$$
$$Y := \{(-1)^n \cdot \mathbf{1}_{[n,\infty)}(t) : n \in \mathbb{N}\}.$$

It is not difficult to show that

$$\mu(X) = 2, \quad \gamma_a(X) = 1, \quad \gamma_b(X) = 2, \quad \gamma_c(X) = 2, \quad \beta_{BG(\mathbb{R}_+, \mathbb{R})}(X) = 1, \tag{39}$$
$$\mu(Y) = 2, \quad \gamma_a(Y) = 1, \quad \gamma_b(Y) = 1, \quad \gamma_c(Y) = 2. \tag{40}$$

Now let $\dim E = \infty$ and we put

$$Z := \{x \in BG(\mathbb{R}_+, E) : x(0) \in B_E(1), \ x(t) = \theta \ \text{ for } \ t > 0\}.$$

We can easily show that

$$\gamma_a(Z) = 1, \quad \gamma_b(Z) = 1, \quad \gamma_c(Z) = 1, \quad \beta_{BG(\mathbb{R}_+, E)}(Z) = 1. \tag{41}$$

Applying (39), (40) and (41) we obtain

$$\mu(X) = 2\gamma_a(X), \quad \mu(Y) = 2\gamma_b(Y), \quad \mu(Y) = \gamma_c(Y),$$
$$\gamma_b(X) = 2\gamma_a(X), \quad \gamma_c(X) = 2\gamma_a(X),$$
$$\beta_{BG(\mathbb{R}_+, \mathbb{R})}(X) = \gamma_a(X), \quad \beta_{BG(\mathbb{R}_+, E)}(Z) = \gamma_b(Z), \quad \beta_{BG(\mathbb{R}_+, E)}(Z) = \gamma_c(Z).$$

4 Superposition operator in $BG(\mathbb{R}_+, E)$

The properties of the superposition operator in the space $G([0, T], E)$ of regulated functions on bounded interval have been studied by several authors (for instance [2,5–7,13,15]). However, there are not any results concerning this operator acting in the space of regulated functions on unbounded interval. The results obtained in this section fill the gap. Let us consider a function $f : \mathbb{R}_+ \times E \to E$. Then, for every function $x : \mathbb{R}_+ \to E$, we can define the function $(F_f x)(t) := f(t, x(t))$, $t \in \mathbb{R}_+$. Operator F_f defined in such a way is called *the superposition* (or Nemytskii) operator generated by the function f (see [1]).

In connection with the space $BG(\mathbb{R}_+, E)$, the natural question appears: what properties must the function f satisfy so that operator F_f maps the space $BG(\mathbb{R}_+, E)$ into itself?

A further part of this section is based on the concepts contained in the paper [15]. Inspired by the result obtained in the paper [13], concerning the space $G([0, T], E)$, we have the following theorem.

Theorem 4.1 *The superposition operator F_f maps $BG(\mathbb{R}_+, E)$ into itself if and only if the function f has the following properties:*

(a) *the limit $\lim_{[0,t) \times E \ni (s,v) \to (t,x)} f(s, v)$ exists for every $(t, x) \in (0, \infty) \times E$,*
(b) *the limit $\lim_{(t,\infty) \times E \ni (s,v) \to (t,x)} f(s, v)$ exists for every $(t, x) \in \mathbb{R}_+ \times E$,*
(c) *$\forall_{r>0} \; \limsup_{t \to \infty} \| f(t, B_E(r)) \| < \infty$.*

Proof As it has been shown in [13], the conditions (a) and (b) are equivalent to the fact that for each regulated function $x : \mathbb{R}_+ \to E$ (unnecessarily bounded) the function $F_f x$ is regulated on $\mathbb{R}_+$, but unnecessarily bounded. If the operator F_f maps $BG(\mathbb{R}_+, E)$ into itself, then the condition (c) has to be satisfied, since in the opposite case the function $x \in BG(\mathbb{R}_+, E)$ such that $F_f x \notin BG(\mathbb{R}_+, E)$ could be constructed. Conversely, if the conditions (a)-(c) hold, then for $x \in BG(\mathbb{R}_+, E)$ the function $F_f x$ is regulated on $\mathbb{R}_+$ and consequently bounded on bounded intervals. Using this fact together with (c) we infer that $F_f x$ is bounded on $\mathbb{R}_+$. □

Using the notation

$$g_t(x) := \lim_{(t,\infty) \times E \ni (s,v) \to (t,x)} f(s, v), \quad t \in \mathbb{R}_+, \; x \in E, \tag{42}$$

the condition (b) from Theorem 4.1 can be described as follows

$$\forall_{x \in E} \forall_{t \in \mathbb{R}_+} \exists_{g_t(x) \in E} \forall_{\varepsilon > 0} \exists_{\delta > 0} \exists_{\tau > 0} \forall_{v \in B_E(x,\delta)} \forall_{s \in (t, t+\tau)} \; \| g_t(x) - f(s, v) \| \le \varepsilon. \tag{43}$$

Similarly, we can describe the condition (a) of Theorem 4.1—we omit the details.

Let us denote by E^E the linear space consisted of all, not necessarily continuous functions $h : E \to E$. In this space we introduce uniform compact convergence, i.e. for a given family $\{h_s\}_{s \in \mathbb{R}_+} \subset E^E$, the uniform compact convergence $\{h_s\}$ to $h \in E^E$ with $s \to t \in \mathbb{R}_+$ means uniform convergence h_s to h on all compact subsets in E when $s \to t$.

Furthermore, for a fixed function $f : \mathbb{R}_+ \times E \to E$ and for each $t \in \mathbb{R}_+$ we shall denote

$$\overline{f}_t(x) := f(t, x), \quad x \in E.$$

Hence, we have $\overline{f}_t \in E^E$ for $t \in \mathbb{R}_+$.

Now we can formulate a theorem that gives (in terms of the function $\overline{f}_t$) necessary conditions for any Banach space E and the sufficient ones, when $\dim E < \infty$ such that the superposition operator F_f maps $BG(\mathbb{R}_+, E)$ into itself.

Theorem 4.2 *Let the superposition operator F_f maps $BG(\mathbb{R}_+, E)$ into itself, where E is a Banach space. Then the family of functions $\{\overline{f}_t\}_{t \in \mathbb{R}_+} \subset E^E$ satisfies the following conditions:*

(a) *The mapping $\mathbb{R}_+ \ni t \mapsto \overline{f}_t \in E^E$ is a regulated function.*
(b) *The following limits of pointwise convergence exist and*

  (b1) *$\lim_{s \to t^+} \overline{f}_s$ is continuous in E for $t \in \mathbb{R}_+$,*
  (b2) *$\lim_{s \to t^-} \overline{f}_s$ is continuous in E for $t \in (0, \infty)$,*

(c) *$\forall_{r > 0} \; \lim \sup_{t \to \infty} \|\overline{f}_t(B_E(r))\| < \infty$.*

Conversely, if additionally E is a finitely dimensional Banach space and conditions (a)*,* (b) *and* (c) *are satisfied, then the superposition operator F_f maps $BG(\mathbb{R}_+, E)$ into itself.*

Proof ($\Rightarrow$) Let us fix $t \in \mathbb{R}_+$. Using condition (b) of Theorem 4.1 and based on notation (42) we have the following equality

$$g_t(x) = \lim_{s \to t^+} \overline{f}_s(x), \quad x \in E.$$

In the sequel we will prove (b1). To do this, let us take an arbitrary $x \in E$ and $\varepsilon > 0$. Keeping in mind (43) we infer that there are $\delta > 0$ and $\tau > 0$ such that

$$\|g_t(x) - \overline{f}_s(v)\| \leq \varepsilon, \quad s \in (t, t + \tau), \; v \in B(x, \delta).$$

Now, letting $s \to t^+$ we have $\|g_t(x) - g_t(v)\| \leq \varepsilon$ which proves continuity of g_t in x and thereby on E. Reasoning similarly, we can prove (b2).

Now, we will prove (a), i.e. that $\overline{f}_s$ tends to g_t in the compact convergence topology on E when $s \to t^+$. Let us fix a non-empty and compact set $K \subset E$ and $\varepsilon > 0$. Then, in view of (43) and already proven continuity of g_t, we have that for each $x \in K$ there exist $\delta_x > 0$, $\tau_x > 0$ such that concurrently

$$\forall_{s \in (t, t + \tau_x)} \; \forall_{v \in B_E(x, \delta_x)} \; \|g_t(x) - \overline{f}_s(v)\| \leq \frac{\varepsilon}{2} \tag{44}$$

and

$$\|g_t(x) - g_t(v)\| \leq \frac{\varepsilon}{2}, \quad v \in B_E(x, \delta_x). \tag{45}$$

From the family $\{B_E(x, \delta_x)\}_{x \in K}$ covering compact set K we choose a finite subcover $\{B_E(x_i, \delta_{x_i})\}_{i=1}^n$. Let us denote $\overline{\tau} := \min\{\tau_{x_i} : i = 1, \ldots, n\}$ and fix arbitrary $v \in K$. Then there exists i such that $v \in B_E(x_i, \delta_{x_i})$. Hence, for $s \in (t, t + \overline{\tau})$ taking into account (44) and (45) we infer that the following estimation

$$\|g_t(v) - \overline{f}_s(v)\| \leq \|g_t(v) - g_t(x_i)\| + \|g_t(x_i) - \overline{f}_s(v)\| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

holds for any $v \in K$. In other words, we have uniformly convergence on K. Similarly, we can prove the existence of the limit $\lim_{s \to t^-} \overline{f}_s$ in the topology of compact convergence.

The condition (c) is a consequence of the Theorem 4.1. Indeed, the condition (c) from Theorem 4.2 is equivalent to (c) from Theorem 4.1.

($\Leftarrow$) Let us assume that $\dim E < \infty$ and fix $t \in \mathbb{R}_+$. Condition (a) implies the existence of the limit $g_t := \lim_{s \to t^+} \overline{f}_s$ which, based on (b) is continuous on E. Let us fix $x \in E$ and $\varepsilon > 0$. Continuity of g_t means that for some $r > 0$ we get

$$\|g_t(x) - g_t(v)\| \leq \frac{\varepsilon}{2}, \quad v \in B_E(x, r). \tag{46}$$

Moreover, (a) implies that for a given compact set $B_E(x, r)$ there is a number $\tau > 0$ such that

$$\|g_t(v) - \overline{f}_s(v)\| \leq \frac{\varepsilon}{2}, \quad v \in B_E(x, r), \ s \in (t, t + \tau).$$

Combining it with (46), for $v \in B_E(x, r), \ s \in (t, t + \tau)$ we derive the following estimate

$$\|g_t(x) - f(s, v)\| \leq \|g_t(x) - g_t(v)\| + \|g_t(v) - f(s, v)\| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

i.e. condition (43) is satisfied and thereby (b) in Theorem 4.1 holds. Similarly, we can prove (a) in Theorem 4.1, so actually F_f maps $BG(\mathbb{R}_+, E)$ into $BG(\mathbb{R}_+, E)$. $\qquad \square$

Corollary 4.3 *If a Banach space E is finitely dimensional then the superposition operator F_f maps $BG(\mathbb{R}_+, E)$ into $BG(\mathbb{R}_+, E)$ if and only if three conditions (a), (b) and (c) in Theorem 4.2 are satisfied.*

Now, we give the compactness criterion for the superposition operator F_f.

Theorem 4.4 *Let $f : \mathbb{R} \times E \rightarrow E$. Assume that E is a separable Banach space and the superposition operator $F_f : BG(\mathbb{R}_+, E) \rightarrow BG(\mathbb{R}_+, E)$ is continuous and compact (i.e. F_f transforms bounded sets in relatively compact sets). Then*

(H1) *there exists a function $g \in BG(\mathbb{R}_+, E)$,*
(H2) *there exists a sequence $\{t_n\} \subset \mathbb{R}_+$,*
(H3) *there exist functions $h_n : E \rightarrow E, n = 1, 2, \ldots$ that are continuous, compact and*

$$\forall_{r>0} \ \lim_{n \to \infty} \|h_n(B_E(r))\| = 0 \tag{47}$$

such that

$$f(t, x) = g(t) + \sum_{n=1}^{\infty} \mathbf{1}_{t_n}(t) h_n(x), \quad t \in \mathbb{R}_+, \ x \in E. \tag{48}$$

Conversely, if the conditions (H1)–(H3) are satisfied and E is a Banach space then the formula (48) gives such a function $f(t, x)$, that operator $F_f : BG(\mathbb{R}_+, E) \rightarrow BG(\mathbb{R}_+, E)$ and it is continuous and compact.

Remark 4.5 Obviously the case when all h_n functions in the previous theorem are equal to θ, that is when $f(t, x) = g(t)$, or only a finite number of them is not equal to θ is also allowed.

The proof of Theorem 4.4 will be preceded by two lemmas. However, in order to make our considerations more transparent, we will give a useful notation. For fixed $x \in BG(\mathbb{R}_+, E)$ we will put

$$\operatorname{supp} x := \{t \in \mathbb{R}_+ : x(t) \neq \theta\}.$$

In contrast to standard definition of a support we do not require the closure.

Lemma 4.6 *If E is a Banach space and $F_f : BG(\mathbb{R}_+, E) \rightarrow BG(\mathbb{R}_+, E)$ is compact, then for each $x, y \in BG(\mathbb{R}_+, E)$ the set $\operatorname{supp}(F_f x - F_f y)$ is at most countable and for each injective sequence $\{t_n\} \subset \mathbb{R}_+$ we have*

$$\lim_{n \to \infty} (f(t_n, x(t_n)) - f(t_n, y(t_n))) = \theta. \tag{49}$$

Proof If the set $\text{supp}(F_f x - F_f y)$ was uncountable or if (49) was not satisfied, then there would exist an injective sequence $\{s_n\} \subset \mathbb{R}_+$ and a number $\varepsilon_0 > 0$ such that

$$\| f(s_n, x(s_n)) - f(s_n, y(s_n)) \| \geq \varepsilon_0, \quad n = 1, 2, \ldots.$$

Let us introduce the sequence of functions $u_n : \mathbb{R}_+ \to E$, $n = 1, 2, \ldots$ defined by the formula

$$u_n(t) := \begin{cases} x(s_n) & \text{for } t = s_n, \\ y(t) & \text{for } t \neq s_n. \end{cases}$$

Obviously $u_n \in BG(\mathbb{R}_+, E)$. For $n \neq m$ we have

$$\begin{aligned} \| F_f u_n - F_f u_m \|_\infty &\geq \| (F_f u_n)(s_n) - (F_f u_m)(s_n) \| \\ &= \| f(s_n, x(s_n)) - f(s_n, y(s_n)) \| \geq \varepsilon_0. \end{aligned}$$

This means that the sequence $\{F_f u_n\}$ is ε_0-separable, hence F_f is not compact, which ends the proof. □

We remind that for arbitrary $u \in E$, the symbol $\widehat{u}$ denotes the function $\widehat{u} : \mathbb{R}_+ \to E$ given by $\widehat{u}(t) \equiv u, t \in \mathbb{R}_+$.

Lemma 4.7 *If E is a separable Banach space and $F_f : BG(\mathbb{R}_+, E) \to BG(\mathbb{R}_+, E)$ is compact and continuous, then there exists a sequence $T = \{t_n\} \subset \mathbb{R}_+$ such that*

$$\forall_{x \in E} \, \text{supp}(F_f \widehat{x} - F_f \widehat{\theta}) \subset T. \tag{50}$$

Proof Let $A = \{a_n : n \in \mathbb{N}\} \subset E$ be a countable dense subset of E. Let us put $T := \cup_{n=1}^\infty \text{supp}(F_f \widehat{a_n} - F_f \widehat{\theta})$. Keeping in mind Lemma 4.6 we conclude that the set T is countable or finite. If there existed $x \in E$ such that (50) did not hold, then there would exist $s \in \mathbb{R}_+$ such that $s \in \text{supp}(F_f \widehat{x} - F_f \widehat{\theta}) \setminus T$. Thus $\| (F_f \widehat{x} - F_f \widehat{\theta})(s) \| = \varepsilon_0$ for some $\varepsilon_0 > 0$ and additionally $(F_f \widehat{a_n} - F_f \widehat{\theta})(s) = \theta$ for $n \in \mathbb{N}$. If we took such a subsequence $\{a_{k_n}\}$ that $a_{k_n} \to x$ in E, we would have

$$\| F_f \widehat{x} - F_f \widehat{a_{k_n}} \|_\infty \geq \| (F_f \widehat{x} - F_f \widehat{\theta})(s) - (F_f \widehat{a_{k_n}} - F_f \widehat{\theta})(s) \| = \varepsilon_0$$

which is in contradiction to the continuity of F_f. □

Proof of Theorem 4.4 ($\Rightarrow$) First let us assume that E is a separable Banach space and $F_f : BG(\mathbb{R}_+, E) \to BG(\mathbb{R}_+, E)$ is compact and continuous. Let $T = \{t_n\}$ be the sequence such as in Lemma 4.7. We define the function $g : \mathbb{R}_+ \to E$ and the sequence of functions $h_n : E \to E$, $n = 1, 2, \ldots$ by the following formulas

$$g(t) := f(t, \theta), \quad t \in \mathbb{R}_+,$$
$$h_n(x) := f(t_n, x) - f(t_n, \theta), \quad x \in E, \ n \in \mathbb{N}.$$

By Lemma 4.7, for each $x \in E$ the mapping $\mathbb{R}_+ \ni t \to f(t, x) - f(t, \theta)$ can be non-zero only on the set T and its formula is given by $\sum_{n=1}^\infty \mathbf{1}_{t_n}(t) h_n(x)$. Therefore

$$f(t, x) = f(t, \theta) + f(t, x) - f(t, \theta) = g(t) + \sum_{n=1}^\infty \mathbf{1}_{t_n}(t) h_n(x). \tag{51}$$

Since F_f is continuous, the functions h_n must be also continuous. Moreover, since F_f is compact, the functions h_n must also be compact. Further, in virtue of Lemma 4.6 we infer that $\lim_{n \to \infty}(f(t_n, x) - f(t_n, \theta)) = \theta$ for $x \in E$, i.e.

$$\lim_{n \to \infty} h_n(x) = \theta, \quad x \in E. \tag{52}$$

We have only (47) left to be proved. Suppose the contrary. Then there would exist $r > 0$, a number $\varepsilon_0 > 0$ and such a subsequence of the sequence $\{h_n\}$, (also denoted by $\{h_n\}$), that

$$\|h_n(B_E(r))\| \geq \varepsilon_0, \quad n = 1, 2, \ldots. \tag{53}$$

Let $k_1 = 1$. We choose $x_1 \in B_E(r)$ such that $\|h_{k_1}(x_1)\| \geq \frac{\varepsilon_0}{2}$. In view of (52) we obtain that there exists $k_2 \in \mathbb{N}$ such that $k_2 > k_1$ and $\|h_i(x_1)\| \leq \frac{\varepsilon_0}{4}$ for $i \geq k_2$. Invoking (53) we conclude that there exists $x_2 \in B_E(r)$ such that $\|h_{k_2}(x_2)\| \geq \frac{\varepsilon_0}{2}$. Further, by (52) we know that there exists $k_3 \in \mathbb{N}$ such that $k_3 > k_2$ and $\|h_i(x_2)\| \leq \frac{\varepsilon_0}{4}$ for $i \geq k_3$. Continuing this reasoning we obtain a strictly increasing sequence $\{k_n\} \subset \mathbb{N}$ and the sequence $\{x_n\} \subset B_E(r)$ such that

$$\|h_{k_n}(x_n)\| \geq \frac{\varepsilon_0}{2}, \quad n = 1, 2, \ldots,$$

$$\|h_i(x_n)\| \leq \frac{\varepsilon_0}{4}, \quad i \geq k_{n+1}, \ n = 1, 2, \ldots.$$

Let $n > m$. Then we have

$$\|F_f \widehat{x_n} - F_f \widehat{x_m}\|_\infty \geq \|(F_f \widehat{x_n})(t_{k_n}) - (F_f \widehat{x_m})(t_{k_n})\|$$
$$= \|h_{k_n}(x_n) - h_{k_n}(x_m)\| \geq \|h_{k_n}(x_n)\| - \|h_{k_n}(x_m)\| \geq \frac{\varepsilon_0}{2} - \frac{\varepsilon_0}{4} = \frac{\varepsilon_0}{4},$$

i.e. $\{F_f \widehat{x_n}\}$ is positively separated and F_f is not relatively compact which contradicts the assumptions.

($\Leftarrow$) Now, let us assume that the conditions (H1)-(H3) hold, E is a Banach space and the function $f(t, x)$ is given by the formula (48). First, we are going to show that $F_f : BG(\mathbb{R}_+, E) \to BG(\mathbb{R}_+, E)$. To do this, let us fix $x \in BG(\mathbb{R}_+, E)$, $t \in \mathbb{R}_+$ and the sequence $\{t'_j\}$ convergent to t from one side, e.g. $t'_j \to t^+$. Since the sequence $\{x(t'_j)\}$ is bounded, then using (47) we get $\lim_{j \to \infty} \sum_{n=1}^\infty \mathbf{1}_{t_n}(t'_j) h_n(x(t'_j)) = \theta$ and in view of (48) we have $(F_f x)(t^+) = \lim_{n \to \infty}(F_f x)(t'_n) = \lim_{n \to \infty} g(t'_n) = g(t^+)$. Similarly we prove that $(F_f x)(t^-) = g(t^-)$ i.e. $F_f x$ is regulated on $\mathbb{R}_+$. Moreover, in virtue of (H1) and (H3) $F_f x$ is bounded and therefore $F_f x \in BG(\mathbb{R}_+, E)$.

In the next step we prove that the operator F_f is compact. Let us fix a bounded sequence $\{x_n\} \subset B_{BG(\mathbb{R}_+, E)}(r)$, where $r > 0$ is a number. Since the operator h_1 is compact, we are able to choose such a subsequence $\{x_{n,1}\}$ of a sequence $\{x_n\}$ that the sequence $\{h_1(x_{n,1}(t_1))\}$ is convergent. Further, the operator h_2 is also compact so we are able to choose a subsequence $\{x_{n,2}\}$ such that the sequence $\{h_2(x_{n,2}(t_2))\}$ is convergent. Moreover, $\{h_1(x_{n,2}(t_1))\}$ is also convergent. Continuing this procedure we have a sequence of sequences $\{x_{n,i}\}_{n=1}^\infty$, $i = 1, 2, \ldots$ for which $\{x_{n,i+1}\}_{n=1}^\infty$ is a subsequence of the sequence $\{x_{n,i}\}_{n=1}^\infty$ and there exists a limit $\lim_{n \to \infty} h_j(x_{n,i}(t_j))$ for $i = 1, 2, \ldots, j = 1, 2, \ldots, i$. Next, using a diagonal method we define the sequence $\{y_n\}$ by $y_n := x_{n,n}$, $n = 1, 2, \ldots$. Obviously $\{y_n\}$ is a subsequence of $\{x_n\}$. Moreover, the limit $\lim_{n \to \infty} h_i(y_n(t_i))$ exists for each $i = 1, 2, \ldots$. This fact together with $(F_f y_n)(t) = g(t)$ for $t \notin T$ implies that the limit $\lim_{n \to \infty}(F_f y_n)(t)$ exists for each $t \in \mathbb{R}_+$. We are able now to introduce the function $z : \mathbb{R}_+ \to E$ defined by the formula $z(t) := \lim_{n \to \infty}(F_f y_n)(t)$, i.e.

$$z(t) = \begin{cases} g(t) & \text{for } t \in \mathbb{R}_+ \setminus T, \\ g(t_i) + \lim_{n \to \infty} h_i(y_n(t_i)) & \text{for } t = t_i, \ i = 1, 2, \ldots. \end{cases}$$

Keeping in mind (47) we get

$$\|z(t_i) - g(t_i)\| = \|\lim_{n \to \infty} h_i(y_n(t_i))\| \leq \|h_i(B_E(r))\| \underset{i \to \infty}{\longrightarrow} 0.$$

This implies that z is regulated on $\mathbb{R}_+$, Moreover, in view of (H1) and (H3) the function z belongs to the space $BG(\mathbb{R}_+, E)$. Using (48) we obtain

$$\|z - F_f y_k\|_\infty = \sup_{t_i \in T} \|\lim_{n \to \infty} h_i(y_n(t_i)) - h_i(y_k(t_i))\|. \tag{54}$$

Let us fix $\varepsilon > 0$. In virtue of (47) we conclude that there exists $m \in \mathbb{N}$ such that for $i > m$ we have $\|h_i(B_E(r))\| \leq \varepsilon$ and therefore $\sup_{i>m} \|\lim_{n \to \infty} h_i(y_n(t_i)) - h_i(y_k(t_i))\| \leq 2\varepsilon$. Since there exists a limit $\lim_{n \to \infty} h_i(y_n(t_i))$, $i = 1, 2, \ldots$, there is $k_0 \in \mathbb{N}$ such that for $k \geq k_0$ and for $i = 1, 2, \ldots, m$ we have

$$\left\| \lim_{n \to \infty} h_i(y_n(t_i)) - h_i(y_k(t_i)) \right\| \leq \varepsilon.$$

Combining the above obtained estimation and (54) we get for $k \geq k_0$ the inequality $\|z - F_f y_k\|_\infty \leq 2\varepsilon$ which proves that $F_f y_k \to z$ in $BG(\mathbb{R}_+, E)$ for $k \to \infty$.

Continuity of the operator F_f is a consequence of condition (H3) and the equalities (47) and (48)—we omit a simple proof of this fact. $\qquad\square$

Corollary 4.8 *Let Banach space E be separable. Then $F_f(BG(\mathbb{R}_+, E)) \subset BG(\mathbb{R}_+, E)$ and the superposition operator $F_f : BG(\mathbb{R}_+, E) \to BG(\mathbb{R}_+, E)$ is continuous and compact if and only if the conditions* (H1)-(H3) *of Theorem* 4.4 *are satisfied and*

$$f(t, x) = g(t) + \sum_{n=1}^{\infty} \mathbf{1}_{t_n}(t) h_n(x), \quad t \in \mathbb{R}_+, \ x \in E,$$

for some function $g \in BG(\mathbb{R}_+, E)$.

5 An application

As an example of application we consider equation

$$x(t) = f(t, x(t)) + \int_0^\infty u(t, s, x(s)) ds, \quad t \geq 0. \tag{55}$$

We will assume that the functions involved in Eq. (55) satisfy the following conditions:

(i) the function $f : \mathbb{R}_+ \times E \to E$ has the following properties:

(i1) f is bounded on $\mathbb{R}_+ \times B_E(r)$ for $r > 0$,

(i2) there is a constant $q \geq 0$ such that

$$\|f(t, x_1) - f(t, x_2)\| \leq q \|x_1 - x_2\|$$

for $t \in \mathbb{R}_+, x_1, x_2 \in E$,

(i3) the family of functions $\{f(\cdot, x) : x \in B_E(r)\}$ is equiregulated on $\mathbb{R}_+$ for all $r > 0$,

(ii) the mapping $u : \mathbb{R}_+ \times \mathbb{R}_+ \times E \to E$ has the following properties:

(ii1) the function $u(t, \cdot, x(\cdot))$ is integrable on $\mathbb{R}_+$ for any $t \in \mathbb{R}_+$ and for all $x \in BG(\mathbb{R}_+, E)$,

(ii2) u is bounded on $\mathbb{R}_+ \times \mathbb{R}_+ \times B_E(r)$ for all $r > 0$,

(ii3) the mapping $u(t, s, x)$ is continuous with respect to the variable x,

(ii4) there exist functions: $\phi \in BG(\mathbb{R}_+, \mathbb{R}_+)$, $g \in L^1(\mathbb{R}_+, \mathbb{R}_+)$ and nondecreasing function $h : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\|u(t_2, s, x) - u(t_1, s, x)\| \leq |\phi(t_2) - \phi(t_1)| g(s) h(\|x\|)$$

for all $t_1, t_2, s \in \mathbb{R}_+, x \in E$,

(ii5) the family of functions $\{u(t, \cdot, x) : t \in \mathbb{R}_+, x \in B_E(r)\}$ is equiregulated on $\mathbb{R}_+$ for all $r > 0$,

(ii6) there is a function $k : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$, integrable with respect to s on $\mathbb{R}_+$ for all $t \in \mathbb{R}_+$, such that

$$\beta_E\big(u(t, s, A)\big) \leq k(t, s)\beta_E(A)$$

for all $t, s \geq 0$, $A \subset E$, A is bounded,

(ii7) there are functions: $k_1 : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$, integrable with respect to s on $\mathbb{R}_+$ for all $t \in \mathbb{R}_+$, and nondecreasing $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\|u(t, s, x)\| \leq k_1(t, s)\psi(\|x\|)$$

for all $t, s \in \mathbb{R}_+, x \in E$, and moreover

$$\lim_{t \to \infty} \int_0^\infty k_1(t, s)ds = 0,$$

(iii) there is a constant $r > 0$ such that

$$\sup \left\{ \left\| f(t, x(t)) + \int_0^\infty u(t, s, x(s))ds \right\| : x \in B_{BG(\mathbb{R}_+, E)}(r), t \in \mathbb{R}_+ \right\} \leq r$$

and

$$q + \sup_{t \in \mathbb{R}_+} \int_0^\infty k(t, s)ds < 1.$$

For our further purposes we will need following fixed point theorem [3].

Theorem 5.1 *Let Q be a nonempty bounded closed convex subset of the Banach space E and let $S : Q \to Q$ be continuous and such that $\mu(SX) \leq q\mu(X)$ for any nonempty subset X of Q, where q is a constant, $q \in [0, 1)$ and μ is a measure of noncompactness. Then S has a fixed point in the set Q.*

Now we formulate our theorem as:

Theorem 5.2 *Under the conditions* (i)–(iii)*, Eq.* (55) *has at least one solution in $BG(\mathbb{R}_+, E)$.*

Proof Let $r > 0$ be as in (iii) and denote $B_r := B_{BG(\mathbb{R}_+, E)}(r)$. For our further purposes we denote by $\overline{U}$ the constant

$$\overline{U} := \sup\{\|u(t, s, x(s))\| : t, s \geq 0, x \in B_r\}.$$

Keeping in mind the assumptions (ii2), we have $\overline{U} < \infty$.

Now we define three operators F_f, U and S for $x \in B_r$:

$$(F_f x)(t) := f(t, x(t)), \qquad (Ux)(t) := \int_0^\infty u(t, s, x(s))ds \ \text{ for } t \geq 0, \qquad Sx := F_f x + Ux.$$

Firstly we show that $F_f x, Ux \in BG(\mathbb{R}_+, E)$. To do this fix $t \in \mathbb{R}_+$. Next assume that $t_1, t_2 \in \mathbb{R}_+$ are such that $t < t_1 < t_2$. Then, in view of (i2) we have

$$\|(F_f x)(t_2) - (F_f x)(t_1)\| \leq q\|x(t_2) - x(t_1)\| + \|f(t_2, x(t_1)) - f(t_1, x(t_1))\|. \quad (56)$$

Using condition (i3) we have that there is limit $\lim_{s \to t^+}(F_f x)(s)$. Similarly we obtain the existence of $\lim_{s \to t^-}(F_f x)(s)$, so $F_f x$ is regulated on $\mathbb{R}_+$. Moreover, in view of (i1), $F_f x$ is bounded and therefore $F_f x \in BG(\mathbb{R}_+, E)$.

By virtue of (ii4) we get

$$\|(Ux)(t_2) - (Ux)(t_1)\| \leq |\phi(t_2) - \phi(t_1)|h(r) \int_0^\infty g(s)ds + (t_2 - t_1)\overline{U}. \quad (57)$$

The condition (ii4) implies that there exists $\lim_{s \to t^+}(Ux)(s)$ (and similarly for $\lim_{s \to t^-}(Ux)(s)$). Then, in view of (ii2), we get $Ux \in BG(\mathbb{R}_+, E)$, and thus by (iii), $S : B_r \to B_r$. From the assumption (i2), (ii3) and Lebesgue's dominated convergence theorem we conclude that the operators F_f and U are continuous, and therefore $S : B_r \to B_r$ is also continuous.

Now we are going to use Theorem 5.1 with the measure γ_c given in (33). Let us take a nonempty set $X \subset B_r$. From (i2), after standard calculation we get

$$\overline{\beta}_\infty(F_f X) \leq q\overline{\beta}_\infty(X), \quad (58)$$
$$c(F_f X) \leq qc(X). \quad (59)$$

Next, fix arbitrarily $T > 0$, $\varepsilon > 0$ and $t \in [0, T)$. Applying (56) we get

$$\omega_T^+(F_f X, t, \varepsilon) \leq q\omega_T^+(X, t, \varepsilon)$$
$$+ \sup\{\|f(t_2, z) - f(t_1, z)\| : t_1, t_2 \in (t, T], |t_2 - t_1| \leq \varepsilon, z \in B_E(r)\}.$$

Then, using (i3) we derive $\omega_T^+(F_f X, t) \leq q\omega_T^+(X, t)$ as $\varepsilon \to 0$, and therefore $\omega_\infty^+(F_f X) \leq q\omega_\infty^+(X)$. Similarly we derive $\omega_\infty^-(F_f X) \leq q\omega_\infty^-(X)$. Thus

$$\max\{\omega_\infty^+(F_f X), \omega_\infty^-(F_f X)\} \leq q \max\{\omega_\infty^+(X), \omega_\infty^-(X)\}.$$

This inequality, together with (58) and (59), imply

$$\gamma_c(F_f X) \leq q\gamma_c(X). \quad (60)$$

Let $X \subset B_r$, $T > 0$, $\varepsilon > 0$, $t \in [0, T)$, $t < t_1 < t_2 \leq T$, $|t_2 - t_1| \leq \varepsilon$. Using (57) we have

$$\omega_T^+(UX, t, \varepsilon) \leq \sup\{|\phi(t_2) - \phi(t_1)| : t < t_1 < t_2 \leq T, |t_2 - t_1| \leq \varepsilon\}$$
$$\cdot h(r) \int_0^\infty g(s)ds + \varepsilon\overline{U}.$$

Keeping in mind that $\phi \in BG(\mathbb{R}_+, \mathbb{R}_+)$ and tending with $\varepsilon \to 0$ we get $\omega_T^+(UX, t) = 0$ and therefore $\omega_\infty^+(UX) = 0$. In similar way we get $\omega_\infty^-(UX) = 0$. Thus

$$\max\{\omega_\infty^+(UX), \omega_\infty^-(UX)\} = 0. \quad (61)$$

Now let $t \in \mathbb{R}_+$ and $x, y \in X$. From (ii7) we have

$$\|(Ux)(t) - (Uy)(t)\| \leq 2\psi(r) \int_0^\infty k_1(t, s) ds.$$

Hence $\limsup_{t \to \infty} \operatorname{diam}(UX)(t) = 0$, i.e.

$$c(UX) = 0. \tag{62}$$

We now introduce an auxiliary operator $U_T : B_r \to BG(\mathbb{R}_+, E)$ for $T > 0$ by formula

$$(U_T x)(t) := \int_0^T u(t, s, x(s)) ds, \quad x \in B_r.$$

Let us fix $X \subset B_r, t \in [0, T]$. Keeping in mind (ii5), Theorem 3.1 [15], (ii6) we have

$$\beta_E\big((U_T X)(t)\big) = \beta_E \left(\int_0^T u(t, s, X(s)) ds \right) \leq \int_0^T \beta_E\big(u(t, s, X(s))\big) ds$$

$$\leq \int_0^T k(t, s) \beta_E(X(s)) ds$$

$$\leq \sup_{t \in \mathbb{R}_+} \int_0^\infty k(t, s) ds \cdot \overline{\beta}_\infty(X). \tag{63}$$

Next, using (ii7) we get for $x \in X, t \in [0, T]$

$$\|(Ux)(t) - (U_T x)(t)\| \leq \psi(r) \int_T^\infty k_1(t, s) ds.$$

Hence we derive

$$d_H\big((UX)(t), (U_T X)(t)\big) \leq \psi(r) \int_T^\infty k_1(t, s) ds \xrightarrow[T \to \infty]{} 0,$$

where $d_H\big((UX)(t), (U_T X)(t)\big)$ is the Hausdorff distance between the sets $(UX)(t)$ and $(U_T X)(t) \subset E$. This implies

$$\beta_E\big((UX)(t)\big) = \lim_{T \to \infty} \beta_E\big((U_T X)(t)\big).$$

Now, taking into account this equality, from (63) we obtain

$$\overline{\beta}_\infty(UX) \leq \sup_{t \in \mathbb{R}_+} \int_0^\infty k(t, s) ds \cdot \overline{\beta}_\infty(X).$$

Linking this estimation with (61) and (62) we get

$$\gamma_c(UX) \leq \sup_{t \in \mathbb{R}_+} \int_0^\infty k(t, s) ds \cdot \gamma_c(X).$$

Thus, joining the above inequality with (60) we infer that

$$\gamma_c(SX) \le \big(q + \sup_{t \in \mathbb{R}_+} \int_0^\infty k(t,s)ds\big)\gamma_c(X),$$

and taking into account Theorem 5.1 we derive that S has at least one fixed point in the ball $B_r \subset BG(\mathbb{R}_+, E)$. □

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