On metrizable subspaces and quotients of non-Archimedean spaces $C_p(X, \mathbb{K})$

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Abstract

Let $\mathbb{K}$ be a non-trivially valued non-Archimedean complete field. Let $\ell_\infty(\mathbb{N}, \mathbb{K})$ [$\ell_c(\mathbb{N}, \mathbb{K})$; $c_0(\mathbb{N}, \mathbb{K})$] be the space of all sequences in $\mathbb{K}$ that are bounded [relatively compact; convergent to 0] with the topology of pointwise convergence (i.e. with the topology induced from $\mathbb{K}^{\mathbb{N}}$). Let X be an infinite ultraregular space and let $C_p(X, \mathbb{K})$ be the space of all continuous functions from X to $\mathbb{K}$ endowed with the topology of pointwise convergence. It is easy to see that $C_p(X, \mathbb{K})$ is metrizable if and only if X is countable. We show that for any X [with an infinite compact subset] the space $C_p(X, \mathbb{K})$ has an infinite-dimensional [closed] metrizable subspace isomorphic to $c_0(\mathbb{N}, \mathbb{K})$. Next we prove that $C_p(X, \mathbb{K})$ has a quotient isomorphic to $c_0(\mathbb{N}, \mathbb{K})$ if and only if it has a complemented subspace isomorphic to $c_0(\mathbb{N}, \mathbb{K})$. It follows that for any extremally disconnected compact space X the space $C_p(X, \mathbb{K})$ has no quotient isomorphic to the space $c_0(\mathbb{N}, \mathbb{K})$; in particular, for any infinite discrete space D the space $C_p(\beta D, \mathbb{K})$ has no quotient isomorphic $c_0(\mathbb{N}, \mathbb{K})$. Finally we investigate the question for which X the space $C_p(X, \mathbb{K})$ has an infinite-dimensional metrizable quotient. We show that for any infinite discrete space D the space $C_p(\beta D, \mathbb{K})$ has an infinite-dimensional metrizable quotient isomorphic to some subspace $\ell_c^0(\mathbb{N}, \mathbb{K})$ of $\mathbb{K}^{\mathbb{N}}$. If $\mathbb{K}$ is locally compact then $\ell_c^0(\mathbb{N}, \mathbb{K}) = \ell_\infty(\mathbb{N}, \mathbb{K})$. If $|n1_{\mathbb{K}}| \neq 1$ for some $n \in \mathbb{N}$, then $\ell_c^0(\mathbb{N}, \mathbb{K}) = \ell_c(\mathbb{N}, \mathbb{K})$. In particular, $C_p(\beta D, \mathbb{Q}_q)$ has a quotient isomorphic to $\ell_\infty(\mathbb{N}, \mathbb{Q}_q)$ and $C_p(\beta D, \mathbb{C}_q)$ has a quotient isomorphic to $\ell_c(\mathbb{N}, \mathbb{C}_q)$ for any prime number q .

Keywords Efimov space · Pseudocompact space · Function space · Closed subspace · Quotient space · Metrizable quotient

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1 Introduction and main problem

Let $\mathbb{K}$ be a non-trivially valued non-Archimedean complete field. The identity element of the multiplicative group $\mathbb{K}^*$ of the field $\mathbb{K}$ is denoted by $1_{\mathbb{K}}$. For all $\alpha, \beta \in \mathbb{K}$ we have $|\alpha + \beta| \leq \max\{|\alpha|, |\beta|\}$ (the strong triangle inequality); if additionally $|\alpha| \neq |\beta|$, then $|\alpha + \beta| = \max\{|\alpha|, |\beta|\}$.

Let E be a linear space over $\mathbb{K}$. A seminorm on E is a function $p : E \rightarrow [0, \infty)$ such that $p(\alpha x) = |\alpha|p(x)$ for all $\alpha \in \mathbb{K}$, $x \in E$ and $p(x + y) \leq \max\{p(x), p(y)\}$ for all $x, y \in E$. For fundamentals of locally convex Hausdorff spaces (lcs) over $\mathbb{K}$ we refer the reader to the monograph [27].

Let X be an infinite ultraregular space i.e. an infinite Hausdorff topological space such that the clopen subsets of X form a basis for the topology of X . It is clear that $C_p(X, \mathbb{K})$ is isomorphic to some dense subspace of $\mathbb{K}^X$ with the product topology. Thus $C_p(X, \mathbb{K})$ is metrizable if and only if X is countable.

This may suggest the following questions.

Problem 1 *Let X be an infinite ultraregular space. Does $C_p(X, \mathbb{K})$ admit*

- (1) *an infinite-dimensional [closed] metrizable subspace?*
- (2) *an infinite-dimensional complemented metrizable subspace?*
- (3) *an infinite-dimensional metrizable quotient?*

2 Results

An lcs E is of *finite type* if $\dim(E/\ker p) < \infty$, for any continuous seminorm p on E . It is easy to see that any subspace and any continuous linear image of an lcs of finite type is of finite type. By [31, Theorems 4 and 7] we note the following

- Remark 2** (1) An lcs E is of finite type if and only if it is isomorphic to some subspace of the product space $\mathbb{K}^A$ for some set A .
- (2) An lcs E of finite type is metrizable if and only if it is isomorphic to some subspace of the Fréchet space $\mathbb{K}^{\mathbb{N}}$.

This yields the following

Corollary 3 *Let X be an infinite ultraregular space. Any metrizable subspace and any metrizable quotient of $C_p(X, \mathbb{K})$ is isomorphic to some subspace of $\mathbb{K}^{\mathbb{N}}$.*

For any sequence (u_n) of continuous linear functionals on an lcs G the linear map $T : G \rightarrow \mathbb{K}^{\mathbb{N}}$, where $T(x) = (u_n(x))$, is continuous and any continuous linear map from G to $\mathbb{K}^{\mathbb{N}}$ is of this form. Moreover, $\dim \text{lin}\{u_n : n \in \mathbb{N}\} = \infty$ if and only if $\dim \text{lin}\{u_n : n \in \mathbb{N}\} = \infty$. Thus any lcs G with an infinite-dimensional topological dual has an infinite-dimensional metrizable continuous linear image isomorphic to some subspace of $\mathbb{K}^{\mathbb{N}}$. In particular, for any infinite ultraregular space X , the space $C_p(X, \mathbb{K})$ has an infinite-dimensional metrizable continuous linear image (isomorphic to some subspace of $\mathbb{K}^{\mathbb{N}}$).

We recall that a regular space X is *pseudocompact* if every continuous function $f : X \rightarrow \mathbb{R}$ is bounded.

For ultraregular spaces which are not pseudocompact we obtain the following.

Proposition 4 *For any ultraregular space X the following conditions are equivalent:*

- (1) X is not pseudocompact;
- (2) There exists a clopen partition (W_n) of X with $W_n \neq \emptyset, n \in \mathbb{N}$;
- (3) There exists an unbounded continuous function $g : X \rightarrow \mathbb{K}$;
- (4) $C_p(X, \mathbb{K})$ has a subspace isomorphic to $\mathbb{K}^{\mathbb{N}}$;
- (5) $C_p(X, \mathbb{K})$ has a complemented subspace isomorphic to $\mathbb{K}^{\mathbb{N}}$;
- (6) $C_p(X, \mathbb{K})$ has a quotient space isomorphic to $\mathbb{K}^{\mathbb{N}}$;
- (7) $C_p(X, \mathbb{K})$ admits a linear continuous map onto $\mathbb{K}^{\mathbb{N}}$.

Let $c_0(\mathbb{N}, \mathbb{K})$ be the space of all sequences in $\mathbb{K}$ that are convergent to 0 with the topology of pointwise convergence.

We shall prove that for any infinite ultraregular space X the space $C_p(X, \mathbb{K})$ has an infinite-dimensional metrizable subspace isomorphic to $c_0(\mathbb{N}, \mathbb{K})$.

Theorem 5 *Let X be an infinite ultraregular space [with an infinite compact subset]. Then $C_p(X, \mathbb{K})$ has an infinite-dimensional [closed] metrizable subspace isomorphic to $c_0(\mathbb{N}, \mathbb{K})$.*

In particular, for any infinite ultraregular compact space X the space $C_p(X, \mathbb{K})$ has an infinite-dimensional closed metrizable subspace isomorphic to $c_0(\mathbb{N}, \mathbb{K})$. On the other hand, if X is discrete, then $C_p(X, \mathbb{K}) = \mathbb{K}^X$, so any closed subspace of $C_p(X, \mathbb{K})$ is isomorphic to $\mathbb{K}^A$ for some subset A of X . Thus any infinite-dimensional closed metrizable subspace of $C_p(X, \mathbb{K})$ is isomorphic to $\mathbb{K}^{\mathbb{N}}$; in particular, $C_p(X, \mathbb{K})$ has no closed subspace isomorphic to $c_0(\mathbb{N}, \mathbb{K})$.

Next we prove that the space $C_p(X, \mathbb{K})$ has a quotient isomorphic to $c_0(\mathbb{N}, \mathbb{K})$ if and only if $C_p(X, \mathbb{K})$ has a complemented subspace isomorphic to $c_0(\mathbb{N}, \mathbb{K})$.

Let X be an ultraregular space. For a point $x \in X$ let $\delta_x : C_p(X, \mathbb{K}) \rightarrow \mathbb{K}, \delta_x(f) = f(x)$, be the Dirac measure concentrated at x . The linear hull $L_p(X, \mathbb{K})$ of the set $\{\delta_x : x \in X\}$ in $\mathbb{K}^{C_p(X, \mathbb{K})}$ can be identified with the topological dual of $C_p(X, \mathbb{K})$.

Each $\mu \in L_p(X, \mathbb{K})$ can be uniquely written as a linear combination of Dirac measures $\mu = \sum_{x \in F} \alpha_x \delta_x$ for some finite subset F of X and some non-zero scalars α_x for $x \in F$; the set F is called the *support* of μ and is denoted by $\text{supp}(\mu)$. The real number $\max_{x \in F} |\alpha_x|$ will be denoted by $\|\mu\|$; for $\mu = 0$ we have $\text{supp}(\mu) = \emptyset$ and $\|\mu\| = 0$.

Any $\mu \in L_p(X, \mathbb{K})$ of the form $\mu = \sum_{x \in F} \alpha_x \delta_x$ determines the function

$$\mu : 2^X \rightarrow \mathbb{K}, \mu(A) = \sum_{x \in A \cap F} \alpha_x.$$

So, μ will be considered both as a continuous linear functional on $C_p(X, \mathbb{K})$ and a finite-additive $\mathbb{K}$ -valued function on the power-set 2^X .

Recall that the famous *Josefson–Nissenzweig theorem* asserts that for each infinite-dimensional Banach space F over $\mathbb{R}$ or $\mathbb{C}$ there exists a sequence (μ_n) in the topological dual F^* of F with $\|\mu_n\| = 1, n \in \mathbb{N}$, such that $\mu_n(f) \rightarrow 0$ for every $f \in F$; see for example [9].

We propose the following corresponding property for spaces $C_p(X, \mathbb{K})$.

Definition 6 Let X be an infinite ultraregular space. We say that the space $C_p(X, \mathbb{K})$ has the *Josefson–Nissenzweig property* (JNP in short) if there exists a sequence $(\mu_n) \subset L_p(X, \mathbb{K})$ such that $\|\mu_n\| = 1, n \in \mathbb{N}$, and $\mu_n(f) \rightarrow_n 0$ for every $f \in C_p(X, \mathbb{K})$.

We prove the following theorem.

Theorem 7 *For any infinite ultraregular space X the following conditions are equivalent:*

- (1) $C_p(X, \mathbb{K})$ contains a complemented subspace isomorphic to $c_0(\mathbb{N}, \mathbb{K})$;

- (2) $C_p(X, \mathbb{K})$ has a quotient isomorphic to $c_0(\mathbb{N}, \mathbb{K})$;
- (3) $C_p(X, \mathbb{K})$ admits a linear continuous map onto $c_0(\mathbb{N}, \mathbb{K})$;
- (4) $C_p(X, \mathbb{K})$ has the JNP.

On the other hand, as follows from [27, Theorem 2.5.22], for every infinite ultraregular compact space X the Banach space $(C(X, \mathbb{K}), \|\cdot\|_\infty)$ contains a complemented subspace isomorphic to the Banach space $(c_0(\mathbb{N}, \mathbb{K}), \|\cdot\|_\infty)$. Hence, according to [8] (see also [29, Theorem 6.2]), there exists a sequence (μ_n) in the dual of $C(X)$ converging to zero in the weak*-topology with $\|\mu_n\| = 1$ for all $n \in \mathbb{N}$. Nevertheless, having in mind Theorem 10 and Corollary 11 below, we have the following

Corollary 8 *There exists a compact ultraregular space X such that for every sequence $(\mu_n) \subset L_p(X, \mathbb{K})$ with $\mu_n(f) \rightarrow_n 0$ for every $f \in C_p(X, \mathbb{K})$ we have $\|\mu_n\| \rightarrow 0$.*

Using Theorem 7 we get the following

Corollary 9 *Assume that an ultraregular space X contains a non-trivial convergent sequence. Then $C_p(X, \mathbb{K})$ has a complemented subspace isomorphic to $c_0(\mathbb{N}, \mathbb{K})$. In particular, $C_p(X, \mathbb{K})$ has an infinite-dimensional metrizable quotient isomorphic to $c_0(\mathbb{N}, \mathbb{K})$.*

First Proof of Corollary 9 Let $(x_n) \subset X, x_0 \in X$ such that $x_n \rightarrow x_0$ and $x_n \neq x_0$ for every $n \in \mathbb{N}$. Let $\mu_n = \delta_{x_n} - \delta_{x_0}$ for $n \in \mathbb{N}$. Then $\|\mu_n\| = 1$ for $n \in \mathbb{N}$ and $\mu_n(f) \rightarrow 0$ for every $f \in C_p(X, \mathbb{K})$. Thus $C_p(X, \mathbb{K})$ satisfies the JNP. By Theorem 7, $C_p(X, \mathbb{K})$ has a complemented subspace isomorphic to $c_0(\mathbb{N}, \mathbb{K})$. $\square$

Second Proof of Corollary 9 Let $(x_n) \subset X, x_0 \in X$ such that $x_n \rightarrow x_0$ and $x_n \neq x_m$ for all $n, m \in \mathbb{N}$ with $n \neq m$. Put $Y = \{x_n : n \in \mathbb{N}\} \cup \{x_0\}$. Then Y is a metrizable compact subset of X , so it is a retract of X (see [24, Theorem 3]). Let $\varphi : X \rightarrow Y$ be a retraction. Put $C_p(X|Y, \mathbb{K}) = \{f \in C_p(X, \mathbb{K}) : f|_Y = 0\}$. The maps

$$T : C_p(X, \mathbb{K}) \rightarrow C_p(Y, \mathbb{K}) \times C_p(X|Y, \mathbb{K}), f \mapsto (f|_Y, f - (f|_Y) \circ \varphi)$$

and

$$S : C_p(Y, \mathbb{K}) \times C_p(X|Y, \mathbb{K}) \rightarrow C_p(X, \mathbb{K}), (g, h) \mapsto g \circ \varphi + h$$

are linear, continuous and mutual inverses. Thus $C_p(X, \mathbb{K})$ has a complemented subspace isomorphic to $C_p(Y, \mathbb{K})$. Clearly the map

$$L : C_p(Y, \mathbb{K}) \rightarrow c_0(\mathbb{N}, \mathbb{K}), g \mapsto (g(x_0), g(x_0) - g(x_1), g(x_0) - g(x_2), g(x_0) - g(x_3), \dots)$$

is an isomorphism. Thus $C_p(X, \mathbb{K})$ has a complemented subspace isomorphic to $c_0(\mathbb{N}, \mathbb{K})$. $\square$

Theorems 5 and 7 and Corollary 9 motivate a question whether for any infinite ultraregular compact space X the space $C_p(X, \mathbb{K})$ has a quotient isomorphic to $c_0(\mathbb{N}, \mathbb{K})$. We prove the following theorem.

A topological space is called *extremally disconnected* if it is regular and the closure of every open set is open.

Theorem 10 *Let X be an extremally disconnected compact space. Then there exists no continuous linear surjection $T : C_p(X, \mathbb{K}) \rightarrow c_0(\mathbb{N}, \mathbb{K})$. In particular, $C_p(X, \mathbb{K})$ has no quotient isomorphic to $c_0(\mathbb{N}, \mathbb{K})$ (and so, no complemented subspace isomorphic to $c_0(\mathbb{N}, \mathbb{K})$).*

Corollary 11 *Let D be a discrete space. Then $C_p(\beta D, \mathbb{K})$ has no quotient isomorphic to $c_0(\mathbb{N}, \mathbb{K})$. In particular, $C_p(\beta \mathbb{N}, \mathbb{K})$ has no quotient isomorphic to $c_0(\mathbb{N}, \mathbb{K})$.*

The last three theorems below provide two partial answers to Problem 1 (3).

Theorem 12 *Let X be an ultraregular compact space. Assume that there exists a sequence (K_n) of non-empty compact subsets of X such that for every $n \in \mathbb{N}$ the set K_n contains two disjoint topological copies of K_{n+1} . Then the space $C_p(X, \mathbb{K})$ has an infinite-dimensional metrizable quotient.*

Corollary 13 *Let D be an infinite discrete space. Then $C_p(\beta D, \mathbb{K})$ has an infinite-dimensional metrizable quotient.*

Any infinite extremally disconnected compact space X contains a closed subspace S homeomorphic to $\beta \mathbb{N}$, so using the last theorem we get the following.

Corollary 14 *Let X be an infinite extremally disconnected compact space. Then $C_p(X, \mathbb{K})$ has an infinite-dimensional metrizable quotient.*

We do not know whether the metrizable quotient obtained in Theorem 12 is isomorphic to $\ell_\infty(\mathbb{N}, \mathbb{K})$, $\ell_c(\mathbb{N}, \mathbb{K})$ or $c_0(\mathbb{N}, \mathbb{K})$.

A compact space X is called an *Efimov space* if X contains neither non-trivial convergent sequences nor copies of $\beta \mathbb{N}$. The famous long-standing open question, called the Efimov problem, asks whether there exists an Efimov space in ZFC. The only known examples of Efimov spaces have been found under additional set-theoretic assumptions, see e.g. [6, 10–13].

By Corollary 9 and Theorem 12 we get the following interesting connection between Problem 1 and the Efimov's problem.

Corollary 15 *If there exists an ultraregular compact space X such that $C_p(X, \mathbb{K})$ does not admit an infinite-dimensional metrizable quotient, then X must be Efimov.*

Example 16 below provides an ultraregular Efimov space X (under $\diamond$) for which Theorem 12 applies and shows also that the converse for Corollary 15 fails.

Example 16 Under $\diamond$ there exists an ultraregular Efimov space X whose function space $C_p(X, \mathbb{K})$ has an infinite-dimensional metrizable quotient.

Proof De la Vega [7, Theorem 3.22] (we refer also to [6]) constructed under $\diamond$ an ultraregular compact hereditary separable space X (hence not containing a copy of $\beta \mathbb{N}$) such that:

- (i) X does not contain non-trivial convergent sequences.
- (ii) X has a base of clopen pairwise homeomorphic sets.

It is easy to see that X admits a sequence (K_n) of infinite compact subsets such that each K_n contains two disjoint subsets homeomorphic to K_{n+1} ; therefore by Theorem 12 the space $C_p(X, \mathbb{K})$ has an infinite-dimensional metrizable quotient. $\square$

Note that in [23, Example 17] we provided (again under $\diamond$) an example of an ultraregular Efimov space X for which Theorem 12 cannot be applied.

A sequence $(x_n) \subset \mathbb{K}$ is said to be *relatively compact* if the set $\{x_n : n \in \mathbb{N}\}$ is relatively compact in $\mathbb{K}$. Denote by $\ell_c(\mathbb{N}, \mathbb{K})$ the space of all relatively compact sequences in $\mathbb{K}$ and

by $\ell_\infty(\mathbb{N}, \mathbb{K})$ the space of all bounded sequences in $\mathbb{K}$ with the topology of pointwise convergence. Clearly, $\ell_c(\mathbb{N}, \mathbb{K})$ and $\ell_\infty(\mathbb{N}, \mathbb{K})$ are linear subspaces of the Fréchet spaces $\mathbb{K}^\mathbb{N}$ and $c_0(\mathbb{N}, \mathbb{K}) \subset \ell_c(\mathbb{N}, \mathbb{K}) \subset \ell_\infty(\mathbb{N}, \mathbb{K})$.

Although Corollary 11 shows that for discrete space D the space $C_p(\beta D, \mathbb{K})$ has no quotient isomorphic to $c_0(\mathbb{N}, \mathbb{K})$, our next theorem describes concrete infinite-dimensional metrizable quotients for spaces $C_p(\beta D, \mathbb{K})$, when D is an infinite discrete space.

Theorem 17 *Let D be an infinite discrete space. Then the space $E = C_p(\beta D, \mathbb{K})$ has an infinite-dimensional metrizable quotient. More precisely, for any sequence $F = (F_n)$ of non-empty, finite and pairwise disjoint subsets of D with $\lim_n |F_n| = \infty$ and the closed linear subspace*

$$Y_F = \left\{ f \in E : \sum_{x \in F_n} f(x) = 0 \text{ for every } n \in \mathbb{N} \right\}$$

of E , the quotient space E/Y_F is isomorphic to the subspace

$$\ell_c^G(\mathbb{N}, \mathbb{K}) = \left\{ \left(\sum_{s \in G_n} y_s \right)_{n=1}^\infty : (y_s) \in \ell_c(\mathbb{N}, \mathbb{K}) \right\}$$

of the Fréchet space $\mathbb{K}^\mathbb{N}$, where $G = (G_n)$ is a partition of $\mathbb{N}$ with $|G_n| = |F_n|$, $n \in \mathbb{N}$. In particular, E has a quotient isomorphic to the subspace

$$\ell_c^0(\mathbb{N}, \mathbb{K}) = \left\{ \left(\sum_{2^{n-1} \leq s < 2^n} y_s \right)_{n=1}^\infty : (y_s) \in \ell_c(\mathbb{N}, \mathbb{K}) \right\}$$

of $\mathbb{K}^\mathbb{N}$.

If $\mathbb{K}$ is locally compact and Y is a compact subset of an ultraregular space X , then every continuous function $g : Y \rightarrow \mathbb{K}$ has an extension to some bounded continuous function $f : X \rightarrow \mathbb{K}$ [14, Theorem]. It follows that the linear map $C_p(X, \mathbb{K}) \rightarrow C_p(Y, \mathbb{K})$, $f \rightarrow f|_Y$ is a quotient map.

This implies the following

Corollary 18 *Let D be an infinite discrete space. If $\mathbb{K}$ is locally compact then the space $C_p(\beta D, \mathbb{K})$ has a quotient isomorphic to $\ell_\infty(\mathbb{N}, \mathbb{K})$. In particular, $C_p(\beta D, \mathbb{Q}_q)$ has a quotient isomorphic to $\ell_\infty(\mathbb{N}, \mathbb{Q}_q)$ for any prime number q .*

Corollary 19 *Let X be an infinite extremally disconnected compact space. If $\mathbb{K}$ is locally compact, then the space $C_p(X, \mathbb{K})$ has an infinite-dimensional metrizable quotient isomorphic to $\ell_\infty(\mathbb{N}, \mathbb{K})$.*

Finally, applying Theorem 17 we prove the following.

Theorem 20 *Let D be an infinite discrete space. If $|n1_\mathbb{K}| \neq 1$ for some $n \in \mathbb{N}$, then $C_p(\beta D, \mathbb{K})$ has a quotient isomorphic to $\ell_c(\mathbb{N}, \mathbb{K})$.*

Corollary 21 *Let D be an infinite discrete space. Then for any prime number q the space $C_p(\beta D, \mathbb{C}_q)$ has a quotient isomorphic to the space $\ell_c(\mathbb{N}, \mathbb{C}_q)$.*

We complete this section with a number of questions motivated by the above results.

Problem 22 Let D be an infinite discrete space.

- (1) Assume that $|n1_{\mathbb{K}}| = 1$ for every $n \in \mathbb{N}$. Does the space $C_p(\beta D, \mathbb{K})$ have a quotient isomorphic to $\ell_\infty(\mathbb{N}, \mathbb{K})$ or $\ell_c(\mathbb{N}, \mathbb{K})$?
- (2) Let q be a prime number. Does the space $C_p(\beta D, \mathbb{C}_q)$ have a quotient isomorphic to $\ell_\infty(\mathbb{N}, \mathbb{C}_q)$?
- (3) Does the space $C_p(\beta D, \mathbb{K})$ have a quotient isomorphic to $\ell_\infty(\mathbb{N}, \mathbb{K})$ for some not locally compact field $\mathbb{K}$?
- (4) Does the space $C_p(\beta D, \mathbb{K})$ have a quotient isomorphic to $\ell_\infty(\mathbb{N}, \mathbb{K})$ for every field $\mathbb{K}$?

3 Proof of Proposition 4

Proof (1) $\Rightarrow$ (2) Let $f : X \rightarrow \mathbb{R}$ be an unbounded continuous function. Then there exists a sequence $(x_n) \subset X$ such that $|f(x_{n+1})| > |f(x_n)| + 2 > 5$, $n \in \mathbb{N}$. Clearly, the sets

$$U_n = \{x \in X : ||f(x)| - |f(x_n)|| < 1\}, n \in \mathbb{N},$$

are open and pairwise disjoint. Moreover, $x_n \in U_n$ and $|f(x)| > 2n$ for any $x \in U_n$, $n \in \mathbb{N}$. Let V_n be a clopen set with $x_n \in V_n \subset U_n$, $n \in \mathbb{N}$. The set $\bigcup_{n=1}^\infty V_n$ is closed. Indeed, let

$$x_0 \in \overline{\bigcup_{n=1}^\infty V_n}.$$

Let

$$(x_\alpha) \subset \bigcup_{n=1}^\infty V_n$$

with $x_\alpha \rightarrow x_0$. Then $|f(x_\alpha)| \rightarrow |f(x_0)|$. If

$$\forall k \in \mathbb{N} \exists \alpha \forall \beta \geq \alpha : x_\beta \in \bigcup_{n=k+1}^\infty V_n, \text{ then } \forall k \in \mathbb{N} \exists \alpha \forall \beta \geq \alpha : |f(x_\beta)| > 2k,$$

so $|f(x_\alpha)| \rightarrow \infty$; a contradiction. Thus

$$\exists k \in \mathbb{N} \forall \alpha \exists \beta \geq \alpha : x_\beta \in \bigcup_{n=1}^k V_n, \text{ then } x_0 \in \bigcup_{n=1}^k V_n \subset \bigcup_{n=1}^\infty V_n.$$

Put $W_1 = X \setminus \bigcup_{n=2}^\infty V_n$ and $W_n = V_n$ for $n > 1$. Then (W_n) is a clopen partition of X with $W_n \neq \emptyset$, $n \in \mathbb{N}$.

(2) $\Rightarrow$ (3) Let $t \in \mathbb{K}$ with $|t| > 1$. Clearly, the function $g : X \rightarrow \mathbb{K}$, $g(x) = \alpha^n$ if $x \in W_n$, $n \in \mathbb{N}$, is continuous and unbounded.

(3) $\Rightarrow$ (1) Let $g : X \rightarrow \mathbb{K}$ be an unbounded continuous function. Then the function $f : X \rightarrow \mathbb{R}$, $f(x) = |g(x)|$ is continuous and unbounded.

(2) $\Rightarrow$ (4) For every $\alpha = (\alpha_n) \in \mathbb{K}^{\mathbb{N}}$ the function $f_\alpha : X \rightarrow \mathbb{K}$, $f_\alpha(x) = \alpha_n$ for $x \in W_n$, $n \in \mathbb{N}$, is continuous. The map

$$T : \mathbb{K}^{\mathbb{N}} \rightarrow C_p(X, \mathbb{K}), \quad \alpha \rightarrow f_\alpha$$

is linear, injective and continuous. Moreover, the linear map

$$T^{-1} : T(\mathbb{K}^{\mathbb{N}}) \rightarrow \mathbb{K}^{\mathbb{N}}, \quad f_\alpha \rightarrow \alpha$$

is continuous. Thus $T(\mathbb{K}^{\mathbb{N}})$ is a subspace of $C_p(X, \mathbb{K})$ isomorphic to $\mathbb{K}^{\mathbb{N}}$.

(4) $\Rightarrow$ (5) The lcs $\mathbb{K}^{\mathbb{N}}$ is minimal [31, Theorem 7], so it is complemented in $C_p(X, \mathbb{K})$ [31, Corollary 9.1].

The implications (5) $\Rightarrow$ (6) and (6) $\Rightarrow$ (7) are trivial.

(7) $\Rightarrow$ (3) Suppose, by contradiction, that every continuous function $g : X \rightarrow \mathbb{K}$ is bounded. Then $C_p(X, \mathbb{K})$ is σ -bounded, since it can be written as the countable union

$$C_p(X, \mathbb{K}) = \bigcup_{n=1}^{\infty} \left\{ g \in C_p(X, \mathbb{K}) : \sup_{x \in X} |g(x)| \leq n \right\}$$

of bounded subsets. Hence $\mathbb{K}^{\mathbb{N}}$, a continuous linear image of $C_p(X, \mathbb{K})$, also is σ -bounded. On the other hand, the Baire Theorem ensures that the space $\mathbb{K}^{\mathbb{N}}$ is not σ -bounded; a contradiction. $\square$

4 Proof of Theorem 5

Proof Put $B(\gamma, r) = \{\beta \in \mathbb{K} : |\beta - \gamma| < r\}$ for $\gamma \in \mathbb{K}$ and $r > 0$. Let (U_n) be a sequence of pairwise disjoint non-empty clopen subsets of X . For every $(\alpha_n) \in c_0(\mathbb{N}, \mathbb{K})$ the function

$$f : X \rightarrow \mathbb{K}, f = \sum_{n=1}^{\infty} \alpha_n \mathbb{I}_{U_n}$$

is continuous. Indeed, if $\varepsilon > 0$, then the set

$$f^{-1}(B(0, \varepsilon)) = \{x \in X : |f(x)| < \varepsilon\} = X \setminus \bigcup \{U_n : n \in \mathbb{N}, |\alpha_n| \geq \varepsilon\}$$

is clopen, since the set $\{n \in \mathbb{N} : |\alpha_n| \geq \varepsilon\}$ is finite; if $\beta_0 \in (\mathbb{K} \setminus \{0\})$ and $0 < \varepsilon < |\beta_0|$, then the set

$$f^{-1}(B(\beta_0, \varepsilon)) = \{x \in X : |f(x) - \beta_0| < \varepsilon\} = \bigcup \{U_n : n \in \mathbb{N}, |\alpha_n - \beta_0| < \varepsilon\}$$

is clopen, since the set $\{n \in \mathbb{N} : |\alpha_n - \beta_0| < \varepsilon\}$ is finite.

Clearly,

$$F = \left\{ \sum_{n=1}^{\infty} \alpha_n \mathbb{I}_{U_n} : (\alpha_n) \in c_0(\mathbb{N}, \mathbb{K}) \right\}$$

is a subspace of $C_p(X, \mathbb{K})$. We shall prove that F is isomorphic to $c_0(\mathbb{N}, \mathbb{K})$. The linear map

$$T : c_0(\mathbb{N}, \mathbb{K}) \rightarrow F, (\alpha_n) \mapsto \sum_{n=1}^{\infty} \alpha_n \mathbb{I}_{U_n}$$

is continuous. Indeed, let $k \in \mathbb{N}$, $x_1, \dots, x_k \in X$, $\varepsilon > 0$ and

$$U = \{f \in F : |f(x_i)| < \varepsilon \text{ for } 1 \leq i \leq k\}.$$

Let $m \in \mathbb{N}$ with $m > \varepsilon^{-1}$ such that

$$\{x_1, \dots, x_k\} \subset U_1 \cup \dots \cup U_m \cup \left(X \setminus \bigcup_{j=1}^{\infty} U_j \right).$$

Clearly, the set

$$V_m = \left\{ (\alpha_n) \in c_0(\mathbb{N}, \mathbb{K}) : \max_{1 \leq n \leq m} |\alpha_n| < m^{-1} \right\}$$

is a neighbourhood of zero in $c_0(\mathbb{N}, \mathbb{K})$ and

$$T(V_m) = \left\{ \sum_{n=1}^{\infty} \alpha_n \mathbb{I}_{U_n} : (\alpha_n) \in c_0(\mathbb{N}, \mathbb{K}), \max_{1 \leq n \leq m} |\alpha_n| < m^{-1} \right\} \subset U.$$

Indeed, let

$$f = \sum_{n=1}^{\infty} \alpha_n \mathbb{I}_{U_n} \in T(V_m)$$

and $1 \leq i \leq k$. If $x_i \in U_1 \cup \dots \cup U_m$, then $x_i \in U_j$ for some $1 \leq j \leq m$, so $|f(x_i)| = |\alpha_j| < m^{-1} < \varepsilon$; if $x_i \in X \setminus \bigcup_{j=1}^{\infty} U_j$, then $|f(x_i)| = 0$. Thus $f \in U$. Hence $V_m \subset T^{-1}(U)$; so T is continuous.

Let $m \in \mathbb{N}$ and let V_m be as above. Let $x_i \in U_i$ for $1 \leq i \leq m$ and $0 < \varepsilon < m^{-1}$. Let

$$U = \{f \in F : |f(x_i)| < \varepsilon \text{ for } 1 \leq i \leq m\}.$$

Let $f \in U$. Then $f = \sum_{n=1}^{\infty} \alpha_n \mathbb{I}_{U_n}$ for some $(\alpha_n) \in c_0(\mathbb{N}, \mathbb{K})$ and $|\alpha_i| < \varepsilon < m^{-1}$ for $1 \leq i \leq m$; so $f \in T(V_m)$. Thus $U \subset T(V_m)$; so T is open. It follows that F is isomorphic to $c_0(\mathbb{N}, \mathbb{K})$.

Now assume that X contains an infinite compact subset W . Without loss of generality we can assume that $W \cap U_n \neq \emptyset$ for every $n \in \mathbb{N}$. Let $M \subset \mathbb{N}$. The set $S_M = \bigcup \{U_n : n \in M\}$ is closed in X if and only if M is finite. Indeed, suppose that S_M is closed for some subset M of $\mathbb{N}$. Then $\{W \cap U_n : n \in M\}$ is an open cover of the compact set $W \cap S_M$, so M is finite. The converse implication is obvious.

We prove that the subspace F of $C_p(X, \mathbb{K})$ (defined in the first part of this proof) is closed. Let

$$(f_\beta) \subset F, \quad f \in C_p(X, \mathbb{K})$$

and $f_\beta \rightarrow f$ in $C_p(X, \mathbb{K})$. Then $f|_{U_n}$ is constant for every $n \in \mathbb{N}$ and $f|(X \setminus \bigcup_{n=1}^{\infty} U_n) = 0$, so $f = \sum_{n=1}^{\infty} \gamma_n \mathbb{I}_{U_n}$ for some $(\gamma_n) \in \mathbb{K}^{\mathbb{N}}$. Suppose that $f \notin F$. Then $(\gamma_n) \notin c_0(\mathbb{N}, \mathbb{K})$. For some $\varepsilon > 0$ the set

$$\mathbb{N}_\varepsilon = \{n \in \mathbb{N} : |\gamma_n| \geq \varepsilon\}$$

is infinite and so the set $\bigcup \{U_n : n \in \mathbb{N}_\varepsilon\}$ is not closed. Then the set

$$\begin{aligned} f^{-1}(B(0, \varepsilon)) &= \{x \in X : |f(x)| < \varepsilon\} \\ &= \left(X \setminus \bigcup_{n=1}^{\infty} U_n \right) \cup \bigcup \{U_n : n \in (\mathbb{N} \setminus \mathbb{N}_\varepsilon)\} = X \setminus \bigcup_{n \in \mathbb{N}_\varepsilon} U_n \end{aligned}$$

is not open; so f is not continuous; a contradiction. Thus $f \in F$; so F is closed in $C_p(X, \mathbb{K})$. $\square$

5 Proof of Theorem 7

We need the following lemma.

Lemma 23 *Let X be an ultraregular space and let $(\mu_n) \subset L_p(X, \mathbb{K})$. Assume that $\|\mu_n\| = 1$, $n \in \mathbb{N}$, and $\mu_n(f) \rightarrow 0$ for every $f \in C_p(X, \mathbb{K})$. Then there exists an increasing sequence $(t_k) \subset \mathbb{N}$ and a sequence (U_n) of pairwise-disjoint clopen subsets of X such that $|\mu_{t_k}(U_k)| = 1$ for every $k \in \mathbb{N}$.*

Proof For every $n \in \mathbb{N}$ there exist a non-empty finite subset G_n of X and $\alpha_x^n \in \mathbb{K}^*$ for $x \in G_n$ such that

$$\mu_n = \sum_{x \in G_n} \alpha_x^n \delta_x.$$

Let $F_n = \{x \in G_n : |\alpha_x^n| = \|\mu_n\|\}$ for $n \in \mathbb{N}$. The set $F = \bigcup_{n=1}^{\infty} F_n$ is infinite. Indeed, suppose by contrary, that F is finite. Let $F = \{x_1, \dots, x_k\}$. Let $(U_1, \dots, U_k)$ be a sequence of pairwise-disjoint clopen subsets of X with $x_i \in U_i$ for $1 \leq i \leq k$. Put

$$N_i = \{n \in \mathbb{N} : x_i \in F_n\}, \quad 1 \leq i \leq k.$$

Then $\mathbb{N} = \bigcup_{i=1}^k N_i$ and $F \cap U_i = \{x_i\}$ for $1 \leq i \leq k$.

For some $1 \leq i \leq k$ the set N_i is infinite. Let $n \in N_i$ and $H_n = G_n \setminus F_n$. Then

$$\mu_n(\mathbb{I}_{U_i}) = \sum_{x \in U_i \cap F_n} \alpha_x^n + \sum_{x \in U_i \cap H_n} \alpha_x^n = \alpha_{x_i}^n + \sum_{x \in U_i \cap H_n} \alpha_x^n, \quad |\alpha_{x_i}^n| = \|\mu_n\|$$

and

$$\left| \sum_{x \in U_i \cap H_n} \alpha_x^n \right| \leq \max_{x \in U_i \cap H_n} |\alpha_x^n| < \|\mu_n\|;$$

so $|\mu_n(\mathbb{I}_{U_i})| = \|\mu_n\|$. Thus $\inf_{n \in N_i} \|\mu_n\| = 0$; a contradiction.

It follows that F is infinite. Thus there exist an increasing sequence $(s_n) \subset \mathbb{N}$ and

$$(x_{s_n}) \in \prod_{n=1}^{\infty} F_{s_n}$$

with $x_{s_n} \neq x_{s_m}$, whenever $n \neq m$. Then there exist an increasing sequence $(n(k)) \subset \mathbb{N}$ and a sequence (V_k) of pairwise-disjoint clopen subsets of X such that $x_{s_{n(k)}} \in V_k$ for $k \in \mathbb{N}$. Let D_k be a clopen subset of X such that

$$G_{s_{n(k)}} \setminus \{x_{s_{n(k)}}\} \subset D_k \subset X \setminus \{x_{s_{n(k)}}\}.$$

Put $U_k = V_k \setminus D_k$ for $k \in \mathbb{N}$. Clearly, (U_k) is a sequence of pairwise-disjoint clopen subsets of X such that $U_k \cap G_{s_{n(k)}} = \{x_{s_{n(k)}}\}$ for $k \in \mathbb{N}$. Then

$$|\mu_{s_{n(k)}}(U_k)| = |\alpha_{x_{s_{n(k)}}}^{s_{n(k)}}| = \|\mu_{s_{n(k)}}\| = 1$$

for every $k \in \mathbb{N}$. □

Proof of Theorem 7 The implications $(1) \Rightarrow (2) \Rightarrow (3)$ are trivial.

$(3) \Rightarrow (4)$

Let T be a continuous linear map from $C_p(X, \mathbb{K})$ onto $c_0(\mathbb{N}, \mathbb{K})$. Then there exists a sequence $(\mu_n) \subset L_p(X, \mathbb{K})$ such that $Tf = (\mu_n(f))$ for every $f \in C_p(X, \mathbb{K})$. Clearly, $\mu_n \neq 0$ for every $n \in \mathbb{N}$ and $\mu_n(f) \rightarrow_n 0$ for every $f \in C_p(X, \mathbb{K})$.

Let $S = \bigcup_{n=1}^{\infty} S_n$, where $S_n = \text{supp}(\mu_n)$ for $n \in \mathbb{N}$. We prove that for every $f \in C_p(X, \mathbb{K})$ the set $f(S)$ is bounded in $\mathbb{K}$.

Suppose, by contrary, that for some $f \in C_p(X, \mathbb{K})$ the set $f(S)$ is unbounded in $\mathbb{K}$. Let $a_n = \max_{x \in S_n} |f(x)|$ and $x_n \in S_n$ with $|f(x_n)| = a_n$ for $n \in \mathbb{N}$. We can choose an increasing sequence $(k_n) \subset \mathbb{N}$ such that (a_{k_n}) is an increasing sequence of positive numbers. Let U_n be a clopen subset of the clopen set $\{x \in X : |f(x)| = a_{k_n}\}$ such that $U_n \cap S_{k_n} = \{x_{k_n}\}$ for $n \in \mathbb{N}$.

Let $n, m \in \mathbb{N}$ with $n \neq m$. For $x \in U_n$ and $y \in U_m$ we have $|f(x)| \neq |f(y)|$, so

$$|f(x) - f(y)| = \max\{|f(x)|, |f(y)|\} = \max\{a_{k_n}, a_{k_m}\} \geq a_{k_1}.$$

It follows that the family $(f(U_n))$ is discrete in $\mathbb{K}$; so (U_n) is discrete in X . We can inductively choose a sequence $(\alpha_n) \subset \mathbb{K}^*$ with

$$|\mu_{k_n}(\alpha_n \mathbb{I}_{U_n})| > \max_{1 \leq m < n} |\mu_{k_n}(\alpha_m \mathbb{I}_{U_m})| + 1, \quad n \in \mathbb{N}.$$

The function

$$g : X \rightarrow \mathbb{K}, \quad g = \sum_{n=1}^{\infty} \alpha_n \mathbb{I}_{U_n}$$

is continuous.

Let $m, n \in \mathbb{N}$ with $m > n$. For $x \in U_m$ and $y \in S_{k_n}$ we have

$$|f(x)| = a_{k_m} > a_{k_n} \geq |f(y)|,$$

so $f(x) \neq f(y)$. Thus $U_m \cap S_{k_n} = \emptyset$ for $m, n \in \mathbb{N}$ with $m > n$. We have

$$|\mu_{k_n}(g)| = \left| \sum_{m=1}^{\infty} \mu_{k_n}(\alpha_m \mathbb{I}_{U_m}) \right| = \left| \sum_{m=1}^n \mu_{k_n}(\alpha_m \mathbb{I}_{U_m}) \right| = |\mu_{k_n}(\alpha_n \mathbb{I}_{U_n})| > 1.$$

This is a contradiction, since $\mu_n(g) \rightarrow_n 0$. Thus for every $f \in C_p(X, \mathbb{K})$ the set $f(S)$ is bounded in $\mathbb{K}$.

The set

$$D = \{(x_n) \in c_0(\mathbb{N}, \mathbb{K}) : |x_n| \leq \|\mu_n\| \text{ for all } n \in \mathbb{N}\}$$

is absorbing. Indeed, let $(x_n) \in c_0(\mathbb{N}, \mathbb{K})$. Then there exists $f \in C_p(X, \mathbb{K})$ with $Tf = (x_n)$. Put $\|f\| = \sup\{|f(x)| : x \in S\}$ and $\gamma \in \mathbb{K}$ with $|\gamma| \geq \|f\|$. Then

$$|x_n| = |\mu_n(f)| \leq \|\mu_n\| \cdot \|f\|,$$

so $(x_n) \in \gamma D$. Thus D is absorbing. Clearly, D is absolutely convex and closed in $c_0(\mathbb{N}, \mathbb{K})$, so it is a barrel in the Banach space $(c_0(\mathbb{N}, \mathbb{K}), \|\cdot\|_{\infty})$. Thus D is a neighbourhood of zero in $(c_0(\mathbb{N}, \mathbb{K}), \|\cdot\|_{\infty})$ and consequently, $\inf_n \|\mu_n\| > 0$. For every $n \in \mathbb{N}$ we put $v_n = \beta_n \mu_n$ where $\beta_n \in \mathbb{K}$ with $|\beta_n| \|\mu_n\| = 1$; clearly,

$$(v_n) \subset L_p(X, \mathbb{K}), \quad \|v_n\| = 1, \quad n \in \mathbb{N},$$

and $v_n(f) \rightarrow_n 0$ for every $f \in C_p(X, \mathbb{K})$. Thus the space $C_p(X, \mathbb{K})$ has the JNP.

(4) $\Rightarrow$ (1)

Let (t_k) and (U_k) be as in Lemma 23. Since the supports of $\mu_n, n \in \mathbb{N}$, are finite and

$$\mu_n(U_k) = \mu_n(\mathbb{I}_{U_k}) \rightarrow_n 0$$

for every $k \in \mathbb{N}$, we can construct inductively an increasing sequence $(k_m) \subset \mathbb{N}$ such that

$$U_{k_n} \cap \text{supp}(\mu_{t_{k_m}}) = \emptyset$$

and

$$|\mu_{t_{k_n}}(U_{k_m})| < 1/4$$

for all $n, m \in \mathbb{N}$ with $n > m$. Passing to subsequences $(\mu_{t_{k_m}})$ and (U_{k_m}) we get a sequence $(\vartheta_n) \subset L_p(X, \mathbb{K})$ and a sequence (V_n) of pairwise disjoint clopen subsets of X such that

- (1) $\|\vartheta_m\| = 1$ for $m \in \mathbb{N}$;
- (2) $\vartheta_m(f) \rightarrow_m 0$ for every $f \in C_p(X, \mathbb{K})$;
- (3) $|\vartheta_m(V_m)| = 1, \vartheta_m(V_n) = 0$ and $|\vartheta_n(V_m)| < 1/4$ for all $m, n \in \mathbb{N}$ with $m < n$.

Put $\psi_k = \mathbb{I}_{V_k}$ for $k \in \mathbb{N}$. Then $(\psi_k) \subset C_p(X, \mathbb{K})$ and

$$|\vartheta_m(\psi_m)| = 1, \vartheta_m(\psi_n) = 0$$

and $|\vartheta_n(\psi_m)| < 1/4$ for all $m, n \in \mathbb{N}$ with $m < n$.

It is easy to see that the map

$$\Phi : \mathbb{K}^{\mathbb{N}} \rightarrow \mathbb{K}^{\mathbb{N}}, (x_n) \rightarrow (y_n), \text{ where } y_n = \sum_{k=1}^n x_k \vartheta_n(\psi_k), n \in \mathbb{N},$$

is linear, injective, surjective and

$$\Phi^{-1} : \mathbb{K}^{\mathbb{N}} \rightarrow \mathbb{K}^{\mathbb{N}}, (y_n) \rightarrow (x_n), \text{ where } x_n = \left[y_n - \sum_{1 \leq k < n} x_k \vartheta_n(\psi_k) \right] / \vartheta_n(\psi_n), n \in \mathbb{N}.$$

Clearly, Φ is continuous; by the open mapping theorem it is open, so Φ^{-1} is continuous.

We shall prove that $\Phi(c_0(\mathbb{N}, \mathbb{K})) = c_0(\mathbb{N}, \mathbb{K})$. Let $y = (y_n) \in c_0(\mathbb{N}, \mathbb{K})$. For some $x = (x_n) \in \mathbb{K}^{\mathbb{N}}$ we have $\Phi x = y$. We prove that $x \in c_0(\mathbb{N}, \mathbb{K})$. Put

$$A = \left\{ n \in \mathbb{N} : |x_n| > \max_{1 \leq k < n} |x_k| \geq 1 \right\}.$$

For $n \in A$ we have $|x_n \vartheta_n(\psi_n)| = |x_n|$ and

$$\left| \sum_{1 \leq k < n} x_k \vartheta_n(\psi_k) \right| \leq \max_{1 \leq k < n} |x_k \vartheta_n(\psi_k)| < |x_n|.$$

Thus

$$|y_n| = \left| \sum_{k=1}^n x_k \vartheta_n(\psi_k) \right| = |x_n \vartheta_n(\psi_n)| = |x_n| > 1$$

for $n \in A$. It follows that the set A is finite, so the sequence (x_n) is bounded.

Put $d = \limsup_n |x_n|$. Suppose that $d > 0$. Then the set $B = \{n \in \mathbb{N} : 2|x_n| > d\}$ is infinite and there exist integers $m_0, m_1 \in \mathbb{N}$ with $m_0 < m_1$ such that

$$\sup_{k \geq m_0} |x_k| < 2d$$

and

$$\max_{1 \leq k < m_0} 2|x_k \vartheta_n(\psi_k)| < d$$

for every $n \geq m_1$.

For $n \in B$ with $n > m_1$ we have

$$|x_n \vartheta_n(\psi_n)| = |x_n| > d/2$$

and

$$\left| \sum_{1 \leq k < m_0} x_k \vartheta_n(\psi_k) \right| \leq \max_{1 \leq k < m_0} |x_k \vartheta_n(\psi_k)| < d/2,$$

$$\left| \sum_{m_0 \leq k < n} x_k \vartheta_n(\psi_k) \right| \leq \max_{m_0 \leq k < n} |x_k \vartheta_n(\psi_k)| < 2d/4 = d/2.$$

Thus

$$|y_n| = \left| \sum_{k=1}^n x_k \vartheta_n(\psi_k) \right| = |x_n \vartheta_n(\psi_n)| = |x_n| > d/2$$

for $n \in B$ with $n > m_1$; a contradiction. It follows that $d = 0$; so $x \in c_0(\mathbb{N}, \mathbb{K})$. Thus

$$\Phi(c_0(\mathbb{N}, \mathbb{K})) = c_0(\mathbb{N}, \mathbb{K}),$$

so

$$\Psi : c_0(\mathbb{N}, \mathbb{K}) \rightarrow c_0(\mathbb{N}, \mathbb{K}), \Psi = \Phi|_{c_0(\mathbb{N}, \mathbb{K})}$$

is a linear-topological isomorphism.

The operator

$$T : c_0(\mathbb{N}, \mathbb{K}) \rightarrow C_p(X, \mathbb{K}), (x_n) \rightarrow \sum_{k=1}^{\infty} x_k \psi_k$$

is well defined, linear, injective and continuous.

The map $S : C_p(X, \mathbb{K}) \rightarrow c_0(\mathbb{N}, \mathbb{K})$, $Sf = (\vartheta_n(f))$, is well defined, linear and continuous. It is easy to see that $\Psi = ST$. The linear operator

$$P : C_p(X, \mathbb{K}) \rightarrow C_p(X, \mathbb{K}), P = T\Psi^{-1}S$$

is continuous and $PP = P$, so P is a continuous linear projection on $C_p(X, \mathbb{K})$ with $\ker P = \ker S$. Clearly, the linear map

$$H = T\Psi^{-1} : c_0(\mathbb{N}, \mathbb{K}) \rightarrow C_p(X, \mathbb{K})$$

is continuous and SH is the identity map on $c_0(\mathbb{N}, \mathbb{K})$. The continuous linear map

$$S : C_p(X, \mathbb{K}) \rightarrow c_0(\mathbb{N}, \mathbb{K})$$

is open since for any open neighbourhood U of zero in $C_p(X, \mathbb{K})$ the set $W = H^{-1}(U)$ is an open neighbourhood of zero in $c_0(\mathbb{N}, \mathbb{K})$ and $W = SH(W) \subset S(U)$. Then the quotient space $C_p(X, \mathbb{K})/\ker P = C_p(X, \mathbb{K})/\ker S$ is isomorphic to $c_0(\mathbb{N}, \mathbb{K})$, so the complemented subspace $P(C_p(X, \mathbb{K}))$ of $C_p(X, \mathbb{K})$ is isomorphic to $c_0(\mathbb{N}, \mathbb{K})$. $\square$

6 Proof of Theorem 10

We shall need the following lemma.

Lemma 24 *Let X be an extremally disconnected compact space. Assume that $(\mu_n) \subset L_p(X, \mathbb{K})$ is weak * convergent to 0 i.e. $\mu_n(f) \rightarrow_n 0$ for every $f \in C_p(X, \mathbb{K})$. Then $\mu_n(U_n) \rightarrow_n 0$ for every sequence (U_n) of pairwise-disjoint open sets in X .*

Proof Suppose, by contrary, that $\mu_n(U_n) \not\rightarrow_n 0$ for some sequence (U_n) of pairwise-disjoint open sets in X . After passing to a subsequence and rescaling we can assume that $|\mu_n(U_n)| > 1$ for every $n \in \mathbb{N}$. Since X is ultraregular and supports of $\mu_n, n \in \mathbb{N}$ are finite, we can assume that the sets $U_n, n \in \mathbb{N}$, are clopen. Clearly, $\mathbb{I}_{U_n} \rightarrow_n 0$ in $C_p(X, \mathbb{K})$, so $\mu_k(\mathbb{I}_{U_n}) \rightarrow_n 0$ for every $k \in \mathbb{N}$. By assumption, $\mu_k(\mathbb{I}_{U_n}) \rightarrow_k 0$ for every $n \in \mathbb{N}$.

We define inductively an increasing sequence $(n_k) \subset \mathbb{N}$ such that

$$|\mu_{n_k}(U_{n_m})| \leq 1$$

for all $k, m \in \mathbb{N}$ with $k \neq m$. Put $n_1 = 1$ and suppose that for some $t \in \mathbb{N}$ we have chosen $n_1, \dots, n_t \in \mathbb{N}$ such that

$$|\mu_{n_k}(U_{n_m})| \leq 1$$

for all $1 \leq k, m \leq t$ with $k \neq m$. For every $1 \leq j \leq t$ the set $\{n \in \mathbb{N} : |\mu_{n_j}(U_n)| > 1\}$ is finite and $\mu_k(U_{n_j}) \rightarrow_k 0$. Thus we can choose $n_{t+1} > n_t$ such that $|\mu_{n_j}(U_{n_{t+1}})| \leq 1$ and $|\mu_{n_{t+1}}(U_{n_j})| \leq 1$ for $1 \leq j \leq t$. In this way we construct the desired sequence. Put $v_k = \mu_{n_k}$ and $V_k = U_{n_k}$ for $k \in \mathbb{N}$. Then $|\nu_k(V_m)| \leq 1$ for all $k, m \in \mathbb{N}$ with $k \neq m$ and $|\nu_k(V_k)| > 1$ for every $k \in \mathbb{N}$.

Let $\{A_t : t \in \mathbb{R}\}$ be a family of infinite subsets of $\mathbb{N}$ such that $A_t \cap A_s$ is finite for all $t, s \in \mathbb{R}$ with $t \neq s$. For each $t \in \mathbb{R}$, the closure W_t of the set $\bigcup\{V_k : k \in A_t\}$ in X is clopen, since X is extremally disconnected. Clearly, the set $V = \bigcup_{k=1}^{\infty} V_k$ is open. The sets $M_t = W_t \setminus V, t \in \mathbb{R}$, are closed and pairwise disjoint. Indeed, for $t, s \in \mathbb{R}$ with $t \neq s$ we have

$$W_t = \overline{\bigcup\{V_k : k \in A_t \setminus A_s\}} \cup \overline{\bigcup\{V_k : k \in A_t \cap A_s\}}$$

and

$$W_s = \overline{\bigcup\{V_k : k \in A_s \setminus A_t\}} \cup \overline{\bigcup\{V_k : k \in A_s \cap A_t\}},$$

so

$$W_t \cap W_s \subset \overline{\bigcup\{V_k : k \in A_t \cap A_s\}} = \bigcup\{V_k : k \in A_t \cap A_s\} \subset V,$$

since X is extremally disconnected and the open sets

$$\bigcup\{V_k : k \in A_s \setminus A_t\}, \bigcup\{V_k : k \in A_s \cap A_t\}$$

and $\bigcup\{V_k : k \in A_s \cap A_t\}$ are pairwise disjoint. For each $k \in \mathbb{N}$ the set $\text{supp}(\nu_k) \cap M_t$ is non-empty only for finite many $t \in \mathbb{R}$. Thus there exists $d \in \mathbb{R}$ such that the set $\text{supp}(\nu_k) \cap M_d$ is empty for every $k \in \mathbb{N}$, so $\nu_k(W_d) = \nu_k(W_d \cap V)$ for every $k \in \mathbb{N}$. For $k \in A_d$ and $A'_d = A_d \setminus \{k\}$ we have

$$\begin{aligned}
|\nu_k(\mathbb{I}_{W_d})| &= |\nu_k(W_d)| = |\nu_k(W_d \cap V)| = |\nu_k\left(\bigcup_{j \in A_d} V_j\right)| \\
&= \left| \sum_{j \in A_d} \nu_k(V_j) \right| = \left| \nu_k(V_k) + \sum_{j \in A'_d} \nu_k(V_j) \right| > 1,
\end{aligned}$$

since $|\nu_k(V_k)| > 1$ and $|\sum_{j \in A'_d} \nu_k(V_j)| \leq \max_{j \in A'_d} |\nu_k(V_j)| \leq 1$. This is a contradiction, since $\nu_k(\mathbb{I}_{W_d}) \rightarrow_k 0$. $\square$

By Lemmas 23 and 24 we get the following.

Proposition 25 *Let X be an extremally disconnected compact space. Then $C_p(X, \mathbb{K})$ has not the JNP.*

Corollary 26 *Let X be an extremally disconnected compact space and let $(\mu_n) \subset L_p(X, \mathbb{K})$. Then (μ_n) is weak * convergent to 0 if and only if $\|\mu_n\| \rightarrow_n 0$.*

Proof of Theorem 10 Using Theorem 7 and Proposition 25 we infer that there exists no continuous linear map from $C_p(X, \mathbb{K})$ onto $c_0(\mathbb{N}, \mathbb{K})$. $\square$

7 Proof of Theorem 12

Proof Let $E = C_p(X, \mathbb{K})$. Replace the sequence (K_n) by (K_{2n}) and put $K_0 = X$. Then for every $n \geq 0$ there exist two embeddings $h_n^0, h_n^1 : K_{n+1} \rightarrow K_n$ with $h_n^0(K_{n+1}) \cap h_n^1(K_{n+1}) = \emptyset$ such that

$$h_n^0(K_{n+1}) \cup h_n^1(K_{n+1}) \neq K_n, n \in \mathbb{N}.$$

Let

$$x_n \in [K_n \setminus (h_n^0(K_{n+1}) \cup h_n^1(K_{n+1}))]$$

for $n \in \mathbb{N}$.

For $k \in \mathbb{N}$ let D_k be the family of all binary sequences of length k , that is

$$D_1 = \{(0), (1)\}, \quad D_2 = \{(0, 0), (0, 1), (1, 0), (1, 1)\}, \dots$$

Let $S_\infty = \bigcup_{k=1}^\infty D_k$ and $S_n = \bigcup_{k=1}^n D_k$ and for $n \in \mathbb{N}$. For a binary sequence $s \in D_n, n \in \mathbb{N}$, we put $|s| = n$.

If $n, m \in \mathbb{N}$ and

$$s = (s_0, \dots, s_{n-1}) \in D_n, \quad p = (p_0, \dots, p_{m-1}) \in D_m,$$

then we put

$$s \hat{\wedge} p = (s_0, \dots, s_{n-1}, p_0, \dots, p_{m-1});$$

clearly $s \hat{\wedge} p \in D_{n+m}$. If $s \in S_\infty$ and $i \in \{0, 1\}$, we set $s \hat{\wedge} i = s \hat{\wedge} (i)$.

Consider the family of embeddings $h_s : K_{|s|} \rightarrow X, s \in S_\infty$ defined by the recursive formula: $h_{(0)} = h_0^0, h_{(1)} = h_0^1$ and

$$h_{s \hat{\wedge} i} = h_s \circ h_{|s|}^i \quad \text{for } s \in S_\infty, i \in \{0, 1\}.$$

For every $s \in S_\infty$, let $K_s = h_s(K_{|s|})$ and $x_s = h_s(x_{|s|}) \in K_s$.

If $n \in \mathbb{N}$, $s \in D_n$, $i \in \{0, 1\}$ and $t = s \hat{i} \in D_{n+1}$, then $K_t \subset K_s$; indeed,

$$K_t = K_{s \hat{i}} = h_{s \hat{i}}(K_{n+1}) = h_s(h_n^i(K_{n+1})) \subset h_s(K_n) = K_s.$$

If $n \in \mathbb{N}$ and $s, t \in D_n$ with $s \neq t$, then $K_s \cap K_t = \emptyset$. Indeed, for $n = 1$ it is obvious. Assume that it is true for some $n \in \mathbb{N}$. Let $s, t \in D_n$ and $i, j \in \{0, 1\}$ with $s \hat{i} \neq t \hat{j}$. If $s \neq t$, then

$$K_{s \hat{i}} \cap K_{t \hat{j}} \subset K_s \cap K_t = \emptyset.$$

If $s = t$, then $i \neq j$, so

$$K_{s \hat{i}} \cap K_{t \hat{j}} = K_{s \hat{i}} \cap K_{s \hat{j}} = h_s(h_n^i(K_{n+1})) \cap h_s(h_n^j(K_{n+1})) = \emptyset$$

since h_s is injective and

$$h_n^i(K_{n+1}) \cap h_n^j(K_{n+1}) = \emptyset.$$

Thus for all $u, v \in D_{n+1}$ with $u \neq v$ we have $K_u \cap K_v = \emptyset$.

If $n \in \mathbb{N}$, $i \in \{0, 1\}$ and $t \in D_n$, then

$$\begin{aligned} x_t &= h_t(x_n) \in h_t(K_n \setminus h_n^i(K_{n+1})) \\ &= h_t(K_n) \setminus h_t(h_n^i(K_{n+1})) = K_t \setminus h_{t \hat{i}}(K_{n+1}) = K_t \setminus K_{t \hat{i}}. \end{aligned}$$

It follows that $x_t \notin K_s$ if $t, s \in S_\infty$ with $|t| < |s|$.

If $s, t, p \in S_\infty$ with $|s| = |t|$, then

$$h_s^{-1} \circ h_{s \hat{p}} = h_t^{-1} \circ h_{t \hat{p}}$$

and

$$h_s^{-1}(x_{s \hat{p}}) = h_s^{-1}(h_{s \hat{p}}(x_{|s|+|p|})) = h_t^{-1}(h_{t \hat{p}}(x_{|t|+|p|})) = h_t^{-1}(x_{t \hat{p}}).$$

For every $n \in \mathbb{N}$ let

$$\mu_n : E \rightarrow \mathbb{K}, \mu_n(f) = \sum_{s \in D_n} f(x_s);$$

clearly, μ_n is a continuous linear functional on E . Let

$$Z = \{f \in E : \mu_n(f) = 0 \text{ for any } n \in \mathbb{N}\}.$$

The set $\{\mu_n : n \in \mathbb{N}\}$ is linearly independent in the dual of the space E , since $x_s \neq x_t$ for $s, t \in S_\infty$ with $s \neq t$. Thus the quotient space E/Z is infinite-dimensional.

Finally we prove that this quotient is metrizable.

Let $X_0 = \{x_s : s \in S_\infty\}$ and $X_n = \bigcup_{s \in D_n} K_s$ for $n \in \mathbb{N}$. Clearly, (X_n) is a decreasing sequence of compact subsets of X and

$$X_0 \setminus X_n \subset \{x_s : s \in S_n\}$$

for any $n \in \mathbb{N}$. The subset

$$K = X_0 \cup \bigcap_{n=1}^{\infty} X_n$$

of X is compact. Indeed,

$$\overline{X_0} = \overline{(X_0 \setminus X_n)} \cup \overline{(X_0 \cap X_n)} \subset X_0 \cup X_n$$

for any $n \in \mathbb{N}$, so

$$\overline{X_0} \subset X_0 \cup \bigcap_{n=1}^{\infty} X_n.$$

Thus

$$\overline{K} = \overline{X_0} \cup \bigcap_{n=1}^{\infty} X_n = X_0 \cup \bigcap_{n=1}^{\infty} X_n = K \subset X,$$

so K is compact.

Consider the quotient map $T : E \rightarrow E/Z$. For every $n \in \mathbb{N}$ consider the open neighborhood

$$V_n = \{f \in E : |f(x_s)| < 2^{-n} \text{ for each } s \in S_n\}$$

of zero in E . We claim that $(T(V_n))_{n \in \mathbb{N}}$ is a neighborhood basis at zero in the quotient space E/Z .

Given any neighborhood U of zero in E , we need to find $n \in \mathbb{N}$ such that $V_n + Z \subset U + Z$. Without loss of generality we can assume that U is of basic form

$$U = \{f \in E : |f(x)| < \varepsilon \text{ for any } x \in A\}$$

for some $\varepsilon > 0$ and some finite set $A \subset X$. Choose $n \in \mathbb{N}$ so large that

- (1) $2^n \varepsilon > 1$;
- (2) $2^{n-1} > |A|$;
- (3) $A \cap X_0 \subset \{x_s : s \in S_{n-1}\}$.

We claim that $V_n \subset U + Z$. Let $f \in V_n$. We need to find $g \in Z$ such that $f - g \in U$.

Consider the set $S = \{s \in D_n : A \cap K_s \neq \emptyset\}$. The condition (2) ensures that $|S| \leq |A| \leq 2^{n-1}$. So, we can find an injective map $v : S \rightarrow D_n \setminus S$. Now define the function $g_0 : K \rightarrow \mathbb{K}$ by the formula

$$g_0(x) = \begin{cases} f(x) & x \in K_s \cap K \text{ for some } s \in S; \\ -f \circ h_s \circ h_{v(s)}^{-1}(x) & \text{if } x \in K_{v(s)} \cap K \text{ for some } s \in S; \\ 0 & \text{otherwise.} \end{cases}$$

The function g_0 is continuous. Indeed, let $B_s = \{x_s\}$ for $s \in S_{n-1}$ and $B_s = K_s \cap K$ for $s \in D_n$. Then $K = \bigcup_{s \in S_n} B_s$ and the sets $B_s, s \in S_n$, are compact and pairwise disjoint, so they are open and closed subsets of K . Since $g_0|_{B_s}$ is continuous for any $s \in S_n$, then g_0 is continuous. As K is compact and A is finite, there exists $g \in E$ with $g|_K = g_0$ and $g(x) = f(x)$ for all $x \in A \setminus K$.

We prove that $g \in Z$.

Let $m \in \mathbb{N}$. If $m < n$ and $t \in 2^m$, then $g(x_t) = 0$, so $\mu_m(g) = 0$. If $m = n$ then

$$\begin{aligned} \sum_{t \in D_m} g(x_t) &= \sum_{s \in D_n} g(x_s) = \sum_{s \in S} g(x_s) + \sum_{s \in S} g(x_{v(s)}) \\ &= \sum_{s \in S} f(x_s) - \sum_{s \in S} f(h_s(h_{v(s)}^{-1}(x_{v(s)}))) \\ &= \sum_{s \in S} f(x_s) - \sum_{s \in S} f(h_s(h_s^{-1}(x_s))) = 0, \end{aligned}$$

so $\mu_m(g) = 0$.

If $m > n$, then

$$\begin{aligned} \sum_{t \in D_m} g(x_t) &= \sum_{p \in D_{m-n}} \sum_{s \in D_n} g(x_{s \wedge p}) = \sum_{p \in D_{m-n}} \left(\sum_{s \in S} g(x_{s \wedge p}) + \sum_{s \in S} g(x_{v(s) \wedge p}) \right) \\ &= \sum_{p \in D_{m-n}} \left(\sum_{s \in S} f(x_{s \wedge p}) - \sum_{s \in S} f(h_s(h_{v(s)}^{-1}(x_{v(s) \wedge p}))) \right) = \\ &= \sum_{p \in D_{m-n}} \left(\sum_{s \in S} f(x_{s \wedge p}) - \sum_{s \in S} f(h_s(h_s^{-1}(x_{s \wedge p}))) \right) = 0, \end{aligned}$$

so $\mu_m(g) = 0$. Thus $g \in Z$.

Finally we prove that $f - g \in U$. For $x \in A \setminus K$ we have $|f(x) - g(x)| = 0 < \varepsilon$. Let $x \in E \cap K$. Then $x = x_t$ for some $t \in D_{n-1}$ or $x \in K_s$ for some $s \in S$. In the first case we have

$$|f(x) - g(x)| = |f(x) - 0| < 2^{-n} < \varepsilon;$$

in the second case we get

$$|f(x) - g(x)| = |f(x) - f(x)| = 0 < \varepsilon.$$

Thus $|f(x) - g(x)| < \varepsilon$ for any $x \in A$, so $f - g \in U$.

We have shown that the quotient space E/Z is infinite-dimensional and metrizable. $\square$

8 Proof of Theorem 17

Proof Let $\varphi : \mathbb{N} \rightarrow D$ be an injective map with $\varphi(G_n) = F_n$, $n \in \mathbb{N}$. For any $n \in \mathbb{N}$, the map

$$\mu_n : E \rightarrow \mathbb{K}, \quad \mu_n(f) = \sum_{x \in F_n} f(x),$$

is a continuous linear functional on E . Let $f \in E$. Let $y_k = f(\varphi(k))$ for $k \in \mathbb{N}$. Then

$$\{y_k : k \in \mathbb{N}\} \subset f(\beta D)$$

and

$$\mu_n(f) = \sum_{k \in G_n} y_k, \quad n \in \mathbb{N}.$$

The set $f(\beta D)$ is compact, so (y_k) is relatively compact and

$$(\mu_n(f))_{n=1}^\infty \in \ell_c^G(\mathbb{N}, \mathbb{K}).$$

Thus the map

$$T : E \rightarrow \ell_c^G(\mathbb{N}, \mathbb{K}), \quad Tf := (\mu_n(f))_{n=1}^\infty$$

is well defined, linear and continuous. We shall prove that the map T is open.

Let W be a neighbourhood of zero in E . Then for some finite subset A of βD and some $\varepsilon > 0$ we have $W \supset U(A, \varepsilon)$, where

$$U(A, \varepsilon) = \{f \in E : |f(x)| < \varepsilon \text{ for any } x \in A\}.$$

Let $q = |A| + 1$. For some $m \in \mathbb{N}$ we have $|F_n| > q(q + 1)$ for every $n > m$. The set

$$V(m, \varepsilon) = \left\{ \left(\sum_{k \in G_n} y_k \right)_{n=1}^\infty : (y_k) \in \ell_c(\mathbb{N}, \mathbb{K}) \text{ with } \left| \sum_{k \in G_n} y_k \right| < \varepsilon \text{ for any } 1 \leq n \leq m \right\}$$

is a neighbourhood of zero in $\ell_c^G(\mathbb{N}, \mathbb{K})$. We prove that

$$T(U(A, \varepsilon)) \supset V(m, \varepsilon).$$

Let $(y_k) \in \ell_c(\mathbb{N}, \mathbb{K})$ with

$$\left| \sum_{k \in G_n} y_k \right| < \varepsilon$$

for any $1 \leq n \leq m$. Let $k > m$. Let $\{S_1^k, \dots, S_q^k\}$ be a partition of F_k such that

$$|S_j^k| \geq (|F_k|/q) - 1 \geq |F_k|/(q + 1)$$

for every $1 \leq j \leq q$. The sets $S_j = \bigcup_{k > m} S_j^k$, $1 \leq j \leq q$, are pairwise disjoint, so they have pairwise disjoint closures in βD . Thus

$$\overline{S_l^k} \cap A = \emptyset$$

for some $1 \leq l \leq q$ since $q > |A|$. Let $t_k = |S_l^k|$ and let

$$\{F_1^k, \dots, F_{t_k}^k\}$$

be a partition of F_k such that $|F_i^k| \leq q + 1$ and

$$F_i^k \cap S_l^k \neq \emptyset$$

for every $1 \leq i \leq t_k$. Let $G_i^k = \varphi^{-1}(F_i^k)$ for $1 \leq i \leq t_k$.

Let $x_1^k \in F_k$ for $1 \leq k \leq m$. Let $x_i^k \in F_i^k \cap S_l^k$ for $1 \leq i \leq t_k$, $k > m$. Consider the function $g : D \rightarrow \mathbb{K}$ defined by

$$g(x) = \begin{cases} \sum_{s \in G_n} y_s & \text{if } x = x_1^n, 1 \leq n \leq m; \\ \sum_{s \in G_i^k} y_s & \text{if } x = x_i^k, k > m, 1 \leq i \leq t_k; \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, the set

$$L = \left\{ \sum_{s \in H} y_s : H \subset \mathbb{N} \text{ with } |H| \leq q + 1 \right\}$$

is relatively compact and

$$g(\{x_i^k : k > m, 1 \leq i \leq t_k\}) = \left\{ \sum_{s \in G_i^k} y_s : k > m, 1 \leq i \leq t_k \right\} \subset L.$$

Thus by [5, Theorem 1] the map g has an extension to some continuous function $f : \beta D \rightarrow \mathbb{K}$. We claim that $f \in U(A, \varepsilon)$ and $T(f) = (\sum_{s \in G_n} y_s)_{n=1}^\infty$.

Clearly,

$$\beta D = \bigcup_{n=1}^m F_n \cup \overline{\bigcup_{n=m+1}^\infty F_n} \cup D \setminus \overline{\bigcup_{n=1}^\infty F_n}.$$

Let $x \in A$. If $x \in F_n$ for some $1 \leq n \leq m$, then $f(x) = \sum_{s \in G_n} y_s$ or $f(x) = 0$, so $|f(x)| < \varepsilon$. If

$$x \in \overline{\bigcup_{n=m+1}^\infty F_n} = \bigcup_{j=1}^q \overline{S_j},$$

then $x \in \overline{S_j}$ for some $1 \leq j \leq q$ with $j \neq l$. By the definition of g we have $g|S_j = 0$, so $f|S_j = 0$. Thus $|f(x)| = 0 < \varepsilon$. If

$$x \in D \setminus \overline{\bigcup_{n=1}^\infty F_n},$$

then $|f(x)| = 0 < \varepsilon$, since

$$g\left(D \setminus \overline{\bigcup_{n=1}^\infty F_n}\right) = 0.$$

We have shown that $f \in U(A, \varepsilon) \subset W$.

For $1 \leq n \leq m$ we have

$$\mu_n(f) = \sum_{s \in G_n} f(\varphi(s)) = f(x_1^n) = \sum_{s \in G_n} y_s.$$

For $n > m$ we have

$$\mu_n(f) = \sum_{s \in G_n} f(\varphi(s)) = \sum_{i=1}^{t_n} \sum_{x \in F_i^n} f(x) = \sum_{i=1}^{t_n} f(x_i^n) = \sum_{i=1}^{t_n} \sum_{s \in G_i^n} y_s = \sum_{s \in G_n} y_s.$$

Thus

$$T(f) = (\mu_n(f)) = \left(\sum_{s \in G_n} y_s \right)_{n=1}^\infty.$$

We have shown that $T(W) \supset V(m, \varepsilon)$. It follows that the map T is open. Since the kernel of T is equal to Y_F , the quotient space E/Y_F is isomorphic to $\ell_c^G(\mathbb{N}, \mathbb{K})$. $\square$

9 Proof of Theorem 20

For a subset A of $\mathbb{K}$ we put $S(A) = \{\sum_{k=1}^n a_k : n \in \mathbb{N}, a_1, \dots, a_n \in A\}$.

Lemma 27 *Assume that $|n1_{\mathbb{K}}| \neq 1$ for some $n \in \mathbb{N}$. If $A \subset \mathbb{K}$ is relatively compact then also $S(A)$ is relatively compact.*

Proof Let $\mathbb{L} = \{n1_{\mathbb{K}} : n \in \mathbb{Z}\}$. If $\text{char } \mathbb{K} \neq 0$, then $\mathbb{L}$ is finite. If $\text{char } \mathbb{K} = 0$, then $\mathbb{L}$ is relatively compact, by the Ostrowski Theorem [32, Theorem 1.2]. Thus for any finite subset B of $\mathbb{K}$ the set $S(B)$ is relatively compact, since

$$S(B) \subset \left\{ \sum_{k=1}^m n_k b_k : n_1, \dots, n_m \in \mathbb{L} \right\}$$

if $B = \{b_1, \dots, b_m\}$. Let $\varepsilon > 0$. Put

$$B(x, \varepsilon) = \{\alpha \in \mathbb{K} : |\alpha - x| < \varepsilon\}$$

for $x \in \mathbb{K}$. A is relatively compact, so there exists a finite subset F_ε of $\mathbb{K}$ such that

$$A \subset \bigcup_{x \in F_\varepsilon} B(x, \varepsilon) = F_\varepsilon + B(0, \varepsilon).$$

Hence

$$S(A) \subset S(F_\varepsilon) + S(B(0, \varepsilon)) = S(F_\varepsilon) + B(0, \varepsilon).$$

$S(F_\varepsilon)$ is relatively compact, so

$$S(F_\varepsilon) \subset \bigcup_{x \in G_\varepsilon} B(x, \varepsilon) = G_\varepsilon + B(0, \varepsilon)$$

for some finite subset G_ε of $\mathbb{K}$. Thus

$$S(A) \subset G_\varepsilon + B(0, \varepsilon) + B(0, \varepsilon) = G_\varepsilon + B(0, \varepsilon) = \bigcup_{x \in G_\varepsilon} B(x, \varepsilon).$$

We have shown that the set $S(A)$ is totally bounded in the complete metric space $\mathbb{K}$, so it is relatively compact. $\square$

Using this lemma we get the following proposition.

Proposition 28 *Let $G = (G_n)$ be a partition of $\mathbb{N}$ into non-empty and finite subsets with $\lim_n |G_n| = \infty$. If $|n1_{\mathbb{K}}| \neq 1$ for some $n \in \mathbb{N}$, then $\ell_c^G(\mathbb{N}, \mathbb{K}) = l_c(\mathbb{N}, \mathbb{K})$. In particular, $\ell_c^G(\mathbb{N}, \mathbb{K}) = \ell_c(\mathbb{N}, \mathbb{K}) \neq \ell_\infty(\mathbb{N}, \mathbb{K})$, if $\mathbb{K} = \mathbb{C}_q$ for some prime number q .*

Proof of Theorem 20 It is enough to use Theorem 17 and Proposition 28. $\square$

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