Singular Felipe

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Abstract

We review the remarkable and singular career of Felipe Cano in the study of Singularities.

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This issue contains original papers dedicated to Felipe Cano after a Conference in occasion of his 60th birthday, which took place in Santander in June 2017. The title of the Conference was “Singular Felipe”, and we are going to explain the reasons for it.

If we consult the Oxford Dictionary, there are several meanings for the word *singular*:

1. Unique.
2. Exceptionally good, remarkable.
3. Strange or eccentric in some aspect.

In our opinion, and we are sure that also in the opinion of all his friends, the three meanings are well adapted to Felipe Cano.

Clearly Felipe is singular, unique, some of us could say hopefully, because if there were two, we would all become crazy.

Seriously, after a lot of years knowing Felipe and working with him, we all know that there are few people comparable to him, he is exceptionally good as person and as mathematician, and this justifies the second meaning of “singular”.

At last if we ask to his students in mathematics in Valladolid, either undergraduate or graduate, all of them would use the word eccentric, or even something worse, to describe the pedagogical methods of Felipe. But as far as we know these eccentric methods work perfectly.

Felipe is also singular by the number of people with whom he has collaborated: he has joint publications with 26 mathematicians, 8 from France, 7 from Brazil, 7 from Spain, 2

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from Peru, 1 from USA and Lev Bribrair. Also by his labor of training of young researchers: he has been advisor of six Phd. Thesis so far, five of them in co-direction.

Another singularity of Felipe is to have been the first Spanish coordinator of two networks in mathematics in a framework program of the European Union [Human Capital and Mobility (HCM) (1992–1995) and Training and Mobility of Researches (TMR) (1996–2001)], composed by the institutions: Limburgs Universitair Centrum (Diepenbeek), Universidad Autónoma de Barcelona, Université de Bourgogne (Dijon), Universidade de Lisboa, Université Louis Pasteur (Strasbourg), Université Paul Sabatier (Toulouse), Université de Rennes I, Universidad de Valladolid.

At last, as it is well known, Felipe's field of work is, since the very beginning, the theory of singularities, a field specially singular in which the work of Felipe has special importance.

1 Introduction

The main part of Felipe's work is devoted to the problems of reduction of singularities. A well known specialist in the field, Vincent Cossart, wrote in 1987:

Il faut bien convenir que pour le moment la résolution des singularités est une théorie caractérisée par la simplicité de ses énoncés et la longueur de leurs démonstrations.

Dominique Cerveau said, with other words, the same thing, this time speaking about Felipe's thesis. (M.R. reviews of F. Cano's Desingularization Strategies ...)

The evolution of numerical invariants and Newton polygons during the explosion process necessitates some long and tedious calculations which hide the beauty of the result a little (the statement of the main theorem is on page 34!)

The reason of this fact has been remarked by Hironaka in an interview published in Notices of A.M.S in 2005:

The resolution of singularities was done by hand. It doesn't use large theory or techniques. It's done just by hand.

This work by hand and with, in some sense, elementary tools, mainly created "ad hoc", demands great tenacity and intellectual fineness to be able to consider and cover all the details. Felipe has done evidence of these qualities along all his career.

2 The origin

The origin for the interest of Felipe in the singularities of differential equations goes back to a conversation that Aroca had with Hironaka in 1974, when Hironaka asked Aroca an easy question:

José Manuel, I propose you a problem: Solve all the differential equations.

In the non-singular situation there is a lot of theorems proving the existence of solutions, then the only thing to do is to reduce the singular situations to non-singular ones and glue the solutions along the exceptional divisor. Two not so easy problems.

In 1968, Seidenberg [28] has proved that in dimension two is possible to reduce, by blowing-up points, the singularities of a first order and first degree complex analytic differential equation to a simple model, proving as a consequence that the equation has formal solutions.

The first order and first degree differential equation can be considered as a derivation or as a vector field. Then Aroca proposed Felipe to do the same construction as Seidenberg's one for complex analytic vector fields in dimension three.

On the other hand, in 1974, Giraud [25] had a new description of maximal contact by means of differential calculus, and he had the idea of solving the worst singularities in the positive characteristic case observing that the singularities of the hypersurface

$$z^{p^e} = f(x_1, \dots, x_n)$$

over a field K of characteristic p are closely related to the singularities of the differential form $\omega = df$. Then, Giraud proposed the reduction of singularities of algebraic varieties in positive characteristic by the following process:

A form ω on K^2 has strong similarity with a curve, and helps us to solve the singularities of a surface, closely connected to a form in K^3 , etc.

Giraud had a big influence in the early Felipe's work, and this observation gave Felipe more motivation for study singularities of vector fields in dimension three.

Roughly speaking, reduction of singularities consists in applying a succession of blow-ups to kill all the singularities. The problem is that, due to the local nature of the blow-up, one follows only a neighborhood, each time smaller, of a point. To find a succession of local blow-ups with non singular end is easy, but you should need to do that for any singular point in the successive transforms and, at the end, to glue together all the local reductions.

In his thesis [7], Felipe found a punctual solution of the problem of reduction of singularities of vector fields in dimension three, by solving a differential version of the classical Hironaka's game. An intermediate problem between finding a way with nonsingular end and proving that any *acceptable* way has non singular end.

The game is a game with two players, Hironaka use to say that the first one is the player A for Abhyankhar and the second, the player B for the bad guys. Beginning with a one dimensional distribution over a three dimensional regular variety X , and a point $P \in X$, the player A chooses a permissible center. Then, after blowing-up this center, the player B chooses a point in the exceptional divisor, and so on. The player A wins if, at some stage there is an improvement of the singularity, measured with an invariant: the adapted order of the distribution along the exceptional divisor. The first important achievement of Felipe is to prove that there is a winner strategy for the player A.

This result is a cornerstone in the way to solve the singularities of differential forms in dimension three, and Felipe needed a lot of hard work to prove it. Hopefully Felipe did not know, until the end of his work, the "encouraging" words of Cerveau and Mattei in a selection of open problems in his book [24]:

Désingulariser (bonne chance) les formes intégrables en dimension supérieure à 2.

Nevertheless in the review of another Felipe's paper about the reduction of singularities of plane vector fields (arbitrary characteristic), the reviewer wrote:

The control method for the singularity is based upon Hironaka's techniques, and seems to have not an essential obstruction for generalization to higher dimensions.

Unfortunately the optimism of the reviewer had no basis at all. And Felipe needed to change a lot his mind and 20 years of hard work to prove the Theorem of Reduction of Singularities of differential forms in dimension three.

In 1985, Felipe's trajectory changes with his attendance to the "Colloque sur les singularités d'équations différentielles", held in Dijon. In this congress Felipe contacted with the who's—who of foliations, as Jean Martinet, Robert Moussu, Jean-François Mattei, Dominique Cerveau, César Camacho, Paolo Sad and Alcides Lins Neto, and many others.

And by this door Felipe enters in the Foliated Heaven.

3 The foliated Heaven

As a consequence of this entrance, Felipe changes distributions for differential forms, objects non exactly equivalent, and started a program of collaboration with french and brazilian mathematicians in research and training of young people, that continues today.

His first important result (published in 1989) in collaboration with César Camacho and Paulo Sad [6], still in the area of vector fields, was the reduction of absolutely isolated singularities of vector fields in any dimension, essentially a generalization of the 2-dimensional case. It consists in the reduction of isolated singularities that after blowing-up gives rise only to a finite number of isolated singularities. They prove that after a finite number of blow-ups centered in points we get a vector field with only elementary singularities, that is, the linear part has at least a non zero eigenvalue.

In the same line, 14 years after, Felipe, Dominique Cerveau and Bruno Scardua [12] characterized the holomorphic codimension one foliations in $\mathbb{C}^3$ that can be desingularized by a sequence of punctual blow-ups producing only simple singularities. This case is studied in analogy with the case of Kleinian singularities of complex surfaces. They proved that a foliation with such a property is given by a closed meromorphic 1-form with simple poles, provided that, along the reduction process, the simple singularities exhibit a hyperbolic transverse type.

Felipe continued working in the reduction problem and around 1990 he gives an important step with the paper [8]. He proved that given a germ $\mathcal{F}$ of non dicritical foliation of codimension one in an ambient space of dimension three, there is a finite sequence of permissible blow-ups such that all singularities of the final transform of $\mathcal{F}$ has adapted multiplicity at most one. A bit later, Felipe and Dominique Cerveau in [11] improved the result, describing the final models of the end of the reduction process, and with this description they were able to give a positive answer to Thom's conjecture concerning the existence of separatrices of codimension one singular non dicritical foliations in dimension three. Camacho and Sad in [5] had proved the conjecture in dimension 2, and Jouanolou in [27] gave a counterexample in dimension 3 for a dicritical foliation.

After that, in collaboration with Mattei in [14], Felipe extended the result to any dimension, proving Thom's conjecture in general under the necessary hypothesis of non dicriticalness. The essential tool was a theorem of generic equisingularity for the non dicritical case.

At the end, in a beautiful paper [9], Felipe finished his work in codimension one foliations in a three dimensional ambient, by finally avoiding the non-dicriticalness hypothesis. This is his most singular, in the sense of remarkable, theorem. A consequence of nearly 20 years of work.

Then after this essential result it follows as the military say:

4 The exploitation of success

Hironaka wrote in 1973 ([26]):

Our aim is not the destruction of singularities by means of savage bombardments. We intend to cultivate them, to take care of them, to grow them, and to benefit from their fruits.

And in Hironaka's spirit, Felipe begins a long process of exploitation of his techniques in several fields.

4.1 Trajectories

First, in the field of real foliations in collaboration with the people from Dijon and Fernando Sanz:

- Felipe, Lion and Moussu proved in [16] that the boundary of a non-spiralling leaf of a codimension one real analytic foliation is a locally finite union of smooth manifolds.
- Felipe, Moussu and Sanz studied in [17] and [18] the behavior of the trajectories around a singular point of a real analytic vector field in dimension three. They prove that:
 1. If a curve possesses iterated tangents and is oscillating with respect to some analytic surface, then it must be spiralling around an analytic curve invariant by the field.
 2. A spiralling axis is always a vortex axis (any trajectory in a neighborhood of the axis also spirals), except for the case when it consists of non-isolated singular points.
 3. If in a pencil of trajectories with the same oriented iterated tangents two of them are sub-analytically separated, each pair of them has this property.
 4. If no pair of curves in the pencil is separated, the pencil contains a formal integral curve $\widehat{F}$ that is divergent if and only if the integral curves of the pencil are non oscillating. Also $\widehat{F}$ is not contained in any germ of analytic surface. As a consequence, it gives rise to a valuation and a Hardy field structure.
- Felipe, Moussu and Sanz proved in [19] that the oscillation of trajectories of vector fields is *visible* by projection in dimension two.

On the other hand, Felipe has worked in the description of the set of secants to a trajectory of a real analytic vector field defined in a neighborhood of the origin in $\mathbb{R}^3$. This question, which can be considered as an infinitesimal version, in dimension three, of the classical Poincaré-Bendixson Theorem, is still open in general. In collaboration with Clemen Alonso and Rudy Rosas, Felipe proved in [4] that given a vector field with an absolutely isolated singularity of Morse-Smale type at the origin, after finitely many blow-ups, the accumulation set of a trajectory is either a single point, a periodic orbit, or a polycycle homeomorphic to $\mathbb{S}^1$.

4.2 Topological classification of real vector fields

In the 6 years between 2001 and 2007, Felipe, Clemen Alonso and Izabel Camacho in the series of papers [1–3], carried out a program of study of invariants and topological classification of real analytic vector fields in a space of dimension three for which we only get hyperbolic singularities after punctual blow-ups.

4.3 Study of dicriticalness property for holomorphic foliations

Still working with singularities that can be improved by blowing-up points, Nuria Corral and Felipe in [13] defined an absolutely dicritical foliation as an holomorphic foliation in $(\mathbb{C}^2, 0)$ having the property that after a finite number of punctual blowing-ups, his strict transform is completely transversal to the exceptional divisor. They gave a positive answer to a Cerveau's question proving that there is always an absolutely dicritical foliation for any given configuration of an exceptional divisor.

One of the last problems addressed by Felipe in the area of foliations is to give an affirmative answer to a local version of Brunella's alternative. It can be stated as follows: Given a codimension one holomorphic foliation $\mathcal{F}$ at $(\mathbb{C}^3, 0)$, either $\mathcal{F}$ admits an invariant analytic surface through p , or any leaf of $\mathcal{F}$ in a punctured neighborhood of the origin contains an analytic curve through 0, except for the point 0 itself. In the joint papers [20,21] with Marianna Ravara and Marcio Soares, Felipe proved that for some special foliations the local version of Brunella's conjecture has a positive answer.

5 The valuations

For many years, valuations were far from the interest of mathematicians, and much further away from the specialists in foliations. In a Seminar at the MIT in the early seventies, a young Spanish mathematician said to André Weil that he was studying valuations and that he had used one year to read Weil's book *Fundaments of Algebraic Geometry*. "You have lost the time, valuations are out of mathematics", was the response of Weil.

Despite the opinion of Weil, nowadays there are many mathematicians working with and over the valuations. And as Spivakovsky, Cossart, Cutkosky, Piltant and many other have shown, the use of Zariski's uniformization techniques simplifies classical results and produces new ones in the area of Reduction of Singularities.

Felipe, too, has exploited that old Zariski's method in order to do reduction of singularities of foliations by uniformization of valuations.

First Felipe proved the Seidenberg theorem on reduction of singularities of vector fields in dimension two by using uniformization (see a recent exposition in [10]).

Then, in collaboration with Robert Moussu and Jean-Philippe Rolin in [15], and connected directly with the results of Felipe, Moussu and Sanz about trajectories, Felipe have proved a theorem of local uniformization of a three-dimensional real analytic vector field along a non-oscillating trajectory. Namely, the vector field becomes elementary after local blow-ups and ramifications following the point where the trajectory accumulates (center of the valuation defined by the trajectory).

After that, in collaboration with Mark Spivakovsky and Claude Roche in [23] proved an analogue of the Zariski local uniformization theorem for three-folds: If M is a three-dimensional algebraic variety over an algebraically closed field k then, for any k -valuation v of the function field $k(M)$ and any line foliation $\mathcal{L}$ on M , there exists a finite sequence of blow-ups with non-singular centers $\pi : \tilde{M} \rightarrow M$ such that $\mathcal{L}$ is log-elementary (a simple reduced form) along the center of v in $\tilde{M}$. A global version of this result would be that there exists a birational morphism $\pi : \tilde{M} \rightarrow M$ such that $\mathcal{L}$ is log-elementary at all points of $\tilde{M}$.

Another interesting result of Felipe and Roche in [22] is that, if a germ of holomorphic vector field at the origin of $\mathbb{C}^3$, having a formal invariant curve $\hat{\Gamma}$ that is totally transcendental, is tangent to a codimension-one holomorphic foliation, then the vector field can be

desingularized by birational blow-ups (no ramifications are needed). This implies that any vector field tangent to a foliation can be desingularized.

Finally, let us comment the projects related with reduction of singularities and valuations that Felipe is working on at this moment:

- With Miguel Fernandez Duque, trying to extend uniformization to dimension bigger than three (Duque proved in his PhD thesis the result for archimedean k -rational valuations).
- With Beatriz Molina, clarifying the combinatorial aspects of the reduction process with interesting results.

The succinct exposition above shows that Felipe is a *singularity* as a mathematician (many of us know well that also as a person). To finish let us remember some works of Hironaka, in an Interview of Notices of the AMS in 2005.

Many phenomena are interesting, or sometimes disastrous, because they have singularities. A singularity might be a crossing or something suddenly changing direction. There are many things like that in the world, and that's why the world is interesting. Otherwise it would be completely flat. If everything were smooth, then there would be no novels or movies. The world is interesting because of the singularities.

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