Hermitian Toeplitz determinants of the second and third-order for classes of close-to-star functions

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Abstract

For some subclasses of close-to-star functions the sharp upper and lower bounds of the second and third-order Hermitian Toeplitz determinants are computed.

Keywords Hermitian Toeplitz determinant · Univalent functions · Close-to-star functions · Carathéodory class

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1 Introduction

Given $r > 0$, let $\mathbb{D}_r := \{z \in \mathbb{C} : |z| < r\}$, $\mathbb{T}_r := \{z \in \mathbb{C} : |z| = r\}$, $\mathbb{D} := \mathbb{D}_1$, $\mathbb{T} := \mathbb{T}_1$ and $\overline{\mathbb{D}} := \{z \in \mathbb{C} : |z| \leq 1\}$. Let $\mathcal{H}$ be the class of all analytic functions in $\mathbb{D}$ and $\mathcal{A}$ be its subclass of functions f normalized by $f(0) = 0$ and $f'(0) = 1$, i.e., of the form

$$f(z) = \sum_{n=1}^{\infty} a_n z^n, \quad a_1 := 1, \quad z \in \mathbb{D}. \quad (1)$$

Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ of all univalent functions.

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Given $q, n \in \mathbb{N}$, the Hermitian Toeplitz determinant $T_{q,n}(f)$ of $f \in \mathcal{A}$ of the form (1) is defined by

$$T_{q,n}(f) := \begin{vmatrix} a_n & a_{n+1} & \dots & a_{n+q-1} \\ \bar{a}_{n+1} & a_n & \dots & a_{n+q-2} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{a}_{n+q-1} & \bar{a}_{n+q-2} & \dots & a_n \end{vmatrix},$$

where $\bar{a}_k := \overline{a_k}$.

Recently, Ali, Thomas and Vasudevarao [2] introduced the concept of the symmetric Toeplitz determinant $T_q(n)$ for $f \in \mathcal{A}$ in the following way:

$$T_q(n)[f] := \begin{vmatrix} a_n & a_{n+1} & \dots & a_{n+q-1} \\ a_{n+1} & a_n & \dots & a_{n+q-2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q-2} & \dots & a_n \end{vmatrix}.$$

They found estimates for $T_2(n)$, $T_3(1)$, $T_3(2)$, and $T_2(3)$ over selected subclasses of $\mathcal{A}$.

In recent years many papers have been devoted to the estimation of determinants whose entries are coefficients of functions in the class $\mathcal{A}$ or its subclasses. Hankel matrices i.e., square matrices which have constant entries along the reverse diagonal and the generalized Zalcman functional $J_{m,n}(f) := a_{m+n-1} - a_m a_n$, $m, n \in \mathbb{N}$, are of particular interest (see e.g., [3,4,9–11,18,20,22–25,28,32,34,35,39]).

In [14,21], research was instigated into the study of Hermitian Toeplitz determinants which elements are coefficients of functions in subclasses of $\mathcal{A}$, observing that Hermitian Toeplitz matrices play an important role in functional analysis, applied mathematics as well as in technical sciences. In this paper we continue this research by finding the sharp upper and lower bounds of the second and third-order Toeplitz determinants over subclasses of close-to-star functions.

Let $\mathcal{F}$ be a subclass of $\mathcal{A}$ such that $\mathcal{F}(2) := \{f \in \mathcal{F} : a_2 = 0\}$ is a nonempty subfamily and $A_2(\mathcal{F}) := \max\{|a_2| : f \in \mathcal{F}\}$ exists. Since for $f \in \mathcal{A}$,

$$T_{2,1}(f) = 1 - |a_2|^2,$$

the result below is clear. Equality for the lower bound is attained by a function in $\mathcal{F}$ which is extremal for $A_2(\mathcal{F})$. Each function from $\mathcal{F}(2)$ makes equality for the upper bound.

Theorem 1 *Let $\mathcal{F}$ be a subclass of $\mathcal{A}$ such that $\mathcal{F}(2) \neq \emptyset$ and $A_2(\mathcal{F})$ exists. Then*

$$1 - A_2^2(\mathcal{F}) \leq T_{2,1}(f) \leq 1.$$

Both inequalities are sharp.

Let $\mathcal{S}^*$ be the subclass of $\mathcal{S}$ of starlike functions ([1]), i.e., $f \in \mathcal{S}^*$ if $f \in \mathcal{A}$ and

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > 0, \quad z \in \mathbb{D}.$$

A function $f \in \mathcal{A}$ is called close-to-star if there exist $g \in \mathcal{S}^*$ and $\beta \in \mathbb{R}$ such that

$$\operatorname{Re} \frac{e^{i\beta} f(z)}{g(z)} > 0, \quad z \in \mathbb{D}. \quad (2)$$

The class $\mathcal{CST}$ of all close-to-star functions which was introduced by Reade [36] bear the same relation to the class of close-to-convex functions as the class of starlike functions bear to

the class of convex functions ([36, p. 61]). This relationship is called Alexander type relation ([1]). Namely, $f \in \mathcal{CST}$ if, and only if, a function

$$F(z) := \int_0^z \frac{f(t)}{t} dt, \quad z \in \mathbb{D}, \quad (3)$$

is close-to-convex ([19], [16, Vol. II, p. 3]). The class of close-to-convex functions was defined by Kaplan [19], where the following geometrical interpretation was given: $f \in \mathcal{A}$ is close-to-convex if, and only if, there are no sections of the curve $f(\mathbb{T}_r)$, for every $r \in (0, 1)$, in which tangent vector turns backward through an angle not less than π (cf. [16, Vol. II, p. 4]). An analogous geometrical interpretation for close-to-star functions was given by Reade, namely, $f \in \mathcal{A}$ is close-to-star if, and only if, there are no sections of the curve $f(\mathbb{T}_r)$, for every $r \in (0, 1)$, in which radius vector to the curve $f(\mathbb{T}_r)$ turns backward through an angle not less than π ([36, p. 61]). Recall also that Lewandowski [29,30] showed that the class of close-to-convex functions is identical with the class of linearly accessible functions introduced by Biernacki [5].

Let us mention that the class $\mathcal{CST}$ and its subclasses were examined by various authors (e.g., MacGregor [31], Sakaguchi [38], Causey and Merkes [8]; for further references, see [16, Vol. II, pp. 97-104]).

Given $g \in \mathcal{S}^*$ and $\beta \in \mathbb{R}$, let $\mathcal{CST}_\beta(g)$ be the subclass of $\mathcal{CST}$ of all f satisfying (2). The classes $\mathcal{CST}_0(g_i)$, $i = 1, 2, 3$, where

$$g_1(z) := \frac{z}{1-z^2}, \quad g_2(z) := \frac{z}{(1-z)^2}, \quad g_3(z) := z, \quad z \in \mathbb{D},$$

are particularly interested and were separately studied by various authors. In this paper we deal with $\mathcal{CST}_0(g_1) =: \mathcal{ST}(i)$ and $\mathcal{CST}_0(g_2) =: \mathcal{ST}(1)$ which elements f in view of (2) satisfy the condition

$$\operatorname{Re} \left\{ (1-z^2) \frac{f(z)}{z} \right\} > 0, \quad z \in \mathbb{D}, \quad (4)$$

and

$$\operatorname{Re} \left\{ (1-z)^2 \frac{f(z)}{z} \right\} > 0, \quad z \in \mathbb{D}, \quad (5)$$

respectively. Let us add that every function f in $\mathcal{ST}(i)$ or in $\mathcal{ST}(1)$ which is analytic in the closed disk $\overline{\mathbb{D}}$ is starlike in one direction as defined by Robertson [37] (see also [16, Vol. I, pp. 207-208]), i.e., intersection of $f(\mathbb{D})$ with the real axis is a line segment. Moreover, every F given by (3) with $f \in \mathcal{ST}(i)$ maps univalently $\mathbb{D}$ onto a domain $F(\mathbb{D})$ convex in the direction of the imaginary axis as defined by Robertson [37] (e.g., [16, p. 199]). Every F given by (3) with $f \in \mathcal{ST}(1)$ maps univalently $\mathbb{D}$ onto a domain $F(\mathbb{D})$ called convex in the positive the direction of the real axis, i.e., $\{w + it : t \geq 0\} \subset f(\mathbb{D})$ for every $w \in f(\mathbb{D})$ ([6,12,13,15,26,27]). Let us remark that the condition (4) was generalized in [17] by replacing the expression $1 - z^2$ by $1 - \alpha^2 z^2$ with $\alpha \in [0, 1]$.

In this paper we compute the sharp upper and lower bounds for

$$T_{3,1}(f) = \begin{vmatrix} 1 & a_2 & a_3 \\ \bar{a}_2 & 1 & a_2 \\ \bar{a}_3 & \bar{a}_2 & 1 \end{vmatrix} = 2 \operatorname{Re} (a_2^2 \bar{a}_3) - 2|a_2|^2 - |a_3|^2 + 1 \quad (6)$$

for f in $\mathcal{ST}(i)$ and $\mathcal{ST}(1)$.

Let $\mathcal{P}$ be the class of all $p \in \mathcal{H}$ of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \quad z \in \mathbb{D}, \quad (7)$$

which have a positive real part in $\mathbb{D}$.

In the proof of the main result we will use the following lemma which contains the Carathéodory results (8) and (9), and the well-known formula for c_2 ([7], [33, p. 166] and further remarks in [11]).

Lemma 1 *If $p \in \mathcal{P}$ is of the form (7), then*

$$|c_n| \leq 2, \quad n \in \mathbb{N}. \quad (8)$$

Moreover

$$c_1 = 2\zeta_1 \quad (9)$$

and

$$c_2 = 2\zeta_1^2 + 2(1 - |\zeta_1|^2)\zeta_2 \quad (10)$$

for some $\zeta_i \in \overline{\mathbb{D}}$, $i \in \{1, 2\}$.

For $\zeta_1 \in \mathbb{T}$, there is a unique function $p \in \mathcal{P}$ with c_1 as in (9), namely,

$$p(z) = \frac{1 + \zeta_1 z}{1 - \zeta_1 z}, \quad z \in \mathbb{D}.$$

For $\zeta_1 \in \mathbb{D}$ and $\zeta_2 \in \mathbb{T}$, there is a unique function $p \in \mathcal{P}$ with c_1 and c_2 as in (9) and (10), namely,

$$p(z) = \frac{1 + (\bar{\zeta}_1 \zeta_2 + \zeta_1)z + \zeta_2 z^2}{1 + (\bar{\zeta}_1 \zeta_2 - \zeta_1)z - \zeta_2 z^2}, \quad z \in \mathbb{D}. \quad (11)$$

2 The class $\mathcal{ST}(i)$

Let $f \in \mathcal{ST}(i)$ be the form (1). By (4) there exists $p \in \mathcal{P}$ of the form (7) such that

$$(1 - z^2) \frac{f(z)}{z} = p(z), \quad z \in \mathbb{D}. \quad (12)$$

Substituting the series (1) and (7) into (12), and equating coefficients we obtain

$$a_2 = c_1, \quad a_3 = c_2 + 1. \quad (13)$$

Hence and from (8) it follows that $A_2(\mathcal{ST}(i)) = 2$ with the Koebe function

$$k(z) := \frac{z}{(1 - z)^2} = z + \sum_{n=2}^{\infty} n z^n, \quad z \in \mathbb{D}, \quad (14)$$

being extremal. Since the identity function belongs to $\mathcal{ST}(i)$, by Theorem 1 we deduce

Theorem 2 *If $f \in \mathcal{ST}(i)$, then*

$$-3 \leq T_{2,1}(f) \leq 1.$$

Both inequalities are sharp.

We next compute the upper and lower bounds of $T_{3,1}(f)$.

Theorem 3 *If $f \in \mathcal{ST}(i)$, then*

$$T_{3,1}(f) \leq 8. \quad (15)$$

The inequality is sharp.

Proof By (13) and (8) we see that $|a_2| \leq 2$ and $|a_3| \leq 3$. Since $\operatorname{Re}(a_2^2 \bar{a}_3) \leq |a_2^2 a_3|$, from (6) we obtain

$$T_{3,1}(f) \leq F(|a_2|, |a_3|), \quad (16)$$

where

$$F(x, y) := 2x^2y - 2x^2 - y^2 + 1, \quad (x, y) \in [0, 2] \times [0, 3].$$

Observe that the point $(1, 1)$ is a unique solution in $(0, 2) \times (0, 3)$ of the system of equations

$$\begin{aligned} \frac{\partial F}{\partial x} &= 4x(y - 1) = 0, \\ \frac{\partial F}{\partial y} &= 2(x^2 - y) = 0. \end{aligned}$$

However

$$\frac{\partial^2 F}{\partial x^2}(1, 1) \frac{\partial^2 F}{\partial y^2}(1, 1) - \left(\frac{\partial F}{\partial x \partial y}(1, 1) \right)^2 = -16 < 0,$$

so $(1, 1)$ is a saddle point of F .

On the boundary of $[0, 2] \times [0, 3]$ we have

1. $F(0, y) = 1 - y^2 \leq 1, \quad y \in [0, 3];$
2. $F(2, y) = -7 + 8y - y^2 \leq 8, \quad y \in [0, 3];$
3. $F(x, 0) = 1 - 2x^2 \leq 1, \quad x \in [0, 2];$
4. $F(x, 3) = -8 + 4x^2 \leq 8, \quad x \in [0, 2].$

Therefore the inequality $F(x, y) \leq 8$ holds for all $(x, y) \in [0, 2] \times [0, 3]$, which in view of (16) shows (15).

For the Koebe function (14), $a_2 = 2$ and $a_3 = 3$, which makes equality in (15). $\square$

Theorem 4 *If $f \in \mathcal{ST}(i)$, then*

$$T_{3,1}(f) \geq -9. \quad (17)$$

The inequality is sharp.

Proof Substituting (9) and (10) into (13) we obtain

$$a_2 = 2\zeta_1, \quad a_3 = 1 + 2\zeta_1^2 + 2(1 - |\zeta_1|^2)\zeta_2$$

for some $\zeta_i \in \overline{\mathbb{D}}, i = 1, 2$. Therefore from (6) we have

$$T_{3,1}(f) = 2(\Psi_1 + \Psi_2), \quad (18)$$

where

$$\Psi_1 := -4|\zeta_1|^2 + 6|\zeta_1|^4 - 2(1 - |\zeta_1|^2)^2 |\zeta_2|^2$$

and

$$\Psi_2 := 2\operatorname{Re}(\zeta_1^2) - 2(1 - |\zeta_1|^2)\operatorname{Re}\zeta_2 + 4(1 - |\zeta_1|^2)\operatorname{Re}(\zeta_1^2 \bar{\zeta}_2).$$

(A) If $\zeta_1 = 0$, then

$$\psi_1 = -2|\zeta_2|^2, \quad \psi_2 = -2\operatorname{Re}(\zeta_2),$$

and therefore

$$T_{3,1}(f) = -4(|\zeta_2|^2 + \operatorname{Re}(\zeta_2)) \geq -8. \quad (19)$$

(B) If $\zeta_2 = 0$, then

$$\psi_1 := -4|\zeta_1|^2 + 6|\zeta_1|^4, \quad \psi_2 = 2\operatorname{Re}(\zeta_1^2),$$

and therefore

$$T_{3,1}(f) = 4[3|\zeta_1|^4 - 2|\zeta_1|^2 + \operatorname{Re}(\zeta_1^2)] \geq -3. \quad (20)$$

(C) Suppose that $\zeta_1\zeta_2 \neq 0$. Then $\zeta_1 = re^{i\theta}$ and $\zeta_2 = se^{i\psi}$, where $r, s \in (0, 1]$ and $\theta, \psi \in [0, 2\pi)$. We have

$$\begin{aligned} \psi_2 &= 2r^2 \cos 2\theta - 2s(1-r^2) \cos \psi + 4r^2s(1-r^2) \cos(2\theta - \psi) \\ &= 2r^2 \sin(2\theta + \alpha) \sqrt{1 + 4s^2(1-r^2)^2 + 4s(1-r^2) \cos \psi} - 2s(1-r^2) \cos \psi, \end{aligned}$$

where $\alpha \in \mathbb{R}$ is a quantity satisfying

$$\cos \alpha = \frac{\kappa_1}{\sqrt{\kappa_1^2 + \kappa_2^2}}, \quad \sin \alpha = \frac{\kappa_2}{\sqrt{\kappa_1^2 + \kappa_2^2}} \quad (21)$$

with

$$\kappa_1 = 2s(1-r^2) \sin \psi, \quad \kappa_2 = 1 + 2s(1-r^2) \cos \psi.$$

Since $\sin(2\theta + \alpha) \geq -1$ and $\cos \psi \leq 1$, we have

$$\begin{aligned} -\psi_2 &\leq 2r^2 \sqrt{1 + 4s^2(1-r^2)^2 + 4s(1-r^2) \cos \psi} + 2s(1-r^2) \cos \psi \\ &\leq 2r^2 + 2(1-r^2)(1+2r^2)s. \end{aligned} \quad (22)$$

Therefore by (18) and (22) for $(r, s) \in (0, 1]^2$ we obtain

$$\frac{1}{4}T_{3,1}(f) = \frac{1}{2}(\psi_1 + \psi_2) \geq F(r, s), \quad (23)$$

where

$$F(x, y) := -(1-x^2)[3x^2 + (1+2x^2)y + (1-x^2)y^2], \quad (x, y) \in [0, 1]^2.$$

Since

$$\frac{\partial F}{\partial y} = -(1-x^2)[1+2x^2+2(1-x^2)y] \leq 0, \quad (x, y) \in [0, 1]^2,$$

we conclude that

$$F(x, y) \geq F(x, 1) = -2 - 2x^2 + 4x^4 \geq F(1/2, 1) = -9/4, \quad (x, y) \in [0, 1]^2.$$

Summarizing, the last inequality together with (23), (19), (20) shows (17).

The function

$$f(z) = \frac{z(1+z^2)}{(1-z^2)(1-iz-z^2)}, \quad z \in \mathbb{D},$$

belongs to $\mathcal{ST}(i)$ having $a_2 = i$ and $a_3 = 2$, for which equality in (17) holds. $\square$

3 The class $\mathcal{ST}(1)$

Let $f \in \mathcal{ST}(1)$ be the form (1). By (5) there exists $p \in \mathcal{P}$ of the form (7) such that

$$(1-z)^2 \frac{f(z)}{z} = p(z), \quad z \in \mathbb{D}. \quad (24)$$

Substituting (1) and (7) into (24), and equating coefficients we obtain

$$a_2 = 2 + c_1, \quad a_3 = 3 + 2c_1 + c_2. \quad (25)$$

Hence and from (8) it follows that $A_2(\mathcal{ST}(1)) = 4$ with the extremal function

$$f(z) = \frac{z(1+z)}{(1-z)^3} = \sum_{n=1}^{\infty} n^2 z^n, \quad z \in \mathbb{D}. \quad (26)$$

Note that the function

$$f(z) = \frac{z}{1-z^2}, \quad z \in \mathbb{D},$$

for which $a_2 = 0$, is an element of $\mathcal{ST}(1)$. Thus by Theorem 1 we deduce

Theorem 5 *If $f \in \mathcal{ST}(1)$, then*

$$-15 \leq T_{2,1}(f) \leq 1.$$

Both inequalities are sharp.

We next compute the upper and lower bounds of $T_{3,1}(f)$.

Theorem 6 *If $f \in \mathcal{ST}(1)$, then*

$$T_{3,1}(f) \leq 176. \quad (27)$$

The inequality is sharp.

Proof By (25) and (8) we see that $|a_2| \leq 4$ and $|a_3| \leq 9$. As in the proof of Theorem 3, the inequality (16) holds with

$$F(x, y) := 2x^2y - 2x^2 - y^2 + 1, \quad (x, y) \in [0, 4] \times [0, 9].$$

It is easy to check that F has no relative maximum in $(0, 4) \times (0, 9)$.

On the boundary of $[0, 4] \times [0, 9]$ we have

1. $F(0, y) = 1 - y^2 \leq 1, \quad y \in [0, 9];$
2. $F(4, y) = -31 + 32y - y^2 \leq F(4, 9) = 176, \quad y \in [0, 9];$
3. $F(x, 0) = 1 - 2x^2 \leq 1, \quad x \in [0, 4];$
4. $F(x, 9) = -80 + 16x^2 \leq F(4, 9) = 176, \quad x \in [0, 4].$

Therefore the inequality $F(x, y) \leq 176$ holds for all $(x, y) \in [0, 4] \times [0, 9]$ which in view of (16) shows (27).

For the function (26), $a_2 = 4$ and $a_3 = 9$, which makes equality in (27). $\square$

Theorem 7 *If $f \in \mathcal{ST}(1)$, then*

$$T_{3,1}(f) \geq \frac{-442 + 208\sqrt{3} + 20\sqrt{28 + 10\sqrt{3}} - 21\sqrt{84 + 30\sqrt{3}}}{9(-1 + 3\sqrt{3})} = -5.0829 \dots \quad (28)$$

The inequality is sharp.

Proof Substituting (9) and (10) into (25) we obtain

$$a_2 = 2(1 + \zeta_1), \quad a_3 = 3 + 4\zeta_1 + 2\zeta_1^2 + 2(1 - |\zeta_1|^2)\zeta_2,$$

for some $\zeta_i \in \overline{\mathbb{D}}$, $i = 1, 2$. Therefore from (6) we have

$$T_{3,1}(f) = \Psi_1 + \Psi_2, \quad (29)$$

where

$$\Psi_1 := 8 + 40|\zeta_1|^2 + 12|\zeta_1|^4 - 4(1 - |\zeta_1|^2)^2|\zeta_2|^2 \quad (30)$$

and

$$\begin{aligned} \Psi_2 := & 40 \operatorname{Re} \zeta_1 + 28 \operatorname{Re}(\zeta_1^2) + 48|\zeta_1|^2 \operatorname{Re} \zeta_1 + 16(1 - |\zeta_1|^2) \operatorname{Re}(\zeta_1 \bar{\zeta}_2) \\ & + 4(1 - |\zeta_1|^2) \operatorname{Re} \zeta_2 + 8(1 - |\zeta_1|^2) \operatorname{Re}(\zeta_1^2 \bar{\zeta}_2). \end{aligned}$$

(A) Suppose that $\zeta_1 \zeta_2 \neq 0$. Thus $\zeta_1 = re^{i\theta}$ and $\zeta_2 = se^{i\psi}$, where $r, s \in (0, 1]$ and $\theta, \psi \in [0, 2\pi)$. Then

$$\Psi_2 = 4(\Psi_3 + \Psi_4), \quad (31)$$

where

$$\begin{aligned} \Psi_3 := & 10r \cos \theta + 7r^2 \cos 2\theta + 12r^3 \cos \theta \\ = & -7r^2 + 10r \cos \theta + 12r^3 \cos \theta + 14r^2 \cos^2 \theta \end{aligned} \quad (32)$$

and

$$\Psi_4 := 4rs(1 - r^2) \cos(\theta - \psi) + s(1 - r^2) \cos \psi + 2r^2s(1 - r^2) \cos(2\theta - \psi). \quad (33)$$

Furthermore, we have

$$\Psi_4 = s(1 - r^2) \sqrt{\kappa_1^2 + \kappa_2^2} \sin(\psi + \alpha),$$

where α is the quantity satisfying (21) with

$$\kappa_1 = 4r \sin \theta + 2r^2 \sin 2\theta, \quad \kappa_2 = 1 + 4r \cos \theta + 2r^2 \cos 2\theta. \quad (34)$$

Since $\sin(\psi + \alpha) \geq -1$ and $s \leq 1$, we have

$$\begin{aligned} \Psi_4 & \geq -(1 - r^2) \sqrt{\kappa_1^2 + \kappa_2^2} \\ & = -(1 - r^2) \sqrt{1 + 12r^2 + 4r^4 + 8r \cos \theta + 16r^3 \cos \theta + 8r^2 \cos^2 \theta}. \end{aligned} \quad (35)$$

Therefore from (31), (32) and (33) we obtain

$$\begin{aligned} \frac{1}{4} \Psi_2 = & \Psi_3 + \Psi_4 \geq 10r \cos \theta + 7r^2 \cos 2\theta + 12r^3 \cos \theta \\ & - (1 - r^2) \sqrt{1 + 12r^2 + 4r^4 + 8r \cos \theta + 16r^3 \cos \theta + 8r^2 \cos^2 \theta}. \end{aligned} \quad (36)$$

Taking into account that $|\zeta_2| = s \leq 1$, from (30) we have

$$\Psi_1 \geq 4(1 + 12r^2 + 2r^4). \quad (37)$$

Thus from (29), (36) and (37) it follows that

$$\frac{1}{4} |T_{3,1}(f)| \geq G(r, \cos \theta), \quad r \in (0, 1], \quad \theta \in [0, 2\pi), \quad (38)$$

where

$$G(t, x) := g_1(t, x) - (1 - t^2) \sqrt{g_2(t, x)}$$

with

$$g_1(t, x) := 1 + 5t^2 + 2t^4 + 10tx + 12t^3x + 14t^2x^2$$

and

$$g_2(t, x) := 1 + 12t^2 + 4t^4 + 8tx + 16t^3x + 8t^2x^2$$

for $t \in [0, 1]$ and $x \in [-1, 1]$.

Let $\Omega := [0, 1] \times [-1, 1]$ and

$$\Theta := \frac{-442 + 208\sqrt{3} + 20\sqrt{28 + 10\sqrt{3}} - 21\sqrt{84 + 30\sqrt{3}}}{36(-1 + 3\sqrt{3})} = -1.27073 \dots$$

We now show that

$$\min\{G(t, x) : (t, x) \in \Omega\} = \Theta.$$

(A1) We first find critical points of G in the interior of Ω , i.e., in $(0, 1) \times (-1, 1)$. From the equation

$$0 = \frac{\partial G}{\partial x}(t, x) = \frac{\partial g_1}{\partial x}(t, x) - \frac{1}{2}(1 - t^2)(g_2(t, x))^{-1/2} \frac{\partial g_2}{\partial x}(t, x),$$

it follows that

$$g_2(t, x)^{1/2} = \frac{(1 - t^2) \frac{\partial g_2}{\partial x}(t, x)}{2 \frac{\partial g_1}{\partial x}(t, x)} = \frac{2(1 - t^2)(1 + 2t^2 + 2tx)}{5 + 6t^2 + 14tx}, \quad (39)$$

or equivalently,

$$\Phi = \Phi(t, x) := (5 + 6t^2 + 14tx)^2 g_2(t, x) - 4(1 - t^2)^2 (1 + 2t^2 + 2tx)^2 = 0. \quad (40)$$

Furthermore, note that by (39),

$$0 \leq \frac{(g_2(t, x))^{1/2}}{2(1 - t^2)} = \frac{1 + 2t^2 + 2tx}{5 + 6t^2 + 14tx}. \quad (41)$$

Differentiating G with respect to t yields

$$\frac{\partial G}{\partial t}(t, x) = \frac{\partial g_1}{\partial t}(t, x) + 2t(g_2(t, x))^{1/2} - \frac{1}{2}(1 - t^2)(g_2(t, x))^{-1/2} \frac{\partial g_2}{\partial t}. \quad (42)$$

By (39) and (42) we obtain

$$\begin{aligned} \frac{\partial g_1}{\partial x} \frac{\partial g_2}{\partial x} \frac{\partial G}{\partial t} \Big|_{(t,x)} &= \frac{\partial g_1}{\partial t} \frac{\partial g_1}{\partial x} \frac{\partial g_2}{\partial x} + t(1 - t^2) \left(\frac{\partial g_2}{\partial x} \right)^2 - \left(\frac{\partial g_1}{\partial x} \right)^2 \frac{\partial g_2}{\partial t} \Big|_{(t,x)} \\ &= -128r^3 H(t, x), \end{aligned}$$

where

$$H(t, x) := (3 + 4t^2 + 8tx) (4 + 5t^2 + 2t^4 + 17tx + 8t^3x + 14t^2x^2).$$

Therefore each critical point of G satisfies

$$3 + 4t^2 + 8tx = 0 \quad (43)$$

or

$$4 + 5t^2 + 2t^4 + (17t + 8t^3)x + 14t^2x^2 = 0. \quad (44)$$

(A1.1.) Assume that (43) holds. Then $x = x(t) = -(3 + 4t^2)/(8t)$, and by (40) we see that

$$\Phi\left(t, -\frac{3 + 4t^2}{8t}\right) = -\frac{1}{32}(1 + 4t^2)^2(39 - 104t^2 + 48t^4) = 0$$

occurs when $t = \hat{t}_1$, where

$$\hat{t}_1 = \frac{1}{2}\sqrt{\frac{1}{3}(13 - 2\sqrt{13})} = 0.694556 \dots$$

Thus

$$x = x(\hat{t}_1) = \hat{x}_1 = -\frac{1}{78}(13 + \sqrt{13})\sqrt{39 - 6\sqrt{13}} = -0.887191 \dots$$

However, it can be seen that

$$\frac{1 + 2\hat{t}_1^2 + 2\hat{t}_1\hat{x}_1}{5 + 6\hat{t}_1^2 + 14\hat{t}_1\hat{x}_1} < 0,$$

which means that the inequality (41) is not satisfied for $t = \hat{t}_1$ and $x = \hat{x}_1$. Therefore G has no critical point in the interior of Ω in the case of (43).

(A1.2.) Suppose now that (44) is satisfied. Then the division algorithm gives $\Phi = 0$ when $\tilde{\Phi} = 0$, where

$$\tilde{\Phi}(t, x) := -53 + 38t^2 + 312t^4 + 48t^6 - 576t^8 + t(-150 + 320t^2 + 496t^4 - 960t^6)x.$$

On the other hand, (44) as a quadratic equation of x with $\Delta := 65 - 8t^2 - 48t^4 > 0$, $t \in (0, 1)$, has two roots, namely,

$$x_i = x_i(t) = \frac{-(17 + 8t^2) + (-1)^{i+1}\sqrt{65 - 8t^2 - 48t^4}}{28t}, \quad i = 1, 2. \quad (45)$$

(I) Let $x = x_1$. Then $\tilde{\Phi}(t, x_1(t)) = 0$ is equivalent to

$$\begin{aligned} (75 - 160t^2 - 248t^4 + 480t^6)\sqrt{65 - 8t^2 - 48t^4} \\ = 533 - 1588t^2 - 1128t^4 + 6848t^6 - 4224t^8. \end{aligned} \quad (46)$$

Squaring both sides of (46) we obtain

$$6272\gamma_1(t)\gamma_2^2(t) = 0, \quad (47)$$

where

$$\gamma_1(t) := -13 + 12t^2 + 72t^4, \quad t \in (0, 1),$$

and

$$\gamma_2(t) := 1 + t^2 - 10t^4 + 8t^6 = (1 - t^2)(1 - 2t^2)(1 + 4t^2), \quad t \in (0, 1).$$

By finding all zeros of polynomials γ_1 and γ_2 we state that there are two roots t_1 and t_2 in $(0, 1)$ of the Eq. (47), namely,

$$t_1 := \frac{1}{\sqrt{2}} = 0.707 \dots, \quad t_2 := \frac{1}{2}\sqrt{\sqrt{3} - \frac{1}{3}} = 0.591 \dots \quad (48)$$

For $t = t_1$, by (45) we have $\tilde{x}_1 := x_1(t_1) = -1/\sqrt{2}$. However, $\tilde{\Phi}(t_1, \tilde{x}_1) = 7 \neq 0$. For $t = t_2$, by (45) we have

$$\tilde{x}_2 := x_1(t_2) = \frac{-7 + \sqrt{3}}{2\sqrt{-3 + 9\sqrt{3}}} = -0.742378 \dots \quad (49)$$

It can be verified that $\tilde{\Phi}(t_2, \tilde{x}_2) = 0$ and the inequality (41) holds for $t = t_2$ and $x = \tilde{x}_2$. Therefore G has a critical point at $(t_2, \tilde{x}_2)$.

II. If $x = x_2$, then a similar method as in the case of $x = x_1$ leads us to state that $G(t, x_2(t))$ is not critical point of G in the interior of Ω for every $t \in (0, 1)$.

Therefore, by A, B1 and B2, the function G has the unique critical point at $(t_2, \tilde{x}_2)$. A numerical computation yields

$$\lambda_1 \lambda_3 - \lambda_2^2 = 87.8004 \dots > 0 \quad \text{and} \quad \lambda_1 = 8.99071 \dots > 0,$$

where

$$\lambda_1 := \frac{\partial^2 G}{\partial t^2}(t_2, \tilde{x}_2), \quad \lambda_2 := \frac{\partial^2 G}{\partial t \partial x}(t_2, \tilde{x}_2), \quad \lambda_3 := \frac{\partial^2 G}{\partial x^2}(t_2, \tilde{x}_2).$$

Thus G has a local minimum at $(t_2, \tilde{x}_2)$.

(A2.) It remains to consider G on the boundary of Ω .

1. On the side $t = 0$,

$$G(0, x) \equiv 0 > \Theta, \quad x \in [-1, 1].$$

2. On the side $t = 1$,

$$G(1, x) = 8 + 22x + 14x^2 \geq G(1, -11/14) = -9/14 > \Theta, \quad x \in [-1, 1].$$

3. On the side $x = 1$,

$$G(t, 1) = 2t(3 + 9t + 8t^2 + 2t^3) \geq G(0, 0) = 0 > \Theta, \quad t \in [0, 1].$$

4.. On the side $x = -1$,

$$\varrho(t) := G(t, -1) = 1 - 10t + 19t^2 - 12t^3 + 2t^4 - (1 - t^2) |1 - 4t + 2t^2|.$$

For $t \in [0, (2 - \sqrt{2})/2] =: I_1$ we have $1 - 4t + 2t^2 \geq 0$. Therefore using inequalities

$$t(1 - t) \leq \frac{1}{2}(-1 + \sqrt{2}), \quad 3 - 6t + 2t^2 \leq 3, \quad t \in I_1,$$

we obtain

$$\varrho(t) = -2t(1 - t)(3 - 6t + 2t^2) \geq 3 - 3\sqrt{2} = -1.24264 \dots > \Theta, \quad t \in I_1.$$

For $t \in [(2 - \sqrt{2})/2, 1] =: I_2$ we have $1 - 4t + 2t^2 \leq 0$ and $\varrho(t) = 2 - 14t + 20t^2 - 8t^3$. Since $\varrho'(t) = 0$ occurs only when $t = 1/2 \in I_2$ and $\varrho''(1/2) = 16 > 0$, we obtain

$$\varrho(t) \geq \varrho(1/2) = -1 > \Theta, \quad t \in I_2.$$

(B) If $\zeta_1 = 0$, then

$$\psi_1 = 8 - 4|\zeta_2|^2, \quad \psi_2 = 4 \operatorname{Re}(\zeta_2),$$

and therefore

$$T_{3,1}(f) = 8 - 4|\zeta_2|^2 + 4 \operatorname{Re}(\zeta_2) \geq 0.$$

(C) If $\zeta_2 = 0$ and $\zeta_1 \neq 0$, then $\psi_4 = 0$ in view of (33). Thus the inequality (35) is true and therefore further consideration of Part A of the proof remains valid.

Summarizing, from Parts A-C it follows that the inequality (28) holds.

Now, we discuss sharpness. From (29), (36), (37) and (38) we see that $T_{3,1}(f) = 4\Theta$ holds when

$$r = t_2, \quad \cos \theta = \tilde{x}_2, \quad s = 1, \quad \sin(\psi + \alpha) = -1, \quad (50)$$

where t_2 and $\tilde{x}_2$ are given by (48) and (49), and α is determined by the condition (21) with κ_1 and κ_2 given in (34). Set $\theta = \arccos(\tilde{x}_2)$, so that it satisfies the second condition in (50). Then $\kappa_1 = 0.889045 \dots > 0$ and $\kappa_2 = -0.0684473 \dots < 0$. Thus (21) is satisfied if we take

$$\alpha = -\arccos\left(\frac{\kappa_1}{\sqrt{\kappa_1^2 + \kappa_2^2}}\right) = -0.656114 \dots$$

Thus, if we put

$$\psi = \frac{3\pi}{2} - \alpha = 5.3685 \dots,$$

then ψ satisfies the fourth condition in (50). Now, consider a function $\tilde{p}$ which has the form (11) with $\zeta_1 = t_2 e^{i\theta}$ and $\zeta_2 = e^{i\psi}$. Since $\zeta_1 \in \mathbb{D}$ and $\zeta_2 \in \mathbb{T}$, in view of Lemma 1, we see that $\tilde{p}$ belongs to $\mathcal{P}$. Finally, define

$$\tilde{f}(z) = \frac{z\tilde{p}(z)}{(1-z)^2}, \quad z \in \mathbb{D}.$$

Since $\tilde{f} \in \mathcal{ST}(1)$ and $T_{3,1}(\tilde{f}) = 4\Theta$, the proof of Theorem 7 is completed. $\square$

Remark 1 Let us mention that in [14] it was proved that $-3 \leq T_{2,1}(f) \leq 1$ and $-3 \leq T_{3,1}(f) \leq 8$ for $f \in \mathcal{S}^*$. Note that the result for $T_{2,1}(f)$ in $\mathcal{S}^*$ is identical as for the class $\mathcal{CST}(i)$ shown in this paper. This is due to the fact that Koebe function k belongs to both classes and is extremal. The difference between these two classes is visible on $T_{3,1}(f)$ which for the class $\mathcal{CST}(i)$ is estimated as $-9 \leq T_{3,1}(f) \leq 8$. This is one reason why studying the Hermitian Toeplitz determinants seems sensible. Their lower and upper sharp bounds carry some information about the richness of the class. Classical estimates of coefficients does not necessarily include such a distinction, e.g., both in the class $\mathcal{S}^*$ and in $\mathcal{CST}(i)$ modulus of n -th coefficient is bounded by n with k as the extremal function (e.g., [16, Vol. I, p. 116] for the class $\mathcal{S}^*$; for the class $\mathcal{CST}(i)$ it is an immediate consequence of Theorem 16 of [16]).

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