Local uniform non-squareness of Orlicz–Lorentz function spaces

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Abstract

We give criteria for local uniform non-squareness of Orlicz–Lorentz function spaces $\Lambda_{\varphi, \omega}$ equipped with the Luxemburg norm, where the widest possible class of convex Orlicz functions is admitted. As immediate consequences, criteria for local uniform non-squareness of Orlicz function spaces L^φ , which complete the already known results, are deduced.

Keywords Local uniform non-squareness · Non-squareness · Uniform non-squareness · Orlicz–Lorentz space · Lorentz space · Local uniform monotonicity

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1 Preliminaries

We say that a Banach space X is non-square if for any x and y from $S(X)$ (the unit sphere of X), we have $\min(\|\frac{x-y}{2}\|, \|\frac{x+y}{2}\|) < 1$. A Banach space X is said to be locally uniformly non-square if for any $x \in S(X)$ there exists $\delta = \delta(x) \in (0, 1)$ such that $\min(\|\frac{x-y}{2}\|, \|\frac{x+y}{2}\|) \leq 1 - \delta$ for any $y \in B(X)$ (the unit ball of X). Finally, we say that a Banach space X is uniformly non-square if there exists $\delta \in (0, 1)$ such that $\min(\|\frac{x-y}{2}\|, \|\frac{x+y}{2}\|) \leq 1 - \delta$ for any $x, y \in B(X)$.

Recall that uniform non-squareness of Banach spaces was defined by James as a geometric property which implies super-reflexivity (see [9,10]). Hence, knowing that a Banach space possesses this property, even without any characterization of its dual space, we can infer that it is super-reflexive, and so reflexive. Additionally, as García-Falset, Llorens-Fuster and Mazcuñan-Navarro have shown, uniformly non-square Banach spaces possess the fixed-point property (see [5]). Therefore, it is natural and interesting to ask about criteria of non-squareness of various well-known classes of Banach spaces. The purpose of this paper is to complete the study started in [3] (see also [2]).

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Let $L^0 = L^0([0, \gamma])$ be the space of all (equivalence classes of) Lebesgue measurable real-valued functions defined on the interval $[0, \gamma]$, where $\gamma \leq \infty$. For any $x, y \in L^0$, we write $x \leq y$ if $x(t) \leq y(t)$ almost everywhere with respect to the Lebesgue measure m on the interval $[0, \gamma]$.

Given any $x \in L^0$, we define its distribution function $\mu_x : [0, +\infty) \rightarrow [0, \gamma]$ by

$$\mu_x(\lambda) = m\{t \in [0, \gamma] : |x(t)| > \lambda\}$$

(see [1, 14, 15]) and the non-increasing rearrangement $x^* : [0, \gamma] \rightarrow [0, \infty]$ of x as

$$x^*(t) = \inf\{\lambda \geq 0 : \mu_x(\lambda) \leq t\}$$

(under the convention $\inf \emptyset = \infty$). We say that two functions $x, y \in L^0$ are equimeasurable if $\mu_x(\lambda) = \mu_y(\lambda)$ for all $\lambda \geq 0$. Then we obviously have $x^* = y^*$.

A Banach space $E = (E, \leq, \|\cdot\|)$, where $E \subset L^0$, is said to be a Köthe space if the following conditions are satisfied:

- (i) if $x \in E$, $y \in L^0$ and $|y| \leq |x|$, then $y \in E$ and $\|y\| \leq \|x\|$,
- (ii) there exists a function x in E that is strictly positive on the whole $[0, \gamma]$.

Recall that the Köthe space E is called a symmetric space if E is rearrangement invariant in the sense that if $x \in E$, $y \in L^0$ and $x^* = y^*$, then $y \in E$ and $\|x\| = \|y\|$. For basic properties of symmetric spaces we refer to [1, 14, 15].

In the whole paper φ will denote an Orlicz function, that is, $\varphi : [-\infty, \infty] \rightarrow [0, \infty]$ (our definition is extended from $\mathbb{R}$ into $\mathbb{R}^e$ by assuming $\varphi(-\infty) = \varphi(\infty) = \infty$) and φ is convex, even, vanishing and continuous at zero, left continuous on $(0, \infty)$ and not identically equal to zero on $(-\infty, \infty)$. Let us denote

$$a_\varphi = \sup\{u \geq 0 : \varphi(u) = 0\} \quad \text{and} \quad b_\varphi = \sup\{u \geq 0 : \varphi(u) < \infty\}$$

and

$$\beta = \sup\left\{u \geq 0 : \varphi\left(\frac{u}{2}\right) = \frac{1}{2}\varphi(u)\right\}. \quad (1)$$

Let us note that if $a_\varphi > 0$, then $\beta = a_\varphi$, while the left-continuity of φ on $(0, \infty)$ is equivalent to the fact that $\lim_{u \rightarrow (b_\varphi)^-} \varphi(u) = \varphi(b_\varphi)$.

Recall that an Orlicz function φ satisfies the condition Δ_2 for all $u \in \mathbb{R}$ ($\varphi \in \Delta_2(\mathbb{R})$, for short) if there exists a constant $K > 0$ such that the inequality

$$\varphi(2u) \leq K\varphi(u) \quad (2)$$

holds for any $u \in \mathbb{R}$ (then we have $a_\varphi = 0$ and $b_\varphi = \infty$). Analogously, we say that an Orlicz function φ satisfies the condition Δ_2 at infinity ($\varphi \in \Delta_2(\infty)$, for short) if there exist a constant $K > 0$ and a constant $u_0 \geq 0$ such that $\varphi(u_0) < \infty$ and inequality (2) holds for any $u \geq u_0$ (then we obtain $b_\varphi = \infty$).

For any Orlicz function φ we define its complementary function in the sense of Young by the formula

$$\psi(u) = \sup_{v>0} \{|u|v - \varphi(v)\} \quad (3)$$

for all $u \in \mathbb{R}$. It is easy to show that ψ is also an Orlicz function.

Let $\omega : [0, \gamma] \rightarrow \mathbb{R}_+$ be a non-increasing and locally integrable function, which will be called a weight function. Define

$$\begin{aligned} \alpha &= \sup\{t \geq 0 : \omega(t) > 0\}, \\ \gamma_0 &= \sup\{t \geq 0 : \omega \text{ is constant on } (0, t)\}. \end{aligned}$$

Recall that Orlicz–Lorentz spaces—a natural generalization of Orlicz and Lorentz spaces—were introduced at the beginning of the 1990s [11, 12, 16–18]. These investigations spurred further investigations of Orlicz–Lorentz spaces.

Given an Orlicz function φ and a weight function ω , we define a convex modular $I_{\varphi, \omega} : L^0 \rightarrow \mathbb{R}_+^e = [0, \infty]$ by the formula

$$I_{\varphi, \omega}(x) := \int_0^\gamma \varphi(x^*(t))\omega(t)dt. \quad (4)$$

The modular space

$$\Lambda_{\varphi, \omega} = \Lambda_{\varphi, \omega}([0, \gamma)) = \{x \in L^0 : I_{\varphi, \omega}(\lambda x) < \infty \text{ for some } \lambda > 0\},$$

generated by the modular $I_{\varphi, \omega}$, is called the Orlicz–Lorentz function space. Since the modular unit ball $\{x \in L^0 : I_{\varphi, \omega}(x) \leq 1\}$ is an absolutely convex and absorbing subset of $\Lambda_{\varphi, \omega}([0, \gamma))$, its Minkowski functional $\|\cdot\|_{\varphi, \omega}$, defined by

$$\|x\|_{\varphi, \omega} = \inf\{\lambda > 0 : I_{\varphi, \omega}(x/\lambda) \leq 1\},$$

is a seminorm in $\Lambda_{\varphi, \omega}([0, \gamma))$. It is easy to check that, thanks to the condition $\varphi(u) \rightarrow \infty$ as $u \rightarrow \infty$, it is a norm in $\Lambda_{\varphi, \omega}([0, \gamma))$, called the Luxemburg norm.

In our investigations we apply the results concerning the monotonicity properties of Lorentz function spaces first presented in [8] and [4]. Let us recall that a Lorentz function space Λ_ω is defined by the formula

$$\Lambda_\omega = \Lambda_\omega([0, \gamma)) = \left\{x \in L^0 : \|x\|_\omega = \int_0^\gamma x^*(t)\omega(t)dt < \infty\right\}.$$

A Banach lattice $E = (E, \leq, \|\cdot\|)$ is said to be locally uniformly monotone, whenever for any $x \in (E)_+$ (the positive cone of E) with $\|x\| = 1$ and any $\varepsilon \in (0, 1)$ there is $\delta(x, \varepsilon) \in (0, 1)$ such that the conditions $0 \leq y \leq x$ and $\|y\| \geq \varepsilon$ imply that $\|x - y\| \leq 1 - \delta(x, \varepsilon)$.

Theorem 1 ([4], Proposition 4.1) *The Lorentz function space Λ_ω is locally uniformly monotone if and only if ω is positive on $[0, \gamma)$ and $\int_0^\gamma \omega(t)dt = \infty$ whenever $\gamma = \infty$.*

In our investigations we will also apply Lemma 1. It is proved analogously as in the case of Orlicz spaces using the convexity of the modular $I_{\varphi, \omega}$ (cf. also [13] for a more general case).

Lemma 1 *Suppose that an Orlicz function φ satisfies a suitable condition Δ_2 , that is, $\varphi \in \Delta_2(\mathbb{R})$ if $\gamma = \infty$ and $\int_0^\infty \omega(t)dt = \infty$, and $\varphi \in \Delta_2(\infty)$ otherwise.*

Then for any $\varepsilon \in (0, 1)$ there exists $\delta = \delta(\varepsilon) \in (0, 1)$ such that $\|x\| \leq 1 - \delta$ for any $x \in \Lambda_{\varphi, \omega}$ whenever $I_{\varphi, \omega}(x) \leq 1 - \varepsilon$.

In particular, for any $x \in \Lambda_{\varphi, \omega}$, we then have $\|x\| = 1$ if and only if $I_{\varphi, \omega}(x) = 1$.

2 Results

We start with the following

Theorem 2 *Let $\gamma = \infty$. Then the Orlicz–Lorentz function space $\Lambda_{\varphi, \omega}$ is locally uniformly non-square if and only if $\int_0^\infty \omega(t)dt = \infty$, $\varphi \in \Delta_2(\mathbb{R})$ and $\varphi(\frac{u}{2}) < \frac{1}{2}\varphi(u)$ for every $u > 0$.*

Proof. Sufficiency. Let us fix $x \in S(\Lambda_{\varphi,\omega})$. Since $\varphi \in \Delta_2(\mathbb{R})$, by Lemma 1 it is enough to show that there exists $\delta \in (0, 1)$ such that $\min(I_{\varphi,\omega}(\frac{x-y}{2}), I_{\varphi,\omega}(\frac{x+y}{2})) \leq 1 - \delta$ for every $y \in B(\Lambda_{\varphi,\omega})$. There exist t_1, t_2 such that $0 \leq t_1 < t_2 < \infty$, $0 < x^*(t_2) \leq x^*(t_1) < \infty$ and

$$\int_{t_1}^{t_2} \varphi(x^*(t))\omega(t)dt \geq \xi \quad (5)$$

for some $\xi \in (0, 1]$. Without loss of generality we may assume that $x^*(s) > x^*(t) > x^*(w)$ for all $s \in (0, t_1)$, $t \in (t_1, t_2)$ and $w > t_2$ if $t_1 > 0$, and $x^*(t) > x^*(w)$ for all $t \in (t_1, t_2)$ and $w > t_2$, otherwise. By [14], property 7°, page 64 there exist sets e_{t_1} and e_{t_2} with $m(e_{t_1}) = t_1$, $m(e_{t_2}) = t_2$,

$$\int_0^{t_1} x^*(t)dt = \int_{e_{t_1}} |x(t)| dt \quad \text{and} \quad \int_0^{t_2} x^*(t)dt = \int_{e_{t_2}} |x(t)| dt$$

(in the case $t_1 = 0$ we have $e_{t_1} = \emptyset$), whence it follows that $e_{t_1} \subsetneq e_{t_2}$. Moreover, by the properties of the function φ , we get

$$\varphi\left(\frac{1}{2}u\right) \leq \frac{1-\eta_x}{2}\varphi(u)$$

for $u \in [x^*(t_1), x^*(t_2)]$, where $\eta_x \in (0, 1)$. For any $y \in B(\Lambda_{\varphi,\omega})$ we define

$$B_1 = \left\{t \in e_{t_2} \setminus e_{t_1} : x(t)y(t) \geq 0\right\},$$

$$B_2 = \left\{t \in e_{t_2} \setminus e_{t_1} : x(t)y(t) < 0\right\}.$$

By the inequality (5), we obtain $I_{\varphi,\omega}(x\chi_{e_{t_2} \setminus e_{t_1}}) \geq \xi$, whence $I_{\varphi,\omega}(x\chi_{B_1}) \geq \frac{\xi}{2}$ or $I_{\varphi,\omega}(x\chi_{B_2}) \geq \frac{\xi}{2}$. Let us assume that the first inequality holds. Then we have

$$\begin{aligned} \varphi\left(\frac{x(t)-y(t)}{2}\right) &\leq \varphi\left(\frac{\max\{|x(t)|, |y(t)|\}}{2}\right) \leq \varphi\left(\frac{x(t)}{2}\right) + \varphi\left(\frac{y(t)}{2}\right) \\ &\leq \frac{1}{2}\varphi(x(t)) - \frac{\eta_x}{2}\varphi(x(t)) + \frac{1}{2}\varphi(y(t)) \end{aligned}$$

for m -a.e. $t \in B_1$. Consequently,

$$\varphi \circ \left(\frac{x-y}{2}\right) \leq \frac{1}{2}\varphi \circ y + \frac{1}{2}\varphi \circ x - \frac{\eta_x}{2}\varphi \circ x\chi_{B_1}.$$

By the local uniform monotonicity of the Lorentz space Λ_ω we get

$$\begin{aligned} I_{\varphi,\omega}\left(\frac{x-y}{2}\right) &= \left\|\varphi \circ \left(\frac{x-y}{2}\right)\right\|_\omega \leq \left\|\frac{1}{2}\varphi \circ y + \frac{1}{2}\varphi \circ x - \frac{\eta_x}{2}\varphi \circ x\chi_{B_1}\right\|_\omega \\ &\leq \frac{1}{2}\|\varphi \circ y\|_\omega + \frac{1}{2}\|\varphi \circ x - \eta_x\varphi \circ x\chi_{B_1}\|_\omega \leq 1 - \frac{1}{2}\delta\left(\varphi \circ x, \frac{\xi \cdot \eta_x}{2}\right), \end{aligned}$$

where $\delta\left(\varphi \circ x, \frac{\xi \cdot \eta_x}{2}\right)$ is a constant from the definition of local uniform monotonicity of the Lorentz space Λ_ω corresponding to $\varphi \circ x$ and $\varepsilon = \frac{\xi \cdot \eta_x}{2}$. In the second case, that is if $I_{\varphi,\omega}(x\chi_{B_2}) \geq \frac{\xi}{2}$, proceeding analogously we can prove that $I_{\varphi,\omega}(\frac{x+y}{2}) \leq 1 - \delta\left(\varphi \circ x, \frac{\xi \cdot \eta_x}{2}\right)$.

Necessity. If $\int_0^\infty \omega(t) dt < \infty$ or $\varphi \notin \Delta_2(\mathbb{R})$, then $\Lambda_{\varphi, \omega}$ contains an order isometric copy of l^∞ (see [11, Theorem 2.4]). Suppose now that $\int_0^\infty \omega(t) dt = \infty$, $\varphi \in \Delta_2(\mathbb{R})$ and there exists $u_0 > 0$ such that $\varphi(\frac{1}{2}u_0) = \frac{1}{2}\varphi(u_0)$. Then $\varphi(\frac{1}{2}u) = \frac{1}{2}\varphi(u)$ for any $u \in [0, u_0)$. Moreover, we can find $t_0 > 0$ such that $\int_0^{t_0} \varphi(u_0)\omega(t) dt = 1$ and t_n such that $\int_0^{t_n} \varphi(u_n)\omega(t) dt = 1$ for all $n \in \mathbb{N}$, where $u_n = \frac{1}{2^n}u_0$. Taking

$$x = u_0 \chi_{[0, t_0)} \quad \text{and} \quad y_n = u_n \chi_{[t_0, t_0+t_n)},$$

we get $I_{\varphi, \omega}(x) = I_{\varphi, \omega}(y_n) = 1$ for all $n \in \mathbb{N}$. Since

$$\int_0^{t_0} \varphi(u_n)\omega(t) dt = \int_0^{t_0} \varphi\left(\frac{u_0}{2^n}\right)\omega(t) dt = \frac{1}{2^n} \int_0^{t_0} \varphi(u_0)\omega(t) dt = \frac{1}{2^n},$$

we get

$$\int_{t_0}^{t_0+t_n} \varphi(u_n)\omega(t) dt \geq \int_{t_0}^{t_n} \varphi(u_n)\omega(t) dt = 1 - \frac{1}{2^n}.$$

Therefore

$$\begin{aligned} I_{\varphi, \omega}\left(\frac{x \pm y_n}{2}\right) &= \int_0^{t_0} \varphi\left(\frac{u_0}{2}\right)\omega(t) dt + \int_{t_0}^{t_0+t_n} \varphi\left(\frac{u_n}{2}\right)\omega(t) dt \\ &= \frac{1}{2} \int_0^{t_0} \varphi(u_0)\omega(t) dt + \frac{1}{2} \int_{t_0}^{t_0+t_n} \varphi(u_n)\omega(t) dt \\ &\geq \frac{1}{2} + \frac{1}{2} \left(1 - \frac{1}{2^n}\right) = 1 - \frac{1}{2^{n+1}}. \end{aligned}$$

Consequently

$$\lim_{n \rightarrow \infty} \min \left(\left\| \frac{x - y_n}{2} \right\|_{\varphi, \omega}, \left\| \frac{x + y_n}{2} \right\|_{\varphi, \omega} \right) = 1,$$

which means that the space $\Lambda_{\varphi, \omega}$ is not locally uniformly non-square.

Theorem 3 Let $\alpha = \gamma < \infty$ and $\int_0^\gamma \varphi(\beta)\omega(t)dt < 1$. Then the following conditions are equivalent:

- (i) $\varphi \in \Delta_2(\infty)$,
- (ii) the Orlicz–Lorentz space $\Lambda_{\varphi, \omega}$ is non-square,
- (iii) the Orlicz–Lorentz space $\Lambda_{\varphi, \omega}$ is locally uniformly non-square.

Proof The implication (iii) $\Rightarrow$ (ii) is obvious. The implication (ii) $\Rightarrow$ (i) has been proved in Theorem 2.2 in [3]. We shall show (i) $\Rightarrow$ (iii). Let us take an arbitrary $x \in S(\Lambda_{\varphi, \omega})$. By the assumption that $\int_0^\gamma \varphi(\beta)\omega(t)dt < 1$ and $\varphi \in \Delta_2(\infty)$, we get

$$m(\{t \in [0, \gamma) : |x(t)| > \beta\}) = t_0 > 0.$$

Hence, there exist t_1, t_2 such that $0 \leq t_1 < t_2 \leq t_0$, $\beta < x^*(t_2) \leq x^*(t_1) < \infty$ and

$$\int_{t_1}^{t_2} \varphi(x^*(t))\omega(t)dt > 0.$$

Proceeding analogously as in the proof of sufficiency in Theorem 2, we can find $\delta > 0$ depending only on x such that $\min \left(\left\| \frac{x-y}{2} \right\|_{\varphi, \omega}, \left\| \frac{x+y}{2} \right\|_{\varphi, \omega} \right) \leq 1 - \delta$ for every $y \in B(\Lambda_{\varphi, \omega})$.

Theorem 4 Let $\alpha = \gamma < \infty$ and $\int_0^\gamma \varphi(\beta)\omega(t)dt \geq 1$. Then the Orlicz–Lorentz function space $\Lambda_{\varphi,\omega}$ is locally uniformly non-square if and only if $\varphi \in \Delta_2(\infty)$, $\psi \in \Delta_2(\infty)$, where ψ denotes the complementary function of φ in the sense of Young (see formula (3)), and $\int_0^{\gamma_0/2} \varphi(\beta)\omega(t)dt < 1$.

Proof Sufficiency. Since $\varphi \in \Delta_2(\infty)$, by Lemma 1, it is sufficient to prove the corresponding inequalities for the modular.

Case 1. Suppose that $\gamma_0 > 0$. Let us take an arbitrary $x \in S(\Lambda_{\varphi,\omega})$. If

$$m\left(\left\{t \in [0, \gamma] : |x(t)| > \beta\right\}\right) = t_0 > 0, \quad (6)$$

then we proceed analogously as in the proof of the implication (i) $\Rightarrow$ (iii) in Theorem 3.

Let us now assume that inequality (6) does not hold or, equivalently, $x^*(0) \leq \beta$. Define

$$c := \frac{1 - \int_0^{\gamma_0/2} \varphi(\beta)\omega(t)dt}{8}.$$

By the assumptions we get $c \in (0, \frac{1}{8})$. Let u_0 be such that

$$\int_0^{\gamma_0/2} \varphi(u_0)\omega(t)dt = 1 - 4c. \quad (7)$$

We have $u_0 > \beta$ and, by the condition $\psi \in \Delta_2(\infty)$, there exists $\eta = \eta(u_0) \in (0, 1)$ such that

$$\varphi\left(\frac{u}{2}\right) \leq \frac{1-\eta}{2}\varphi(u) \quad (8)$$

for $u \geq u_0$. Moreover, in virtue of $\varphi \in \Delta_2(\infty)$, for $K = \frac{2-\eta}{2(1-\eta)}$ we can find $\tau \in (0, 1)$ such that

$$\varphi((1+\tau)u) \leq K\varphi(u)$$

for $u \geq u_0$. Thus, for $u \geq u_1$, where $u_1 = \max\left(u_0, \frac{\beta}{\tau}\right)$, we get

$$\varphi(u+\beta) \leq \varphi(u+u_1\tau) \leq \varphi(u+u\tau) \leq K\varphi(u). \quad (9)$$

In virtue of (8) and (9), we obtain

$$\varphi\left(\frac{u+\beta}{2}\right) \leq \frac{1-\eta}{2}\varphi(u+\beta) \leq \frac{1-\eta}{2}K\varphi(u) = \frac{2-\eta}{4}\varphi(u) \quad (10)$$

for $u \geq u_1$. Let us take an arbitrary $y \in S(\Lambda_{\varphi,\omega})$ and define

$$\begin{aligned} e\left(\frac{u_0+\beta}{2}\right) &= \left\{t \in [0, \gamma] : |y(t)| \geq \frac{u_0+\beta}{2}\right\}, \\ e(u_0) &= \left\{t \in [0, \gamma] : |y(t)| \geq u_0\right\}, \\ e(u_1+2\beta) &= \left\{t \in [0, \gamma] : |y(t)| \geq u_1+2\beta\right\}. \end{aligned} \quad (11)$$

Case 1.1. Suppose

$$\int_0^\gamma \varphi\left((y\chi_{e(u_1+2\beta)})^*(t)\right)\omega(t)dt = \int_0^{m(e(u_1+2\beta))} \varphi(y^*(t))\omega(t)dt \geq \frac{c}{2}. \quad (12)$$

Define

$$f(u_1 + \beta) = \left\{ t \in [0, \gamma) : |x(t) + y(t)| \geq u_1 + \beta \right\}.$$

We have $|y(t)| \geq u_1$ for m -a.e. $t \in f(u_1 + \beta)$ and $e(u_1 + 2\beta) \subset f(u_1 + \beta)$. Moreover, in virtue of (10), we obtain

$$\varphi\left(\frac{x(t) + y(t)}{2}\right) \leq \varphi\left(\frac{|y(t)| + \beta}{2}\right) \leq \frac{2 - \eta}{4} \varphi(y(t))$$

for m -a.e. $t \in f(u_1 + \beta)$, whence

$$\varphi\left(\left(\frac{x + y}{2} \chi_{f(u_1 + \beta)}\right)^*(t)\right) \leq \frac{2 - \eta}{4} \varphi\left((y \chi_{f(u_1 + \beta)})^*(t)\right)$$

and in consequence

$$\begin{aligned} & \int_0^{m(f(u_1 + \beta))} \varphi\left(\left(\frac{x + y}{2}\right)^*(t)\right) \omega(t) dt \\ &= \int_0^{m(f(u_1 + \beta))} \varphi\left(\left(\frac{x + y}{2} \chi_{f(u_1 + \beta)}\right)^*(t)\right) \omega(t) dt \\ &\leq \frac{2 - \eta}{4} \int_0^{m(f(u_1 + \beta))} \varphi\left((y \chi_{f(u_1 + \beta)})^*(t)\right) \omega(t) dt \\ &\leq \frac{2 - \eta}{4} \int_0^{m(e(u_1 + 2\beta))} \varphi\left((y \chi_{e(u_1 + 2\beta)})^*(t)\right) \omega(t) dt \\ &\quad + \frac{2 - \eta}{4} \int_{m(e(u_1 + 2\beta))}^{m(f(u_1 + \beta))} \varphi\left((y \chi_{f(u_1 + \beta)})^*(t)\right) \omega(t) dt. \end{aligned} \quad (13)$$

Define

$$\begin{aligned} t(x) &= m\left(\left([0, \gamma) \setminus f(u_1 + \beta)\right) \cap \text{supp } x\right), \\ t(y) &= m\left(\left([0, \gamma) \setminus f(u_1 + \beta)\right) \cap \text{supp } y\right). \end{aligned}$$

By [3, Remark 1.1], we get

$$\begin{aligned} & \int_{m(f(u_1 + \beta))}^{\gamma} \varphi\left(\left(\frac{x + y}{2}\right)^*(t)\right) \omega(t) dt \\ &= \int_{m(f(u_1 + \beta))}^{\gamma} \varphi\left(\left(\frac{x + y}{2} \chi_{([0, \gamma) \setminus f(u_1 + \beta))}\right)^*(t - m(f(u_1 + \beta)))\right) \omega(t) dt \\ &\leq \frac{1}{2} \int_{m(f(u_1 + \beta))}^{m(f(u_1 + \beta)) + t(y)} \varphi\left((y \chi_{([0, \gamma) \setminus f(u_1 + \beta))})^*(t - m(f(u_1 + \beta)))\right) \omega(t) dt \\ &\quad + \frac{1}{2} \int_{m(f(u_1 + \beta))}^{m(f(u_1 + \beta)) + t(x)} \varphi\left((x \chi_{([0, \gamma) \setminus f(u_1 + \beta))})^*(t - m(f(u_1 + \beta)))\right) \omega(t) dt. \end{aligned} \quad (14)$$

By (12), (13) and (14) we obtain

$$\begin{aligned}
 I_{\varphi, \omega} \left(\frac{x+y}{2} \right) &= \int_0^\gamma \varphi \left(\left(\frac{x+y}{2} \right)^* (t) \right) \omega(t) dt \\
 &\leq \frac{2-\eta}{4} \int_0^{m(e(u_1+2\beta))} \varphi \left((y\chi_{e(u_1+2\beta)})^* (t) \right) \omega(t) dt \\
 &\quad + \frac{2-\eta}{4} \int_{m(e(u_1+2\beta))}^{m(f(u_1+\beta))} \varphi \left((y\chi_{f(u_1+\beta)})^* (t) \right) \omega(t) dt \\
 &\quad + \frac{1}{2} \int_{m(f(u_1+\beta))}^{m(f(u_1+\beta))+t(y)} \varphi \left((y\chi_{([0,\gamma]\setminus f(u_1+\beta))})^* (t - m(f(u_1+\beta))) \right) \omega(t) dt \\
 &\quad + \frac{1}{2} \int_{m(f(u_1+\beta))}^{m(f(u_1+\beta))+t(x)} \varphi \left((x\chi_{([0,\gamma]\setminus f(u_1+\beta))})^* (t - m(f(u_1+\beta))) \right) \omega(t) dt \\
 &\leq \frac{2-\eta}{4} \int_0^{m(e(u_1+2\beta))} \varphi (y^*(t)) \omega(t) dt \\
 &\quad + \frac{1}{2} \int_{m(e(u_1+2\beta))}^{m(f(u_1+\beta))} \varphi \left((y\chi_{(f(u_1+\beta)\setminus e(u_1+2\beta))})^* (t - m(e(u_1+2\beta))) \right) \omega(t) dt \\
 &\quad + \frac{1}{2} \int_{m(f(u_1+\beta))}^{m(f(u_1+\beta))+t(y)} \varphi \left((y\chi_{([0,\gamma]\setminus f(u_1+\beta))})^* (t - m(f(u_1+\beta))) \right) \omega(t) dt \\
 &\quad + \frac{1}{2} \int_{m(f(u_1+\beta))}^{m(f(u_1+\beta))+t(x)} \varphi \left((x\chi_{([0,\gamma]\setminus f(u_1+\beta))})^* (t - m(f(u_1+\beta))) \right) \omega(t) dt \quad (15)
 \end{aligned}$$

Since the function

$$\bar{y}(t) := \begin{cases} (y\chi_{(f(u_1+\beta)\setminus e(u_1+2\beta))})^* (t - m(e(u_1+2\beta))) & \text{for } t \in [m(e(u_1+2\beta)), m(f(u_1+\beta))], \\ (y\chi_{([0,\gamma]\setminus f(u_1+\beta))})^* (t - m(f(u_1+\beta))) & \text{for } t \in [m(f(u_1+\beta)), m(f(u_1+\beta)) + t(y)], \\ 0 & \text{otherwise,} \end{cases}$$

is equimeasurable with $y\chi_{([0,\gamma]\setminus e(u_1+2\beta))}$, by Hardy-Littlewood inequality, we have

$$\begin{aligned}
 &\int_{m(e(u_1+2\beta))}^\gamma \varphi (\bar{y}(t)) \omega(t) dt \\
 &= \int_{m(e(u_1+2\beta))}^{m(f(u_1+\beta))} \varphi \left((y\chi_{(f(u_1+\beta)\setminus e(u_1+2\beta))})^* (t - m(e(u_1+2\beta))) \right) \omega(t) dt \\
 &\quad + \int_{m(f(u_1+\beta))}^{m(f(u_1+\beta))+t(y)} \varphi \left((y\chi_{([0,\gamma]\setminus f(u_1+\beta))})^* (t - m(f(u_1+\beta))) \right) \omega(t) dt \\
 &\leq \int_{m(e(u_1+2\beta))}^{m(f(u_1+\beta))+t(y)} \varphi \left((y\chi_{([0,\gamma]\setminus e(u_1+2\beta))})^* (t - m(e(u_1+2\beta))) \right) \omega(t) dt \\
 &= \int_{m(e(u_1+2\beta))}^\gamma \varphi (y^*(t)) \omega(t) dt. \quad (16)
 \end{aligned}$$

Simultaneously, by

$$\varphi \circ (x\chi_{([0,\gamma]\setminus f(u_1+\beta))}) \leq \varphi \circ x,$$

we get

$$\varphi \circ \left(\left(x \chi_{([0, \gamma] \setminus f(u_1 + \beta))} \right)^* \right) \leq \varphi \circ x^*,$$

whence

$$\begin{aligned} & \int_{m(f(u_1 + \beta))}^{m(f(u_1 + \beta)) + t(x)} \varphi \left(\left(x \chi_{([0, \gamma] \setminus f(u_1 + \beta))} \right)^* (t - m(f(u_1 + \beta))) \right) \omega(t) dt \\ & \leq \int_0^{t(x)} \varphi \left(\left(x \chi_{([0, \gamma] \setminus f(u_1 + \beta))} \right)^* (t) \right) \omega(t) dt \leq \int_0^\gamma \varphi(x^*(t)) \omega(t) dt. \end{aligned} \quad (17)$$

Eventually, by (12), (15), (16) and (17), we have

$$\begin{aligned} I_{\varphi, \omega} \left(\frac{x + y}{2} \right) &= \int_0^\gamma \varphi \left(\left(\frac{x + y}{2} \right)^* (t) \right) \omega(t) dt \\ &\leq \frac{2 - \eta}{4} \int_0^{m(e(u_1 + 2\beta))} \varphi(y^*(t)) \omega(t) dt \\ &\quad + \frac{1}{2} \int_{m(e(u_1 + 2\beta))}^\gamma \varphi(y^*(t)) \omega(t) dt \\ &\quad + \frac{1}{2} \int_0^\gamma \varphi(x^*(t)) \omega(t) dt \\ &= \frac{1}{2} \int_0^\gamma \varphi(y^*(t)) \omega(t) dt + \frac{1}{2} \int_0^\gamma \varphi(x^*(t)) \omega(t) dt \\ &\quad - \frac{\eta}{4} \int_0^{m(e(u_1 + 2\beta))} \varphi(y^*(t)) \omega(t) dt \leq 1 - \frac{\eta c}{8}. \end{aligned} \quad (18)$$

Case 1.2. Let now

$$\int_0^\gamma \varphi \left((y \chi_{e(u_1 + 2\beta)})^* (t) \right) \omega(t) dt = \int_0^{m(e(u_1 + 2\beta))} \varphi(y^*(t)) \omega(t) dt < \frac{c}{2} \quad (19)$$

and

$$\int_0^\gamma \varphi \left((y \chi_{e(u_0)})^* (t) \right) \omega(t) dt = \int_0^{m(e(u_0))} \varphi(y^*(t)) \omega(t) dt \geq c. \quad (20)$$

Let t_1 and t_2 be such that the following equalities hold:

$$\begin{aligned} \int_0^{t_1} \varphi(u_1 + 2\beta) \omega(t) dt &= \frac{c}{2}, \\ \int_0^{t_2} \varphi \left(\frac{u_0 + \beta}{2} \right) \omega(t) dt &= 1. \end{aligned} \quad (21)$$

We have $0 < t_1 < t_2 < \gamma$. Moreover, by (19) and (20), we get

$$\int_{m(e(u_1 + 2\beta))}^{m(e(u_0))} \varphi(u_1 + 2\beta) \omega(t) dt \geq \int_{m(e(u_1 + 2\beta))}^{m(e(u_0))} \varphi(y^*(t)) \omega(t) dt > \frac{c}{2},$$

whence, by the definition of t_1 , we obtain $m(e(u_0)) \geq t_1$. Define

$$\begin{aligned} A_y^1 &= \left\{ t \in e(u_0) : y(t)x(t) \geq 0 \right\}, \\ A_y^2 &= \left\{ t \in e(u_0) : y(t)x(t) < 0 \right\}. \end{aligned}$$

We have $m\left(A_y^1\right) \geq \frac{t_1}{2}$ or $m\left(A_y^2\right) \geq \frac{t_1}{2}$. Let us assume that the first inequality holds. Then, by inequality (8),

$$\varphi\left(\frac{x(t)-y(t)}{2}\right) \leq \varphi\left(\frac{y(t)}{2}\right) \leq \frac{1-\eta}{2} \varphi(y(t)) \leq \frac{1}{2} \varphi(x(t)) + \frac{1}{2} \varphi(y(t)) - \frac{\eta}{2} \varphi(y(t))$$

for m -a.e. $t \in A_y^1$. Moreover,

$$\varphi\left(\frac{x(t)-y(t)}{2}\right) \leq \frac{1}{2} \varphi(x(t)) + \frac{1}{2} \varphi(y(t))$$

for m -a.e. $t \in [0, \gamma) \setminus A_y^1$. Hence

$$\varphi \circ \left(\frac{x-y}{2}\right) \leq \frac{1}{2} \varphi \circ x + \frac{1}{2} \varphi \circ y - \frac{\eta}{2} \varphi \circ y \chi_{A_y^1} \leq \frac{1}{2} \varphi \circ x + \frac{1}{2} \varphi \circ y - \frac{\nu}{2} \varphi \circ y \chi_{A_y^1},$$

where $\nu = \min\left(\eta, 1 - \frac{\varphi\left(\frac{u_0+\beta}{2}\right)}{\varphi(u_0)}\right)$. Since $y(t) \geq u_0$ for m -a.e. $t \in A_y^1$, we obtain

$$\begin{aligned} \varphi(y(t)) - \nu \varphi(y(t)) &\geq \varphi(u_0)(1-\nu) \geq \varphi(u_0) \left(1 - \left(1 - \frac{\varphi\left(\frac{u_0+\beta}{2}\right)}{\varphi(u_0)}\right)\right) \\ &= \varphi\left(\frac{u_0+\beta}{2}\right) \end{aligned} \quad (22)$$

for the same t . It follows from the inequality $0 \leq \varphi \circ y - \nu \varphi \circ y \chi_{A_y^1} \leq \varphi \circ y$ that

$$\left(\varphi \circ y - \nu \varphi \circ y \chi_{A_y^1}\right)^* \leq (\varphi \circ y)^* = \varphi \circ y^*. \quad (23)$$

Furthermore, it follows from (22) that for any $t \in [0, \gamma)$ such that $y^*(t) < \frac{u_0+\beta}{2}$ we have equality in (23). Simultaneously

$$\begin{aligned} \int_0^\gamma \left(\varphi \circ y - \nu \varphi \circ y \chi_{A_y^1}\right)(t) dt &\leq \int_0^\gamma \left(\varphi \circ y - \nu \varphi(u_0) \chi_{A_y^1}\right)(t) dt \\ &= \int_0^\gamma \varphi(y(t)) dt - \int_0^\gamma \nu \varphi(u_0) \chi_{A_y^1}(t) dt \leq \int_0^\gamma \varphi(y(t)) dt - \nu \varphi(u_0) \cdot \frac{t_1}{2}. \end{aligned}$$

Hence, by [14, property 1°, p. 63] we obtain

$$\int_0^\gamma \left(\varphi \circ y - \nu \varphi \circ y \chi_{A_y^1}\right)^*(t) dt \leq \int_0^\gamma \varphi(y^*(t)) dt - \frac{\nu t_1 \varphi(u_0)}{2}.$$

Since $I_{\varphi, \omega}(y) = 1$, by the definitions of $e\left(\frac{u_0+\beta}{2}\right)$ and t_2 (see formulas (11) and (21), respectively), we have $m(e\left(\frac{u_0+\beta}{2}\right)) \leq t_2$. Hence, by the remark concerning inequality (23), we obtain

$$\int_{t_2}^\gamma \left(\varphi \circ y - \nu \varphi \circ y \chi_{A_y^1}\right)^*(t) dt = \int_{t_2}^\gamma \varphi(y^*(t)) dt$$

and in consequence

$$\begin{aligned}
 & \int_0^\gamma \left(\varphi(y^*(t)) - \left(\varphi \circ y - v\varphi \circ y\chi_{A_y^1} \right)^*(t) \right) \omega(t) dt \\
 &= \int_0^{t_2} \left(\varphi(y^*(t)) - \left(\varphi \circ y - v\varphi \circ y\chi_{A_y^1} \right)^*(t) \right) \omega(t) dt \\
 &\geq \int_0^{t_2} \left(\varphi(y^*(t)) - \left(\varphi \circ y - v\varphi \circ y\chi_{A_y^1} \right)^*(t) \right) \omega(t_2) dt \\
 &\geq \frac{vt_1\varphi(u_0)\omega(t_2)}{2}.
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 I_{\varphi,\omega} \left(\frac{x-y}{2} \right) &= \left\| \varphi \circ \left(\frac{x-y}{2} \right) \right\|_\omega \leq \left\| \frac{1}{2}\varphi \circ x + \frac{1}{2}\varphi \circ y - \frac{v}{2}\varphi \circ y\chi_{A_y^1} \right\|_\omega \\
 &\leq \frac{1}{2}\|\varphi \circ x\|_\omega + \frac{1}{2}\|\varphi \circ y - v\varphi \circ y\chi_{A_y^1}\|_\omega \\
 &\leq \frac{1}{2}\|\varphi \circ x\|_\omega + \frac{1}{2}\|\varphi \circ y\|_\omega - \frac{vt_1\varphi(u_0)\omega(t_2)}{4} \\
 &= 1 - \frac{vt_1\varphi(u_0)\omega(t_2)}{4}.
 \end{aligned} \tag{24}$$

If $m(A_y^1) < \frac{t_1}{2}$, then $m(A_y^2) \geq \frac{t_1}{2}$, and proceeding analogously we obtain

$$I_{\varphi,\omega} \left(\frac{x+y}{2} \right) \leq 1 - \frac{vt_1\varphi(u_0)\omega(t_2)}{4}. \tag{25}$$

Case 1.3. Suppose that

$$\int_0^\gamma \varphi((y\chi_{e(u_0)})^*(t)) \omega(t) dt = \int_0^{m(e(u_0))} \varphi(y^*(t)) \omega(t) dt < c. \tag{26}$$

Let t_3 and u_2 be such that

$$\int_0^{t_3} \varphi(u_0)\omega(t) dt = 1 - 2c$$

and

$$\int_0^\gamma \varphi(u_2)\omega(t) dt = c.$$

By the definition of u_0 (see (7)) we get $\frac{1}{2}\gamma_0 < t_3 < \frac{3}{4}\gamma_0$. Simultaneously, since $\int_0^\gamma \varphi(\beta)\omega(t) dt \geq 1$, we have $0 < u_2 < \beta < u_0$. Define

$$B_x = \{t \in [0, \gamma) : |x(t)| \geq u_2\}, \tag{27}$$

$$B_y = \{t \in [0, \gamma) : |y(t)| \geq u_2\}. \tag{28}$$

By the inequalities $x^*(0) \leq \beta$ and (26) we obtain $m(B_x) \geq t_3$ and $m(B_y) \geq t_3$. Define

$$t_4 := \frac{t_3 - \frac{\gamma_0}{2}}{2} < \frac{1}{8}\gamma_0$$

and

$$C_{x,y}^+ = \left\{ t \in [0, \gamma) : \min(|x(t)|, |y(t)|) \geq \frac{u_2}{4} \wedge x(t)y(t) > 0 \right\}, \quad (29)$$

$$C_{x,y}^- = \left\{ t \in [0, \gamma) : \min(|x(t)|, |y(t)|) \geq \frac{u_2}{4} \wedge x(t)y(t) < 0 \right\}. \quad (30)$$

Case 1.3.1. Assume that $m(C_{x,y}^+) \geq \frac{t_4}{2}$. Then we have

$$\varphi\left(\frac{x(t) - y(t)}{2}\right) \leq \frac{1}{2}\varphi(y(t)) + \frac{1}{2}\varphi(x(t)) - \frac{1}{2} \cdot \frac{u_2}{4\beta}\varphi(x(t))$$

for m -a.e. $t \in C_{x,y}^+$. Indeed, if $|x(t)| \geq |y(t)|$, then

$$\begin{aligned} \varphi\left(\frac{x(t) - y(t)}{2}\right) &\leq \varphi\left(\frac{x(t) - \frac{u_2}{4} \operatorname{sgn}(x(t))}{2}\right) \leq \varphi\left(\frac{x(t) - \frac{u_2}{4} \cdot \frac{x(t)}{\beta}}{2}\right) \\ &= \varphi\left(\frac{1}{2}\left(1 - \frac{u_2}{4\beta}\right)x(t)\right) \leq \frac{1}{2}\left(1 - \frac{u_2}{4\beta}\right)\varphi(x(t)) \\ &= \frac{1}{2}\varphi(x(t)) - \frac{1}{2} \cdot \frac{u_2}{4\beta}\varphi(x(t)). \end{aligned}$$

On the other hand, if $|x(t)| < |y(t)|$, we get

$$\varphi\left(\frac{x(t) - y(t)}{2}\right) \leq \varphi\left(\frac{y(t)}{2}\right) \leq \frac{1}{2}\varphi(y(t)) + \frac{1}{2}\varphi(x(t)) - \frac{1}{2} \cdot \frac{u_2}{4\beta}\varphi(x(t)).$$

It is obvious that for $t \in [0, \gamma) \setminus C_{x,y}^+$ we have

$$\varphi\left(\frac{x(t) - y(t)}{2}\right) \leq \frac{1}{2}\varphi(x(t)) + \frac{1}{2}\varphi(y(t)).$$

Thus,

$$\varphi \circ \left(\frac{x(t) - y(t)}{2}\right) \leq \frac{1}{2}\varphi \circ y + \frac{1}{2}\varphi \circ x - \frac{1}{2} \cdot \frac{u_2}{4\beta}\varphi \circ x \chi_{C_{x,y}^+}.$$

Since

$$I_{\varphi,\omega}\left(x \chi_{C_{x,y}^+}\right) = \left\| \varphi \circ x \chi_{C_{x,y}^+} \right\|_{\omega} \geq \int_0^{t_4/2} \varphi\left(\frac{u_2}{4}\right) \omega(t) dt =: \zeta,$$

by the local uniform monotonicity of the Lorentz space Λ_{ω} , we get

$$\begin{aligned} I_{\varphi,\omega}\left(\frac{x - y}{2}\right) &= \left\| \varphi \circ \left(\frac{x(t) - y(t)}{2}\right) \right\|_{\omega} \leq \frac{1}{2} \|\varphi \circ y\|_{\omega} + \frac{1}{2} \left\| \varphi \circ x - \frac{u_2}{4\beta} \varphi \circ x \chi_{C_{x,y}^+} \right\|_{\omega} \\ &\leq 1 - \frac{1}{2} \delta \left(\varphi \circ x, \frac{u_2 \zeta}{4\beta} \right), \end{aligned} \quad (31)$$

where $\delta\left(\varphi \circ x, \frac{u_2 \zeta}{4\beta}\right)$ is a constant from the definition of the local uniform monotonicity of the Lorentz space Λ_{ω} corresponding to $\varphi \circ x$ and $\varepsilon = \frac{u_2 \zeta}{4\beta}$.

Analogously, if $m(C_{x,y}^-) > \frac{t_4}{2}$, we have

$$I_{\varphi,\omega}\left(\frac{x + y}{2}\right) \leq 1 - \frac{1}{2} \delta \left(\varphi \circ x, \frac{u_2 \zeta}{4\beta} \right). \quad (32)$$

Case 1.3.2. Finally assume that $m(C_{x,y}^+) < \frac{t_4}{2}$ and $m(C_{x,y}^-) < \frac{t_4}{2}$. The proof of this case is analogous to the proof of Case 2.2.2 in Theorem 2.4 in [3]. We shall present it for completeness' sake. Let us define

$$C_x = B_x \setminus (C_{x,y}^+ \cup C_{x,y}^-), \quad (33)$$

$$C_y = B_y \setminus (C_{x,y}^+ \cup C_{x,y}^-), \quad (34)$$

where B_x and B_y are defined by formulas (27) and (28). We have $C_x \cap C_y = \emptyset$ and

$$\min(m(C_x), m(C_y)) \geq t_3 - t_4 = \frac{1}{2}t_3 + \frac{1}{4}\gamma_0 > \frac{1}{2}\gamma_0, \quad (35)$$

whence

$$m(C_x \cup C_y) \geq t_3 + \frac{1}{2}\gamma_0 > \gamma_0. \quad (36)$$

Define

$$a := \frac{t_3 - \frac{\gamma_0}{2}}{8} < \frac{1}{32}\gamma_0 \quad \text{and} \quad t_5 := \gamma_0 + a.$$

By [14, property 7°, p. 64] there exist sets $e_{\gamma_0} = e_{\gamma_0}(\frac{x+y}{2})$ and $e_{t_5} = e_{t_5}(\frac{x+y}{2})$ such that $m(e_{\gamma_0}) = \gamma_0$, $m(e_{t_5}) = t_5$,

$$\int_0^{\gamma_0} \left(\frac{x+y}{2}\right)^*(t) dt = \int_{e_{\gamma_0}} \left|\frac{x+y}{2}\right|(t) dt$$

and

$$\int_0^{t_5} \left(\frac{x+y}{2}\right)^*(t) dt = \int_{e_{t_5}} \left|\frac{x+y}{2}\right|(t) dt.$$

Without loss of generality we may assume that $e_{\gamma_0} \subset e_{t_5}$. Denoting $D_{\gamma_0} = e_{t_5} \setminus e_{\gamma_0}$ and $D_{t_5} = [0, \gamma) \setminus e_{t_5}$, by [3, Remark 1.1], we obtain

$$\begin{aligned} I_{\varphi, \omega} \left(\frac{x+y}{2}\right) &= \int_0^{\gamma_0} \varphi \left(\left(\frac{x+y}{2}\right)^*(t) \right) \omega(t) dt + \int_{\gamma_0}^{t_5} \varphi \left(\left(\frac{x+y}{2}\right)^*(t) \right) \omega(t) dt \\ &\quad + \int_{t_5}^{\gamma} \varphi \left(\left(\frac{x+y}{2}\right)^*(t) \right) \omega(t) dt \\ &= \int_0^{\gamma_0} \varphi \left(\left(\frac{x+y}{2} \chi_{e_{\gamma_0}}\right)^*(t) \right) \omega(t) dt \\ &\quad + \int_{\gamma_0}^{t_5} \varphi \left(\left(\frac{x+y}{2} \chi_{D_{\gamma_0}}\right)^*(t - \gamma_0) \right) \omega(t) dt \\ &\quad + \int_{t_5}^{\gamma} \varphi \left(\left(\frac{x+y}{2} \chi_{D_{t_5}}\right)^*(t - t_5) \right) \omega(t) dt \\ &\leq \frac{1}{2} \int_0^{\gamma_0} \varphi \left((x \chi_{e_{\gamma_0}})^*(t) \right) \omega(t) dt + \frac{1}{2} \int_0^{\gamma_0} \varphi \left((y \chi_{e_{\gamma_0}})^*(t) \right) \omega(t) dt \\ &\quad + \frac{1}{2} \int_{\gamma_0}^{t_5} \varphi \left((x \chi_{D_{\gamma_0}})^*(t - \gamma_0) \right) \omega(t) dt \\ &\quad + \frac{1}{2} \int_{\gamma_0}^{t_5} \varphi \left((y \chi_{D_{\gamma_0}})^*(t - \gamma_0) \right) \omega(t) dt \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2} \int_{t_5}^{\gamma} \varphi \left(\left(x \chi_{D_{t_5}} \right)^* (t - t_5) \right) \omega(t) dt \\
& + \frac{1}{2} \int_{t_5}^{\gamma} \varphi \left(\left(y \chi_{D_{t_5}} \right)^* (t - t_5) \right) \omega(t) dt.
\end{aligned}$$

By inequality (36) we have

$$\begin{aligned}
m((C_x \cup C_y) \cap D_{t_5}) &= m((C_x \cup C_y) \setminus e_{t_5}) \geq \left(t_3 + \frac{\gamma_0}{2} \right) - (\gamma_0 + a) \\
&= t_3 - \frac{\gamma_0}{2} - a = 7a.
\end{aligned}$$

Hence, without loss of generality, it may be assumed that $m(C_x \cap D_{t_5}) > a$. Then, since $C_x \subset B_x$ we have $m(\{t : |x| \chi_{D_{t_5}}(t) \geq u_2\}) > a$ and thus $\left(x \chi_{D_{t_5}} \right)^*(a) \geq u_2$. If $\left(x \chi_{e_{\gamma_0}} \right)^*(\gamma_0 - a) \leq \frac{u_2}{4}$, then

$$\begin{aligned}
& \int_{\gamma_0-a}^{\gamma_0} \left[\varphi \left(\left(x \chi_{D_{t_5}} \right)^* (t - \gamma_0 + a) \right) - \varphi \left(\left(x \chi_{e_{\gamma_0}} \right)^* (t) \right) \right] \omega(t) dt \\
& - \int_{t_5}^{t_5+a} \left[\varphi \left(\left(x \chi_{D_{t_5}} \right)^* (t - t_5) \right) - \varphi \left(\left(x \chi_{e_{\gamma_0}} \right)^* (t - (t_5 - \gamma_0 + a)) \right) \right] \omega(t) dt \\
& = \int_{\gamma_0-a}^{\gamma_0} \left[\varphi \left(\left(x \chi_{D_{t_5}} \right)^* (t - \gamma_0 + a) \right) - \varphi \left(\left(x \chi_{e_{\gamma_0}} \right)^* (t) \right) \right] (\omega(t) - \omega(t + t_5 - \gamma_0 + a)) dt \\
& \geq (\omega_0 - \omega(t_5)) \int_{\gamma_0-a}^{\gamma_0} \left(\varphi \left(\left(x \chi_{D_{t_5}} \right)^* (t - \gamma_0 + a) \right) - \varphi \left(\left(x \chi_{e_{\gamma_0}} \right)^* (t) \right) \right) dt \\
& \geq a \left(\varphi(u_2) - \varphi \left(\frac{u_2}{4} \right) \right) (\omega_0 - \omega(t_5)) \geq \frac{3 a p u_2 (\omega_0 - \omega(t_5))}{4}, \tag{37}
\end{aligned}$$

where p is in fact the constant value of the derivative of φ on the interval $(0, \beta)$ (the interval of linearity of φ) and $\omega_0 = \omega(t)$ for any $t \in (0, \gamma_0)$. By the definition of γ_0 we get $\omega_0 - \omega(t_5) > 0$. Hence and by Hardy-Littlewood inequality (proceeding analogously as in formula (16)), we obtain

$$\begin{aligned}
& \int_0^{\gamma_0} \varphi \left(\left(x \chi_{e_{\gamma_0}} \right)^* (t) \right) \omega(t) dt + \int_{\gamma_0}^{t_5} \varphi \left(\left(x \chi_{D_{\gamma_0}} \right)^* (t - \gamma_0) \right) \omega(t) dt \\
& + \int_{t_5}^{\gamma} \varphi \left(\left(x \chi_{D_{t_5}} \right)^* (t - t_5) \right) \omega(t) dt \\
& \leq \int_0^{\gamma_0-a} \varphi \left(\left(x \chi_{e_{\gamma_0}} \right)^* (t) \right) \omega(t) dt + \int_{\gamma_0-a}^{\gamma_0} \varphi \left(\left(x \chi_{D_{t_5}} \right)^* (t - (\gamma_0 - a)) \right) \omega(t) dt \\
& + \int_{\gamma_0}^{t_5} \varphi \left(\left(x \chi_{D_{\gamma_0}} \right)^* (t - \gamma_0) \right) \omega(t) dt \\
& + \int_{t_5}^{t_5+a} \varphi \left(\left(x \chi_{e_{\gamma_0}} \right)^* (t - (t_5 - \gamma_0 + a)) \right) \omega(t) dt \\
& + \int_{t_5+a}^{\gamma} \varphi \left(\left(x \chi_{D_{t_5}} \right)^* (t - t_5) \right) \omega(t) dt - \frac{3 a p u_2 (\omega_0 - \omega(t_5))}{4} \\
& \leq \int_0^{\gamma} \varphi(x^*(t)) \omega(t) dt - \frac{3 a p u_2 (\omega_0 - \omega(t_5))}{4} = 1 - \frac{3 a p u_2 (\omega_0 - \omega(t_5))}{4}. \tag{38}
\end{aligned}$$

Suppose now that $\left(x\chi_{e_{\gamma_0}}\right)^*(\gamma_0 - a) > \frac{u_2}{4}$. Since $m(C_{x,y}^+ \cup C_{x,y}^-) < t_4 < \frac{1}{8}\gamma_0$, we have $m\left(\left\{t \in e_{\gamma_0} : |y(t)| > \frac{u_2}{4}\right\}\right) \leq t_4 + a \leq \frac{5}{32}\gamma_0$. Thus $\left(y\chi_{e_{\gamma_0}}\right)^*(\gamma_0 - a) \leq \frac{u_2}{4}$. Simultaneously,

$$m(C_y \cap e_{\gamma_0}) \leq m(B_y \cap e_{\gamma_0}) \leq t_4 + a \leq \frac{5}{32}\gamma_0.$$

Hence, by (35) we get

$$m(C_y \cap D_{t_5}) \geq \frac{1}{2}t_3 + \frac{1}{4}\gamma_0 - (t_4 + a) - a = \frac{\gamma_0}{2} - 2a$$

and we conclude that $(y\chi_{D_{t_5}})^*(a) \geq u_2$. Repeating the procedure from (37) and (38) for y we obtain

$$I_{\varphi,\omega}\left(\frac{x+y}{2}\right) \leq 1 - \frac{3a p u_2(\omega_0 - \omega(t_5))}{8}. \quad (39)$$

Summarizing Case 1, for any $x \in S(A_{\varphi,\omega})$ such that $x^*(0) \leq \beta$, by (18), (24), (25), (31), (32) and (39) we have

$$\begin{aligned} & \min\left(I_{\varphi,\omega}\left(\frac{x+y}{2}\right), I_{\varphi,\omega}\left(\frac{x-y}{2}\right)\right) \\ & \leq 1 - \min\left(\frac{\eta c}{8}, \frac{v t_1 \varphi(u_0) \omega(t_2)}{4}, \frac{1}{2}\delta\left(\varphi \circ x, \frac{u_2 \zeta}{4\beta}\right), \frac{3a p u_2(\omega_0 - \omega(t_5))}{8}\right). \end{aligned}$$

Case 2. Let $\gamma_0 = 0$. If $x^*(0) > \beta$, we can proceed analogously as in the proof of Case 1. Assume now that $x^*(0) \leq \beta$. Let $c = \frac{1}{8}$ and $u_0 > \beta$. Since $\psi \in \Delta_2(\infty)$, there exists $\eta \in (0, 1)$ such that inequality (8) holds for $u \geq u_0$. We define u_1 , $e(u_0)$ and $e(u_1 + 2\beta)$ as in Case 1. If

$$\int_0^\gamma \varphi\left((y\chi_{e(u_1+2\beta)})^*(t)\right) \omega(t) dt \geq \frac{1}{16},$$

then proceeding analogously as in Case 1.1 we obtain

$$I_{\varphi,\omega}\left(\frac{x+y}{2}\right) \leq 1 - \frac{\eta}{64}. \quad (40)$$

Suppose now that

$$\int_0^\gamma \varphi\left((y\chi_{e(u_1+2\beta)})^*(t)\right) \omega(t) dt < \frac{1}{16}$$

and

$$\int_0^\gamma \varphi\left((y\chi_{e(u_0)})^*(t)\right) \omega(t) dt \geq \frac{1}{8}.$$

We define t_1 and t_2 by the equalities

$$\int_0^{t_1} \varphi(u_1 + 2\beta) \omega(t) dt = \frac{1}{16}$$

and

$$\int_0^{t_2} \varphi\left(\frac{u_0 + \beta}{2}\right) \omega(t) dt = 1.$$

We have $0 < t_1 < t_2 < \gamma$. Proceeding analogously as in Case 1.2, we get

$$\min\left(I_{\varphi,\omega}\left(\frac{x-y}{2}\right), I_{\varphi,\omega}\left(\frac{x+y}{2}\right)\right) = 1 - \frac{v t_1 \varphi(u_0) \omega(t_2)}{4}, \quad (41)$$

where $v = \min \left(\eta, 1 - \frac{\varphi\left(\frac{u_0+\beta}{2}\right)}{\varphi(u_0)} \right)$.

Finally let us assume that

$$\int_0^\gamma \varphi \left((y\chi_{e(u_0)})^*(t) \right) \omega(t) dt < \frac{1}{8}$$

and define t_3 and u_2 by the equalities

$$\int_0^{t_3} \varphi(u_0) \omega(t) dt = \frac{3}{4}$$

and

$$\int_0^\gamma \varphi(u_2) \omega(t) dt = \frac{1}{8}.$$

It is obvious that $0 < u_2 < \beta < u_0$. Defining the sets B_x and B_y as in Case 1.3 (see (27) and (28)), we obtain $m(B_x) \geq t_3$ and $m(B_y) \geq t_3$. By the assumption that $\gamma_0 = 0$, we can find t_γ such that $0 < t_\gamma < t_3$ and $\omega(t_\gamma) > \omega(t_3)$. Let now $t_4 := \frac{t_\gamma}{4} < \frac{t_3}{4}$ and let $C_{x,y}^+$ and $C_{x,y}^-$ be defined by formulas (29) and (30). If $m(C_{x,y}^+) \geq \frac{t_4}{2}$ or $m(C_{x,y}^-) \geq \frac{t_4}{2}$, then, analogously as in Case 1.3.1, we obtain

$$\min \left(I_{\varphi,\omega} \left(\frac{x-y}{2} \right), I_{\varphi,\omega} \left(\frac{x+y}{2} \right) \right) \leq 1 - \frac{1}{2} \delta \left(\varphi \circ x, \frac{u_2 \zeta}{4\beta} \right), \quad (42)$$

where $\zeta = \int_0^{t_4/2} \varphi\left(\frac{u_2}{4}\right) \omega(t) dt$ and $\delta\left(\varphi \circ x, \frac{u_2 \zeta}{4\beta}\right)$ is the constant from the definition of local uniform monotonicity of the Lorentz space Λ_ω corresponding to $\varphi \circ x$ and $\varepsilon = \frac{u_2 \zeta}{4\beta}$.

In the case when $\max \left(m(C_{x,y}^+), m(C_{x,y}^-) \right) < \frac{t_4}{2}$, we define the sets C_x and C_y as in Case 1.3.2 (see (33) and (34)). We have $C_x \cap C_y = \emptyset$ and, moreover,

$$\min \left(m(C_x), m(C_y) \right) \geq t_3 - t_4 \geq t_3 - \frac{t_3}{4} = \frac{3}{4} t_3,$$

whence

$$m(C_x \cup C_y) \geq \frac{3}{2} t_3.$$

Defining $a = \frac{1}{32} t_\gamma < \frac{1}{32} t_3$ and putting t_γ in place of γ_0 and t_3 in place of t_5 , we can repeat the procedure from Case 1.3.2. We get

$$I_{\varphi,\omega} \left(\frac{x+y}{2} \right) \leq 1 - \frac{3apu_2 (\omega(t_\gamma) - \omega(t_3))}{8}, \quad (43)$$

where p is the derivative of φ on the interval $(0, \beta)$.

Recapitulating Case 2, for any x such that $x^*(0) \leq \beta$, by (40), (41), (42) and (43) we get

$$\begin{aligned} & \min \left(I_{\varphi,\omega} \left(\frac{x-y}{2} \right), I_{\varphi,\omega} \left(\frac{x+y}{2} \right) \right) \\ & \leq 1 - \min \left(\frac{\eta}{64}, \frac{vt_1 \varphi(u_0) \omega(t_2)}{4}, \frac{1}{2} \delta \left(\varphi \circ x, \frac{u_2 \zeta}{4\beta} \right), \frac{3apu_2 (\omega(t_\gamma) - \omega(t_3))}{8} \right). \end{aligned}$$

Necessity. The necessity of conditions $\varphi \in \Delta_2(\infty)$ and $\int_0^{\gamma_0/2} \varphi(\beta) \omega(t) dt < 1$ has been shown in Theorem 2.2 of [3]. Suppose that $\psi \notin \Delta_2(\infty)$. Then there exists a sequence $(u_n)_{n=1}^\infty$

such that $u_n \rightarrow \infty$ as $n \rightarrow \infty$ and

$$\varphi\left(\frac{u_n}{2}\right) > \frac{1 - \frac{1}{n}}{2} \varphi(u_n)$$

for any $n \in \mathbb{N}$. We can assume that $u_1 > 2\beta$.

There exist $t_0 > 0$ such that $\int_0^{t_0} \varphi(\beta)\omega(t)dt = 1$ and $t_n > 0$ such that $\int_0^{t_n} \varphi(u_n)\omega(t)dt = 1$ for all $n \in \mathbb{N}$. Then we obtain $t_n < t_0$ for all $n \in \mathbb{N}$ and $t_n \rightarrow 0$ as $n \rightarrow \infty$. Define

$$\begin{aligned} x &= \beta \chi_{[0, t_0)} \\ y_n &= u_n \chi_{[0, t_n)}, \quad \text{for any } n \in \mathbb{N}. \end{aligned}$$

For any $n \in \mathbb{N}$ consider a straight line passing through the points $(\frac{1}{2}u_n, \frac{n-1}{2n}\varphi(u_n))$ and $(u_n, \varphi(u_n))$, given by the equation

$$y(t) = \frac{n+1}{n} \frac{\varphi(u_n)}{u_n} t - \frac{1}{n} \varphi(u_n) \quad \text{for } t \in \mathbb{R}.$$

It is easy to observe that for any $u \in (0, u_n)$ the graph of the function φ lies above the graph of this line. Hence, we obtain

$$\begin{aligned} \varphi\left(\frac{u_n - \beta}{2}\right) &\geq \frac{n+1}{n} \frac{\varphi(u_n)}{u_n} \left(\frac{u_n - \beta}{2}\right) - \frac{1}{n} \varphi(u_n) \\ &= \frac{n-1}{2n} \varphi(u_n) - \frac{n+1}{2n} \varphi(u_n) \frac{\beta}{u_n} \end{aligned}$$

and

$$\begin{aligned} \lim_{n \rightarrow \infty} I_{\varphi, \omega}\left(\frac{y_n - x}{2}\right) &= \lim_{n \rightarrow \infty} \left(\int_0^{t_n} \varphi\left(\frac{u_n - \beta}{2}\right) \omega(t) dt + \int_{t_n}^{t_0} \varphi\left(\frac{\beta}{2}\right) \omega(t) dt \right) \\ &\geq \lim_{n \rightarrow \infty} \left(\frac{n-1}{2n} \int_0^{t_n} \varphi(u_n) \omega(t) dt - \frac{n+1}{2n} \frac{\beta}{u_n} \int_0^{t_n} \varphi(u_n) \omega(t) dt + \frac{1}{2} \int_{t_n}^{t_0} \varphi(\beta) \omega(t) dt \right) \\ &= \frac{1}{2} + \frac{1}{2} = 1. \end{aligned}$$

In consequence,

$$\lim_{n \rightarrow \infty} \min \left(\left\| \frac{x - y_n}{2} \right\|_{\varphi, \omega}, \left\| \frac{x + y_n}{2} \right\|_{\varphi, \omega} \right) = 1.$$

3 Applications to Orlicz function spaces

Note that in the case when $\omega(t) = 1$ for any $t \in (0, \gamma)$ the corresponding Orlicz–Lorentz function spaces become the well-known Orlicz function spaces $L^\varphi = L^\varphi([0, \gamma))$. On the basis of the results presented in [3] and the foregoing part of the present paper we can easily deduce criteria for non-squariness properties of Orlicz function spaces L^φ .

In the case when $\gamma = \infty$, by Theorem 2.1 from [3] and Theorem 2, we get the following

Corollary 1 ([7], Theorem 2) *Let $\gamma = \infty$. Then the following conditions are equivalent:*

1. $\varphi \in \Delta_2(\mathbb{R})$ and $\varphi(\frac{u}{2}) < \frac{1}{2}\varphi(u)$ for any $u > 0$,
2. the Orlicz space L^φ is non-square,
3. the Orlicz space L^φ is locally uniformly non-square.

In turn, by Theorem 2.3 from [3], we have

Corollary 2 ([6], Theorem 1.2.(i)) *Let $\gamma = \infty$. Then the Orlicz space L^φ is uniformly non-square if and only if $\varphi \in \Delta_2(\mathbb{R})$ and $\psi \in \Delta_2(\mathbb{R})$, where ψ denotes the complementary function of φ in the sense of Young (see formula (3)).*

Let now $\gamma < \infty$ and let β be defined by formula (1) (in the paper [3] the same constant was denoted by δ). Then by Theorems 2.2 and 2.4 from [3] and Theorems 3 and 4, we obtain

Corollary 3 *Let $\gamma < \infty$. Then the following assertions are true:*

1. ([7], Theorem 3) *The Orlicz space L^φ is non-square if and only if $\varphi \in \Delta_2(\infty)$ and $\varphi(\beta) \cdot \gamma < 2$.*
2. *The Orlicz space L^φ is locally uniformly non-square if and only if $\varphi \in \Delta_2(\infty)$ and $(\varphi(\beta) \cdot \gamma < 1$ or $\varphi(\beta) \cdot \gamma \in [1, 2)$ and $\psi \in \Delta_2(\infty))$.*
3. [6], Theorem 1.2.(ii) *The Orlicz space L^φ is uniformly non-square if and only if $\varphi \in \Delta_2(\infty)$, $\psi \in \Delta_2(\infty)$ and $\varphi(\beta) \cdot \gamma < 2$.*

Therefore, if $\varphi(\beta) \cdot \gamma < 1$, then the Orlicz space L^φ is locally uniformly non-square if and only if it is non-square, while if $\varphi(\beta) \cdot \gamma \in [1, 2)$, then the Orlicz space L^φ is locally uniformly non-square if and only if it is uniformly non-square.

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Compliance with Ethical Standards

Conflict of interest The authors declare that they have no conflict of interest.

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