Check for
updates

Lusin and Suslin properties of function spaces

Taras Banakh[1,2] · **Leijie Wang[3]**

Abstract

A topological space is *Suslin* (*Lusin*) if it is a continuous (and bijective) image of a Polish space. For a Tychonoff space X let $C_p(X)$, $C_k(X)$ and $C_{\downarrow\mathsf{F}}(X)$ be the space of continuous real-valued functions on X, endowed with the topology of pointwise convergence, the compact-open topology, and the Fell hypograph topology, respectively. For a metrizable space X we prove the equivalence of the following statements: (1) X is σ-compact, (2) $C_p(X)$ is Suslin, (3) $C_k(X)$ is Suslin, (4) $C_{\downarrow\mathsf{F}}(X)$ is Suslin, (5) $C_p(X)$ is Lusin, (6) $C_k(X)$ is Lusin, (7) $C_{\downarrow\mathsf{F}}(X)$ is Lusin, (8) $C_p(X)$ is F_σ-Lusin, (9) $C_k(X)$ is F_σ-Lusin, (10) $C_{\downarrow\mathsf{F}}(X)$ is $C_{\delta\sigma}$-Lusin. Also we construct an example of a sequential $\aleph_0$-space X with a unique non-isolated point such that the function spaces $C_p(X)$, $C_k(X)$ and $C_{\downarrow\mathsf{F}}(X)$ are non-Suslin.

1 Introduction

In this paper we study the descriptive properties of the spaces $C_p(X)$, $C_k(X)$ and $C_{\downarrow\mathsf{F}}(X)$ of continuous real-valued functions on a Tychonoff space X.

The function space $C_p(X)$ is the space $C(X)$ of continuous real-valued functions on X, endowed with the topology of pointwise convergence. This topology is generated by the

✉ Taras Banakh
  t.o.banakh@gmail.com

  Leijie Wang
  leijie.wan@qq.com

1   Jan Kochanowski University in Kielce, Kielce, Poland

2   Ivan Franko National University in Lviv, Lviv, Ukraine

3   Department of Mathematics, Shantou University, Shantou 515063, Guangdong, People's Republic of China

subbase consisting of the sets

$$\lfloor x; r \rfloor := \{ f \in C(X) : f(x) > r \} \quad \text{and} \quad \lceil x; r \rceil := \{ f \in C(X) : f(x) < r \}$$

where $x \in X$ and r is a real number. The function spaces $C_p(X)$ were thoroughly studied in the monographs [1] and [16–18].

The function space $C_k(X)$ is the space $C(X)$ endowed with the compact-open topology. This topology is generated by the subbase consisting of the sets

$$\lfloor K; r \rfloor := \{ f \in C(X) : \min f[K] > r \} \text{ and } \lceil K; r \rceil := \{ f \in C(X) : \max f[K] < r \}$$

where K is a nonempty compact set in X and r is a real number. The function spaces $C_k(X)$ are also well-studied in General Topology [7,12, §3.4] and Functional Analysis [9].

Our third object of study is the function space $C_{\downarrow \mathsf{F}}(X)$. It is the space $C(X)$ endowed with the *Fell hypograph topology*. This topology is generated by the subbase consisting of the sets

$$\lceil K; r \rceil := \{ f \in C(X) : \max f[K] < r \} \text{ and } \lceil U; r \rfloor := \{ f \in C(X) : \sup f[U] > r \}$$

where K is a nonempty compact subset of X, U is a nonempty open set in X, and r is a real number.

The study of the function spaces $C_{\downarrow \mathsf{F}}(X)$ was initiated by McCoy and Ntantu [11] and continued in [19–25].

The function spaces $C_p(X)$ and $C_k(X)$ are Tychonoff for any topological space X. In contrast, $C_{\downarrow \mathsf{F}}(X)$ is Tychonoff only for weakly locally compact spaces X.

A space X is defined to be *weakly locally compact* if for every compact set K in X there exists an open set U in X such that $K \subseteq \overline{U}$ and $\overline{U}$ is compact. The Fréchet–Urysohn fan (see [1, III.1.8]) is an example of a weakly locally compact space, which is not locally compact.

The following characterization of the (complete) regularity of the function spaces $C_{\downarrow \mathsf{F}}(X)$ can be found in [11, Theorem 3.7].

Theorem 1.1 (McCoy, Ntantu) *For a Tychonoff space X, the following statements are equivalent:*

(1) $C_{\downarrow \mathsf{F}}(X)$ *is a Tychonoff space.*
(2) $C_{\downarrow \mathsf{F}}(X)$ *is a regular space.*
(3) *The space X is weakly locally compact.*

In this paper we shall be interested in descriptive properties of the function spaces, i.e., properties that can be described in terms of Borel sets.

Let us recall that a set A in a topological space X is called

- *Borel* if A belongs to the smallest σ-algebra of sets in X, containing all open subsets in X;
- *constructible* if A belongs to the smallest algebra of sets in X, containing all open sets in X;
- *clopen* if it is both open and closed;
- an F_σ-*set* if A is a countable union of closed sets in X;
- a G_δ-*set* if A is a countable intersection of open sets in X;
- a C_σ-*set* if A is a countable union of constructible sets in X;
- a C_δ-*set* if A is a countable intersection of constructible sets in X;
- an $C_{\sigma\delta}$-*set* (resp. $F_{\sigma\delta}$-*set*) if A is a countable intersection of C_σ-sets (resp. F_σ-sets) in X;

– an $C_{\delta\sigma}$-*set* (resp. $G_{\delta\sigma}$-*set*) if A is a countable union of C_δ-sets (resp. G_δ-sets) in X.

De Morgan's laws imply that each constructible set in a topological space can be written as a finite union $(U_1 \cap F_1) \cup \cdots \cup (U_n \cap F_n)$ of intersections $U_i \cap F_i$ of open and closed sets.

For any set in a topological space we have the following implications.
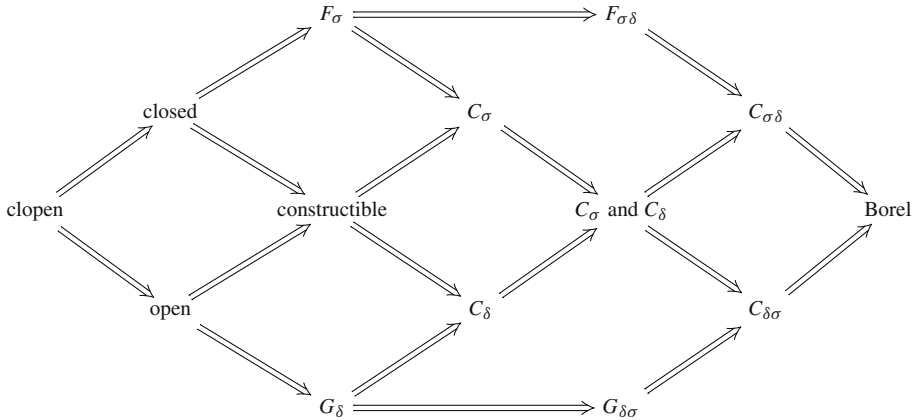
A topological space X is called *perfect* if each open set in X is of type F_σ. For example, each metrizable space is perfect. In perfect spaces the above diagram simplifies to the following form.
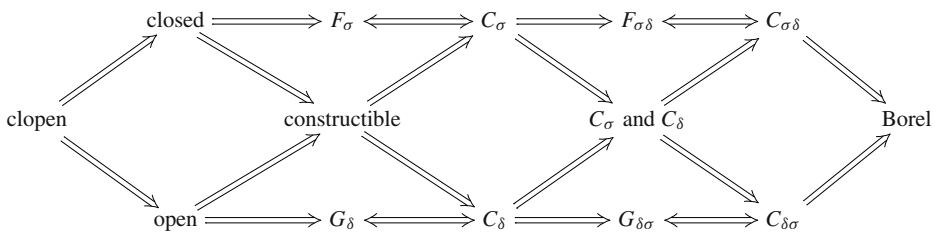
Let Γ be a class of Borel sets in topological spaces. A function $f : X \to Y$ between topological spaces is called Γ-*measurable* if for any open set $U \subseteq Y$ the preimage $f^{-1}[U]$ is Borel of class Γ in X. In particular, a function $f : X \to Y$ is continuous if and only if it is G-measurable for the class G of open sets in topological spaces.

A topological space X is defined to be

– *Polish* if it is homeomorphic to a separable complete metric space;
– *Suslin* if it is the image of a Polish space under a continuous map;
– *Lusin* if it is the image of a Polish space under a continuous bijective map;
– Γ-*Lusin* for a Borel class Γ if it is the image of a Polish space P under a continuous bijective map $f : P \to X$ such that the inverse map $f^{-1} : X \to P$ is Γ-measurable.

In the role of the class Γ we shall consider the (additive) Borel classes G, F_σ, C_σ, $C_{\delta\sigma}$, $G_{\delta\sigma}$, B of open sets, F_σ-sets, C_σ-sets, $C_{\delta\sigma}$-sets, $G_{\delta\sigma}$-sets, Borel sets in topological spaces, respectively.

For any topological space we have the implications:

$$\text{Polish} \Leftrightarrow G\text{-Lusin} \Rightarrow F_\sigma\text{-Lusin} \Rightarrow C_\sigma\text{-Lusin} \Rightarrow C_{\delta\sigma}\text{-Lusin} \Rightarrow \text{Lusin} \Rightarrow \text{Suslin}.$$

For regular spaces some implications in this chain turn into equivalences (see Theorem 3.4):

$$F_\sigma\text{-Lusin} \Leftrightarrow C_\sigma\text{-Lusin} \Rightarrow G_{\delta\sigma}\text{-Lusin} \Leftrightarrow C_{\delta\sigma}\text{-Lusin} \Rightarrow B\text{-Lusin} \Leftrightarrow \text{Lusin} \Rightarrow \text{Suslin}.$$

By Lusin–Suslin Theorem [10, 15.1], a subspace X of a Polish space P is Lusin if and only if X is a Borel subset of P. By the famous result of Suslin [10, 14.2], each uncountable Polish space contains a Suslin subset, which is not Borel and hence is not Lusin. In the class of metrizable spaces, Lusin and Suslin spaces were defined by Bourbaki in [5].

It is well-known [7, 4.3.26] that each Polish (= G-Lusin) space X is a G_δ-set in any Tychonoff space containing X as a subspace. In Theorem 3.1 we shall prove that each F_σ-Lusin space X is a $C_{\sigma\delta}$-set in each Hausdorff space containing X as a subspace.

Let us observe that each Suslin space has a countable network, being a continuous image of a Polish space (which has a countable base).

We recall that a family $\mathcal{N}$ of subsets of a topological space X is called

– a *network* if for every open set $U \subseteq X$ and point $x \in U$, there is a set $N \in \mathcal{N}$ such that $x \in N \subseteq U$;
– a *k-network* if for every open set $U \subseteq X$ and compact subset $K \subseteq U$, there is a finite subfamily $\mathcal{F} \subseteq \mathcal{N}$ such that $K \subseteq \bigcup \mathcal{F} \subseteq U$.

A topological space X is called

– *cosmic* if X is regular and has a countable network;
– an $\aleph_0$-*space* if X is regular and has a countable k-network.

For any topological space we have the following implications (see [8, §4]):

$$\text{metrizable separable space} \Rightarrow \aleph_0\text{-space} \Rightarrow \text{cosmic space}.$$

Function spaces $C_{\downarrow\mathsf{F}}(X)$ and $C_k(X)$ possessing countable networks were characterized in [11, Theorem 3.7 and 4.5] and [13] (see also [12, §4.1]).

Theorem 1.2 (McCoy, Ntantu, Michael) *For a Tychonoff space X the following conditions are equivalent:*

(1) *the function space $C_{\downarrow\mathsf{F}}(X)$ has a countable network;*
(2) *the function space $C_k(X)$ is cosmic;*
(3) *the function space $C_k(X)$ is an $\aleph_0$-space;*
(4) *X is an $\aleph_0$-space.*

The following characterization of cosmic spaces $C_p(X)$ is well-known and can be found in [12, 4.1.3] or [1, I.1.3].

Theorem 1.3 *A Tychonoff space X is cosmic if and only if $C_p(X)$ is cosmic.*

The following fundamental result is due to Calbrix [6] (see also Theorem 9.7 in [9, p.208]).

Theorem 1.4 (Calbrix) *If for a Tychonoff space X the function space $C_p(X)$ is Suslin, then X is σ-compact.*

For an Ascoli space X the Suslin property of the function space $C_k(X)$ can be characterized in terms of $\mathbb{R}$-universal ω^ω-based uniformities.

Let us recall that a topological space X is called

– *Fréchet–Urysohn* if for each set $A \subseteq X$ and point $a \in \bar{A}$ there exists a sequence $\{a_n\}_{n\in\omega} \subseteq A$ that converges to a;

– *sequential* if a subset $A \subseteq X$ is closed in X if and only if A contains the limits of all
  sequences $\{a_n\}_{n \in \omega} \subseteq A$ that converge in X;
– a *k-space* if a subset $A \subseteq X$ is closed in X if and only if for any compact subset $K \subseteq X$
  the intersection $A \cap K$ is closed in K;
– a $k_{\mathbb{R}}$*-space* if a function $f : X \to \mathbb{R}$ is continuous if and only if for every compact subset
  $K \subseteq X$ the restriction $f \restriction K$ is continuous;
– *Ascoli* if the canonical map $\delta : X \to C_k(C_k(X))$, $\delta : x \mapsto \delta_x$, is continuous.
  The canonical map assigns to each point x the Dirac functional $\delta_x : C_k(X) \to \mathbb{R}$,
  $\delta_x : f \mapsto f(x)$.

By [15], each Tychonoff $k_{\mathbb{R}}$-space is Ascoli. Therefore, for any Tychonoff space we obtain
the following implications:

first-countable $\Rightarrow$ Fréchet-Urysohn $\Rightarrow$ sequential $\Rightarrow k$-space $\Rightarrow k_{\mathbb{R}}$-space $\Rightarrow$ Ascoli.

None of these implications can be reversed, see Examples 1.6.18, 1.6.19 in [7,14], and [4,
6.7].
   Next, we recall some information on ω^{ω}-based uniformities. Here we consider ω^{ω} as a
partially ordered space endowed with the partial order $\leq$ defined by $\alpha \leq \beta$ iff $\alpha(n) \leq \beta(n)$
for all $n \in \omega$.
   A uniformity $\mathcal{U}$ on a set X is called ω^{ω}-*based* if it has a base $(U_{\alpha})_{\alpha \in \omega^{\omega}}$ such that $U_{\beta} \subseteq U_{\alpha}$
for any $\alpha \leq \beta$ in ω^{ω}. For example, any metrizable uniformity is ω^{ω}-based.
   A uniformity $\mathcal{U}$ on a topological space X is called $\mathbb{R}$-*universal* if it generates the topology
of X and every continuous function $f : X \to \mathbb{R}$ is uniformly continuous in the uniformity
$\mathcal{U}$.
   A topological space X is called *universally* ω^{ω}-*based* if its universal uniformity is
ω^{ω}-based. The universal uniformity of X is generated by the family of all continuous pseu-
dometrics on X.

Theorem 1.5 *For an* (*Ascoli*) *Tychonoff space X the function space $C_k(X)$ is Suslin if* (*and
only if*) *X is separable and has an $\mathbb{R}$-universal ω^{ω}-based uniformity.*

   This theorem will be proved in Sect. 2. Now we discuss the descriptive properties of
function spaces on Γ-quotient spaces.
   We say that a topological space X is a *quotient* of a topological space M if there exists a
surjective quotient map $f : M \to X$. The quotient property of f means that a subset $U \subseteq X$
is open if and only if its preimage $f^{-1}[U]$ in open in M.
   A topological space X is called Γ-*quotient* for a Borel class Γ if X is a quotient of some
space of class Γ in a compact metrizable space.
   In particular, a *G-quotient* space is a quotient of a locally compact Polish space and an
F_{σ}-*quotient* space is a quotient of a σ-compact metrizable space.
   We recall that a topological space is *Lashnev* if it is the image of a metrizable space under a
continuous closed map. It is known (see [8, 11.3] or [2, §8.2]) that separable Lashnev spaces
are Fréchet–Urysohn $\aleph_0$-spaces.
   The following characterization is proved in [2, 7.8.10 and 8.3.1].

Proposition 1.6 *A* (*separable Lashnev*) *Tychonoff space X is universally ω^{ω}-based if* (*and
only if*) *X is F_{σ}-quotient.*

   We recall that a Tychonoff space is universally ω^{ω}-based if its universal uniformity of X
has an ω^{ω}-base.

Therefore, for any Tychonoff space X we have the implications:

$$G\text{-quotient} \Rightarrow F_\sigma\text{-quotient} \Rightarrow \text{universally } \omega^\omega\text{-based,}$$

where the last implication can be reversed for separable Lashnev spaces.

The following theorem is the main result of this paper.

Theorem 1.7 *For a Tychonoff space X consider the following statements:*

  (1) *X is G-quotient;*
  (2) *$C_k(X)$ is Polish;*
  (3) *$C_{\downarrow F}(X)$ is C_σ-Lusin;*
  (4) *X is F_σ-quotient;*
  (5) *$C_p(X)$, $C_k(X)$ are F_σ-Lusin and $C_{\downarrow F}(X)$ is $C_{\delta\sigma}$-Lusin;*
  (6) *$C_p(X)$ is Lusin and X is an $\aleph_0$-space;*
  (7) *$C_k(X)$ is Lusin;*
  (8) *$C_{\downarrow F}(X)$ is Lusin;*
  (9) *$C_p(X)$ is Suslin and X is an $\aleph_0$-space;*
 (10) *$C_k(X)$ is Suslin;*
 (11) *$C_{\downarrow F}(X)$ is Suslin;*
 (12) *X is σ-compact.*

Then (3) $\Leftarrow$ (2) $\Leftrightarrow$ (1) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (6) $\Leftrightarrow$ (7) $\Leftrightarrow$ (8) $\Rightarrow$ (9) $\Leftrightarrow$ (10) $\Leftrightarrow$ (11) $\Rightarrow$ (12).

Theorem 1.7 will be proved in Sect. 6 after some preliminary work done in Sects. 3, 4 and 5. This theorem implies the following characterization.
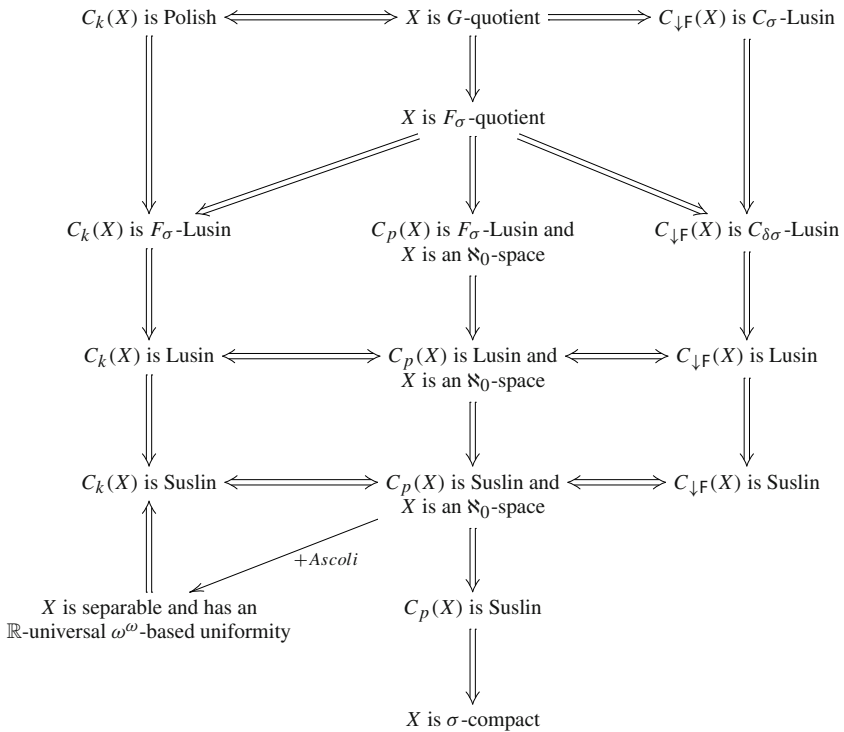
Corollary 1.8 *For a metrizable space X the following statements are equivalent:*

  (1) *X is σ-compact;*
  (2) *$C_p(X)$ is F_σ-Lusin;*
  (3) *$C_k(X)$ is F_σ-Lusin;*
  (4) *$C_{\downarrow F}(X)$ is $C_{\delta\sigma}$-Lusin;*
  (5) *$C_p(X)$ is Lusin;*
  (6) *$C_k(X)$ is Lusin;*
  (7) *$C_{\downarrow F}(X)$ is Lusin;*
  (8) *$C_p(X)$ is Suslin;*
  (9) *$C_k(X)$ is Suslin;*
 (10) *$C_{\downarrow F}(X)$ is Suslin.*

Theorem 1.7(3), Corollary 1.8(3) (and Theorem 1.1) suggest the following open problem.

Problem 1.9 *Is the function space $C_{\downarrow F}(X)$ F_σ-Lusin for any metrizable σ-compact (weakly locally compact) space X?*

The implications of Theorems 1.7, 1.4 and 1.5 are represented in the following diagram holding for any Tychonoff space X.

$$C_k(X) \text{ is Polish} \Longleftrightarrow X \text{ is } G\text{-quotient} \Longrightarrow C_{\downarrow\mathsf{F}}(X) \text{ is } C_\sigma\text{-Lusin}$$

$$X \text{ is } F_\sigma\text{-quotient}$$

$$C_k(X) \text{ is } F_\sigma\text{-Lusin} \qquad C_p(X) \text{ is } F_\sigma\text{-Lusin and} \qquad C_{\downarrow\mathsf{F}}(X) \text{ is } C_{\delta\sigma}\text{-Lusin}$$
$$X \text{ is an } \aleph_0\text{-space}$$

$$C_k(X) \text{ is Lusin} \Longleftrightarrow C_p(X) \text{ is Lusin and} \Longleftrightarrow C_{\downarrow\mathsf{F}}(X) \text{ is Lusin}$$
$$X \text{ is an } \aleph_0\text{-space}$$

$$C_k(X) \text{ is Suslin} \Longleftrightarrow C_p(X) \text{ is Suslin and} \Longleftrightarrow C_{\downarrow\mathsf{F}}(X) \text{ is Suslin}$$
$$X \text{ is an } \aleph_0\text{-space}$$

$$+Ascoli$$

$$X \text{ is separable and has an} \qquad\qquad C_p(X) \text{ is Suslin}$$
$$\mathbb{R}\text{-universal } \omega^\omega\text{-based uniformity}$$

$$X \text{ is } \sigma\text{-compact}$$

The following proposition shows that the last implication in the diagram (established by Calbrix's Theorem 1.4) cannot be reversed even for countable Lashnev (and hence $\aleph_0$-spaces) with a unique non-isolated point. This proposition also implies that Theorem 5.7.4 in [12] is incorrect (that theorem claims that for any sequential σ-compact $\aleph_0$-space X the function space $C_k(X)$ is Suslin).

Proposition 1.10 *Let $X = M/A$ be the quotient space of a metrizable space M by a closed nowhere dense subset $A \subset M$. If the function space $C_k(X)$ is Suslin, then the space M is σ-compact and hence X is an F_σ-quotient space.*

Proof The quotient space $X = M/A$ of the metrizable space M is a sequential Tychonoff space and hence Ascoli. If the function space $C_k(X)$ is Suslin, then X cosmic by Theorem 1.3, and by Theorem 1.5, the topology of X is generated by some ω^ω-based uniformity. By Corollary 8.2.3 [2], the set A is σ-compact. By Theorem 1.4, the cosmic space X is σ-compact and so is its open subspace $X \setminus \{A\}$, which is homeomorphic to $M \setminus A$. Then $M = A \cup (M \setminus A)$ is σ-compact and hence X is F_σ-quotient.   □

Example 1.11 Let $\omega^{<\omega} = \bigcup_{n \in \omega} \omega^n$ be the family of all functions $x : n \to \omega$ defined on finite ordinals $n \in \omega$. Let $\omega^{\leq\omega} = \omega^{<\omega} \cup \omega^\omega$. For any function $x \in \omega^{\leq\omega}$ defined on an ordinal $n \leq \omega$ and any ordinal $k \leq \omega$ denote by $x{\restriction}k$ the restriction of x to the ordinal $k \cap n = \min\{n, k\}$. The set $\omega^{\leq\omega}$ carries a natural partial order $\leq$ defined by $x \leq y$ iff there exists an ordinal $n \leq \omega$ such that $x = y{\restriction}n$. The space $\omega^{\leq\omega}$ is endowed with the topology τ generated by

the countable base consisting of the sets $\uparrow x = \{y \in \omega^{\leq\omega} : x \leq y\}$ where $x \in \omega^{<\omega}$. It is easy to see that $(\omega^{\leq\omega}, \tau)$ is a Polish space, $\omega^{<\omega}$ is a dense discrete subspace in $\omega^{\leq\omega}$ and ω^ω is a closed nowhere dense subset in $\omega^{\leq\omega}$. Consider the quotient space $X = \omega^{\leq\omega}/\omega^\omega$ and observe that X is a countable sequential $\aleph_0$-space with a unique non-isolated point. Since the space $\omega^{\leq\omega}$ is not σ-compact, the function spaces $C_p(X)$, $C_k(X)$ and $C_{\downarrow F}(X)$ are not Suslin according to Proposition 1.10. In Sect. 7 we shall present an alternative self-contained proof of this fact.

Observe that the space $X = M/A$ in Proposition 1.10 is Lashnev, i.e., the image of a metrizable space under a closed continuous map.

Problem 1.12 *Assume that a Tychonoff space X is Lashnev and its function space $C_k(X)$ is Suslin. Is X an F_σ-quotient space?*

2 The Suslin property of the function spaces on Ascoli spaces

In this section we shall prove Theorem 1.5. To prove the "if" part of this theorem, assume that a Tychonoff space X is separable and has an $\mathbb{R}$-universal ω^ω-based uniformity. By Theorem 7.5.1(18) of [2], the function space $C_k(X)$ is Suslin.

To prove the "only if" part of Theorem 1.5, assume that the space X is Ascoli and the function space $C_k(X)$ is Suslin. By Theorem 1.2, the space X is separable. By the definition of an Ascoli space, the canonical map $\delta : X \to C_k(C_k(X))$ is continuous. Since the space $C_k(X)$ is Suslin, there exists a continuous surjective map $\xi : \omega^{\mathbb{N}} \to C_k(X)$. Here $\mathbb{N} = \omega \setminus \{0\}$. For any $\alpha \in \omega^{\mathbb{N}}$ the subspace $\downarrow\alpha = \{\beta \in \omega^{\mathbb{N}} : \beta \leq \alpha\}$ of $\omega^{\mathbb{N}}$ is compact and so is its continuous image $K_\alpha = \xi(\downarrow\alpha) \subseteq C_k(X)$. Observe that $K_\alpha \subseteq K_\beta$ for any $\alpha \leq \beta$ in ω^ω.

For every $\alpha \in \omega^\omega$ consider the entourage

$$U_\alpha = \left\{ (x, y) \in X \times X : \sup_{f \in K_\alpha \restriction \mathbb{N}} |f(x) - f(y)| < 2^{-\alpha(0)} \right\}$$

of the diagonal in $X \times X$. The continuity of the map $\delta : X \to C_k(C_k(X))$ implies that U_α is an open neighborhood of the diagonal in $X \times X$. It is easy to see that $(U_\alpha)_{\alpha \in \omega^\omega}$ is an ω^ω-base of some uniformity $\mathcal{U}$ on X. To see that this uniformity in $\mathbb{R}$-universal, we need to show that every function $f \in C(X)$ is uniformly continuous with respect to the uniformity $\mathcal{U}$. Given any $\varepsilon > 0$, find $\alpha \in \omega^\omega$ such that $2^{-\alpha(0)} < \varepsilon$ and $f = \xi(\alpha \restriction \mathbb{N})$. Then for any pair $(x, y) \in U_\alpha \in \mathcal{U}$, we get

$$|f(x) - f(y)| \leq \sup_{g \in K_\alpha \restriction \mathbb{N}} |g(x) - g(y)| < 2^{-\alpha(0)} < \varepsilon,$$

which means that f is uniformly continuous.

3 Some results on Lusin and Suslin spaces

Theorem 3.1 *Each F_σ-Lusin subspace X of a Hausdorff topological space Y is a $C_{\sigma\delta}$-set in Y. More precisely, $X = A \cap B$ for some G_δ-set $A \subseteq Y$ and some $F_{\sigma\delta}$-set $B \subseteq Y$.*

Proof Write X as the image of a Polish space P under a continuous bijective map $f : P \to X$ with F_σ-measurable inverse $f^{-1} : X \to P$. Fix a complete metric d that generates the topology of the Polish space X.

For every $n \in \omega$ fix a countable open cover $\mathcal{U}_n$ of X by sets of d-diameter $< 2^{-n}$. By the choice of the map f, for every $U \in \mathcal{U}_n$ the image $f[U]$ is an F_σ-set in X. Consequently, $f[U] = \bigcup \mathcal{F}_{n,U}$ for some countable family $\mathcal{F}_{n,U}$ of closed sets in X. Let $\mathcal{F}_n = \bigcup_{U \in \mathcal{U}_n} \mathcal{F}_{n,U}$ and $\mathcal{F} = \bigcup_{n \in \omega} \mathcal{F}_n$. Therefore, $\mathcal{F}$ is a countable family of closed sets in X. It follows that each set $F \in \mathcal{F}$ is equal to the intersection $X \cap \bar{F}$ of X and the closure $\bar{F}$ of F in the space Y. Observe that for any sets $F_1, \ldots, F_n \in \mathcal{F}$ we get $\bar{F}_1 \cap \ldots \bar{F}_n \cap X = F_1 \cap \cdots \cap F_n$.

Consequently,

$$A := Y \setminus \bigcup \{\bar{F}_1 \cap \cdots \cap \bar{F}_n : F_1, \ldots, F_n \in \mathcal{F}, \ F_1 \cap \cdots \cap F_n = \emptyset\}$$

is a G_δ-set in Y containing X. Now consider the $F_{\sigma\delta}$-set $B = \bigcap_{n \in \omega} \bigcup_{F \in \mathcal{F}_n} \bar{F}$ in Y and observe that $A \cap B$ is a $C_{\sigma\delta}$-set in Y. We claim that $X = A \cap B$. The inclusion $X \subseteq A \cap B$ is obvious. Assuming that $X \neq A \cap B$, find a point $y \in A \cap B \setminus X$.

Then for every $n \in \omega$ there exists a set $F_n \in \mathcal{F}_n$ such that $y \in \bar{F}_n$. For the set F_n find an open set $U_n \in \mathcal{U}_n$ such that $F_n \subseteq f[U_n]$ and hence $f^{-1}[F_n] \subseteq U_n$ has d-diameter $< 2^{-n}$.

We claim that for every $n \in \omega$ the intersection $F_0 \cap \cdots \cap F_n$ is not empty. Assuming that this intersection is empty, we would conclude that $y \in \bar{F}_0 \cap \cdots \cap \bar{F}_n$ is contained in $Y \setminus A$, which contradicts the choice of y. Therefore, the family of closed sets $(F_n)_{n \in \omega}$ is centered and so is the family $(f^{-1}[F_n])_{n \in \omega}$. Since each set $f^{-1}[F_n]$ has d-diameter $< 2^{-n}$, the completeness of the metric d ensures that the intersection $\bigcap_{n \in \omega} f^{-1}[F_n]$ contains a unique point $x \in X$.

By the Hausdorff property of Y, the point $f(x)$ has an open neighborhood $V \subseteq Y$ whose closure does not contain the point y. By the continuity of f at x, there exists $n \in \omega$ such that $f^{-1}[V]$ contain the ball $B(x; 2^{-n}) = \{z \in X : d(x, z) < 2^{-n}\}$. Then $f^{-1}[F_n] \subseteq B(x; 2^{-n})$ and hence $F_n = f[f^{-1}[F_n]] \subseteq V$. Finally, $y \in \bar{F}_n \subseteq \bar{V}$, which contradicts the choice of V. This contradiction completes the proof of the equality $X = A \cap B$. □

We do not know if Theorem 3.1 generalizes to higher Borel classes.

Problem 3.2 *Let X be a $G_{\delta\sigma}$-Lusin subspace of a regular topological space Y. Is X a $C_{\delta\sigma\delta}$-set (=a countable intersection of $C_{\delta\sigma}$-sets) in Y?*

Let us recall that a map $f : X \to Y$ between topological spaces is *Borel* if for any open set $U \subseteq Y$ the preimage $f^{-1}[U]$ is a Borel subset of X.

The following characterization of Suslin spaces was proved in [3, 2.5].

Theorem 3.3 *A cosmic space X is Suslin if and only if it is the image of a Suslin space Z under a surjective Borel map $f : Z \to X$.*

A similar characterization holds for Lusin spaces.

Theorem 3.4 *For a cosmic space X the following conditions are equivalent:*

(1) *X is Lusin;*
(2) *X is B-Lusin;*
(3) *X is the image of a Lusin space Z under a bijective Borel map $f : Z \to X$.*

Proof $(1) \Rightarrow (2)$: Assuming that X is Lusin, find a Polish space P and a continuous bijective map $f : P \to X$. The space X is cosmic, being a continuous image of the Polish space. By [8, 2.9], the cosmic space X is submetrizable and hence admits a continuous injective map $g : X \to Y$ to a Polish space Y. By Lusin–Suslin Theorem [10, 15.1], for every open set $U \subseteq P$ the image $g \circ f[U]$ is a Borel subset of the Polish space Y. The continuity of the

map g implies that the set $f[U] = g^{-1}[g \circ f[U]]$ is Borel in X. This means that the space X is B-Lusin.

The implication (2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (1): Assume that the cosmic space X is the image of a Lusin space L under a bijective Borel map $f : L \to X$. By the definition, the Lusin space L is the image of a Polish space P under a continuous bijective map $g : P \to L$. Then X is the image of the Polish space under the Borel bijective map $h = f \circ g : P \to X$. Being cosmic, the space X has a countable network $\mathcal{N}$ consisting of closed subsets of X. Since the map h is Borel, for every $N \in \mathcal{N}$ the preimage $h^{-1}[N]$ is a Borel subset of the Polish space P. By [10, 13.5], there exists a continuous bijective map $\xi : Z \to P$ from a Polish space Z such that for every $N \in \mathcal{N}$ the Borel set $\xi^{-1}[h^{-1}[N]]$ is open in Z. Consider the map $\varphi = h \circ \xi : Z \to X$ and observe that for any open set $U \subseteq X$ the preimage $\varphi^{-1}[U] = \bigcup \{\varphi^{-1}[N] : N \in \mathcal{N}, \ N \subseteq U\}$ is open in Z, witnessing that the bijective map $\varphi : Z \to X$ is continuous and hence X is Lusin.   □

4 Borel properties of the identity maps between various function spaces

It is clear that for any Tychonoff space X the identity maps $C_k(X) \to C_p(X)$ and $C_k(X) \to C_{\downarrow\mathsf{F}}(X)$ are continuous.

Lemma 4.1 *For any $\aleph_0$-space X, the identity map $C_p(X) \to C_k(X)$ is F_σ-measurable.*

Proof By Theorem 1.2, the function space $C_k(X)$ has a countable network and hence is hereditarily Lindelöf. Then it suffices to find a subbase of the topology of $C_k(X)$ consisting of the sets which are Borel in the topology of the space $C_p(X)$. We claim that the standard subbase of $C_k(X)$ has this property. Given a compact set $K \subseteq X$ and a real number r, we need to check that the open sets

$$\lceil K; r \rceil := \{f \in C_k(X) : \max f[K] < r\} \quad \text{and}$$
$$\lfloor K; r \rfloor := \{f \in C_k(X) : \min f[K] > r\}$$

are Borel in $C_p(X)$. The compact subset K of the $\aleph_0$-space has a countable network and hence is separable. Consequently, we can fix a countable dense set $\{x_m\}_{m\in\omega}$ in K.

Now observe that the sets

$$\lceil K; r \rceil = \bigcup_{n=1}^{\infty} \bigcap_{m\in\omega} (C_p(X) \setminus \lfloor x_m; r - \tfrac{1}{n} \rfloor) \quad \text{and} \quad \lfloor K; r \rfloor = \bigcup_{n=1}^{\infty} \bigcap_{m\in\omega} (C_p(X) \setminus \lceil x_m; r + \tfrac{1}{n} \rceil)$$

are Borel of type F_σ in $C_p(X)$.   □

Lemma 4.2 *For any cosmic space X, the identity map $C_{\downarrow\mathsf{F}}(X) \to C_p(X)$ is C_σ-measurable.*

Proof By Theorem 1.3, the function space $C_p(X)$ has a countable network and hence is hereditarily Lindelöf. Then it suffices to find a subbase of the topology of $C_p(X)$ consisting of sets which are of type C_σ in the Fell hypograph topology. We claim that the standard subbase of $C_p(X)$ has this property. Given a point $x \in X$ and a real number r, we need to check that the open sets

$$\lceil x; r \rceil := \{f \in C(X) : f(x) < r\} \quad \text{and} \quad \lfloor x; r \rfloor := \{f \in C(X) : f(x) > r\}$$

are Borel of type C_σ in $C_{\downarrow F}(X)$. The set $\lceil x; r \rceil$ is open in $C_{\downarrow F}(X)$ and hence C_σ by the definition of the Fell hypograph topology. Since

$$C(X) \setminus \lfloor x; r \rfloor = \{f \in C(X) : f(x) \le r\} = \bigcap_{n=1}^{\infty} \{f \in C(X) : f(x) < r + \tfrac{1}{n}\} = \bigcap_{n=1}^{\infty} \lceil x; r + \tfrac{1}{n} \rceil,$$

the set $\lfloor K; r \rfloor$ is Borel of type F_σ in $C_{\downarrow F}(X)$. □

Lemma 4.3 *For any $\aleph_0$-space X, the identity map $C_{\downarrow F}(X) \to C_k(X)$ is C_σ-measurable.*

Proof By Theorem 1.2, the function space $C_k(X)$ has a countable network and hence is hereditarily Lindelöf. Then it suffices to find a subbase of the topology of $C_k(X)$ consisting of sets which are of type F_σ in the Fell hypograph topology. We claim that the standard subbase of $C_k(X)$ has this property. Given a nonempty compact set $K \subseteq X$ and a real number r, we need to check that the open sets

$$\lceil K; r \rceil := \{f \in C(X) : \max f[K] < r\} \quad \text{and} \quad \lfloor K; r \rfloor := \{f \in C(X) : \min f[K] > r\}$$

are Borel of class C_σ in $C_{\downarrow F}(X)$. The set $\lceil K; r \rceil$ is open in $C_{\downarrow F}(X)$ by the definition of the Fell hypograph topology.

The compact subset K of the $\aleph_0$-space has a countable network $\{K_n\}_{n \in \omega}$ consisting of closed (and hence compact) sets in K.

Since

$$\lfloor K; r \rfloor = \bigcup_{n \in \omega} \{f \in C(X) : \min f[K] \ge r + \tfrac{1}{2^n}\} = \bigcup_{n \in \omega} \Big(C(X) \setminus \bigcup_{m \in \omega} \lceil K_m; r + \tfrac{1}{2^n} \rceil \Big),$$

the set $\lfloor K; r \rfloor$ is Borel of type F_σ in $C_{\downarrow F}(X)$. □

5 Function spaces on F_σ-quotient spaces

Lemma 5.1 *For any F_σ-quotient Tychonoff space X, the function spaces $C_p(X)$ and $C_k(X)$ are F_σ-Lusin and $C_{\downarrow F}(X)$ is $C_{\delta\sigma}$-Lusin.*

Proof By the definition of an F_σ-quotient space, there exists a quotient surjective map $q : M \to X$ defined on a σ-compact metrizable space M. First we establish two properties of the quotient map q.

Claim 5.2 *For any sequence $\{x_n\}_{n \in \omega} \subseteq X$ that accumulates at some point $x \in X$ there exists a sequence $\{z_k\}_{k \in \omega} \subseteq q^{-1}[\{x_n\}_{n \in \omega}]$ that converges to a point $z \in q^{-1}(x)$.*

Proof If the set $\Omega = \{n \in \omega : x_n = x\}$ is infinite, then take any point $z \in q^{-1}(x)$ and put $z_k = z$ for all $k \in \Omega$. It is clear that the sequence $(z_k)_{k \in \Omega}$ converges to z and $\{z_k\}_{k \in \Omega} \subseteq q^{-1}[\{x_n\}_{n \in \Omega}] \subseteq q^{-1}[\{x_n\}_{n \in \omega}]$.

So, we assume that the set Ω is finite. Then the set $A = \{x_n : n \in \omega \setminus \Omega\}$ is not closed in X and by the quotient property of q, the preimage $q^{-1}(A)$ is not closed in M. Since M is metrizable, there exists a sequence $\{z_k\}_{k \in \omega} \subseteq q^{-1}[A]$, convergent to a point $z \notin q^{-1}[A]$. The continuity of q implies that $q(z) \in \bar{A} \setminus A = \{x\}$. □

Claim 5.3 *For every compact set $K \subseteq X$ and a cover $\mathcal{U}$ of the set $q^{-1}[K]$ by open subsets of M there exists a finite subfamily $\mathcal{F} \subseteq \mathcal{U}$ such that $K \subseteq \bigcup_{U \in \mathcal{F}} q[U]$.*

Proof Since the σ-compact space M is Lindelöf, the open cover $\mathcal{U}$ of the closed set $q^{-1}[K]$ contains a countable subcover $\mathcal{V}$. We can choose an enumeration $\{U_n\}_{n\in\omega}$ of the countable family $\mathcal{V}$ such that $q^{-1}[K] \subseteq \bigcup_{n=k}^{\infty} U_n$ and hence $K \subseteq \bigcup_{n=k}^{\infty} q[U_n]$ for every $k \in \omega$.

To finish the proof of the claim, it suffices to find $n \in \omega$ such that $K \subseteq \bigcup_{k\leq n} q[U_k]$. Assuming that no such number n exists, for every $n \in \omega$ we can choose a point $x_n \in K \setminus \bigcup_{k\leq n} q[U_k]$. By the compactness of K, the sequence $(x_n)_{n\in\omega}$ accumulates at some point $x \in K$. Since $K \subseteq \bigcup_{n\in\omega} q[U_n]$, there exists a number $k \in \omega$ such that $x \in q[U_k]$. Then $x \notin \{x_n\}_{n>k}$.

By Claim 5.2, there exists a sequence $\{z_n\}_{n\in\omega} \subseteq q^{-1}[\{x_n\}_{n>k}]$ that converges to a point $z \in q^{-1}(x) \subseteq q^{-1}[K]$. Since $\{U_n\}_{n>k}$ is a cover of $q^{-1}[K]$, the point z belongs to some set U_n with $n > k$. Since the sequence $(z_m)_{m\in\omega}$ converges to the point $z \in U_n$, there exists a number $m > n$ such that $z_m \in U_n \setminus q^{-1}[\{x_{k+1}, \ldots, x_n\}]$. Then $q(z_m) = x_i$ for some $i > n$ and hence $x_i = q(z_m) \in q[U_n]$, which contradicts the choice of x_i. $\qquad\square$

Write the σ-compact space M as the countable union $M = \bigcup_{n\in\omega} M_n$ of an increasing sequence $(M_n)_{n\in\omega}$ of compact subsets of M. Fix a countable subset $D \subseteq M$ such that for any $n \in \omega$ the intersection $D \cap M_n$ is dense in M_n. Also fix a metric d generating the topology of M. For a point $x \in M$ and a positive real number ε let $B(x; \varepsilon) = \{y \in M : d(y, x) < \varepsilon\}$ be the ε-ball centered at x and $D(x; \varepsilon) := D \cap B(x; \varepsilon)$ be the trace of the ball $B(x, \varepsilon)$ on D. For every $n \in \omega$ let $Q_n = \{(x, y) \in M_n \times M_n : q(x) = q(y)\}$.

In the Polish space $\omega^\omega \times \mathbb{R}^D$ consider the G_δ-subset

$$P := \{(\alpha, f) \in \omega^\omega \times \mathbb{R}^D : \forall n \in \omega \; \forall x \in D \cap M_n \; \forall y \in D(x; \tfrac{1}{2^{\alpha(n)}}) \; (|f(x) - f(y)| \leq \tfrac{1}{2^n})\}$$

$$\cap \{(\alpha, f) \in \omega^\omega \times \mathbb{R}^D : \forall n \in \omega \; \exists x \in D \cap M_n \; \exists y \in D(x; \tfrac{2}{2^{\alpha(n)}}) \; (|f(x) - f(y)| > \tfrac{1}{2^n})\}$$

$$\cap \{(\alpha, f) \in \omega^\omega \times \mathbb{R}^D : \forall n \in \omega \; \forall (x, y) \in Q_n \; \forall x' \in D(x; \tfrac{1}{2^{\alpha(n)}})$$

$$\forall y' \in D(y; \tfrac{1}{2^{\alpha(n)}}) \; |f(x') - f(y')| \leq \tfrac{3}{2^n}\}.$$

Observe that for every $(\alpha, f) \in P$ and every $n \in \omega$ the restriction $f{\restriction}D \cap M_n$ is a uniformly continuous function, which admits a uniformly continuous extension $\bar{f}_n : M_n \to \mathbb{R}$ to M_n (by the density of $D \cap M_n$ in M_n). Taking into account that $D \cap M_n \subseteq D \cap M_{n+1}$, we conclude that $\bar{f}_n = \bar{f}_{n+1}{\restriction}M_n$, which allows us to define a function $\bar{f} : M \to \mathbb{R}$ such that $\bar{f}{\restriction}M_n = \bar{f}_n$ for all $n \in \omega$. We claim that the function $\bar{f}$ is continuous. Indeed, for any $x \in M$ and any $\varepsilon > 0$, we can find $n \in \omega$ such that $x \in M_n$ and $\frac{1}{2^n} < \frac{1}{3}\varepsilon$. We claim that $|\bar{f}(x) - \bar{f}(y)| < \varepsilon$ for any $y \in M$ with $d(x, y) < \frac{1}{2^{\alpha(n)}}$. Find a number $m \geq n$ such that $y \in M_m$. By the continuity of the map $\bar{f}{\restriction}M_m$ and the density of $D \cap M_k$ in M_k for $k \in \{n, m\}$, there exist points $x' \in D \cap M_n$ and $y' \in D \cap M_m$ such that $d(x', y') < \frac{1}{2^{\alpha(n)}}, |\bar{f}(x) - \bar{f}(x')| < \frac{1}{3}\varepsilon$ and $|\bar{f}(y) - \bar{f}(y')| < \frac{1}{3}\varepsilon$. Then

$$|\bar{f}(x) - \bar{f}(y)| \leq |\bar{f}(x) - \bar{f}(x')| + |\bar{f}(x') - \bar{f}(y')| + |\bar{f}(y') - \bar{f}(y)| < \tfrac{1}{3}\varepsilon + \tfrac{1}{2^n} + \tfrac{1}{3}\varepsilon < \varepsilon.$$

Therefore the function $\bar{f}$ is continuous.

Next, we show that $\bar{f}(x) = \bar{f}(y)$ for any $x, y \in M$ with $q(x) = q(y)$. Assuming that $\bar{f}(x) \neq \bar{f}(y)$, we can find $n \in \omega$ such that $x, y \in M_n$ and $|\bar{f}(x) - \bar{f}(y)| > \frac{3}{2^n}$. Then $(x, y) \in Q_n$. By the density of D in M, there exist points $x' \in D(x; \frac{1}{2^{\alpha(n)}})$ and $y' \in D(y; \frac{1}{2^{\alpha(n)}})$ such that $|f(x') - f(y')| = |\bar{f}(x') - \bar{f}(y')| > \frac{3}{2^n}$. But this contradicts the inclusion $(\alpha, f) \in P$. This contradiction shows that $\bar{f} = \tilde{f} \circ q$ for some function $\tilde{f} : X \to \mathbb{R}$. Since the map q is quotient, the function $\tilde{f} : X \to \mathbb{R}$ is continuous.

So, we can consider the function $\xi : P \to C(X)$ assigning to each pair (α, f) the (unique) continuous function $\tilde{f} : X \to \mathbb{R}$ such that $f = \tilde{f} \circ q \upharpoonright D$.

Claim 5.4 *The function $\xi : P \to C(X)$ is surjective.*

Proof Let $\varphi : X \to \mathbb{R}$ be any continuous function. For every $n \in \omega$, the continuity of the function $\psi = \varphi \circ q$ at points of the compact set M_n yields a number $\alpha(n) \in \omega$ such that $|\psi(x) - \psi(y)| \leq 2^{-n}$ for any $x \in M_n$ and $y \in M$ with $d(x, y) < 2^{-\alpha(n)}$. We can assume that $\alpha(n)$ is the smallest possible number with this property. Then there exists $x \in M_n$ and $y \in M$ such that $d(x, y) < 2^{1-\alpha(n)}$ and $|\psi(x) - \psi(y)| > 2^{-n}$. By the density of the sets $D \cap M_n$ in M_n and D in M, there are points $x' \in M_n \cap D$ and $y' \in D$ such that $d(x', y') < 2^{1-\alpha(n)}$ and $|\psi(x') - \psi(y')| > 2^{-n}$. It is easy to see that the pair $(\alpha, \psi \upharpoonright D)$ belongs to the first two sets in the definition of the set P.

Let us show that $(\alpha, \psi \upharpoonright D)$ also belongs to the third set. Assuming that this is not true, we can find $n \in \omega$, $(x, y) \in Q_n$ and points $x' \in D(x; 2^{-\alpha(n)})$, $y' \in D(y; 2^{-\alpha(n)})$ such that $|\psi(x') - \psi(y')| > \frac{3}{2^n}$. It follows that $x \in M_n \cap D(x'; 2^{-\alpha(n)})$ and $y \in M_n \cap D(y'; 2^{-\alpha(n)})$. By the density of the set $D \cap M_n$ in M, there are points $x'' \in M_n \cap D(x'; 2^{-\alpha(n)})$ and $y'' \in M_n \cap D(y'; 2^{-\alpha(n)})$ such that $\max\{|\psi(x) - \psi(x'')|, |\psi(y) - \psi(y'')|\} < \frac{1}{2^{n+1}}$. Since $x' \in D(x''; 2^{-\alpha(n)})$ and $y' \in D(y''; 2^{-\alpha(n)})$, the choice of $\alpha(n)$ ensures that $\max\{|\psi(x'') - \psi(x')|, |\psi(y'') - \psi(y')|\} \leq \frac{1}{2^n}$. It follows from $(x, y) \in Q_n$ that $q(x) = q(y)$ and hence $\psi(x) = \varphi(q(x)) = \varphi(q(y)) = \psi(y)$. Then

$$\frac{3}{2^n} < |\psi(x') - \psi(y')| \leq |\psi(x') - \psi(x'')| + |\psi(x'') - \psi(x)| + |\psi(x) - \psi(y)|$$
$$+ |\psi(y) - \psi(y'')| + |\psi(y'') - \psi(y')| < \frac{1}{2^n} + \frac{1}{2^{n+1}} + 0 + \frac{1}{2^{n+1}} + \frac{1}{2^n} = \frac{3}{2^n},$$

which is a contradiction finishing the proof of the inclusion $(\alpha, \psi \upharpoonright D) \in P$.

Observe that for the function $f = \psi \upharpoonright D$, we get $\tilde{f} = \psi$ and $\varphi = \tilde{f} = \xi(\alpha, f)$, which means that the function ξ is surjective. $\qquad \square$

Claim 5.5 *The function $\xi : P \to C(X)$ is injective.*

Proof Assume that $(\alpha, f), (\beta, g) \in P$ are two pairs such that $\tilde{f} = \tilde{g}$, where $\tilde{f} = \xi(\alpha, f)$ and $\tilde{g} = \xi(\beta, g)$. Then $f = \tilde{f} \circ q \upharpoonright D = \tilde{g} \circ q \upharpoonright D = g$.

It remains to prove that $\alpha = \beta$. Assuming that $\alpha \neq \beta$, we can find $n \in \omega$ such that $\alpha(n) \neq \beta(n)$. We lose no generality assuming that $\alpha(n) < \beta(n)$. Since $(\beta, g) \in P$, there exist points $x \in D \cap M_n$ and $y \in D(x; 2^{1-\beta(n)})$ such that $|g(x) - g(y)| > 2^{-n}$. The strict inequality $\alpha(n) < \beta(n)$ implies $2^{-\alpha(n)} \geq 2^{1-\beta(n)}$ and hence $y \in D(x; 2^{1-\beta(n)}) \subseteq D(x; 2^{-\alpha(n)})$. Then we get points $x \in D \cap M_n$ and $y \in D(x; 2^{-\alpha(n)})$ such that $|f(x) - f(y)| = |g(x) - g(y)| > 2^{-n}$. But this contradicts the inclusion $(\alpha, f) \in P$. $\qquad \square$

Claim 5.6 *The function $\xi : P \to C_k(X)$ is continuous.*

Proof By [8, 11.3], the F_σ-quotient space X is an $\aleph_0$-space. By Theorem 1.2, the function space $C_k(X)$ is cosmic and hence hereditarily Lindelöf. So, it suffices to show that for any nonempty compact set $K \subseteq X$ and any real number r the sets $\xi^{-1}\big[\lfloor K; r \rfloor\big]$ and $\xi^{-1}\big[\lceil K; r \rceil\big]$ are open in P.

To see that $\xi^{-1}\big[\lfloor K; r \rfloor\big]$ is open, take any pair $(\alpha, f) \in \xi^{-1}\big[\lfloor K; r \rfloor\big]$. Consider the function $\tilde{f} = \xi(\alpha, f)$. Since $\tilde{f} \in \lfloor K; r \rfloor$, there exists an open neighborhood $U \subseteq X$ of K and a number $n \in \omega$ such that $r + \frac{2}{2^n} < \inf \tilde{f}(U)$.

For every $z \in q^{-1}[K]$ we can find a number $k \geq n$ such that $z \in M_k$. By the density of the set $D \cap M_k$ in M_k, there exists a point $x \in D \cap M_k \cap q^{-1}[U] \cap B(z; 2^{-\alpha(k)})$. Then $z \in B(x; 2^{-\alpha(k)})$. Therefore,

$$q^{-1}[K] \subseteq \bigcup_{k=n}^{\infty} \{B(x; 2^{-\alpha(k)}) : x \in D \cap M_k \cap q^{-1}[U]\}.$$

By Claim 5.3, there exists $m \geq n$ and a finite subset $F \subseteq D \cap q^{-1}[U]$ such that

$$K \subseteq \bigcup_{k=n}^{m} \bigcup_{x \in F \cap M_k} q[B(x; 2^{-\alpha(k)})].$$

Consider the open neighborhood

$$W := \bigcap_{k \leq m} \bigcap_{x \in F} \left\{ (\beta, g) \in P : \beta(k) = \alpha(k), \ g(x) > r + \tfrac{1}{2^n} \right\}$$

of (α, f) in P.

We claim that $\xi[W] \subseteq \lfloor K; r \rfloor$. Take any pair $(\beta, g) \in W$ and consider the function $\tilde{g} = \xi(\beta, g)$. Given any point $y \in K$ we should prove that $\tilde{g}(y) > r$. Find $k \in [n, m]$ and $x \in F \cap M_k$ such that $y \in q[B(x, 2^{-\alpha(k)})]$. Then $y = q(z)$ for some $z \in B(x, 2^{-\alpha(k)})$. The inclusion $(\beta, g) \in W$ ensures that $g(x) > r + \tfrac{1}{2^n}$. Let $\bar{g} : X \to \mathbb{R}$ be the (unique) continuous function extending the function g. We claim that $|\bar{g}(z) - g(x)| \leq \tfrac{1}{2^k}$. To derive a contradiction, assume that $|\bar{g}(z) - g(x)| > \tfrac{1}{2^k}$. By the continuity of $\bar{g}$ and the density of D in M, there exists a point $t \in D$ such that $d(t, z) < 2^{-\alpha(k)} - d(z, x)$ and $|\bar{g}(t) - \bar{g}(z)| < |\bar{g}(z) - g(x)| - \tfrac{1}{2^k}$. Then $t \in B(x; 2^{-\alpha(k)})$ and $|\bar{g}(t) - g(x)| > \tfrac{1}{2^k}$. The inclusion $(\beta, g) \in W$ guarantees that $\beta(k) = \alpha(k)$. Then $x \in D \cap M_k$ and $t \in D \cap B(x; 2^{-\beta(k)})$ are two points with $|g(t) - g(x)| > \tfrac{1}{2^k}$, which contradicts the inclusion $(\beta, g) \in P$. This contradiction shows that $|\bar{g}(z) - g(x)| \leq \tfrac{1}{2^k} \leq \tfrac{1}{2^n}$. Then

$$\tilde{g}(y) = \tilde{g} \circ q(z) = \bar{g}(z) > g(x) - |\bar{g}(z) - g(x)| \geq r + \tfrac{1}{2^n} - \tfrac{1}{2^n} = r.$$

Therefore, $W \subseteq \xi^{-1}\big[\lfloor K; r \rfloor\big]$ and the set $\xi^{-1}\big[\lfloor K; r \rfloor\big]$ is open in P. By analogy we can prove that the set $\xi^{-1}\big[\lceil K; r \rceil\big]$ is open in P. $\qquad \square$

Claim 5.7 *The function* $\xi^{-1} : C_p(X) \to P$ *is* F_σ-*measurable.*

Proof Since the Polish space P is hereditarily Lindelöf, it suffices to show that for any $(n, m) \in \omega$, point $x \in D$ and real number r, the images of the subbasic open sets

$$P_{n \leq m} := \{(\alpha, f) \in P : \alpha(n) \leq m\}, \quad P_{n \geq m} := \{(\alpha, f) \in P : \alpha(n) \geq m\}$$

and

$$P_{x < r} := \{(\alpha, f) \in P : f(x) < r\}, \quad P_{x > r} := \{(\alpha, f) \in P : f(x) > r\}$$

under the map ξ are F_σ-sets in $C_p(X)$. We shall prove that the images of these sets are open or closed (and hence F_σ) in $C_p(X)$.

Observe that the set

- $\xi[P_{n \leq m}] = \{f \in C(X) : \forall x \in D \cap X_n \ \forall y \in D(x, \tfrac{1}{2^m}) \ |f \circ q(x) - f \circ q(y)| \leq \tfrac{1}{2^n}\}$ is closed in $C_p(X)$,

- $\xi[P_{n\geq m}] = \{f \in C(X) : \exists x \in D \cap X_n \; \exists y \in D(x, \frac{2}{2^m}) \; |f \circ q(x) - f \circ q(y)| > \frac{1}{2^n}\}$ is open in $C_p(X)$,
- $\xi[P_{x<r}] = \{f \in C(X) : f \circ q(x) < r\}$ and $\xi(P_{x>r}) = \{f \in C(X) : f \circ q(x) > r\}$ are open in $C_p(X)$. $\qquad\square$

Claim 5.8 *For every $x, y \in X$ and $\varepsilon > 0$ the set $E = \{f \in C_{\downarrow F}(X) : |f(x) - f(y)| > \varepsilon\}$ is of type C_σ in $C_{\downarrow F}(X)$.*

Proof Let $\mathbb{Q}$ be the set of rational numbers and $Q = \{(p, q) \in \mathbb{Q} \times \mathbb{Q} : p + \varepsilon < q\}$. Observe that

$$E = \bigcup_{(p,q)\in Q} \left(\{f \in C_{\downarrow F}(X) : f(x) < p, \; q \leq f(y)\} \cup \{f \in C_{\downarrow F}(X) : f(y) < p, \; q \leq f(x)\}\right)$$

$$= \bigcup_{(p,q)\in Q} (\lceil x; p\rceil \setminus \lceil y; q\rceil) \cup (\lceil y; p\rceil \setminus \lceil x; q\rceil)$$

is of type C_σ in $C_{\downarrow F}(X)$. $\qquad\square$

Claim 5.9 *The function $\xi^{-1} : C_{\downarrow F}(X) \to P$ is $C_{\delta\sigma}$-measurable.*

Proof Similarly as in Claim 5.7, it suffices to check that for any $(n, m) \in \omega$, point $x \in D$ and real number r, the images of the subbasic open sets $P_{n\leq m}$, $P_{n\geq m}$, $P_{x>r}$, $P_{x<r}$ under the map ξ are $C_{\delta\sigma}$-sets in $C_{\downarrow F}(X)$. We shall prove that the images of these sets are of type C_σ or C_δ in $C_{\downarrow F}(X)$.

By Claim 5.8, the set

- $\xi[P_{n\leq m}] = \{f \in C(X) : \forall x \in D \cap X_n \; \forall y \in D(x, \frac{1}{2^m}) \; |f \circ q(x) - f \circ q(y)| \leq \frac{1}{2^n}\}$ is of type C_δ in $C_{\downarrow F}(X)$,
- $\xi[P_{n\geq m}] = \{f \in C_p(X) : \exists x \in D \cap X_n \; \exists y \in D(x, \frac{2}{2^m}) \; |f(x) - f(y)| > \frac{1}{2^n}\}$ is of type C_σ in $C_{\downarrow F}(X)$,
- $\xi[P_{x<r}] = \{f \in C(X) : f(q(x)) < r\}$ is open in $C_{\downarrow F}(X)$,
- $\xi[P_{x>r}] = \{f \in C(X) : f(q(x)) > r\} = \bigcup_{n\in\omega}\{f \in C(X) : f(q(x)) \geq r + \frac{1}{2^n}\} = \bigcup_{n\in\omega}(C(X) \setminus \lceil q(x); r + \frac{1}{2^n}\rceil)$ is of type F_σ in $C_{\downarrow F}(X)$. $\qquad\square$

Claim 5.10 *The function space $C_p(X)$ is F_σ-Lusin.*

Proof By Claims 5.4, 5.5 and 5.6, the map $\xi : P \to C_k(X)$ is bijective and continuous. Since the identity map $C_k(X) \to C_p(X)$ is continuous, the map $\xi : P \to C_p(X)$ is continuous as the composition of two continuous maps. By Claim 5.7, the inverse map $\xi^{-1} : C_p(X) \to P$ is F_σ-measurable, which implies that the space $C_p(X)$ is F_σ-Lusin. $\qquad\square$

Claim 5.11 *The function space $C_k(X)$ is F_σ-Lusin.*

Proof By Claims 5.4, 5.5 and 5.6, the map $\xi : P \to C_k(X)$ is bijective and continuous. By Claim 5.7 the map $\xi^{-1} : C_p(X) \to P$ is F_σ-measurable. The continuity of the identity map $C_k(X) \to C_p(X)$ implies that the map $\xi^{-1} : C_k(X) \to P$ is F_σ-measurable (as the composition of a continuous and F_σ-measurable maps). Now we see that the Polish space P and the map $\xi : P \to C_k(X)$ witness that the space $C_k(X)$ is F_σ-Lusin. $\qquad\square$

Claim 5.12 *The function space $C_{\downarrow F}(X)$ is $C_{\delta\sigma}$-Lusin.*

Proof By Claims 5.4, 5.5 and 5.6, the map $\xi : P \to C_k(X)$ is bijective and continuous. Since the identity map $C_k(X) \to C_{\downarrow F}(X)$ is continuous, the map $\xi : P \to C_{\downarrow F}(X)$ is continuous (as the composition of two continuous maps). By Claim 5.9, the inverse map $\xi^{-1} : C_{\downarrow F}(X) \to P$ is $C_{\delta\sigma}$-measurable, which implies that the space $C_p(X)$ is $C_{\delta\sigma}$-Lusin. $\square$

6 Proof of Theorem 1.7

In this section, for a Tychonoff space X we shall prove the implications

$$(3) \Leftarrow (2) \Leftrightarrow (1) \Rightarrow (4) \Rightarrow (6)$$
$$\Rightarrow (7) \Rightarrow (8) \Rightarrow (6) \Rightarrow (7) \Rightarrow (10) \Rightarrow (11) \Rightarrow (9) \Rightarrow (10) \Rightarrow (12)$$

of Theorem 1.7.

$(2) \Rightarrow (3)$ Assume that the function space $C_k(X)$ is Polish. The continuity of the identity map $C_k(X) \to C_{\downarrow F}(X)$ implies that the space $C_{\downarrow F}(X)$ is Lusin. Since the identity map $C_{\downarrow F}(X) \to C_k(X)$ is C_σ-measurable (by Lemma 4.3), the space $C_{\downarrow F}(X)$ is C_σ-Lusin.

$(2) \Rightarrow (1)$ If the function space $C_k(X)$ is Polish, then by Corollary 5.2.5 in [12], X is a cosmic hemicompact k-space. The hemicompactness of X yields an increasing sequence $(K_n)_{n \in \omega}$ of compact sets in X such that each compact subset of X is contained in some set K_n. Consider the locally compact subspace $M = \bigcup_{n \in \omega}(K_n \times \{n\})$ of the product $X \times \omega$ where the ordinal ω is endowed with the discrete topology. Let $q : M \to X$ be the natural projection. We claim that the map q is quotient. Indeed, take any subset $A \subseteq X$ such that the preimage $q^{-1}[A]$ is closed in X. Then $A \cap K_n$ is closed in K_n for every $n \in \omega$. Since each compact set $K \subseteq X$ is contained in some K_n, the intersection $A \cap K = (A \cap K_n) \cap K$ is closed in K. Since X is a k-space, the set A is closed in X. Therefore, the map q is quotient. Since M is open in its one-point compactification, the space X is G-quotient.

$(1) \Rightarrow (2)$ Assume that X is G-quotient and find a quotient surjective map $q : M \to X$ defined on an open subspace M of a compact metrizable space. Write the locally compact space M as the countable union $M = \bigcup_{n \in \omega} U_n$ of an increasing sequence $(U_n)_{n \in \omega}$ of open sets such that each set U_n has compact closure $\overline{U_n}$, contained in U_{n+1}. By Claim 5.3, for every compact set $K \subseteq X$ there exists a number $n \in \omega$ such that $K \subseteq q[U_n] \subseteq q[\overline{U_n}]$. Now we see that the sequence of compact sets $(q[\overline{U_n}])_{n \in \omega}$ witnesses that the space X is hemicompact. By Theorem 11.3 [8], X is a cosmic k-space, and by Corollary 5.2.5 of [12], for the cosmic hemicompact k-space X, the function space $C_k(X)$ is Polish.

The implication $(1) \Rightarrow (4)$ is trivial, $(4) \Rightarrow (5)$ is proved in Lemma 5.1 and $(5) \Rightarrow (7)$ is trivial. The implication $(7) \Rightarrow (8)$ follows from the continuity of the identity map $C_k(X) \to C_{\downarrow F}(X)$.

$(8) \Rightarrow (6)$ Assume that the space $C_{\downarrow F}(X)$ is Lusin. Then $C_{\downarrow F}(X)$ has a countable network and X is an $\aleph_0$-space by Theorem 1.2. By Theorem 1.3, the function space $C_p(X)$ is cosmic. By Lemma 4.2, the identity map $C_{\downarrow F}(X) \to C_p(X)$ is Borel and by Theorem 3.4, the cosmic space $C_p(X)$ is Lusin.

$(6) \Rightarrow (7)$ Assume that $C_p(X)$ is Lusin and X is an $\aleph_0$-space. By Theorem 1.2, the function space $C_k(X)$ is cosmic. By Lemma 4.1, the identity map $C_p(X) \to C_k(X)$ is Borel and by Theorem 3.4, the cosmic space $C_k(X)$ is Lusin.

The implication $(7) \Rightarrow (10)$ is trivial and $(10) \Rightarrow (11)$ follows from the continuity of the identity map $C_k(X) \to C_{\downarrow F}(X)$.

$(11) \Rightarrow (9)$ Assume that the space $C_{\downarrow F}(X)$ is Suslin. Then $C_{\downarrow F}(X)$ has a countable network and X is an $\aleph_0$-space by Theorem 1.2. By Theorem 1.3, the function space $C_p(X)$ is cosmic. By Lemma 4.2, the identity map $C_{\downarrow F}(X) \to C_p(X)$ is Borel and by Theorem 3.3, the space $C_p(X)$ is Suslin.

(9) $\Rightarrow$ (10) Assume that $C_p(X)$ is Suslin and X is an $\aleph_0$-space. By Theorem 1.2, the function space $C_k(X)$ is cosmic. By Lemma 4.1, the identity map $C_p(X) \to C_k(X)$ is Borel and by Theorem 3.3, the space $C_k(X)$ is Suslin.

(10) $\Rightarrow$ (12) If the space $C_k(X)$ is Suslin, then so is the space $C_p(X)$ (being a continuous image of $C_k(X)$). By Calbrix's Theorem 1.4, the space X is σ-compact.

7 Discussing Example 1.11

In this section we prove that for the quotient space $X = \omega^{\leq\omega}/\omega^\omega$ from Example 1.11, the function spaces $C_p(X)$, $C_k(X)$ and $C_{\downarrow F}(X)$ are non-Suslin.

For a topological space T denote by T' the set of non-isolated points of T and observe that

$$C'_p(T) := \{f \in C_p(T) : f(T') \subseteq \{0\}\}$$

is a closed linear subspace of $C_p(T)$.

We recall that the discrete subspace $\omega^{<\omega}$ of the Polish space $\omega^{\leq\omega} = \omega^{<\omega} \cup \omega^\omega$ carries the partial order $\leq$ defined by $x \leq y$ iff $x = y{\restriction}n$ for some $n \in \omega$. Endowed with this partial order, the set $\omega^{<\omega}$ is a tree (which means that for any $x \in \omega^{<\omega}$ the set $\downarrow x = \{y \in \omega^{<\omega} : y \leq x\}$ is finite and linearly ordered). A subtree T of $\omega^{<\omega}$ is *well-founded* if it contains no infinite linearly ordered subsets.

In the function space $C_p(\omega^{\leq\omega})$ consider the closed subspace

$$M_0(\omega^{\leq\omega}) = \big\{ f \in C_p(\omega^{\leq\omega}) : \ f[\omega^\omega] \subseteq \{0\},$$
$$f[\omega^{<\omega}] \subseteq \{0, 1\}, \ \ \forall x, y \in \omega^{<\omega} \ \big(x \leq y \ \Rightarrow f(x) \geq f(y)\big)\big\}$$

consisting of non-increasing continuous functions $f : \omega^{\leq\omega} \to \{0, 1\}$ that vanish on the subspace ω^ω.

For any function $f \in M_0(\omega^{\leq\omega})$ the preimage $f^{-1}(1)$ is a well-founded subtree of the tree $\omega^{<\omega}$. So, the space $M_0(\omega^{\leq\omega})$ can be identified with the space WF of well-founded trees on ω. By [10, 32.B], the space WF is coanalytic but not analytic and so is the space $M_0(\omega^{\leq\omega})$. Let us recall that a subset A of a Polish space P is *analytic* (resp. *coanalytic*) if the space A (resp. $P \setminus A$) is Suslin.

Consider the quotient space $X = \omega^{\leq\omega}/\omega^\omega$ of the Polish space $\omega^{\leq\omega}$ by its closed nowhere dense subspace ω^ω. It is clear that X is a countable Tychonoff space with a unique non-isolated point $\{\omega^\omega\}$. By [8, 11.3], X is a sequential $\aleph_0$-space.

Claim 7.1 *The function space* $C'_p(X) = \big\{f \in C_p(X) : f[X'] \subseteq \{0\}\big\}$ *is not Suslin.*

Proof Let $q : \omega^{\leq\omega} \to X$ be the quotient map. It induces a continuous map $q^* : C_p(X) \to C_p(\omega^{\leq\omega})$, $q^* : f \mapsto f \circ q$. Observe that $q^*[C'_p(X)] = C'_p(\omega^{\leq\omega})$, where $C'_p(\omega^{\leq\omega}) = \{f \in C_p(\omega^{\leq\omega}) : f[\omega^\omega] \subseteq \{0\}\}$. Assuming that the space $C'_p(X)$ is Suslin, we would conclude that its continuous image $C'_p(\omega^{\leq\omega})$ is Suslin. On the other hand, $C'_p(\omega^{\leq\omega})$ contains the non-analytic space $M_0(\omega^{\leq\omega})$ as a closed subspace and hence $C'_p(\omega^{\leq\omega})$ cannot be Suslin. $\square$

Claim 7.2 *The function spaces* $C_p(X)$, $C_k(X)$ *and* $C_{\downarrow F}(X)$ *are non-Suslin.*

Proof The space $C_p(X)$ is not Suslin since it contains the closed subspace $C'_p(X)$ which is not Suslin by Claim 7.1. By Theorem 1.7, the space $C_k(X)$ and $C_{\downarrow F}(X)$ are not Suslin, too. $\square$

References

1. Arkhangel'skiĭ, A.V.: Topological Function Spaces. Kluwer Academic Publ. Group, Dordrecht (1992)
2. Banakh, T.: Topological spaces with an ω^ω-base. Dissert. Math. **538**, 1–141 (2019)
3. Banakh, T.: Banalytic spaces and characterization of Polish groups. Lett. Math. Sci. **(in press)**. http://www.prior-sci-pub.com/lims_2019art1.html
4. Banakh, T., Gabriyelyan, S.: On the C_k-stable closure of the class of (separable) metrizable spaces. Monatsh. Math. **180**(1), 39–64 (2016)
5. Bourbaki, N.: General Topology. Chapters 5–10, Elements of Mathematics. Springer, Berlin (1989)
6. Calbrix, J.: Espaces K_σ et espaces des applications continues. Bull. Soc. Math. Fr. **113**(2), 183–203 (1985)
7. Engelking, R.: General Topology. Heldermann, Berlin (1989)
8. Gruenhage, G.: Generalized metric spaces. In: Handbook of Set-theoretic Topology, pp. 423–501. North-Holland, Amsterdam (1984)
9. Kąkol, J., Kubiś, W., López-Pellicer, M.: Descriptive topology in selected topics of functional analysis, Developments in Mathematics, vol. 24. Springer, Berlin (2011)
10. Kechris, A.: Classical Descriptive Set Theory. Springer, New York (1995)
11. McCoy, R., Ntantu, I.: Properties of C(X) with the epi-topology. Boll. Unione Mat. Ital. B (7) **6**, 507–532 (1992)
12. McCoy, R., Ntantu, I.: Topological properties of spaces of continuous functions, Lecture Notes in Mathematics, vol. 1315. Springer, Berlin (1988)
13. Michael, E.: $\aleph_0$-spaces. J. Math. Mech. **15**, 983–1002 (1966)
14. Michael, E.: On k-spaces, k_R-spaces and $k(X)$. Pac. J. Math. **47**, 487–498 (1973)
15. Noble, N.: Ascoli theorems and the exponential map. Trans. Am. Math. Soc. **143**, 393–411 (1969)
16. Tkachuk, V.: A C_p-Theory Problem Book. Special Features of Function Spaces. Springer, Cham (2014)
17. Tkachuk, V.: A C_p-Theory Problem Book. Compactness in Function Spaces. Springer, Cham (2015)
18. Tkachuk, V.: A C_p-Theory Problem Book. Functional Equivalencies. Springer, Cham (2016)
19. Wang, L., Banakh, T.: Baire category properties of function spaces with the Fell hypograph topology. Topol. Appl. **265**, 106817 (2019)
20. Yang, Z., Chen, L., Zheng, Y.: Topological classification of function spaces with the Fell topology III. Topol. Appl. **197**, 112–132 (2016)
21. Yang, Z., Hu, S., Wei, G.: Topological structures of the space of continuous functions on a non-compact space with the Fell topology. Topol. Proc. **41**, 17–38 (2013)
22. Yang, Y., Zhang, B.: The hyperspace of the hypographs of continuous maps with the Fell topology. Acta Math. Sin. Engl. Ser. **28**, 57–66 (2012)
23. Yang, Y., Yan, P.: Topological classification of function spaces with the Fell topology I. Topol. Appl. **178**, 146–159 (2014)
24. Yang, H., Yang, Y., Zheng, Y.: Topological classification of function spaces with the Fell topology IV. Topol. Appl. **228**, 222–235 (2017)
25. Yang, Z., Zheng, Y., Chen, J.: Topological classification of function spaces with the Fell topology II. Topol. Appl. **187**, 82–96 (2015)