Characterizations of a class of Pilipović spaces by powers of harmonic oscillator

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Abstract

We show that a smooth function f on $\mathbf{R}^d$ belongs to the Pilipović space $\mathcal{H}_{b_\sigma}(\mathbf{R}^d)$ or the Pilipović space $\mathcal{H}_{0,b_\sigma}(\mathbf{R}^d)$, if and only if the L^p norm of $H_d^N f$ for $N \geq 0$, satisfy certain types of estimates. Here $H_d = |x|^2 - \Delta_x$ is the harmonic oscillator.

Keywords Harmonic oscillator · Pilipović spaces

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0 Introduction

In the paper we characterize Pilipović spaces of the form $\mathcal{H}_{b_\sigma}(\mathbf{R}^d)$ and $\mathcal{H}_{0,b_\sigma}(\mathbf{R}^d)$, considered in [3, 11], in terms of estimates of powers of the harmonic oscillator, on the involved functions.

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The set of Pilipović spaces is a family of Fourier invariant spaces, containing any Fourier invariant (standard) Gelfand-Shilov space. The (standard) Pilipović spaces $\mathcal{H}_s(\mathbf{R}^d)$ and $\mathcal{H}_{0,s}(\mathbf{R}^d)$ with respect to $s \in \mathbf{R}_+$, are the sets of all formal Hermite series expansions

$$f(x) = \sum_{\alpha \in \mathbf{N}^d} c_\alpha(f) h_\alpha(x) \quad (0.1)$$

such that

$$|c_\alpha(f)| \lesssim e^{-r|\alpha|^{\frac{1}{2s}}} \quad (0.2)$$

holds true for some $r > 0$ respective for every $r > 0$. Here $f(\theta) \lesssim g(\theta)$ means that $f(\theta) \leq cg(\theta)$ for some constant $c > 0$ which is independent of θ in the domain of f and g (see also [6] and Sect. 1 for notations). Evidently, $\mathcal{H}_s(\mathbf{R}^d)$ and $\mathcal{H}_{0,s}(\mathbf{R}^d)$ increase with s . It is proved in [7] that if $\mathcal{S}_s(\mathbf{R}^d)$ and $\Sigma_s(\mathbf{R}^d)$ are the Gelfand-Shilov spaces of Roumieu respective Beurling type of order s , then

$$\mathcal{H}_s(\mathbf{R}^d) = \mathcal{S}_s(\mathbf{R}^d), \quad s \geq \frac{1}{2}, \quad (0.3)$$

$$\mathcal{H}_{0,s}(\mathbf{R}^d) = \Sigma_s(\mathbf{R}^d), \quad s > \frac{1}{2}, \quad (0.4)$$

and

$$\mathcal{H}_{0,s}(\mathbf{R}^d) \neq \Sigma_s(\mathbf{R}^d) = \{0\}, \quad s = \frac{1}{2}.$$

It is also well-known that $\mathcal{S}_s(\mathbf{R}^d) = \{0\}$ when $s < \frac{1}{2}$ and $\Sigma_s(\mathbf{R}^d) = \{0\}$ when $s \leq \frac{1}{2}$. These relationships are completed in [11] by the relations

$$\mathcal{H}_s(\mathbf{R}^d) \neq \mathcal{S}_s(\mathbf{R}^d) = \{0\}, \quad s < \frac{1}{2}$$

and

$$\mathcal{H}_{0,s}(\mathbf{R}^d) \neq \Sigma_s(\mathbf{R}^d) = \{0\}, \quad s \leq \frac{1}{2}.$$

In particular, each Pilipović space is contained in the Schwartz space $\mathcal{S}(\mathbf{R}^d)$.

For $\mathcal{H}_s(\mathbf{R}^d)$ ($\mathcal{H}_{0,s}(\mathbf{R}^d)$) we also have the characterizations

$$f \in \mathcal{H}_s(\mathbf{R}^d) \quad (f \in \mathcal{H}_{0,s}(\mathbf{R}^d)) \Leftrightarrow \|H_d^N f\|_{L^\infty} \lesssim r^N N!^{2s} \quad (0.5)$$

for some $r > 0$ (for every $r > 0$) concerning estimates of powers of the harmonic oscillator

$$H_d = |x|^2 - \Delta_x, \quad x \in \mathbf{R}^d,$$

acting on the involved functions. These relations were obtained in [7] for $s \geq \frac{1}{2}$, and in [11] in the general case $s > 0$.

In [3, 11] characterizations of $\mathcal{H}_s(\mathbf{R}^d)$ and $\mathcal{H}_{0,s}(\mathbf{R}^d)$ were also obtained by certain spaces of analytic functions on $\mathbf{C}^d$, via the Bargmann transform. From these mapping properties it follows that near $s = \frac{1}{2}$ there is a jump concerning these Bargmann images. More precisely, if $s = \frac{1}{2}$, then the Bargmann image of $\mathcal{H}_s(\mathbf{R}^d)$ (of $\mathcal{H}_{0,s}(\mathbf{R}^d)$) is the set of all entire functions F on $\mathbf{C}^d$ such that F obeys the condition

$$|F(z)| \lesssim e^{(\frac{1}{2}-r)|z|^2} \quad (|F(z)| \lesssim e^{r|z|^2}) \quad (0.6)$$

for some $r > 0$ (for every $r > 0$). For $s < \frac{1}{2}$, this estimate is replaced by

$$|F(z)| \lesssim e^{r(\log(1+|z|))^{1-\frac{1}{2s}}} \quad (0.7)$$

for some $r > 0$ (for every $r > 0$), which is indeed a stronger condition compared to the case $s = \frac{1}{2}$.

An important motivation for considering the spaces $\mathcal{H}_{b_\sigma}(\mathbf{R}^d)$ and $\mathcal{H}_{0,b_\sigma}(\mathbf{R}^d)$ is to make this gap smaller. More precisely, $\mathcal{H}_{b_\sigma}(\mathbf{R}^d)$ and $\mathcal{H}_{0,b_\sigma}(\mathbf{R}^d)$, which are Pilipović spaces of Roumieu respectively Beurling type, is a family of function spaces, which increases with σ and such that

$$\mathcal{H}_{s_1}(\mathbf{R}^d) \subseteq \mathcal{H}_{0,b_\sigma}(\mathbf{R}^d) \subseteq \mathcal{H}_{b_\sigma}(\mathbf{R}^d) \subseteq \mathcal{H}_{0,s_2}(\mathbf{R}^d), \quad s_1 < \frac{1}{2}, \quad s_2 \geq \frac{1}{2}.$$

The spaces $\mathcal{H}_{b_\sigma}(\mathbf{R}^d)$ and $\mathcal{H}_{0,b_\sigma}(\mathbf{R}^d)$ consist of all formal Hermite series expansions (0.1) such that

$$|c_\alpha(f)| \lesssim r^{|\alpha|} |\alpha|!^{-\frac{1}{2\sigma}} \quad (0.8)$$

hold true for some $r > 0$ respectively for every $r > 0$. For the Bargmann images of $\mathcal{H}_{b_\sigma}(\mathbf{R}^d)$ and $\mathcal{H}_{0,b_\sigma}(\mathbf{R}^d)$, the conditions (0.6) and (0.7) above are replaced by

$$|F(z)| \lesssim e^{r|z|^{\frac{2\sigma}{\sigma+1}}},$$

for some $r > 0$ respectively for every $r > 0$. It follows that the gaps of the Bargmann images of $\mathcal{H}_s(\mathbf{R}^d)$ and $\mathcal{H}_{0,s}(\mathbf{R}^d)$ between the cases $s < \frac{1}{2}$ and $s \geq \frac{1}{2}$ are drastically decreased by including the spaces $\mathcal{H}_{b_\sigma}(\mathbf{R}^d)$ and $\mathcal{H}_{0,b_\sigma}(\mathbf{R}^d)$, $\sigma > 0$, in the family of Pilipović spaces.

In [3], characterizations of $\mathcal{H}_{b_1}(\mathbf{R}^d)$ and $\mathcal{H}_{0,b_1}(\mathbf{R}^d)$ in terms of estimates of powers of the harmonic oscillator acting on the involved functions which corresponds to (0.5) are deduced. On the other hand, apart from the case $\sigma = 1$, it seems that no such characterizations for $\mathcal{H}_{b_\sigma}(\mathbf{R}^d)$ and $\mathcal{H}_{0,b_\sigma}(\mathbf{R}^d)$ have been obtained so far.

In Sect. 2 we fill this gap in the theory, and deduce such characterizations. In particular, as a consequence of our main result, Theorem 2.1 in Sect. 2, we have

$$\begin{aligned} f \in \mathcal{H}_{b_\sigma}(\mathbf{R}^d) \quad (f \in \mathcal{H}_{0,b_\sigma}(\mathbf{R}^d)) \\ \Leftrightarrow \\ \|H_d^N f\|_{L^\infty} \lesssim 2^N r^{\frac{N}{\log(N\sigma)}} \left(\frac{2N\sigma}{\log(N\sigma)} \right)^{N(1-\frac{1}{\log(N\sigma)})} \end{aligned}$$

for some (every) $r > 0$. By choosing $\sigma = 1$ we regain the corresponding characterizations in [3] for $\mathcal{H}_{b_1}(\mathbf{R}^d)$ and $\mathcal{H}_{0,b_1}(\mathbf{R}^d)$.

1 Preliminaries

In this section we recall some facts about Gelfand-Shilov spaces, Pilipović spaces and modulation spaces.

Let $s > 0$. Then the (Fourier invariant) Gelfand-Shilov spaces $\mathcal{S}_s(\mathbf{R}^d)$ and $\Sigma_s(\mathbf{R}^d)$ of Roumieu and Beurling type, respectively, consist of all $f \in C^\infty(\mathbf{R}^d)$ such that

$$\|f\|_{\mathcal{S}_{s;r}} \equiv \sup_{\alpha, \beta \in \mathbf{N}^d} \left(\frac{\|x^\alpha D^\beta f\|_{L^\infty(\mathbf{R}^d)}}{r^{|\alpha+\beta|} (\alpha! \beta!)^s} \right) \quad (1.1)$$

is finite, for some $r > 0$ respectively for every $r > 0$. The topologies of $\mathcal{S}_s(\mathbf{R}^d)$ and $\Sigma_s(\mathbf{R}^d)$ are the inductive limit topology and the projective limit topology, respectively, supplied by the norms (1.1). We refer to [1, 5] for more facts about Gelfand-Shilov spaces.

For $\mathcal{H}_s(\mathbf{R}^d)$ and $\mathcal{H}_{0,s}(\mathbf{R}^d)$ we consider the norms

$$\|f\|_{\mathcal{H}_{s;r}} \equiv \sup_{\alpha \in \mathbf{N}^d} \left(|c_\alpha(f)| e^{r|\alpha|^{\frac{1}{2s}}} \right) \quad \text{when } s \in \mathbf{R}_+$$

and

$$\|f\|_{\mathcal{H}_{s;r}} \equiv \sup_{\alpha \in \mathbf{N}^d} \left(|c_\alpha(f)| r^{-|\alpha|} \alpha^{\frac{1}{2\sigma}} \right) \quad \text{when } s = b_\sigma,$$

when $r > 0$ is fixed. Then the set $\mathcal{H}_{s;r}(\mathbf{R}^d)$ consists of all $f \in C^\infty(\mathbf{R}^d)$ such that $\|f\|_{\mathcal{H}_{s;r}}$ is finite. It follows that $\mathcal{H}_{s;r}(\mathbf{R}^d)$ is a Banach space.

The Pilipović spaces $\mathcal{H}_s(\mathbf{R}^d)$ and $\mathcal{H}_{0,s}(\mathbf{R}^d)$ are the inductive limit and the projective limit, respectively, of $\mathcal{H}_{s;r}(\mathbf{R}^d)$ with respect to $r > 0$. In particular,

$$\mathcal{H}_s(\mathbf{R}^d) = \bigcup_{r>0} \mathcal{H}_{s;r}(\mathbf{R}^d) \quad \text{and} \quad \mathcal{H}_{0,s}(\mathbf{R}^d) = \bigcap_{r>0} \mathcal{H}_{s;r}(\mathbf{R}^d)$$

and it follows that $\mathcal{H}_s(\mathbf{R}^d)$ is complete, and that $\mathcal{H}_{0,s}(\mathbf{R}^d)$ is a Fréchet space. It is well-known that the identities (0.3) and (0.4) also hold in topological sense (cf. [7]).

By extending $\mathbf{R}_+$ into $\mathbf{R}_b \equiv \mathbf{R}_+ \cup \{b_\sigma\}_{\sigma>0}$ and letting

$$s_1 < b_{\sigma_1} < b_{\sigma_2} < s_2 \quad \text{when } s_2 \geq \frac{1}{2}, \quad s_1 < \frac{1}{2} \quad \text{and } \sigma_1 < \sigma_2,$$

we have

$$\mathcal{H}_{s_1}(\mathbf{R}^d) \subseteq \mathcal{H}_{0,s_2}(\mathbf{R}^d) \subseteq \mathcal{H}_{s_2}(\mathbf{R}^d), \quad s_1, s_2 \in \mathbf{R}_b \text{ and } s_1 < s_2.$$

We also need some facts about weights and modulation spaces, a family of (quasi-)Banach spaces, introduced by Feichtinger in [2]. A weight on $\mathbf{R}^d$ is a function $\omega \in L_{loc}^\infty(\mathbf{R}^d)$ such that $\omega(x) > 0$ for every $x \in \mathbf{R}^d$ and $1/\omega \in L_{loc}^\infty(\mathbf{R}^d)$. The weight ω on $\mathbf{R}^d$ is called moderate of polynomial type, if there is an integer $N \geq 0$ such that

$$\omega(x+y) \lesssim \omega(x)(1+|y|)^N, \quad x, y \in \mathbf{R}^d.$$

The set of moderate weights of polynomial type on $\mathbf{R}^d$ is denoted by $\mathcal{P}(\mathbf{R}^d)$.

Let $p, q \in (0, \infty]$, $\phi \in \mathcal{S}(\mathbf{R}^d) \setminus 0$ and $\omega \in \mathcal{P}(\mathbf{R}^{2d})$ be fixed. Then the modulation space, $M_{(\omega)}^{p,q}(\mathbf{R}^d)$ consists of all $f \in \mathcal{S}'(\mathbf{R}^d)$ such that

$$\|f\|_{M_{(\omega)}^{p,q}} \equiv \|V_\phi f \cdot \omega\|_{L^{p,q}}$$

is finite. Here $V_\phi f$ is the short-time Fourier transform of f with respect to ϕ , given by

$$V_\phi f(x, \xi) = (2\pi)^{-\frac{d}{2}} \langle f, e^{i(\cdot, \xi)} \overline{\phi(\cdot - x)} \rangle$$

and

$$\|F\|_{L^{p,q}} = \|F\|_{L^{p,q}(\mathbf{R}^{2d})} \equiv \|g_F\|_{L^q(\mathbf{R}^d)} \quad \text{when } g_F(\xi) = \|F(\cdot, \xi)\|_{L^p(\mathbf{R}^d)}$$

and F is measurable on $\mathbf{R}^{2d}$.

Modulation spaces possess several convenient properties. For example we have the following proposition (see [2, 4] for proofs).

Proposition 1.1 Let $p, q \in (0, \infty]$ and $\omega \in \mathcal{P}(\mathbf{R}^{2d})$. Then the following is true:

- (1) $M_{(\omega)}^{p,q}(\mathbf{R}^d)$ is a quasi-Banach space under the quasi-norm $\|\cdot\|_{M^{p,q}}$ above. If in addition $p, q \geq 1$, then $\|\cdot\|_{M^{p,q}}$ is a norm and $M_{(\omega)}^{p,q}(\mathbf{R}^d)$ is a Banach space;
- (2) the definition of $M_{(\omega)}^{p,q}(\mathbf{R}^d)$ is independent of the choice of ϕ above and different $\phi \in \mathcal{S}(\mathbf{R}^d) \setminus 0$ gives rise to equivalent quasi-norms;
- (3) $M_{(\omega)}^{p,q}(\mathbf{R}^d)$ increases with p and q (also in topological sense).

2 Characterizations of $\mathcal{H}_{b_\sigma}(\mathbf{R}^d)$ and $\mathcal{H}_{0,b_\sigma}(\mathbf{R}^d)$ in terms of powers of the harmonic oscillator

In this section we deduce characterizations of the test function spaces $\mathcal{H}_{0,b_\sigma}(\mathbf{R}^d)$ and $\mathcal{H}_{b_\sigma}(\mathbf{R}^d)$.

More precisely we have the following.

Theorem 2.1 Let $\sigma > 0$, $N, N_0 \in \mathbf{N}$ be such that $N_0\sigma > 1$, $p_0 \in [1, \infty]$, $p, q \in (0, \infty]$, $\omega \in \mathcal{P}(\mathbf{R}^{2d})$ and let $f \in C^\infty(\mathbf{R}^d)$ be given by (0.1). Then the following conditions are equivalent:

- (1) $f \in \mathcal{H}_{b_\sigma}(\mathbf{R}^d)$ ($f \in \mathcal{H}_{0,b_\sigma}(\mathbf{R}^d)$);
- (2) for some $r > 0$ (for every $r > 0$) it holds

$$\{c_\alpha(f)r^{-|\alpha|}(\alpha!)^{\frac{1}{2\sigma}}\}_{\alpha \in \mathbf{N}^d} \in \ell^q(\mathbf{N}^d);$$

- (3) for some $r > 0$ (for every $r > 0$) it holds

$$\|H_d^N f\|_{L^{p_0}} \lesssim 2^N r^{\frac{N}{\log(N\sigma)}} \left(\frac{2N\sigma}{\log(N\sigma)} \right)^{N(1-\frac{1}{\log(N\sigma)})}, \quad N \geq N_0; \quad (2.1)$$

- (4) for some $r > 0$ (for every $r > 0$) it holds

$$\|H_d^N f\|_{M_{(\omega)}^{p,q}} \lesssim 2^N r^{\frac{N}{\log(N\sigma)}} \left(\frac{2N\sigma}{\log(N\sigma)} \right)^{N(1-\frac{1}{\log(N\sigma)})}, \quad N \geq N_0. \quad (2.2)$$

We need some preparations for the proof. In the following proposition we treat separately the equivalence between (3) and (4) in Theorem 2.1.

Proposition 2.2 Let $p_0 \in [1, \infty]$, $p, q \in (0, \infty]$, $\sigma > 0$, $N_0 > \sigma^{-1}$ be an integer and let $\omega \in \mathcal{P}(\mathbf{R}^{2d})$. Then the following conditions are equivalent:

- (1) (2.1) holds for some $r > 0$ (for every $r > 0$);
- (2) (2.2) holds for some $r > 0$ (for every $r > 0$).

We need the following lemma for the proof of Proposition 2.2.

Lemma 2.3 Let $R \geq e$, $I = (0, R]$,

$$g(r, t_1, t_2) \equiv \frac{r^{\frac{t_2}{\log t_2}}}{r^{\frac{t_1}{\log t_1}}} \quad \text{and} \quad h(t_1, t_2) \equiv \frac{\left(\frac{2t_2}{\log t_2}\right)^{t_2\left(1-\frac{1}{\log t_2}\right)}}{\left(\frac{2t_1}{\log t_1}\right)^{t_1\left(1-\frac{1}{\log t_1}\right)}},$$

when $t_1, t_2 > e$ and $r > 0$. Then

$$0 \leq g(r, t_1, t_2) \leq C \quad \text{and} \quad 0 \leq h(t_1, t_2) \leq \left(\frac{2t_1}{\log t_1} \right)^C \quad (2.3)$$

when

$$t_1, t_2 > R, \quad 0 \leq t_2 - t_1 \leq R, \quad r \in I,$$

for some constant $C > 0$ which only depends on R .

Proof Since $t \mapsto \frac{t}{\log t}$ is increasing when $t \geq e$, g is upper bounded by one when $r \leq 1$, and the boundedness of g follows in this case.

If $r \geq 1$, $t = t_1$, $u = t_2 - t_1 > 0$ and $\rho = \log r$, then

$$\begin{aligned} 0 \leq \log g(r, t_1, t_2) &= \left(\frac{t+u}{\log(t+u)} - \frac{t}{\log t} \right) \rho \\ &= \frac{t}{\log t} \left(\frac{1 + \frac{u}{t}}{1 + \frac{\log(1+\frac{u}{t})}{\log t}} - 1 \right) \rho = \frac{t}{\log t} \left(\frac{\frac{u}{t} - \frac{\log(1+\frac{u}{t})}{\log t}}{1 + \frac{\log(1+\frac{u}{t})}{\log t}} \right) \rho \\ &< \frac{t}{\log t} \cdot \frac{u}{t} \cdot \rho \\ &= \frac{u\rho}{\log t} \leq C \end{aligned}$$

for some constant C which only depends on R . This shows the boundedness of g .

Next we show the estimates for $h(t_1, t_2)$ in (2.3). By taking the logarithm of $h(t_1, t_2) = h(t, t_2)$ we get

$$\log h(t, t_2) = t_2 \log \left(\frac{2t_2}{\log t_2} \right) - t \log \left(\frac{2t}{\log t} \right) - b(t, t_2),$$

where

$$b(t, t_2) = \left(\frac{t_2}{\log t_2} \log \left(\frac{2t_2}{\log t_2} \right) - \frac{t}{\log t} \log \left(\frac{2t}{\log t} \right) \right).$$

Since $b(t, t_2) > 0$ when $t_2 > t$, we get

$$\begin{aligned} \log h(t_1, t_2) &< t_2 \log \left(\frac{2t_2}{\log t_2} \right) - t \log \left(\frac{2t}{\log t} \right) \\ &= (t+u) \left(\log \left(\frac{2t}{\log t} \right) + \log \left(\frac{1 + \frac{u}{t}}{1 + \frac{\log(1+\frac{u}{t})}{\log t}} \right) \right) - t \log \left(\frac{2t}{\log t} \right) \\ &\leq u \log \left(\frac{2t}{\log t} \right) + t \log \left(1 + \frac{u}{t} \right) + C \\ &\leq u \log \left(\frac{2t}{\log t} \right) + u + C \end{aligned}$$

for some constant $C \geq 0$. Here we have used that $t_1, t_2 > R \geq e$ and the fact that $t \mapsto \frac{t}{\log t}$ increases for $t \geq R$. $\square$

Proof of Proposition 2.2. First we prove that (2.2) is independent of $N_0 > \sigma^{-1}$ when $p, q \geq 1$. Evidently, if (2.2) is true for N_0 , then it is true for any larger replacement of N_0 . On the other hand, the map

$$H_d^N : M_{(v_N \omega)}^{p,q}(\mathbf{R}^d) \rightarrow M_{(\omega)}^{p,q}(\mathbf{R}^d), \quad v_N(x, \xi) = (1 + |x|^2 + |\xi|^2)^N, \quad (2.4)$$

and its inverse are continuous and bijective (cf. e. g. [8, Theorem 3.10]). Hence, if $\sigma^{-1} < N_1 \leq N_0$, $N_2 = N_0 - N_1 \geq 0$ and (2.2) holds for N_0 , then

$$\|H_d^{N_1} f\|_{M_{(\omega)}^{p,q}} \lesssim \|H_d^{N_0} f\|_{M_{(\omega/v_{N_2})}^{p,q}} \lesssim \|H_d^{N_0} f\|_{M_{(\omega)}^{p,q}} < \infty,$$

and a straight-forward combination of these estimates and (2.3) shows that (2.2) holds for N_1 in place of N_0 . This implies that (2.2) is independent of $N_0 > \sigma^{-1}$ when $p, q \geq 1$.

Next we prove that (2.2) is independent of the choice of $\omega \in \mathcal{P}(\mathbf{R}^{2d})$. By the first part of the proof, we may assume that $N_0 \sigma > e$. For every $\omega_1, \omega_2 \in \mathcal{P}(\mathbf{R}^{2d})$, we may find an integer $N_0 > \sigma^{-1}e$ such that

$$\frac{1}{v_{N_0}} \lesssim \omega_1, \omega_2 \lesssim v_{N_0},$$

and then

$$\|f\|_{M_{(1/v_{N_0})}^{p,q}} \lesssim \|f\|_{M_{(\omega_1)}^{p,q}}, \|f\|_{M_{(\omega_2)}^{p,q}} \lesssim \|f\|_{M_{(v_{N_0})}^{p,q}}. \quad (2.5)$$

Hence the stated invariance follows if we prove that (2.2) holds for $\omega = v_{N_0}$, if it is true for $\omega = 1/v_{N_0}$.

Therefore, assume that (2.2) holds for $\omega = 1/v_{N_0}$. Let $f_N = H_d^N f$, $u = 2N_0\sigma$, $t = t_1 = N\sigma$, $N_2 = N + 2N_0$ and $t_2 = t_1 + u = N_2\sigma$. If $N \geq 2N_0$, then the bijectivity of (2.4) gives

$$\begin{aligned} \frac{\|f_N\|_{M_{(v_{N_0})}^{p,q}}^\sigma}{2^{N\sigma} r^{\frac{N\sigma}{\log(N\sigma)}} \left(\frac{2N\sigma}{\log(N\sigma)}\right)^{N\sigma \left(1 - \frac{1}{\log(N\sigma)}\right)}} &= \frac{\|f_N\|_{M_{(v_{N_0})}^{p,q}}^\sigma}{2^t r^{\frac{t}{\log t}} \left(\frac{2t}{\log t}\right)^{t \left(1 - \frac{1}{\log t}\right)}} \\ &\lesssim \frac{\|f_{N+2N_0}\|_{M_{(1/v_{N_0})}^{p,q}}^\sigma}{2^t r^{\frac{t}{\log t}} \left(\frac{2t}{\log t}\right)^{t \left(1 - \frac{1}{\log t}\right)}} \\ &= 2^u g(r, t_1, t_2) h(t_1, t_2) \cdot \frac{\|f_{N_2}\|_{M_{(1/v_{N_0})}^{p,q}}^\sigma}{2^{N_2\sigma} r^{\frac{t_2}{\log(t_2)}} \left(\frac{2t_2}{\log t_2}\right)^{t_2 \left(1 - \frac{1}{\log t_2}\right)}}, \end{aligned} \quad (2.6)$$

where $g(r, t_1, t_2)$ and $h(t_1, t_2)$ are the same as in Lemma 2.3. A combination of Lemma 2.3, (2.6) and the fact that $N\sigma > e$ shows that (2) is independent of $\omega \in \mathcal{P}(\mathbf{R}^{2d})$. For general $p, q > 0$, the invariance of (2.2) with respect to ω , p and q is a consequence of the embeddings

$$M_{(v_N \omega)}^\infty(\mathbf{R}^d) \subseteq M_{(\omega)}^{p,q}(\mathbf{R}^d) \subseteq M_{(\omega)}^\infty(\mathbf{R}^d), \quad N > d \left(\frac{1}{p} + \frac{1}{q} \right)$$

(see e. g. [4, Theorem 3.4] or [10, Proposition 3.5]).

The equivalence between (1) and (2) now follows from these invariance properties and the continuous embeddings

$$M^{p_0, q_1} \subseteq L^{p_0} \subseteq M^{p_0, q_2}, \quad q_1 = \min(p_0, p'_0), \quad q_2 = \max(p_0, p'_0),$$

which can be found in e.g. [9, Proposition 1.7]. $\square$

Proposition 2.4 *Let $f \in C^\infty(\mathbf{R}^d)$ and $\sigma > 0$. If*

$$\|H_d^N f\|_{L^2} \lesssim 2^N r^{\frac{N}{\log(N\sigma)}} \left(\frac{2N\sigma}{\log(N\sigma)} \right)^{N(1 - \frac{1}{\log(N\sigma)})}, \quad N \in \mathbf{N}, \quad N\sigma \geq e, \quad (2.7)$$

for some $r > 0$ (for every $r > 0$), then

$$|c_\alpha(f)| \lesssim r^{|\alpha|} |\alpha|^{-\frac{|\alpha|}{2\sigma}}, \quad \alpha \in \mathbf{N}^d, \quad (2.8)$$

for some $r > 0$ (for every $r > 0$).

Proposition 2.5 *Let $f \in C^\infty(\mathbf{R}^d)$ and $\sigma > 0$. If (2.8) holds for some $r > 0$ (for every $r > 0$), then (2.7) holds for some $r > 0$ (for every $r > 0$).*

For the proofs we need some preparation lemmas.

Lemma 2.6 *Let $\sigma > 0$, $\sigma_0 \in [0, \sigma]$ and let*

$$F(r, t) = \left(\frac{2t}{\log t} \right)^{t(1 - \frac{1}{\log t})} r^{\frac{t}{\log t}}, \quad r \geq 0, \quad t \geq e \cdot \max(1, \sigma).$$

Then

$$F(r, t) \leq F(r, t + \sigma_0), \quad r \in [1, \infty), \quad (2.9)$$

and

$$F(r, t) \leq F(r^{1 - \frac{1}{e}}, t + \sigma_0), \quad r \in (0, 1]. \quad (2.10)$$

Proof If $r \geq 1$, then it follows by straight-forward tests with derivatives that $F(r, t)$ is increasing with respect to $t \geq e$. This gives (2.9).

In order to prove (2.10), let $t_1 = t + \sigma_0$ and

$$h(t_1, \sigma_0) = \frac{1 - \frac{\sigma_0}{t_1}}{\log\left(1 - \frac{\sigma_0}{t_1}\right)}, \quad 1 + \frac{\log\left(1 - \frac{\sigma_0}{t_1}\right)}{\log t_1}$$

where $0 \leq \sigma_0 \leq \sigma$. Then

$$\left(\frac{2t}{\log t} \right)^{t(1 - \frac{1}{\log t})} r^{\frac{t}{\log t}} \leq \left(\frac{2t_1}{\log t_1} \right)^{t_1(1 - \frac{1}{\log t_1})} r^{\frac{t}{\log t}} \quad (2.11)$$

and

$$\frac{2t}{\log t} = h(t_1, \sigma_0) \cdot \frac{2t_1}{\log t_1}.$$

Since

$$0 \leq \frac{\sigma_0}{t_1} \leq \frac{1}{e} \quad \text{and} \quad -1 < \frac{\log\left(1 - \frac{\sigma_0}{t_1}\right)}{\log t_1} \leq 0$$

we get

$$h(t_1, \sigma_0) \geq 1 - \frac{\sigma_0}{t_1} \geq 1 - \frac{1}{e}.$$

Hence the facts $\frac{t_1}{\log t_1} \geq 1$ and $0 < r \leq 1$ give

$$r^{\frac{t}{\log t}} = r^{h(t_1, \sigma_0) \frac{t_1}{\log t_1}} \leq r^{\left(1 - \frac{1}{e}\right) \frac{t_1}{\log t_1}}.$$

A combination of the latter inequality with (2.11) gives

$$F(r, t) \leq \left(\frac{2t_1}{\log t_1}\right)^{t_1 \left(1 - \frac{1}{\log t_1}\right)} \left(r^{1 - \frac{1}{e}}\right)^{\frac{t_1}{\log t_1}} = F\left(r^{1 - \frac{1}{e}}, t_1\right).$$

□

Lemma 2.7 Let $s \geq \sigma(e + 1) + e^2$

$$\Omega_1 = [e, \infty) \cap (\sigma \cdot \mathbf{N}) \quad \text{and} \quad \Omega_2 = [e, \infty).$$

Then the following is true:

(1) for any $r_2 > 0$, there is an $r_1 > 0$ such that

$$\inf_{t \in \Omega_j} \left(s^{-t} \left(\frac{2t}{\log t} \right)^{t \left(1 - \frac{1}{\log t}\right)} r_1^{\frac{t}{\log t}} \right) \lesssim r_2^s s^{-\frac{s}{2}}, \quad j = 1, 2; \quad (2.12)$$

(2) for any $r_1 > 0$, there is an $r_2 > 0$ such that (2.12) holds.

Proof First prove the result for $j = 2$. Let

$$x = \log t, \quad y = \log s \geq \log(\sigma(e + 1) + e^2) > 2, \quad \rho_j = \log r_j, \quad j = 1, 2.$$

By applying the logarithm on (2.12), the statements (1) and (2) follow if we prove:

(1)' for any $\rho_2 \in \mathbf{R}$, there is a $\rho_1 \in \mathbf{R}$ such that

$$\inf_{x \geq x_0} F(x) \leq 0, \quad x_0 = \log(\sigma(e + 1) + e^2) \quad (2.13)$$

where

$$F(x) = -e^x y + e^x \left(1 - \frac{1}{x}\right) (x + \log 2 - \log x) + \rho_1 \frac{e^x}{x} - \rho_2 e^y + \frac{e^y y}{2} \quad (2.14)$$

(2)' for any $\rho_1 \in \mathbf{R}$, there is a $\rho_2 \in \mathbf{R}$ such that (2.13) holds.

We choose

$$x = y + \log y - \log 2 \geq \log s \geq x_0 \quad \text{and let} \quad h = g(y),$$

where

$$g(u) = \frac{\log u - \log 2}{u}.$$

Obviously, x increases with y , and by function investigations it follows that

$$0 = g(2) < g(u) \leq g(2e) = \frac{1}{2e}, \quad u > 2,$$

giving that $0 < h \leq \frac{1}{2e} < 1$. Then (2.14) becomes

$$\begin{aligned} e^{-y} F(y + \log y - \log 2) &= -\frac{y^2}{2} + \frac{y}{2} \left(1 - \frac{1}{y + \log \frac{y}{2}} \right) \left(y + \log y - \log \left(y + \log \frac{y}{2} \right) \right) \\ &\quad + \frac{\rho_1 y}{2(y + \log \frac{y}{2})} - \rho_2 + \frac{y}{2} \\ &= -\frac{y}{2} \log(1+h) + \frac{\log y + \log(1+h)}{2(1+h)} + \frac{\rho_1 - \log 2}{2(1+h)} - \rho_2. \end{aligned}$$

If $\rho_1 \in \mathbf{R}$ is fixed, then we choose $\rho_2 \in \mathbf{R}$ such that

$$\frac{\rho_1 - \log 2}{2(1+h)} - \rho_2 \leq -C_0 \quad (2.15)$$

for some large number $C_0 > 0$. In the same way, if $\rho_2 \in \mathbf{R}$ is fixed, then we choose $\rho_1 \in \mathbf{R}$ such that (2.15) holds. For such choices and the fact that $0 < h < 1$, the inequalities

$$0 < h - \frac{h^2}{2} \leq \log(1+h) \leq h$$

give

$$\begin{aligned} F(y + \log y - \log 2) &\leq e^y \left(-\frac{y}{2} \log(1+h) + \frac{\log y + \log(1+h)}{2(1+h)} - C_0 \right) \\ &\leq e^y \left(-\frac{y}{2} \log(1+h) + \frac{\log y + \log(1+h)}{2} - C_0 \right) \\ &\leq e^y \left(-\frac{\log y - \log 2}{2} + \frac{(\log y - \log 2)^2}{4y} + \frac{1}{2} (\log y + h) - C_0 \right) \\ &\leq e^y \left(\frac{1}{2} \log 2 + \frac{(\log y - \log 2)^2}{4y} + \frac{h}{2} - C_0 \right) < 0, \end{aligned}$$

provided C_0 was chosen large enough. This gives the result in the case $j = 2$.

Next we prove the result for $j = 1$. Let $r_2 > 0$. By the first part of the proof, there are $t_1 \geq e(\sigma + 1) + \sigma$ and $r_0 > 0$ such that

$$s^{-t_1} \left(\frac{2t_1}{\log t_1} \right)^{t_1 \left(1 - \frac{1}{\log t_1} \right)} r_0^{\frac{t_1}{\log t_1}} \leq r_2^s s^{-\frac{s}{2}}.$$

Let $r_1 = r_0$ if $r_0 \geq 1$ and $r_1 = r_0^{\frac{e}{e-1}}$ otherwise. By Lemma 2.6 it follows that

$$s^{-t} \left(\frac{2t}{\log t} \right)^{t \left(1 - \frac{1}{\log t} \right)} r_1^{\frac{t}{\log t}} \leq r_2^s s^{-\frac{s}{2}}$$

holds when $t = N\sigma$ and $N \in \mathbf{N}$ is chosen such that $0 \leq t_1 - N\sigma \leq \sigma$. Observe that Lemma 2.6 can be applied since $N\sigma \geq e(\sigma + 1)$. This gives (1) for $j = 1$.

By similar arguments, (2) for $j = 1$ follows from (2) in the case $j = 2$. The details are left for the reader. $\square$

Proof of Proposition 2.4. Suppose that (2.7) holds for some $r = r_1 > 0$. By

$$c_\alpha(H_d^N f) = (2|\alpha| + d)^N c_\alpha(f), \quad |c_\alpha(H_d^N f)| \leq \|H_d^N f\|_{L^2} \quad (2.16)$$

and (2.7) we get

$$\begin{aligned} |c_\alpha(f)| &= \frac{|c_\alpha(H_d^N f)|}{(2|\alpha| + d)^N} \\ &\lesssim \left(|\alpha| + \frac{d}{2}\right)^{-N} r_1^{\frac{N}{\log(N\sigma)}} \left(\frac{2N\sigma}{\log(N\sigma)}\right)^{N\left(1 - \frac{1}{\log(N\sigma)}\right)} \\ &\leq \left(|\alpha|^{-N\sigma} r_1^{\frac{N\sigma}{\log(N\sigma)}} \left(\frac{2N\sigma}{\log(N\sigma)}\right)^{N\sigma\left(1 - \frac{1}{\log(N\sigma)}\right)}\right)^{\frac{1}{\sigma}}. \end{aligned}$$

By taking the infimum over all $N \geq 0$, it follows from Lemma 2.7 (2) that

$$|c_\alpha(f)| \lesssim \left(r_2^{|\alpha|} |\alpha|^{-\frac{|\alpha|}{2}}\right)^{\frac{1}{\sigma}} = r^{|\alpha|} |\alpha|^{-\frac{|\alpha|}{2\sigma}}, \quad |\alpha| \geq 2\sigma(e+1) + e^2,$$

for some $r_2 > 0$, where $r = r_2^{\frac{1}{\sigma}}$. Hence (2.8) holds for some $r > 0$.

By similar arguments, using (1) instead of (2) in Lemma 2.7, it follows that if (2.7) holds for every $r > 0$, then (2.8) holds for every $r > 0$. $\square$

For the proof of Proposition 2.5 we will use the following result which is essentially a slight clarification of [3, Lemma 2]. The proof is therefore omitted.

Lemma 2.8 *Let $r > 0$ and*

$$f(s, t, r) = \frac{s^{2t}(2re)^s}{s^s}, \quad s > 1, t \geq 0.$$

Then there exist a positive increasing function θ on $[0, \infty)$ and a constant $t_0 = t_0(r) > e$ which only depends on r such that

$$\max_{s>0} f(s, t, r) \leq \left(\frac{2t}{\log t}\right)^{2t\left(1 - \frac{1}{\log t}\right)} (\theta(r)r)^{\frac{2t}{\log t}}, \quad t \geq t_0(r). \quad (2.17)$$

Remark 2.9 The constants s , t and $t_0(r)$ in Lemma 2.8 are denoted by t , N and $N_0(r)$, respectively in Lemmas 1 and 2 in [3]. In the latter results it is understood that N and $N_0(r)$ are integers. On the other hand, it is evident from the proofs of these results that they also hold when N and $N_0(r)$ are allowed to be in $\mathbf{R}_+$.

Proof of Proposition 2.5. Let θ be as in Lemma 2.8 and let $\rho \in (0, 1)$. Suppose that (2.8) holds for some $r > 0$ and let $r_2 > r^\sigma$. From (2.8) and (2.16) we get

$$\begin{aligned} \|H_d^N f\|_{L^2}^2 &= \sum_{\alpha \in \mathbf{N}^d} |(2|\alpha| + d)^N c_\alpha(f)|^2 \\ &\lesssim \sup_{|\alpha| \geq 1} \left((2|\alpha| + d)^{2N} r_2^{\frac{2|\alpha|}{\sigma}} |\alpha|^{-\frac{|\alpha|}{\sigma}} \right) \\ &= \sup_{s \geq 1} \left(2^{2t} \left(s + \frac{d}{2}\right)^{2t} r_2^{2s} s^{-s} \right)^{\frac{1}{\sigma}}, \end{aligned}$$

where $s = |\alpha|$ and $t = N\sigma$. Since $0 < \rho < 1$ we have

$$\begin{aligned} s^s &= \left(s - \frac{d}{2} + \frac{d}{2}\right)^{s-\frac{d}{2}} s^{\frac{d}{2}} = \left(s - \frac{d}{2}\right)^{s-\frac{d}{2}} s^{\frac{d}{2}} \left(1 + \frac{d}{2s-d}\right)^{s-\frac{d}{2}} \\ &\leq \left(s - \frac{d}{2}\right)^{s-\frac{d}{2}} (se)^{\frac{d}{2}} \lesssim \left(s - \frac{d}{2}\right)^{s-\frac{d}{2}} \rho^{-2s}. \end{aligned}$$

This gives

$$\begin{aligned} \|H_d^N f\|_{L^2}^2 &\lesssim \sup_{s \geq 1} \left(2^{2t} \left(s + \frac{d}{2}\right)^{2t} r_2^{2s} s^{-s}\right)^{\frac{1}{\sigma}} \\ &= \sup_{s \geq 1 + \frac{d}{2}} \left(2^{2t} s^{2t} r_2^{2s-d} \left(s - \frac{d}{2}\right)^{-(s-\frac{d}{2})}\right)^{\frac{1}{\sigma}} \\ &\lesssim \sup_{s \geq 1 + \frac{d}{2}} \left(2^{2t} s^{2t} \left(\frac{r_2}{\rho}\right)^{2s} s^{-s}\right)^{\frac{1}{\sigma}}. \end{aligned} \quad (2.18)$$

Using (2.18) and Lemma 2.8 we obtain

$$\begin{aligned} \|H_d^N f\|_{L^2}^2 &\lesssim \sup_{s \geq 1 + \frac{d}{2}} \left(2^{2t} s^{2t} \left(\frac{r_2}{\rho}\right)^{2s} s^{-s}\right)^{\frac{1}{\sigma}} \\ &= \sup_{s \geq 1 + \frac{d}{2}} \left(2^{2t} s^{2t} (2r_3 e)^s s^{-s}\right)^{\frac{1}{\sigma}} \\ &\lesssim \left(2^{2t} \left(\frac{2t}{\log t}\right)^{2t(1-\frac{1}{\log t})} (\theta(r_3)r_3)^{\frac{2t}{\log t}}\right)^{\frac{1}{\sigma}} \\ &= 2^{2N} (r_3 \theta(r_3))^{\frac{2N}{\log(N\sigma)}} \left(\frac{2N\sigma}{\log(N\sigma)}\right)^{2N(1-\frac{1}{\log(N\sigma)})} \end{aligned}$$

when

$$r_3 = \frac{r_2^2}{2\rho^2 e} \quad \text{and} \quad N\sigma \geq t_0(r_3).$$

This gives the result in the Roumieu case.

By similar argument, using the fact that the non-negative function θ is increasing, it also follows that (2.7) holds for every $r > 0$ when (2.8) holds for every $r > 0$, and the result follows. $\square$

Proof of Theorem 2.1. We have

$$\begin{aligned} \|\{c_\alpha(f)r^{-|\alpha|}(\alpha!)^{\frac{1}{2\sigma}}\}_{\alpha \in \mathbb{N}^d}\|_{\ell^\infty(\mathbb{N}^d)} &\leq \|\{c_\alpha(f)r^{-|\alpha|}(\alpha!)^{\frac{1}{2\sigma}}\}_{\alpha \in \mathbb{N}^d}\|_{\ell^q(\mathbb{N}^d)} \\ &\lesssim \|\{c_\alpha(f)(cr)^{-|\alpha|}(\alpha!)^{\frac{1}{2\sigma}}\}_{\alpha \in \mathbb{N}^d}\|_{\ell^\infty(\mathbb{N}^d)} \end{aligned}$$

when $c \in (0, 1)$, which shows that (2) is independent of the choice of q . The equivalence between (1) and (2) now follows by the definitions and choosing $q = \infty$ in (2).

By Proposition 2.2 we may assume that $p = 2$. The result now follows from Propositions 2.4 and 2.5, together with the fact that

$$(d \cdot e)^{-|\alpha|} |\alpha|^{|\alpha|} \leq \alpha! \leq |\alpha|^{|\alpha|}, \quad \alpha \in \mathbb{N}^d.$$

□

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References

1. Chung, J., Chung, S.-Y., Kim, D.: Characterizations of the Gelfand-Shilov spaces via Fourier transforms. *Proc. Am. Math. Soc.* **124**, 2101–2108 (1996)
2. Feichtinger, H.G.: Modulation spaces on locally compact abelian groups. Technical report, University of Vienna, Vienna, 1983. In: Krishna, M., Radha, R., Thangavelu, S. (eds.) *Wavelets and Their Applications*, pp. 99–140. Allied Publishers Private Limited, New Delhi (2003)
3. Fernández, C., Galbis, A., Toft, J.: The Bargmann transform and powers of harmonic oscillator on Gelfand-Shilov subspaces. *RACSAM* **111**, 1–13 (2017)
4. Galperin, Y.V., Samarah, S.: Time-frequency analysis on modulation spaces $M_m^{p,q}$, $0 < p, q \leq \infty$. *Appl. Comput. Harmon. Anal.* **16**, 1–18 (2004)
5. Gelfand, I.M., Shilov, G.E.: *Generalized Functions, II–III*. Academic Press, New York (1968)
6. Hörmander, L.: *The Analysis of Linear Partial Differential Operators*, vol. I. Springer, Berlin (1983)
7. Pilipović, S.: Tempered ultradistributions. *Boll. UMI.* **7**, 235–251 (1988)
8. Signahl, M., Toft, J.: Mapping properties for the Bargmann transform on modulation spaces. *J. Pseudo Differ. Oper. Appl.* **3**, 1–30 (2012)
9. Toft, J.: Continuity properties for modulation spaces, with applications to pseudo-differential operators. *I. J. Funct. Anal.* **207**, 399–429 (2004)
10. Toft, J.: Gabor analysis for a broad class of quasi-Banach modulation spaces. In: Pilipović, S., Toft, J. (eds.) *Pseudo-differential Operators, Generalized Functions, Operator Theory: Advances and Applications*, pp. 249–278. Birkhäuser, Berlin (2015)
11. Toft, J.: Images of function and distribution spaces under the Bargmann transform. *J. Pseudo Differ. Oper. Appl.* **8**, 83–139 (2017)

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