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ORIGINAL PAPER

On the third logarithmic coefficient in some subclasses of close-to-convex functions

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Abstract
For analytic functions f in the unit disk $\mathbb{D}$ normalized by $f(0) = 0$ and $f'(0) = 1$ satisfying in $\mathbb{D}$ respectively the conditions $\operatorname{Re}\{(1-z)f'(z)\} > 0$, $\operatorname{Re}\{(1-z^2)f'(z)\} > 0$, $\operatorname{Re}\{(1-z+z^2)f'(z)\} > 0$, $\operatorname{Re}\{(1-z)^2 f'(z)\} > 0$, the sharp upper bound of the third logarithmic coefficient in case when $f''(0)$ is real was computed.

Keywords Univalent functions · Close-to-convex functions · Functions convex in the direction of the imaginary axis · Logarithmic coefficients · Carathéodory class

Mathematics Subject Classification 30C45

1 Introduction

Let $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, $\overline{\mathbb{D}} := \{z \in \mathbb{C} : |z| \le 1\}$ and $\mathbb{T} := \partial\mathbb{D}$. Let $\mathcal{H}$ be the class of all analytic functions in $\mathbb{D}$, $\mathcal{A}$ be its subclass of f normalized by $f(0) := 0$ and $f'(0) := 1$,

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i.e., of the form

$$f(z)=\sum_{n=1}^{\infty}a_n z^n,\quad a_1:=1,\ z\in\mathbb{D}. \tag{1}$$

and $\mathcal{S}$ be the subclass of $\mathcal{A}$ of all univalent functions.

Given $f\in\mathcal{S}$ let

$$\log\frac{f(z)}{z}=2\sum_{n=1}^{\infty}\gamma_n z^n,\quad z\in\mathbb{D}\backslash\{0\},\ \log 1:=0. \tag{2}$$

The numbers γ_n are called logarithmic coefficients of f. As is well known, the logarithmic coefficients play a crucial role in Milin conjecture ([23], see also [10, p. 155]), namely that for $f\in\mathcal{S}$,

$$\sum_{m=1}^{n}\sum_{k=1}^{n}\left(k|\gamma_k|^2-\frac{1}{k}\right)\le 0.$$

De Branges [8] showing Milin conjecture confirmed the famous Bieberbach conjecture (e.g., [10, p. 37]). It is surprising that for the class $\mathcal{S}$ the sharp estimates of single logarithmic coefficients $\mathcal{S}$ are known only for γ_1 and γ_2, namely,

$$|\gamma_1|\le 1,\quad |\gamma_2|\le\frac{1}{2}+\frac{1}{e}=0.635\ldots$$

and are unknown for $n\ge 3$.

As usual, instead of the whole class $\mathcal{S}$ one can take into account their subclasses for which the problem of finding sharp estimates of logarithmic coefficients can be studied. When $f\in\mathcal{S}^*$, the class of starlike functions, the inequality $|\gamma_n|\le 1/n$ holds for $n\in\mathbb{N}$ (see e.g. [30, p. 42]). Moreover, for $f\in\mathcal{SS}^*(\beta)$, the class of strongly starlike function of order β $(0<\beta\le 1)$, it holds that $|\gamma_n|\le\beta/n$ $(n\in\mathbb{N})$ (see [28]). Also, the bounds of γ_n for functions in the class of gamma-starlike functions, close-to-convex functions and Bazilevič functions were examined in [30, p. 116], [9,27,29], respectively. In two recent papers, namely, in [15] the bounds of early logarithmic coefficients of the subclasses $\mathcal{F}_1,\mathcal{F}_2,\mathcal{F}_3$ of $\mathcal{S}$ and in [1] of the subclass $\mathcal{F}_4$ of $\mathcal{S}$ of functions f satisfying respectively the condition

$$\operatorname{Re}\left\{(1-z)f'(z)\right\}>0,\quad z\in\mathbb{D}, \tag{3}$$

$$\operatorname{Re}\left\{(1-z^2)f'(z)\right\}>0,\quad z\in\mathbb{D}, \tag{4}$$

$$\operatorname{Re}\left\{(1-z+z^2)f'(z)\right\}>0,\quad z\in\mathbb{D}, \tag{5}$$

$$\operatorname{Re}\left\{(1-z)^2f'(z)\right\}>0,\quad z\in\mathbb{D}, \tag{6}$$

were computed. Let us note that each class defined above is the subclass of the well known class of close-to-convex functions, so therefore families $\mathcal{F}_i$, $i=1,\ldots,4$, contain only univalent functions (e.g., [12, Vol. II, p. 2]). Both cited paper contains sharp bounds of γ_1 and γ_2 and partial results for γ_3 only. The first three results in theorem below were shown in [15], and the last one in [1].

Theorem 1 *Let $f\in\mathcal{A}$ be of the form* (1). *Then*

1. *if $f\in\mathcal{F}_1$ and $1\le a_2\le 3/2$, then*

$$|\gamma_3|\le\frac{1}{288}(11+15\sqrt{30})=0.323466\ldots;$$

2. *if* $f \in \mathcal{F}_2$ *and* $0 \le a_2 \le 1$, *then*

$$|\gamma_3| \le \frac{1}{972}(95 + 23\sqrt{46}) = 0.258223\ldots;$$

3. *if* $f \in \mathcal{F}_3$ *and* $1/2 \le a_2 \le 3/2$, *then*

$$|\gamma_3| \le \frac{1}{7776}(743 + 131\sqrt{262}) = 0.368238\ldots;$$

4. *if* $f \in \mathcal{F}_4$ *and* $1 \le a_2 \le 2$, *then*

$$|\gamma_3| \le \frac{1}{243}(28 + 19\sqrt{19}) = 0.456045\ldots$$

In this paper we improve all results in Theorem 1 for γ_3 for the general case when a_2 is real. Differentiating (2) and using (1) we get

$$\gamma_3 = \frac{1}{2}\left(a_4 - a_2 a_3 + \frac{1}{3}a_2^3\right). \tag{7}$$

Since each class $\mathcal{F}_i$, $i = 1, \ldots, 4$, has a representation by using the Carathéodory class $\mathcal{P}$, i.e., the class of functions $p \in \mathcal{H}$ of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \quad z \in \mathbb{D}, \tag{8}$$

having a positive real part in $\mathbb{D}$, the coefficients of functions in $\mathcal{F}_i$, so γ_3 has a suitable representation expressed by the coefficients of functions in $\mathcal{P}$. Therefore to get the upper bound of γ_3 our computing is based on parametric formulas for the second and third coefficients in $\mathcal{P}$. The proof of results of Theorem 1 are based on the well known formula on c_2 and on the formula c_3 due to Libera and Zlotkiewicz [21,22] with the restriction that $c_1 \ge 0$. Since all classes $\mathcal{F}_i$ are not rotation invariant, to omit the assumption $c_1 \ge 0$. we will use a general formula for c_3, which was found in [4]. However to be self contained we present a proof for c_3 here. Moreover in our computation of the sharp bound of γ_3 we use a lemma due to Ohno and Sugawa [24].

Let us mention that the conditions (3), (4) and (6) were discovered by Ozaki [25] as useful criteria of univalence. Recall also that the classes $\mathcal{F}_2$ and $\mathcal{F}_4$ have nice geometrical interpretations, and therefore they play an important role in the geometric function theory. Each function $f \in \mathcal{F}_2$ maps univalently $\mathbb{D}$ onto a domain $f(\mathbb{D})$ convex in the direction of the imaginary axis, i.e., for every $w_1, w_2 \in f(\mathbb{D})$ such that $\operatorname{Re} w_1 = \operatorname{Re} w_2$ the line segment $[w_1, w_2]$ lies in $f(\mathbb{D})$, with the additional property that there exist two points $\omega_1, \omega_2 \in \partial f(\mathbb{D})$ for which $\{\omega_1 + \mathrm{i}t : t > 0\} \subset \mathbb{C}\backslash f(\mathbb{D})$ and $\{\omega_2 - \mathrm{i}t : t > 0\} \subset \mathbb{C}\backslash f(\mathbb{D})$ (see e.g., [12, p. 199]). Each function in the class $\mathcal{F}_4$ maps univalently $\mathbb{D}$ onto a domain $f(\mathbb{D})$ called convex in the positive direction of the real axis, i.e., $\{w + it : t \ge 0\} \subset f(\mathbb{D})$ for every $w \in f(\mathbb{D})$ [2,6,7,11,18,19].

At the end, let us say that the conditions (3)–(6) were generalized by replacing polynomials standing at f' by any quadratic polynomial [16,17], and by any polynomial of any degree having their roots in $\mathbb{C}\backslash\mathbb{D}$ [13,14].

2 Lemmas

The formula (9) is due to Carathéodory [3] (see e.g., [10, p. 41]). The formula (10) can be found in [26, p. 166]. In a recent paper [4] the formula (11) was shown and the extremal functions (13) and (14) were computed also. When $c_1 \geq 0$ the formula (11) was found by Libera and Zlotkiewicz [21,22] (see also [20]).

Lemma 1 *If $p \in \mathcal{P}$ is of the form (8), then*

$$c_1 = 2\zeta_1, \tag{9}$$
$$c_2 = 2\zeta_1^2 + 2(1 - |\zeta_1|^2)\zeta_2 \tag{10}$$

and

$$c_3 = 2\zeta_1^3 + 4(1 - |\zeta_1|^2)\zeta_1\zeta_2 - 2(1 - |\zeta_1|^2)\overline{\zeta_1}\zeta_2^2 + 2(1 - |\zeta_1|^2)(1 - |\zeta_2|^2)\zeta_3 \tag{11}$$

for some $\zeta_i \in \overline{\mathbb{D}}$, $i \in \{1, 2, 3\}$.

For $\zeta_1 \in \mathbb{T}$, there is a unique function $p \in \mathcal{P}$ with c_1 as in (9), namely,

$$p(z) = \frac{1 + \zeta_1 z}{1 - \zeta_1 z}, \quad z \in \mathbb{D}. \tag{12}$$

For $\zeta_1 \in \mathbb{D}$ and $\zeta_2 \in \mathbb{T}$, there is a unique function $p \in \mathcal{P}$ with c_1 and c_2 as in (9)–(10), namely,

$$p(z) = \frac{1 + \left(\overline{\zeta_1}\zeta_2 + \zeta_1\right) z + \zeta_2 z^2}{1 + \left(\overline{\zeta_1}\zeta_2 - \zeta_1\right) z - \zeta_2 z^2}, \quad z \in \mathbb{D}. \tag{13}$$

For $\zeta_1, \zeta_2 \in \mathbb{D}$ and $\zeta_3 \in \mathbb{T}$, there is a unique function $p \in \mathcal{P}$ with c_1, c_2 and c_3 as in (9)–(11), namely,

$$p(z) = \frac{1 + \left(\overline{\zeta_2}\zeta_3 + \overline{\zeta_1}\zeta_2 + \zeta_1\right) z + \left(\overline{\zeta_1}\zeta_3 + \zeta_1\overline{\zeta_2}\zeta_3 + \zeta_2\right) z^2 + \zeta_3 z^3}{1 + \left(\overline{\zeta_2}\zeta_3 + \overline{\zeta_1}\zeta_2 - \zeta_1\right) z + \left(\overline{\zeta_1}\zeta_3 - \zeta_1\overline{\zeta_2}\zeta_3 - \zeta_2\right) z^2 - \zeta_3 z^3}, \quad z \in \mathbb{D}. \tag{14}$$

The next lemma is a special case of more general results due to Choi, Kim and Sugawa [5] (see also [24]). Define

$$Y(a, b, c) := \max_{z \in \overline{\mathbb{D}}} \left(|a + bz + cz^2| + 1 - |z|^2\right), \quad a, b, c \in \mathbb{R}.$$

Lemma 2 [5] *If $ac \geq 0$, then*

$$Y(a, b, c) = \begin{cases} |a| + |b| + |c|, & |b| \geq 2(1 - |c|), \\ 1 + |a| + \dfrac{b^2}{4(1 - |c|)}, & |b| < 2(1 - |c|). \end{cases}$$

If $ac < 0$, then

$$Y(a, b, c) = \begin{cases} 1 - |a| + \dfrac{b^2}{4(1 - |c|)}, & -4ac(c^{-2} - 1) \leq b^2 \wedge |b| < 2(1 - |c|), \\ 1 + |a| + \dfrac{b^2}{4(1 + |c|)}, & b^2 < \min\left\{4(1 + |c|)^2, -4ac(c^{-2} - 1)\right\}, \\ R(a, b, c), & \text{otherwise}, \end{cases}$$

where

$$R(a, b, c) = \begin{cases} |a| + |b| - |c|, & |c|(|b| + 4|a|) \leq |ab|, \\ -|a| + |b| + |c|, & |ab| \leq |c|(|b| - 4|a|), \\ (|c| + |a|)\sqrt{1 - \dfrac{b^2}{4ac}}, & \text{otherwise}. \end{cases}$$

3 Logarithmic coefficients

Now we will prove the main results of this paper.

3.1 The class $\mathcal{F}_1$

Recall that $f \in \mathcal{F}_1$ if $f \in \mathcal{A}$ and

$$\mathrm{Re}\{(1-z)f'(z)\} > 0, \quad z \in \mathbb{D}.$$

Theorem 2 *If $f \in \mathcal{F}_1$ is of the form (1) with $a_2 \in \mathbb{R}$, then*

$$|\gamma_3| \leq \frac{1}{288}(11 + 15\sqrt{30}) = 0.323466\ldots \tag{15}$$

The inequality is sharp with the extremal function

$$f(z) = \int_0^z \frac{p(t)}{1-t}\mathrm{d}t, \quad z \in \mathbb{D}, \tag{16}$$

where

$$p(z) = \frac{(1+z)(6+(7-2\sqrt{30})z+6z^2)}{(1-z)(6+(1+\sqrt{30})z+6z^2)}, \quad z \in \mathbb{D}. \tag{17}$$

Proof Let $f \in \mathcal{F}_1$ be of the form (1) with $a_2 \in \mathbb{R}$. Then there exists $p \in \mathcal{P}$ of the form (8) such that

$$(1-z)f'(z) = p(z), \quad z \in \mathbb{D}. \tag{18}$$

Substituting the series (1) and (8) into (18) and equating the coefficients we get

$$a_2 = \frac{1}{2}(1+c_1), \quad a_3 = \frac{1}{3}(1+c_1+c_2), \quad a_4 = \frac{1}{4}(1+c_1+c_2+c_3). \tag{19}$$

Note first that since a_2 is real, so is c_1, and (9) holds with some $\zeta_1 \in [-1, 1]$. Moreover, from (19) it follows that $a_2 \in [-1/2, 3/2]$.

By (7) and (19) we get

$$48\gamma_3 = 3 + c_1 - c_1^2 + c_1^3 - 4c_1c_2 + 2c_2 + 6c_3.$$

Using now (9)–(11) we have

$$\begin{aligned} 48\gamma_3 =& 3 + 2\zeta_1 + 4\zeta_1^3 + 4(1-\zeta_1^2)\zeta_2 + 8(1-\zeta_1^2)\zeta_1\zeta_2 \\ &- 12(1-\zeta_1^2)\zeta_1\zeta_2^2 + 12(1-\zeta_1^2)(1-|\zeta_2|^2)\zeta_3, \end{aligned} \tag{20}$$

with $\zeta_1 \in [-1, 1]$ and $\zeta_2, \zeta_3 \in \overline{\mathbb{D}}$.

Hence for $\zeta_1 = 1$ and $\zeta_1 = -1$ we respectively have

$$\gamma_3 = \frac{3}{16} = 0.1875, \quad \gamma_3 = -\frac{1}{16} = -0.0625. \tag{21}$$

Let now $\zeta_1 \in (-1, 1)$. Then from (20) we obtain

$$48|\gamma_3| \leq 12(1-\zeta_1^2)\Psi(A, B, C), \tag{22}$$

where

$$\Psi(A, B, C) := |A + B\zeta_2 + C\zeta_2^2| + 1 - |\zeta_2|^2, \tag{23}$$

with

$$A := \frac{3 + 2\zeta_1 + 4\zeta_1^3}{12(1 - \zeta_1^2)}, \quad B := \frac{1}{3}(1 + 2\zeta_1), \quad C := -\zeta_1.$$

Note that

$$AC < 0, \quad \zeta_1 \in [-1, \zeta') \cup (0, 1],$$

and

$$AC \geq 0, \quad \zeta_1 \in [\zeta', 0],$$

where $\zeta' = -0.72808\ldots$ is the zero of the equation $3 + 2x + 4x^3 = 0$, $x \in (-1, 0)$.

A. Let $\zeta_1 \in [\zeta', 0]$. Then the inequality $|B| < 2(1 - |C|)$ holds, so by (22) and Lemma 2 we have

$$48|\gamma_3| \leq 12(1 - \zeta_1^2)\left(1 + |A| + \frac{B^2}{4(1 - |C|)}\right) = \varphi(\zeta_1), \tag{24}$$

where

$$\varphi(x) := \frac{1}{3}(46 + 9x - 36x^2 + 8x^3), \quad x \in [-1, 1]. \tag{25}$$

Since φ is increasing on $[\zeta', 0]$, so $\varphi(x) \leq \varphi(0) = 46/3$ for all $x \in [\zeta', 0]$. Therefore from (24) we get

$$|\gamma_3| \leq \frac{23}{72} = 0.319444\ldots$$

B. Let $\zeta_1 \in (0, 1)$. Then the following inequalities hold:

$$B^2 + 4AC(C^{-2} - 1) = \frac{1}{9\zeta_1}(-9 - 5\zeta_1 + 4\zeta_1^2 - 8\zeta_1^3) < 0$$

and

$$B^2 - 4(1 + |C|)^2 = -\frac{1}{9}(35 + 68\zeta_1 + 32\zeta_1^2) < 0.$$

Therefore from (22) and Lemma 2 it follows that

$$48|\gamma_3| \leq 12(1 - \zeta_1^2)\left(1 + |A| + \frac{B^2}{4(1 + |C|)}\right) = \varphi(\zeta_1), \tag{26}$$

where φ is the function defined by (25). Since $\varphi'(x) = 0$ occurs only at $x = (6 - \sqrt{30})/4 =: x_0$ in $(0, 1)$ and $\varphi''(x_0) = -4\sqrt{30} < 0$, it follows that

$$\varphi(x) \leq \varphi(x_0) = \frac{1}{6}(11 + 15\sqrt{30}), \quad x \in (0, 1).$$

Thus by (26) we get

$$|\gamma_3| \leq \frac{1}{288}(11 + 15\sqrt{30}) = 0.323466\ldots$$

C. Let $\zeta_1 \in (-1, \zeta')$. Note that then $B^2 < 4(1 + |C|)^2$. Furthermore, $B^2 + 4AC(C^{-2} - 1) < 0$ holds if and only if $\zeta_1 \in [-1, \zeta'']$, where $\zeta'' = -0.73448\ldots$ is the zero of the equation $9+5x-4x^2+8x^3=0,\ x\in(-1,1)$. Therefore, when $\zeta_1\in(-1,\zeta'']$, by (22) and Lemma 2 we have

$$\begin{aligned}|\gamma_3| &\le \frac{1}{4}(1-\zeta_1^2)\left(1+|A|+\frac{B^2}{4(1+|C|)}\right)\\ &= \frac{1}{144}(28-\zeta_1-28\zeta_1^2-8\zeta_1^3) < \frac{1}{8} = 0.125.\end{aligned} \tag{27}$$

For $\zeta_1\in(\zeta'',\zeta']$ it holds $B^2+4AC(C^{-2}-1)>0$ and $|B|<2(1-|C|)$. Hence by (22) and Lemma 2 we get

$$\begin{aligned}|\gamma_3| &\le \frac{1}{4}(1-\zeta_1^2)\left(1-|A|+\frac{B^2}{4(1-|C|)}\right)\\ &= \frac{1}{144}(46+9\zeta_1-36\zeta_1^2+8\zeta_1^3) < \frac{7}{48} = 0.145833\ldots.\end{aligned} \tag{28}$$

Summarizing, from (21) and parts A-C it follows that the inequality (15) is true.

By tracking back the above proof, we see that equality in (15) holds when it is satisfied that

$$\zeta_1=\frac{1}{4}(6-\sqrt{30}),\quad \zeta_3=1 \tag{29}$$

and

$$|A+B\zeta_2+C\zeta_2^2|+1-|\zeta_2|^2=1+|A|+\frac{B^2}{4(1+|C|)}, \tag{30}$$

where

$$A=\frac{-2490+731\sqrt{30}}{5460},\quad B=\frac{8-\sqrt{30}}{6},\quad C=-\frac{1}{4}(6-\sqrt{30}).$$

Indeed we can easily check that one of the solutions of Eq. (30) is

$$\zeta_2=\frac{1}{105}(25-\sqrt{30}). \tag{31}$$

By Lemma 1 a function p of the form (14) with ζ_i ($i\in\{1,2,3\}$) given by (29) and (31), i.e., the function (17) belongs to $\mathcal{P}$. Thus the function (16) belongs to $\mathcal{F}_1$. Substituting (29) and (31) into (20) we get equality in (15). This ends the proof of the theorem. □

3.2 The class $\mathcal{F}_2$

Recall that $f\in\mathcal{F}_2$ if $f\in\mathcal{A}$ and

$$\operatorname{Re}\{(1-z^2)f'(z)\}>0,\quad z\in\mathbb{D}.$$

Theorem 3 *If $f\in\mathcal{F}_2$ is of the form (1) with $a_2\in\mathbb{R}$, then*

$$|\gamma_3|\le\frac{1}{972}(95+23\sqrt{46})=0.258223\ldots \tag{32}$$

The inequality is sharp with the extremal function

$$f(z)=\int_0^z\frac{p(t)}{1-t^2}\mathrm{d}t,\quad z\in\mathbb{D}, \tag{33}$$

where

$$p(z) = \frac{(1+z)(9+(7-2\sqrt{46})z+9z^2)}{(1-z)(9+(1+\sqrt{46})z+9z^2)}, \quad z \in \mathbb{D}. \tag{34}$$

Proof Let $f \in \mathcal{F}_2$ be of the form (1). Then there exists $p \in \mathcal{P}$ of the form (8) such that

$$(1-z^2)f'(z) = p(z), \quad z \in \mathbb{D}. \tag{35}$$

Substituting the series (1) and (8) into (35) by equating the coefficients we get

$$a_2 = \frac{1}{2}c_1, \quad a_3 = \frac{1}{3}(1+c_2), \quad a_4 = \frac{1}{4}(c_1+c_3). \tag{36}$$

Note first that since a_2 is real, so is c_1 and (9) holds with some $\zeta_1 \in [-1, 1]$. Moreover, from (36) it follows that $a_2 \in [-1, 1]$.

By (7) and (36) we get

$$48\gamma_3 = 2c_1 + c_1^3 - 4c_1c_2 + 6c_3.$$

Using now (9)–(11) we have

$$\begin{aligned} 12\gamma_3 = {} & \zeta_1^3 + \zeta_1 + 2(1-\zeta_1^2)\zeta_1\zeta_2 \\ & - 3(1-\zeta_1^2)\zeta_1\zeta_2^2 + 3(1-\zeta_1^2)(1-|\zeta_2|^2)\zeta_3. \end{aligned} \tag{37}$$

with $\zeta_1 \in [-1, 1]$ and $\zeta_2, \zeta_3 \in \overline{\mathbb{D}}$.

Hence for $\zeta_1 = 1$, $\zeta_1 = -1$ and $\zeta_1 = 0$ we respectively have

$$\gamma_3 = \frac{1}{6}, \quad \gamma_3 = -\frac{1}{6}, \quad |\gamma_3| \le \frac{1}{4}(1-|\zeta_2|^2) \le \frac{1}{4}. \tag{38}$$

Now let $\zeta_1 \in (-1, 1)\setminus\{0\} =: I$. Then from (37) we obtain

$$12|\gamma_3| \le 3(1-\zeta_1^2)\Psi(A, B, C), \tag{39}$$

where Ψ is defined by (23) with

$$A := \frac{\zeta_1(1+\zeta_1^2)}{3(1-\zeta_1^2)}, \quad B := \frac{2}{3}\zeta_1, \quad C := -\zeta_1.$$

Note now that $AC < 0$ for $\zeta_1 \in I$. Moreover,

$$B^2 \le -4AC(C^{-2}-1), \quad \zeta_1 \in I,$$

since

$$B^2 + 4AC(C^{-2}-1) = -\frac{4}{9}(3+2\zeta_1^2) < 0, \quad \zeta_1 \in I,$$

and

$$B^2 < 4(1+|C|)^2, \quad \zeta_1 \in I,$$

since

$$B^2 - 4(1+|C|)^2 = -\frac{4}{9}(8\zeta_1^2 + 18|\zeta_1| + 9) < 0, \quad \zeta_1 \in I.$$

Therefore by Lemma 2 we get

$$\begin{aligned}\Psi(A,B,C) &\le 1+|A|+\frac{B^2}{4(1+|C|)}\\ &=1+\frac{|\zeta_1|(1+\zeta_1^2)}{3(1-\zeta_1^2)}+\frac{\zeta_1^2}{9(1+|\zeta_1|)}.\end{aligned}$$

Hence and from (39) it follows that

$$12|\gamma_3|\le\varphi(\zeta_1), \tag{40}$$

where

$$\varphi(x):=\frac{1}{3}(9+3|x|-8x^2+2|x|^3),\quad x\in I.$$

We note that the function φ is even in I. As easy to verify

$$\varphi(x)\le\varphi(x_0)=\frac{1}{81}(95+23\sqrt{46}),\quad x\in I,$$

where $x_0:=(8-\sqrt{46})/6=0.202945\ldots$ Hence and by (40) we obtain

$$|\gamma_3|\le\frac{1}{12}\varphi(x_0)\le\frac{1}{972}(95+23\sqrt{46}).$$

This and (38) show that the inequality (32) is true.

By tracking back the above proof, we see that equality in (32) holds when it is satisfied that

$$\zeta_1=\frac{1}{6}(8-\sqrt{46}),\quad \zeta_3=1 \tag{41}$$

and

$$|A+B\zeta_2+C\zeta_2^2|+1-|\zeta_2|^2=1+|A|+\frac{B^2}{4(1+|C|)}, \tag{42}$$

where

$$A=\frac{-1688+283\sqrt{46}}{3150},\quad B=\frac{1}{9}(8-\sqrt{46}),\quad C=-\frac{1}{6}(8-\sqrt{46}).$$

Indeed we can easily check that one of the solutions of the equation (42) is

$$\zeta_2=\frac{1}{75}(11-\sqrt{46}). \tag{43}$$

By Lemma 1 a function p of the form (14) with ζ_i $(i\in\{1,2,3\})$ given by (41) and (43), i.e., the function (34) belongs to $\mathcal{P}$. Thus the function (33) belongs to $\mathcal{F}_2$. Substituting (41) and (43) into (37) we get equality in (32). This ends the proof of the theorem. □

3.3 The class $\mathcal{F}_3$

Recall that $f\in\mathcal{F}_3$ if $f\in\mathcal{A}$ and

$$\mathrm{Re}\{(1-z+z^2)f'(z)\}>0,\quad z\in\mathbb{D}.$$

Theorem 4 *If $f \in \mathcal{F}_3$ is of the form* (1) *with $a_2 \in \mathbb{R}$, then*

$$|\gamma_3| \le \frac{1}{7776}(743 + 131\sqrt{262}) = 0.368238\ldots \tag{44}$$

This result is sharp.

Proof Let $f \in \mathcal{F}_3$ be of the form (1) with $a_2 \in \mathbb{R}$. Then there exists $p \in \mathcal{P}$ of the form (8) such that

$$(1 - z + z^2) f'(z) = p(z), \quad z \in \mathbb{D}. \tag{45}$$

Substituting the series (1) and (8) into (45) by equating the coefficients we get

$$a_2 = \frac{1}{2}(1 + c_1), \quad a_3 = \frac{1}{3}(c_1 + c_2), \quad a_4 = \frac{1}{4}(-1 + c_2 + c_3). \tag{46}$$

Note first that since a_2 is real, so is c_1 and (9) holds with some $\zeta_1 \in [-1, 1]$. Moreover, from (46) it follows that $a_2 \in [-1/2, 3/2]$.

By (7) and (46) we get

$$48\gamma_3 = -5 - c_1 - c_1^2 + c_1^3 - 4c_1c_2 + 2c_2 + 6c_3.$$

Using now (9)–(11) we have

$$\begin{aligned} 48\gamma_3 = & -5 - 2\zeta_1 + 4\zeta_1^3 + 4(1 - \zeta_1^2)\zeta_2 + 8(1 - \zeta_1^2)\zeta_1\zeta_2 \\ & - 12(1 - \zeta_1^2)\zeta_1\zeta_2^2 + 12(1 - \zeta_1^2)(1 - |\zeta_2|^2)\zeta_3, \end{aligned} \tag{47}$$

with $\zeta_1 \in [-1, 1]$ and $\zeta_2, \zeta_3 \in \overline{\mathbb{D}}$.

Hence for $\zeta_1 = 1$ and $\zeta_1 = -1$ we respectively have

$$\gamma_3 = -\frac{1}{16}, \quad \gamma_3 = -\frac{7}{48}. \tag{48}$$

Now let $\zeta_1 \in (-1, 1)$. Then from (47) we get

$$48|\gamma_3| \le 12(1 - \zeta_1^2)\Psi(A, B, C), \tag{49}$$

where Ψ is defined by (23) with

$$A := \frac{-5 - 2\zeta_1 + 4\zeta_1^3}{12(1 - \zeta_1^2)}, \quad B := \frac{1}{3}(1 + 2\zeta_1), \quad C := -\zeta_1.$$

Note that $A < 0$ for $\zeta_1 \in (-1, 1)$.

Let $\zeta_1 \in (-1, 0)$. Then $AC < 0$ and it can be easily checked that the following inequalities are true:

$$B^2 + 4AC(C^{-2} - 1) = \frac{1}{9\zeta_1}(15 + 7\zeta_1 + 4\zeta_1^2 - 8\zeta_1^3) < 0$$

and

$$B^2 - 4(1 + |C|)^2 = -\frac{1}{9}(35 - 76\zeta_1 + 32\zeta_1^2) < 0.$$

Hence from (49) and Lemma 2 and it follows that

$$\begin{aligned} |\gamma_3| &\le \frac{1}{4}(1 - \zeta_1^2)\left(1 + |A| + \frac{B^2}{4(1 + |C|)}\right) \\ &= \frac{1}{144}(52 + 11\zeta_1 - 28\zeta_1^2 - 8\zeta_1^3) \le \frac{13}{36} = 0.361111\ldots \end{aligned} \tag{50}$$

Let now $\zeta_1 \in [0,1)$. Then $AC \geq 0$ and we consider two subcases, i.e., $\zeta_1 \in [5/8, 1)$ and $\zeta_1 \in [0, 5/8)$. For $\zeta_1 \in [5/8, 1)$, it holds $|B| \geq 2(1-|C|)$. Thus by (49) and Lemma 2 we have

$$\begin{aligned} |\gamma_3| &\leq \frac{1}{4}(1-\zeta_1^2)(|A|+|B|+|C|) \\ &= \frac{1}{48}\left(9+22\zeta_1-4\zeta_1^2-24\zeta_1^3\right) \leq \frac{327}{1024} = 0.319335\ldots \end{aligned} \tag{51}$$

For $\zeta_1 \in [0, 5/8)$ it holds $|B| < 2(1-|C|)$. Thus (49) and Lemma 2 lead to

$$48|\gamma_3| \leq 12(1-\zeta_1^2)\left(1+|A|+\frac{B^2}{4(1-|C|)}\right) = \varphi(\zeta_1), \tag{52}$$

where

$$\varphi(x) := \frac{1}{3}(52+11x-28x^2-8x^3), \quad x \in [0, 5/8).$$

As easy to verify, for $x \in [0, 5/8]$,

$$\varphi(x) \leq \varphi(x_0) = \frac{1}{162}(743+131\sqrt{262}) = 17.675433\ldots,$$

where $x_0 = (-14+\sqrt{262})/12 = 0.182201\ldots \in [0, 5/8)$. Hence and by (52) it follows that

$$|\gamma_3| \leq \frac{1}{7776}(743+131\sqrt{262}). \tag{53}$$

Summarizing (48), (50), (51) and (53) show that the inequality (44) is true.

By tracking back the above proof, we see that equality in (44) holds when it is satisfied that

$$\zeta_1 = \frac{1}{12}(-14+\sqrt{262}), \quad \zeta_3 = 1 \tag{54}$$

and

$$|A+B\zeta_2+C\zeta_2^2|+1-|\zeta_2|^2 = 1+|A|+\frac{B^2}{4(1-|C|)}, \tag{55}$$

where

$$A = \frac{9526-1601\sqrt{262}}{35604}, \quad B = -\frac{1}{18}(8-\sqrt{262}), \quad C = \frac{1}{12}(14-\sqrt{262}).$$

Indeed we can check that ζ_2 defined by

$$\zeta_2 = \frac{4924-269\sqrt{262} \pm 2\mathrm{i}\sqrt{57399872-3438382\sqrt{262}}}{11946-555\sqrt{262}} \tag{56}$$

satisfies the equation (55).

By Lemma 1 a function p of the form (14) with ζ_i $(i \in \{1,2,3\})$ given by (54) and (56) belongs to $\mathcal{P}$. Note that $|\zeta_2| = 0.912\ldots$ Thus the corresponding function function

$$f(z) = \int_0^z \frac{p(t)}{1-t+t^2}\mathrm{d}t, \quad z \in \mathbb{D},$$

belongs to $\mathcal{F}_3$. Substituting such chosen ζ_1, ζ_2 and ζ_3 into (47) we get equality in (44). This ends the proof of the theorem. □

3.4 The class $\mathcal{F}_4$

Recall that $f \in \mathcal{F}_4$ if $f \in \mathcal{A}$ and

$$\operatorname{Re}\{(1-z)^2 f'(z)\} > 0, \quad z \in \mathbb{D}.$$

Theorem 5 *If $f \in \mathcal{F}_4$ is of the form (1) with $a_2 \in \mathbb{R}$, then*

$$|\gamma_3| \le \frac{1}{243}(28 + 19\sqrt{19}) = 0.456045\ldots \tag{57}$$

The inequality is sharp with the extremal function

$$f(z) = \int_0^z \frac{p(t)}{(1-t)^2}\mathrm{d}t, \quad z \in \mathbb{D}, \tag{58}$$

where

$$p(z) = \frac{(1+z)\left(9 + (14 - 4\sqrt{19})z + 9z^2\right)}{(1-z)\left(9 + 2(1+\sqrt{19})z + 9z^2\right)}, \quad z \in \mathbb{D}. \tag{59}$$

Proof Let $f \in \mathcal{F}_4$ be of the form (1) with $a_2 \in \mathbb{R}$. Then there exists $p \in \mathcal{P}$ of the form (8) such that

$$(1-z)^2 f'(z) = p(z), \quad z \in \mathbb{D}. \tag{60}$$

Putting the series (1) and (8) into (60) by equating the coefficients we get

$$\begin{aligned} a_2 &= \frac{1}{2}(2 + c_1), \quad a_3 = \frac{1}{3}(3 + 2c_1 + c_2),\\ a_4 &= \frac{1}{4}(4 + 3c_1 + 2c_2 + c_3). \end{aligned} \tag{61}$$

As in earlier consideration, $\zeta_1 \in [-1, 1]$ and from (61) it follows that $a_2 \in [0, 1]$.

By (7) and (61) we get

$$12\gamma_3 = \frac{1}{4}(8 + 2c_1 - 2c_1^2 + c_1^3 - 4c_1c_2 + 4c_2 + 6c_3).$$

Using now (9)–(11) we have

$$\begin{aligned} 12\gamma_3 = &\zeta_1^3 + \zeta_1 + 2 + 2(1-\zeta_1^2)\zeta_2 + 2(1-\zeta_1^2)\zeta_1\zeta_2\\ &- 3(1-\zeta_1^2)\zeta_1\zeta_2^2 + 3(1-\zeta_1^2)(1-|\zeta_2|^2)\zeta_3, \end{aligned} \tag{62}$$

with $\zeta_1 \in [-1, 1]$ and $\zeta_2, \zeta_3 \in \overline{\mathbb{D}}$.

Hence for $\zeta_1 = 1$ and $\zeta_1 = -1$ we respectively have

$$\gamma_3 = 1/3, \quad \gamma_3 = 0. \tag{63}$$

Let now $\zeta_1 \in (-1, 1)$. Then

$$12|\gamma_3| \le 3(1-\zeta_1^2)\Psi(A, B, C), \tag{64}$$

where Ψ is defined by (23) with

$$A := \frac{2 - \zeta_1 + \zeta_1^2}{3(1-\zeta_1)}, \quad B := \frac{2}{3}(1+\zeta_1), \quad C := -\zeta_1.$$

For $\zeta_1 \in (-1, 0]$ it holds $AC \geq 0$ and $|B| < 2(1 - |C|)$. Thus by (64) and Lemma 2 we have

$$|\gamma_3| \leq \frac{1}{4}(1 - \zeta_1^2)\left(1 + |A| + \frac{B^2}{4(1 - |C|)}\right)$$
$$= \frac{1}{18}(8 + 2\zeta_1 - 5\zeta_1^2 + \zeta_1^3) \leq \frac{4}{9}. \tag{65}$$

For $\zeta_1 \in (0, 1)$ it can be easily checked that

$$AC < 0, \quad B^2 < -4AC(C^{-2} - 1), \quad B^2 < 4(1 + |C|)^2.$$

Therefore by (64) and Lemma 2 we get

$$12|\gamma_3| \leq 3(1 - \zeta_1^2)\left(1 + |A| + \frac{B^2}{4(1 + |C|)}\right) = \varphi(\zeta_1), \tag{66}$$

where

$$\varphi(x) := \frac{2}{3}(8 + 2x - 5x^2 + x^3), \quad x \in (0, 1).$$

As easy to verify

$$\varphi(x) \leq \varphi(x_0) = \frac{4}{81}(28 + 19\sqrt{19}), \quad x \in (0, 1),$$

where $x_0 := (5 - \sqrt{19})/3$. Hence and by (66) it follows that

$$|\gamma_3| \leq \frac{1}{243}(28 + 19\sqrt{19}). \tag{67}$$

Summarizing, (63), (65) and (67) show that the inequality (57) is true.

A simlar method used for the proof of Theorem 2, the equality in (57) when it is satisfied that

$$\zeta_1 = \frac{1}{3}(5 - \sqrt{19}), \quad \zeta_2 = \frac{1}{3}, \quad \zeta_3 = 1. \tag{68}$$

By Lemma 1 a function p of the form (14) with ζ_i ($i \in \{1, 2, 3\}$) given by (68), i.e., the function (59) belongs to $\mathcal{P}$. Thus the function (58) belongs to $\mathcal{F}_4$. Substituting (68) into (62) we get equality in (57). This ends the proof of the theorem. □

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