A note on universal operators between separable Banach spaces

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Abstract

We compare two types of universal operators constructed relatively recently by Cabello Sánchez, and the authors. The first operator Ω acts on the Gurarii space, while the second one P_S has values in a fixed separable Banach space S . We show that if S is the Gurarii space, then both operators are isometric. We also prove that, for a fixed space S , the operator P_S is isometrically unique. Finally, we show that Ω is generic in the sense of a natural infinite game.

Keywords Isometrically universal operator · Gurarii space · Gurarii property · Almost homogeneity

Mathematics Subject Classification 47A05 · 47A65 · 46B04

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1 Universal operators

The purpose of this note is to discuss two constructions of universal operators between separable Banach spaces. We are interested in isometric universality. Namely, an operator U is

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universal if its restrictions to closed subspaces are, up to linear isometries, *all* linear operators of norm not exceeding $\|U\|$. To be more precise, a bounded linear operator $U: V \rightarrow W$ acting between separable Banach spaces is *universal* if for every linear operator $T: X \rightarrow Y$ with X, Y separable and $\|T\| \leq \|U\|$, there exist linear isometric embeddings $i: X \rightarrow V, j: Y \rightarrow W$ such that the diagram

$$\begin{array}{ccc} V & \xrightarrow{U} & W \\ i \uparrow & & \uparrow j \\ X & \xrightarrow{T} & Y \end{array}$$

is commutative, that is, $U \circ i = j \circ T$. Such an operator has been relatively recently constructed by the authors [5]. Another recent work [2], due to Cabello Sánchez and the present authors, contains in particular a construction of a linear operator that is universal in a different sense. Namely, let us say that a bounded linear operator $U: V \rightarrow W$ is *left-universal* (for operators into W) if for every linear operator $T: X \rightarrow W$ with X separable and $\|T\| \leq \|U\|$ there exists a linear isometric embedding $i: X \rightarrow V$ for which the diagram

$$\begin{array}{ccc} V & \xrightarrow{U} & W \\ i \uparrow & \nearrow T & \\ X & & \end{array}$$

is commutative, that is, $U \circ i = T$. Note that if W is separable, then U is right-invertible (i.e. a projection), as one can take the identity of W in place of T . Clearly, if W is isometrically universal in the class of all separable Banach spaces then a left-universal operator with values into W is universal. The left-universal operator U constructed in [2] had been later essentially used (with a suitable space W) for finding an isometrically universal graded Fréchet space [1]. There exist other concepts of universality in operator theory, see the introduction of [5] for more details and references.

Let us note the following simple facts related to universal operators.

Proposition 1.1 *Let $U: V \rightarrow W$ be a bounded linear operator acting between separable Banach spaces.*

- (1) *If U is universal then both V and W are isometrically universal among the class of separable Banach spaces.*
- (2) *Assume U is left-universal. Then $\ker U$ is isometrically universal among the class of separable Banach spaces. Furthermore, U is right-invertible, that is, there exists an isometric embedding $e: W \rightarrow V$ such that $U \circ e = \text{id}_W$.*
- (3) *Assume U is (left-)universal. Then λU is (left-)universal for every $\lambda > 0$.*

Proof (1) Fix a separable Banach space X . Taking the zero operator $T: X \rightarrow 0$, we see that V contains an isometric copy of X . Taking the identity id_X , we see that W contains an isometric copy of X .

(2) The same argument as above, using the zero operators, shows that $\ker U$ is isometrically universal. Taking the identity id_W , we obtain the required isometric embedding $e: W \rightarrow V$.

(3) Assume U is universal, fix $\lambda > 0$ and fix $T: X \rightarrow Y$ with $\|T\| \leq \lambda \|U\|$. Then $\|\lambda^{-1}T\| \leq \|U\|$, therefore there are isometric embeddings $i: X \rightarrow V, j: Y \rightarrow W$ such that $U \circ i = j \circ (\lambda^{-1}T)$. Finally, $(\lambda U) \circ i = j \circ T$. If U is left-universal, the argument is the same, the only difference is that $j = \text{id}_W$. $\square$

By (3) above, we may restrict attention to non-expansive operators. It turns out that there is an easy way of constructing left-universal operators, once we have in hand an isometrically universal space. The argument below was pointed out to us by Przemysław Wojtaszczyk.

Example 1.2 Let V be an isometrically universal Banach space and let W be an arbitrary Banach space. Consider $V \oplus W$ with the maximum norm and let

$$\pi: V \oplus W \rightarrow W$$

be the canonical projection. Given a non-expansive operator $T: X \rightarrow W$ with X separable, choose an isometric embedding $e: X \rightarrow V$ and define $j: X \rightarrow V \oplus W$ by $j(x) = (e(x), T(x))$. Then j is an isometric embedding and $\pi \circ j = T$, showing that π is left-universal. Of course, if additionally W is isometrically universal, then π is a universal operator.

Perhaps the most well known universal Banach space is $C([0, 1])$, the space of all continuous (real or complex) valued functions on the unit interval, endowed with the maximum norm. In view of the example above, there exists a universal operator from $C([0, 1]) \oplus C([0, 1])$ onto $C([0, 1])$. This leads to (at least potentially) many other universal operators, namely:

Proposition 1.3 *Let V, W be isometrically universal separable Banach spaces. Then there exists a universal operator from V into W .*

Proof Fix a universal operator $\pi: E \rightarrow F$ (for instance, $E = C([0, 1]) \oplus C([0, 1])$ and $F = C([0, 1])$) and fix a linear isometric embedding $e: E \rightarrow V$. Using the amalgamation property for linear operators (see, e.g. [4, Lemma 1.2] where it is proved for almost isometries, however exactly the same construction works for arbitrary operators), we find a separable Banach space V' , a linear isometric embedding $e': F \rightarrow V'$, and a non-expansive linear operator $\Omega: V \rightarrow V'$ for which the diagram

$$\begin{array}{ccc} V & \xrightarrow{\Omega} & V' \\ e \uparrow & & \uparrow e' \\ E & \xrightarrow{\pi} & F \end{array}$$

is commutative. As W is isometrically universal, we may additionally assume that $V' = W$, replacing Ω by $i \circ \Omega$ and e' by $i \circ e'$, where i is a fixed isometric embedding of V' into W . It is evident that now Ω is a universal operator, because of the universality of π . $\square$

As a consequence, there exists a universal operator on $C([0, 1])$. We do not know whether there exists a left-universal operator on $C([0, 1])$. The situation changes when replacing $[0, 1]$ with the Cantor set $2^{\mathbb{N}}$. The space $C(2^{\mathbb{N}})$ is linearly isomorphic (but not isometric) to $C([0, 1])$ and it is isometrically universal, too. Furthermore, $C(2^{\mathbb{N}}) \oplus C(2^{\mathbb{N}})$ with the maximum norm is linearly isometric to $C(2^{\mathbb{N}})$, because the disjoint sum of two copies of the Cantor set is homeomorphic to the Cantor set. Thus, Example 1.2 provides a left-universal operator on $C(2^{\mathbb{N}})$.

Another, not so well known, universal Banach space is the *Gurarii space*. This is the unique, up to a linear isometry, separable Banach space $\mathbb{G}$ satisfying the following condition:

For every $\varepsilon > 0$, for every finite-dimensional spaces $X_0 \subseteq X$, for every linear isometric embedding $f_0: X_0 \rightarrow \mathbb{G}$ there exists a linear ε -isometric embedding $f: X \rightarrow \mathbb{G}$ such that $f \upharpoonright X_0 = f_0$.

By an ε -isometric embedding (briefly: ε -embedding) we mean a linear operator f satisfying

$$(1 - \varepsilon)\|x\| \leq \|f(x)\| \leq (1 + \varepsilon)\|x\|$$

for every x in the domain of f . The space $\mathbb{G}$ was constructed by Gurariĭ [6]; its uniqueness was proved by Lusky [9].

The universal operator constructed in [5] has a special property that actually makes it unique, up to linear isometries. Below we quote the precise result.

Theorem 1.4 ([5]) *There exists a non-expansive linear operator $\Omega: \mathbb{G} \rightarrow \mathbb{G}$ with the following property:*

- (G) *Given $\varepsilon > 0$, given a non-expansive operator $T: X \rightarrow Y$ between finite-dimensional spaces, given $X_0 \subseteq X$, $Y_0 \subseteq Y$ and isometric embeddings $i: X_0 \rightarrow \mathbb{G}$, $j: Y_0 \rightarrow \mathbb{G}$ such that $\Omega \circ i = j \circ (T \upharpoonright X_0)$, there exist ε -embeddings $i': X \rightarrow \mathbb{G}$, $j': Y \rightarrow \mathbb{G}$ satisfying*

$$\|i' \upharpoonright X_0 - i\| \leq \varepsilon, \quad \|j' \upharpoonright Y_0 - j\| \leq \varepsilon, \quad \text{and} \quad \|\Omega \circ i' - j' \circ T\| \leq \varepsilon.$$

Furthermore, Ω is a universal operator and property (G) specifies it uniquely, up to a linear isometry.

According to [5], we shall call condition (G) the *Gurariĭ property*. What makes this operator of particular interest is perhaps its *almost homogeneity*:

Theorem 1.5 ([5]) *Given finite-dimensional subspaces X_0, X_1, Y_0, Y_1 of $\mathbb{G}$, given linear isometries $i: X_0 \rightarrow X_1$, $j: Y_0 \rightarrow Y_1$ such that $\Omega \circ i = j \circ \Omega$, for every $\varepsilon > 0$ there exist bijective linear isometries $I: \mathbb{G} \rightarrow \mathbb{G}$, $J: \mathbb{G} \rightarrow \mathbb{G}$ satisfying $\Omega \circ I = J \circ \Omega$ and $\|I \upharpoonright X_0 - i\| < \varepsilon$, $\|J \upharpoonright Y_0 - j\| < \varepsilon$.*

We now describe the left-universal operators constructed in [2]. Fix a separable Banach space $\mathbb{S}$.

Theorem 1.6 ([2, Section 6]) *There exists a non-expansive linear operator $\mathbf{P}_{\mathbb{S}}: V_{\mathbb{S}} \rightarrow \mathbb{S}$ with $V_{\mathbb{S}}$ a separable Banach space, satisfying the following condition:*

- ($\ddagger$) *For every finite-dimensional spaces $X_0 \subseteq X$, for every non-expansive linear operator $T: X \rightarrow \mathbb{S}$, for every linear isometric embedding $e: X_0 \rightarrow V_{\mathbb{S}}$ such that $\mathbf{P}_{\mathbb{S}} \circ e = T \upharpoonright X_0$, for every $\varepsilon > 0$ there exists an ε -embedding $f: X \rightarrow V_{\mathbb{S}}$ satisfying*

$$\|f \upharpoonright X_0 - e\| \leq \varepsilon \quad \text{and} \quad \|\mathbf{P}_{\mathbb{S}} \circ f - T\| \leq \varepsilon.$$

Furthermore, $\mathbf{P}_{\mathbb{S}}$ is left-universal for operators into $\mathbb{S}$.

We shall say that an operator P has the *left-Gurariĭ property* if it satisfies ($\ddagger$) in place of $\mathbf{P}_{\mathbb{S}}$. Of course, unlike the Gurariĭ property, the left-Gurariĭ property involves a parameter $\mathbb{S}$, namely, the common range of the operators. The left-universality of $\mathbf{P}_{\mathbb{S}}$ implies that it is right-invertible, i.e., there is an isometric embedding $J: \mathbb{S} \rightarrow V_{\mathbb{S}}$ such that $\mathbf{P}_{\mathbb{S}} \circ J = \text{id}_{\mathbb{S}}$. The operator $J \circ \mathbf{P}_{\mathbb{S}}$ is a non-expansive projection of $V_{\mathbb{S}}$ onto an isometric copy of $\mathbb{S}$.

Actually, the operator $\mathbf{P}_{\mathbb{S}}$ was constructed in [2] in case where $\mathbb{S}$ had some additional property, needed only for determining the domain of $\mathbf{P}_{\mathbb{S}}$. Moreover, [2] deals with p -Banach spaces, where $p \in (0, 1]$, however $p = 1$ gives exactly the result stated above. Operators $\mathbf{P}_{\mathbb{S}}$ have the following property which can be called *almost left-homogeneity*.

Theorem 1.7 *Given finite-dimensional subspaces X_0, X_1 of $V_{\mathbb{S}}$, a linear isometry $h: X_0 \rightarrow X_1$ such that $\mathbf{P}_{\mathbb{S}} \circ h = \mathbf{P}_{\mathbb{S}} \upharpoonright X_0$, for every $\varepsilon > 0$ there exists a bijective linear isometry $H: V_{\mathbb{S}} \rightarrow \mathbb{S}$ satisfying $\mathbf{P}_{\mathbb{S}} \circ H = \mathbf{P}_{\mathbb{S}}$ and $\|H \upharpoonright X_0 - h\| < \varepsilon$.*

Two operators U, V are *isometric* if there are bijective linear isometries i, j such that $U \circ j = i \circ V$. In this note we present a proof that condition $(\ddagger)$ determines $\mathbf{P}_{\mathbb{S}}$ uniquely, up to linear isometries. The arguments will also provide a proof of Theorem 1.7. Furthermore, we show that $\Omega = \mathbf{P}_{\mathbb{G}}$ and that Ω is a generic operator in the space of all non-expansive operators on the Gurarii space into itself, in the sense of a natural variant of the Banach-Mazur game.

2 Properties of Ω and $\mathbf{P}_{\mathbb{S}}$

Let us recall the following easy fact concerning finite-dimensional normed spaces (cf. [4, Thm. 2.7] or [1, Claim 2.3]). It actually says that the strong operator topology is equivalent to the norm topology in the space of linear operators with a fixed finite-dimensional domain.

Lemma 2.1 *Let A be a vector basis of a finite-dimensional normed space E . For every $\varepsilon > 0$ there exists $\delta > 0$ such that for every Banach space X , for every linear operator $f: E \rightarrow X$ the following implication holds:*

$$\max_{a \in A} \|f(a)\| \leq \delta \implies \|f\| \leq \varepsilon.$$

Proof Fix $M > 0$ satisfying the following condition:

$$(*) \max_{a \in A} |\lambda_a| \leq M \text{ whenever } x = \sum_{a \in A} \lambda_a a \text{ and } \|x\| \leq 1.$$

Such M clearly exists, because of compactness of the unit ball of E . Now, given $\varepsilon > 0$, let $\delta = \varepsilon / (M \cdot |A|)$. Suppose $\max_{a \in A} \|f(a)\| \leq \delta$. Then, given $x = \sum_{a \in A} \lambda_a a$ with $\|x\| \leq 1$, we have

$$\|f(x)\| \leq \sum_{a \in A} |\lambda_a| \cdot \|f(a)\| \leq |A| \cdot M \cdot \delta = \varepsilon.$$

We conclude that $\|f\| \leq \varepsilon$. □

The following result, in the case $\mathbb{S} = \mathbb{G}$, can be found in [1].

Theorem 2.2 *Let $P: V \rightarrow \mathbb{S}$ be a linear operator. The following conditions are equivalent.*

- (a) *P has the left-Gurarii property $(\ddagger)$.*
- (b) *For every finite-dimensional spaces $X_0 \subseteq X$, for every non-expansive linear operator $T: X \rightarrow \mathbb{S}$, for every linear isometric embedding $e: X_0 \rightarrow V$ such that $P \circ e = T \upharpoonright X_0$, for every $\varepsilon > 0$ there exists an ε -embedding $f: X \rightarrow V$ satisfying*

$$f \upharpoonright X_0 = e \quad \text{and} \quad P \circ f = T.$$

Proof Obviously, (b) is stronger than $(\ddagger)$.

Fix $\varepsilon > 0$ and fix a vector basis A of X such that $A_0 = X_0 \cap A$ is a basis of X_0 . We may assume that $\|a\| = 1$ for every $a \in A$. Fix $\delta > 0$ and apply the left-Gurarii property for δ instead of ε . We obtain a δ -embedding $f: X \rightarrow V$ such that $\|f \upharpoonright X_0 - e\| \leq \delta$ and $\|P \circ f - T\| \leq \delta$. Define $f': X \rightarrow V$ by the conditions $f'(a) = e(a)$ for $a \in A_0$ and $f'(a) = f(a)$ for $a \in A \setminus A_0$. Note that $\|f'(a) - f(a)\| \leq \delta$ for every $a \in A$. Thus, if δ is small enough, then by Lemma 2.1, we can obtain that f' is an ε -embedding. Furthermore, $\|P \circ f' - P \circ f\| \leq \varepsilon$ (recall that δ depends on ε and the norm of X only), therefore $\|P \circ f' - T\| \leq \varepsilon + \delta$.

The arguments above show that for every $\varepsilon > 0$ there exists an ε -embedding $f': X \rightarrow V$ extending e and satisfying $\|P \circ f' - T\| \leq \varepsilon$.

Let us apply this property for δ instead of ε , where δ is taken from Lemma 2.1. We obtain a δ -embedding $f: X \rightarrow V$ extending e and satisfying $\|P \circ f - T\| \leq \delta$.

Now recall that P is left-universal (Theorem 1.6), therefore there is an isometric embedding $J: \mathbb{S} \rightarrow V$ such that $P \circ J$ is the identity (take the identity of $\mathbb{S}$ in the definition of left-universality). Given $a \in A \setminus A_0$, the vector

$$w_a = P(f(a)) - T(a)$$

has norm $\leq \delta$. Define $f': X \rightarrow V$ by the conditions $f' \upharpoonright X_0 = e$ and

$$f'(a) = f(a) - J(w_a)$$

for $a \in A \setminus A_0$. Lemma 2.1 implies that f' is an ε -embedding, because $\|f'(a) - f(a)\| = \|w_a\| \leq \delta$ for $a \in A \setminus A_0$. Finally, given $a \in A \setminus A_0$, we have

$$Pf'(a) = Pf(a) - w_a = T(a)$$

and the same obviously holds for $a \in A_0$. Thus $P \circ f' = T$. $\square$

The proof of the next result is just a suitable adaptation of the arguments above, therefore we skip it.

Proposition 2.3 *Let $\Omega: V \rightarrow W$ be a linear operator. The following conditions are equivalent.*

- (a) Ω has the Gurarii property (G).
- (b) Given $\varepsilon > 0$, given a non-expansive operator $T: X \rightarrow Y$ between finite-dimensional spaces, given $X_0 \subseteq X$, $Y_0 \subseteq Y$ and isometric embeddings $i_0: X_0 \rightarrow V$, $j_0: Y_0 \rightarrow W$ such that $\Omega \circ i_0 = j_0 \circ (T \upharpoonright X_0)$, there exist ε -embeddings $i: X \rightarrow V$, $j: Y \rightarrow W$ satisfying

$$i \upharpoonright X_0 = i_0, \quad j \upharpoonright Y_0 = j_0, \quad \text{and} \quad \Omega \circ i = j \circ T.$$

The last result of this section is the key step towards identifying Ω with $\mathbf{P}_{\mathbb{G}}$.

Theorem 2.4 *The operator Ω has the left-Gurarii property (i.e., it satisfies condition $(\ddagger)$ of Theorem 1.6 with $\mathbb{S} = \mathbb{G}$). In particular, it is left-universal.*

Proof Fix a non-expansive linear operator $T: X \rightarrow \mathbb{G}$ with X finite-dimensional, and fix an isometric embedding $e: X_0 \rightarrow \mathbb{G}$, where X_0 is a linear subspace of X and $T \upharpoonright X_0 = \Omega \circ e$. Let $Y_0 = Y = T[X] \subseteq \mathbb{G}$ and consider T as an operator from X to Y . Applying the Gurarii property with $i = e$ and j the inclusion $Y_0 \subseteq \mathbb{G}$, we obtain an ε -embedding $e': X \rightarrow \mathbb{G}$ which is ε -close to e and satisfies $\|\Omega \circ e' - T\| \leq \varepsilon$. This is precisely condition $(\ddagger)$ from Theorem 1.6. $\square$

In order to conclude that $\Omega = \mathbf{P}_{\mathbb{G}}$, it remains to show that $(\ddagger)$ determines the operator uniquely. This is done in the next section.

3 Uniqueness of $\mathbf{P}_{\mathbb{S}}$

Before proving that the left-Gurariĭ property determines the operator uniquely, we quote the following crucial lemma from [3].

Lemma 3.1 *Let $\varepsilon > 0$ and let $f: X \rightarrow Y$ be an ε -embedding, where X, Y are Banach spaces. Let $\pi: X \rightarrow \mathbb{S}$, $\varrho: Y \rightarrow \mathbb{S}$ be non-expansive linear operators such that $\|\varrho \circ f - \pi\| \leq \varepsilon$. Then there exists a norm on $Z = X \oplus Y$ such that the canonical embeddings $i: X \rightarrow Z$, $j: Y \rightarrow Z$ are isometric, $\|j \circ f - i\| \leq \varepsilon$ and the operator $t: Z \rightarrow \mathbb{S}$ defined by $t(x, y) = \pi(x) + \varrho(y)$ is non-expansive.*

Note that the operator t satisfies $t \circ i = \pi$ and $t \circ j = \varrho$. Actually, the norm mentioned in the lemma above does not depend on the operators π, ϱ . It is defined by the following formula:

$$\|(x, y)\| = \inf \left\{ \|x - w\|_X + \|y - f(w)\|_Y + \varepsilon \|w\|_X : w \in X \right\}, \quad (*)$$

where $\|\cdot\|_X, \|\cdot\|_Y$ denote the norm of X and Y , respectively. An easy exercise shows that $(*)$ is the required norm, proving Lemma 3.1.

Theorem 3.2 *Let $\mathbb{S}$ be a separable Banach space and let $\pi: E \rightarrow \mathbb{S}$, $\pi': E' \rightarrow \mathbb{S}$ be non-expansive linear operators, both with the left-Gurariĭ property. If E, E' are separable Banach spaces, then there exists a linear isometry $i: E \rightarrow E'$ such that $\pi = \pi' \circ i$. In particular, π and π' are linearly isometric to $\mathbf{P}_{\mathbb{S}}$.*

Proof It suffices to prove the following

Claim 3.3 *Let $E_0 \subseteq E$ be a finite-dimensional space, $0 < \varepsilon < 1$, let $i_0: E_0 \rightarrow E'$ be an ε -embedding such that $\pi' \circ i_0 = \pi \upharpoonright E_0$. Then for every $v \in E$, $v' \in E'$, for every $\eta > 0$ there exists an η -embedding $i_1: E_1 \rightarrow E'$ such that E_1 is finite-dimensional, $E_0 \subseteq E_1 \subseteq E$ and the following conditions are satisfied:*

- (1) $v \in E_1$ and $\text{dist}(v', i_1[E_1]) < \eta$;
- (2) $\|i_0 - i_1 \upharpoonright E_0\| < \varepsilon + \eta$ and $\pi' \circ i_1 = \pi$.

Using Claim 3.3 together with the separability of E and E' , we can construct a sequence $i_n: E_n \rightarrow E'$ of linear operators such that i_n is a 2^{-n} -embedding, $\bigcup_{n \in \omega} E_n$ is dense in E , $\bigcup_{n \in \omega} i_n[E_n]$ is dense in E' and

$$\|i_n - i_{n+1} \upharpoonright E_n\| \leq 2^{-n} + 2^{-n-1} \quad \text{and} \quad \pi' \circ i_{n+1} = \pi$$

for every $n \in \omega$. It is evident that $\{i_n\}_{n \in \omega}$ converges pointwise to a linear isometry whose completion i is the required bijection from E onto E' satisfying $\pi' \circ i = \pi$. Thus, it remains to prove Claim 3.3.

This will be carried out by making two applications of Lemma 3.1.

Fix $0 < \delta < 1$; more precise estimations for δ will be given later. Let $E'_0 \subseteq E'$ be a finite-dimensional space containing v' and such that $i_0[E_0] \subseteq E'_0$. Applying Lemma 3.1, we obtain linear isometric embeddings $e_1: E_0 \rightarrow W_0$, $f_1: E'_0 \rightarrow W_0$ and a non-expansive operator $t_0: W_0 \rightarrow \mathbb{S}$ such that $t_0 \circ e_1 = \pi \upharpoonright E_0$, $t_0 \circ f_1 = \pi' \upharpoonright E'_0$, and $\|e_1 - f_1 \circ i_0\| \leq \varepsilon$. Knowing that π has the left-Gurariĭ property, by Theorem 2.2 applied to the isometric embedding e_1 , we obtain a δ -embedding $g_1: W_0 \rightarrow E$ such that $g_1 \circ e_1$ is the identity on E_0 and $\pi \circ g_1 = t_0$.

Now note that $g_1 \circ f_1$ is a δ -embedding of E'_0 into a finite-dimensional subspace E_1 of E . Without loss of generality, we may assume that $v \in E_1$. Applying Lemma 3.1 again to

$g_1 \circ f_1$, we obtain linear isometric embeddings $e_2: E_1 \rightarrow W_1$, $f_2: E'_0 \rightarrow W_1$ and a non-expansive linear operator $t_1: W_1 \rightarrow \mathbb{S}$ such that $t_1 \circ e_2 = \pi \upharpoonright E_1$, $t_1 \circ f_2 = \pi' \upharpoonright E'_0$, and $\|e_2 \circ g_1 \circ f_1 - f_2\| \leq \delta$. Knowing that π' has the left-Gurariĭ property and using Theorem 2.2 for the isometric embedding f_2 , we obtain a δ -embedding $g_2: W_1 \rightarrow E'$ such that $g_2 \circ f_2$ is the identity on E'_0 and $\pi' \circ g_2 = t_1$. The configuration is described in the following diagram, where the horizontal arrows are inclusions, the triangle $E_0 E'_0 W_0$ is ε -commutative, and the triangle $E'_0 E_1 W_1$ is δ -commutative.

$$\begin{array}{ccccc}
 E_0 & \xrightarrow{\quad} & E_1 & \xrightarrow{\quad} & E \\
 \downarrow i_0 & \searrow e_1 & \nearrow g_1 & \searrow e_2 & \\
 & W_0 & & W_1 & \\
 E'_0 & \nearrow f_1 & & \nearrow f_2 & \\
 & & & & E' \\
 & & & & \nwarrow g_2
 \end{array}$$

It remains to check that $i_1 := g_2 \circ e_2$ is the required δ -embedding.

First, recall that $v \in E_1$, $v' \in E'_0$ and $v' = g_2(f_2(v'))$. Thus, using the fact that $\|g_2\| \leq 1 + \delta$, we get

$$\begin{aligned}
 \|i_1 g_1 f_1(v') - v'\| &= \|g_2 e_2 g_1 f_1(v') - g_2 f_2(v')\| \\
 &\leq (1 + \delta) \|e_2 g_1 f_1(v') - f_2(v')\| \\
 &\leq (1 + \delta) \delta \|v'\|.
 \end{aligned}$$

Now if $(1 + \delta) \delta \|v'\| < \eta$, then we conclude that $\text{dist}(v', i_1[E_1]) < \eta$, therefore condition (1) is satisfied.

Given $x \in E_1$, note that

$$\pi' i_1(x) = \pi' g_2 e_2(x) = t_1 e_2(x) = \pi(x).$$

Here we have used the fact that $\pi' \circ g_2 = t_1$ and $t_1 \circ e_2 = \pi \upharpoonright E_1$.

Furthermore, given $x \in E_0$, we have

$$\begin{aligned}
 \|i_1(x) - i_0(x)\| &= \|g_2 e_2(x) - i_0(x)\| = \|g_2 e_2 g_1 e_1(x) - g_2 f_2 i_0(x)\| \\
 &\leq (1 + \delta) \|e_2 g_1 e_1(x) - f_2 i_0(x)\|,
 \end{aligned}$$

because $\|g_2\| \leq 1 + \delta$. On the other hand,

$$\begin{aligned}
 \|e_2 g_1 e_1(x) - f_2 i_0(x)\| &\leq \|e_2 g_1 e_1(x) - e_2 g_1 f_1 i_0(x)\| + \|e_2 g_1 f_1 i_0(x) - f_2 i_0(x)\| \\
 &= \|g_1 e_1(x) - g_1 f_1 i_0(x)\| + \|e_2 g_1 f_1 i_0(x) - f_2 i_0(x)\| \\
 &\leq (1 + \delta) \|e_1(x) - f_1 i_0(x)\| + \delta \|i_0(x)\| \\
 &\leq (1 + \delta) \varepsilon \|x\| + \delta(1 + \varepsilon) \|x\| \leq (\varepsilon + 3\delta) \|x\|.
 \end{aligned}$$

Here we have used the following facts: e_2 is an isometric embedding, g_1 is a δ -embedding, i_0 is an ε -embedding, $\|e_2 g_1 f_1 - f_2\| \leq \delta$, $\|e_1 - f_1 i_0\| \leq \varepsilon$ and $\varepsilon < 1$.

Finally, $\|i_1(x) - i_0(x)\| \leq (1 + \delta)(\varepsilon + 3\delta) \|x\| \leq (\varepsilon + 7\delta) \|x\|$. Summarizing, if $(1 + \delta) \delta \|v'\| < \eta$ and $7\delta < \eta$ then conditions (1), (2) are satisfied. This completes the proof. $\square$

Note that if $\mathbb{S}$ is the trivial space, the proof above reduces to the well known uniqueness of the Gurariĭ space, shown by this way in [8]. Furthermore, the arguments above can be applied to $\pi = \pi' = \mathbf{P}_{\mathbb{S}}$ and $i_0 = h$, thus proving Theorem 1.7. Theorems 2.4 and 3.2 yield the following result, announced before.

Corollary 3.4 $\Omega = \mathbf{P}_{\mathbb{G}}$.

In particular, $V_{\mathbb{G}} = \mathbb{G}$. It has been shown in [2] that $V_{\mathbb{S}} = \mathbb{G}$ as long as $\mathbb{S}$ is a (separable) *Lindenstrauss space*, namely, an isometric L_1 predual or (equivalently) a locally almost 1-injective space. Instead of going into details, let us just say that Lindenstrauss spaces are those (separable) Banach spaces that are linearly isometric to a 1-complemented subspace of the Gurarii space. The non-trivial direction was proved by Wojtaszczyk [10]. Thus, since $\mathbf{P}_{\mathbb{S}}$ is a projection, if $V_{\mathbb{S}}$ is linearly isometric to $\mathbb{G}$ then $\mathbb{S}$ is necessarily a Lindenstrauss space.

4 Generic operators

Inspired by the result of [7], let us consider the following infinite game for two players *Eve* and *Adam*. Namely, Eve starts by choosing a non-expansive linear operator $T_0: E_0 \rightarrow F_0$, where E_0, F_0 are finite-dimensional normed spaces. Adam responds by a non-expansive linear operator $T_1: E_1 \rightarrow F_1$, such that $E_1 \supseteq E_0, F_1 \supseteq F_0$ are again finite-dimensional and T_1 extends T_0 . Eve responds by a further non-expansive linear extension $T_2: E_2 \rightarrow F_2$, and so on. So at each stage of the game we have a linear operator between finite-dimensional normed spaces. After infinitely many steps we obtain a chain of non-expansive operators $\{T_n: E_n \rightarrow F_n\}_{n \in \omega}$. Let $T_{\infty}: E_{\infty} \rightarrow F_{\infty}$ denote the completion of its union, namely, E_{∞} is the completion of $\{E_n\}_{n \in \omega}$, F_{∞} is the completion of $\{F_n\}_{n \in \omega}$ and $T_{\infty} \upharpoonright E_n = T_n$ for every $n \in \omega$. So far, we cannot say who wins the game.

Let us say that a (necessarily non-expansive) linear operator $U: X \rightarrow Y$ is *generic* if Adam has a strategy making the operator T_{∞} isometric to U . Recall that operators U, V are *isometric* if there are bijective linear isometries i, j such that $U \circ j = i \circ V$.

Theorem 4.1 *The operator Ω is generic.*

Proof Let us fix a non-expansive linear operator $U: \mathbb{G} \rightarrow \mathbb{G}$ between separable Banach spaces satisfying (G). Adam's strategy can be described as follows.

Fix a countable set $\{v_n: a_n \rightarrow b_n\}_{n \in \mathbb{N}}$ linearly dense in $U: \mathbb{G} \rightarrow \mathbb{G}$. Let $T_0: E_0 \rightarrow F_0$ be the first move of Eve. Adam finds isometric embeddings $i_0: E_0 \rightarrow \mathbb{G}, j_0: F_0 \rightarrow \mathbb{G}$ and finite-dimensional spaces $E_0 \subset E_1, F_0 \subset F_1$ together with isometric embeddings $i_1: E_1 \rightarrow \mathbb{G}, j_1: F_1 \rightarrow \mathbb{G}$ and non-expansive linear operators $T_1: E_1 \rightarrow F_1$ such that T_1 extends $T_0, a_0 \in i_1[E_1], b_0 \in j_1[F_1]$.

Suppose now that $n = 2k > 0$ and $T_n: E_n \rightarrow F_n$ was the last move of Eve. We assume that linear isometric embeddings $i_{n-1}: E_{n-1} \rightarrow \mathbb{G}, j_{n-1}: F_{n-1} \rightarrow \mathbb{G}$ have already been fixed. Using (G) from Theorem 1.4 we choose linear isometric embeddings $i_n: E_n \rightarrow \mathbb{G}, j_n: F_n \rightarrow \mathbb{G}$ such that $i_n \upharpoonright E_{n-1}$ is 2^{-k} -close to $i_{n-1}, j_n \upharpoonright F_{n-1}$ is 2^{-k} -close to j_{n-1} and $U \circ i_n$ is 2^{-k} -close to $j_n \circ T_n$.

Let $\{T_n: E_n \rightarrow F_n\}_{n \in \mathbb{N}}$ be the chain of non-expansive operators between finite-dimensional normed spaces resulting from a fixed play, when Adam was using his strategy. In particular, Adam has recorded sequences $\{T_n: E_n \rightarrow F_n\}_{n \in \mathbb{N}}, \{i_n: E_n \rightarrow \mathbb{G}\}_{n \in \mathbb{N}}, \{j_n: F_n \rightarrow \mathbb{G}\}_{n \in \mathbb{N}}$ of linear isometric embeddings such that $i_{2n+1} \upharpoonright E_{2n-1}$ is 2^{-n} -close to i_{2n-1} and $j_{2n+1} \upharpoonright F_{2n-1}$ is 2^{-n} -close to j_{2n-1} for each $n \in \mathbb{N}$.

Let $T_{\infty}: E_{\infty} \rightarrow F_{\infty}$ denote the completion of those unions, namely, E_{∞} is the completion of $\{E_n\}_{n \in \omega}$, F_{∞} is the completion of $\{F_n\}_{n \in \omega}$ and $T_{\infty} \upharpoonright E_n = T_n$ for every $n \in \omega$. The assumptions that $i_{2n+1}[E_{2n+1}]$ contains all the vectors $a_0, \dots, a_n$ and $j_{2n+1}[F_{2n+1}]$

contains all the vectors $b_0, \dots, b_n$ ensures that both $i_\infty[E_\infty]$, $j_\infty[F_\infty]$ are dense in $\mathbb{G}$, where $i_\infty: E_\infty \rightarrow \mathbb{G}$, $j_\infty: F_\infty \rightarrow \mathbb{G}$ are pointwise limits of $\{i_n\}_{n \in \mathbb{N}}$ and $\{j_n\}_{n \in \mathbb{N}}$, respectively. More precisely, $i_\infty \upharpoonright E_k$ is the pointwise limit of $\{i_n \upharpoonright E_k\}_{n \geq k}$ and $j_\infty \upharpoonright F_k$ is the pointwise limit of $\{j_n \upharpoonright F_k\}_{n \geq k}$ for every $k \in n \in \mathbb{N}$. In particular, both i_∞ and j_∞ are surjective linear isometries.

Finally, $U \circ i_\infty = j_\infty \circ T_\infty$, because $U \circ i_{2k}$ is 2^{-k} -close to $j_{2k} \circ T_{2k}$ for every $k \in \mathbb{N}$. This completes the proof. $\square$

Question 1 Is Ω generic in the space of all non-expansive operators on the Gurarii space? In this case, being “generic” means that the set

$$\{i \circ \Omega \circ j : i, j \text{ bijective linear isometries of } \mathbb{G}\}$$

is residual in the space of all non-expansive operators on $\mathbb{G}$. Here, it is natural to consider the pointwise convergence (i.e., strong operator) topology.

One could also consider a “parametrized” variant of the game above, where the two players build a chain of non-expansive operators from finite-dimensional normed spaces into a fixed Banach space $\mathbb{S}$. If $\mathbb{S}$ is separable then similar arguments as in the proof of Theorem 4.1 show that the second player has a strategy leading to $\mathbf{P}_\mathbb{S}$. Thus, a variant of Question 1 makes sense: Is it true that isometric copies of $\mathbf{P}_\mathbb{S}$ form a residual set in a suitable space of operators?

After concluding that $\Omega = \mathbf{P}_\mathbb{G}$, it seems that the “parametrized” construction of universal projections is better in the sense that it “captures” both the Gurarii space $\mathbb{G}$ (when the range is the trivial space $\{0\}$) and the universal operator Ω (when the range equals $\mathbb{G}$), but also other examples, including projections from the Gurarii space onto any separable Lindenstrauss space (see [10] and [2]).

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