Josefson–Nissenzweig property for C_p -spaces

T. Banakh^{1,2} · J. Kąkol^{3,4} · W. Śliwa⁵

Received: 6 March 2019 / Accepted: 2 April 2019 / Published online: 13 April 2019
© The Author(s) 2019

Abstract

The famous Rosenthal–Lacey theorem asserts that for each infinite compact space K the Banach space $C(K)$ admits a quotient isomorphic to Banach spaces c or ℓ_2 . The aim of the paper is to study a natural variant of this result for the space $C_p(X)$ of continuous real-valued maps on a Tychonoff space X with the pointwise topology. Following Josefson–Nissenzweig theorem for infinite-dimensional Banach spaces we introduce a corresponding property (called Josefson–Nissenzweig property, briefly, the JNP) for $C_p(X)$ -spaces. We prove: for a Tychonoff space X the space $C_p(X)$ satisfies the JNP if and only if $C_p(X)$ has a quotient isomorphic to $c_0 := \{(x_n)_{n \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}} : x_n \rightarrow 0\}$ (with the product topology of $\mathbb{R}^{\mathbb{N}}$) if and only if $C_p(X)$ contains a complemented subspace isomorphic to c_0 . The last statement provides a C_p -version of the Cembranos theorem stating that the Banach space $C(K)$ is not a Grothendieck space if and only if $C(K)$ contains a complemented copy of the Banach space c_0 with the sup-norm topology. For a pseudocompact space X the space $C_p(X)$ has the JNP if and only if $C_p(X)$ has a complemented metrizable infinite-dimensional subspace. An example of a compact space K without infinite convergent sequences with $C_p(K)$ containing a complemented subspace isomorphic to c_0 is given.

Keywords The separable quotient problem · Spaces of continuous functions · Quotient spaces · The Josefson–Nissenzweig theorem · Efimov space

To the memory of our Friend Professor Mati Rubin.

J. Kąkol supported by GAČR Project 16-34860L and RVO: 67985840.

✉ W. Śliwa
sliwa@amu.edu.pl; wśliwa@ur.edu.pl

T. Banakh
t.o.banakh@gmail.com

J. Kąkol
kakol@amu.edu.pl

¹ Ivan Franko National University of Lviv, Lviv, Ukraine

² Jan Kochanowski University in Kielce, Kielce, Poland

³ Faculty of Mathematics and Informatics, A. Mickiewicz University, 61-614 Poznań, Poland

⁴ Institute of Mathematics Czech Academy of Sciences, Prague, Czech Republic

⁵ Faculty of Mathematics and Natural Sciences, University of Rzeszów, 35-310 Rzeszów, Poland

1 Introduction, motivations and two main problems

Let X be a Tychonoff space. By $C_p(X)$ we denote the space of real-valued continuous functions on X endowed with the pointwise topology.

We will need the following fact stating that each metrizable (linear) quotient $C_p(X)/Z$ of $C_p(X)$ by a closed vector subspace Z of $C_p(X)$ is *separable*. Indeed, this follows from the separability of metrizable spaces of countable cellularity and the fact that $C_p(X)$ has countable cellularity, being a dense subspace of $\mathbb{R}^X$, see [2].

The classic Rosenthal–Lacey theorem, see [19,23,27], asserts that the Banach space $C(K)$ of continuous real-valued maps on an infinite compact space K has a quotient isomorphic to Banach spaces c or ℓ_2 , or equivalently, there exists a continuous linear (and open; by the open mapping Banach theorem) map from $C(K)$ onto c or ℓ_2 , see also a survey paper [14].

This theorem motivates the following natural question for spaces $C_p(X)$.

Problem 1 For which compact spaces K any of the following equivalent conditions holds:

- (1) The space $C_p(K)$ has an infinite dimensional metrizable quotient.
- (2) The space $C_p(K)$ has an infinite dimensional metrizable separable quotient.
- (3) The space $C_p(K)$ has a quotient isomorphic to a dense subspace of $\mathbb{R}^{\mathbb{N}}$.

In [21] it was shown that $C_p(K)$ has an *infinite-dimensional separable quotient algebra* if and only if K contains an infinite countable closed subset. Hence $C_p(\beta\mathbb{N})$ lacks infinite-dimensional separable quotient algebras. Nevertheless, as proved in [22, Theorem 4], the space $C_p(K)$ has infinite-dimensional separable quotient for any compact space K containing a copy of $\beta\mathbb{N}$.

Problem 1 has been already partially studied in [3], where we proved that for a Tychonoff space X the space $C_p(X)$ has an infinite-dimensional metrizable quotient if X either contains an infinite discrete C^* -embedded subspace or else X has a sequence $(K_n)_{n \in \mathbb{N}}$ of compact subsets such that for every n the space K_n contains two disjoint topological copies of K_{n+1} . If fact, the first case (for example if compact X contains a copy of $\beta\mathbb{N}$) asserts that $C_p(X)$ has a quotient isomorphic to the subspace $\ell_\infty = \{(x_n) \in \mathbb{R}^{\mathbb{N}} : \sup_n |x_n| < \infty\}$ of $\mathbb{R}^{\mathbb{N}}$ or to the product $\mathbb{R}^{\mathbb{N}}$.

Consequently, this theorem reduces Problem 1 to the case when K is an *Efimov space* (i.e. K is an infinite compact space that contains neither a non-trivial convergent sequence nor a copy of $\beta\mathbb{N}$). Although, it is unknown if Efimov spaces exist in ZFC (see [7–10,12,13,16,18]) we showed in [22] that under $\diamond$ for some Efimov spaces K the function space $C_p(K)$ has an infinite dimensional metrizable quotient.

By c_0 we mean the subspace $\{(x_n)_{n \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}} : x_n \rightarrow 0\}$ of $\mathbb{R}^{\mathbb{N}}$ endowed with the product topology. The term “the Banach space c_0 ” means the classic Banach space of null-sequences with the sup-norm topology.

It is known that the Banach space $C(K)$ over an infinite compact K contains a copy of the Banach space c_0 , see for example [6]. By a result of Cembranos, see [4, Theorem, page 74], the space $C(K)$ is not a Grothendieck space if and only if $C(K)$ contains a complemented copy of the Banach space c_0 . Recall a Banach space E is a *Grothendieck space* if every weak* converging sequence in the dual E^* weakly converges in E^* . It is well-known that if a compact space K contains a non-trivial converging sequence, $C(K)$ is not a Grothendieck space; hence $C(K)$ contains a complemented copy of the Banach space c_0 . It is also easy to

see that for every infinite compact space K the space $C_p(K)$ contains a closed copy of the space c_0 endowed with the product topology of $\mathbb{R}^{\mathbb{N}}$.

Cembranos theorem motivates the following next problem (connected with Problem 1).

Problem 2 Characterize those spaces $C_p(K)$ which contain a complemented copy of c_0 with the product topology of $\mathbb{R}^{\mathbb{N}}$.

2 The main results

For a Tychonoff space X and a point $x \in X$ let $\delta_x : C_p(X) \rightarrow \mathbb{R}$, $\delta_x : f \mapsto f(x)$, be the Dirac measure concentrated at x . The linear hull $L_p(X)$ of the set $\{\delta_x : x \in X\}$ in $\mathbb{R}^{C_p(X)}$ can be identified with the dual space of $C_p(X)$. We refer also the reader to [15] for more information about the dual $L_p(X)$.

Elements of the space $L_p(X)$ will be called *finitely supported sign-measures* (or simply *sign-measures*) on X .

Each $\mu \in L_p(X)$ can be uniquely written as a linear combination of Dirac measures $\mu = \sum_{x \in F} \alpha_x \delta_x$ for some finite set $F \subset X$ and some non-zero real numbers α_x . The set F is called the *support* of the sign-measure μ and is denoted by $\text{supp}(\mu)$. The measure $\sum_{x \in F} |\alpha_x| \delta_x$ will be denoted by $|\mu|$ and the real number $\|\mu\| = \sum_{x \in F} |\alpha_x|$ coincides with the *norm* of μ (in the dual Banach space $C(\beta X)^*$).

The sign-measure $\mu = \sum_{x \in F} \alpha_x \delta_x$ determines the function $\mu : 2^X \rightarrow \mathbb{R}$ defined on the power-set of X and assigning to each subset $A \subset X$ the real number $\sum_{x \in A \cap F} \alpha_x$. So, a finitely supported sign-measure will be considered both as a linear functional on $C_p(X)$ and an additive function on the power-set 2^X .

The famous *Josefson–Nissenzweig theorem* asserts that for each infinite-dimensional Banach space E there exists a null sequence in the weak*-topology of the topological dual E^* of E and which is of norm one in the dual norm, see for example [6].

We propose the following corresponding property for spaces $C_p(X)$.

Definition 1 For a Tychonoff space X the space $C_p(X)$ satisfies the *Josefson–Nissenzweig property* (JNP in short) if there exists a sequence (μ_n) of finitely supported sign-measures on X such that $\|\mu_n\| = 1$ for all $n \in \mathbb{N}$, and $\mu_n(f) \rightarrow_n 0$ for each $f \in C_p(X)$.

Concerning the JNP of function spaces $C_p(X)$ on compacta we have the following:

- (1) If a compact space K contains a non-trivial convergent sequence, say $x_n \rightarrow x$, then $C_p(K)$ satisfies the JNP. This is witnessed by the weak* null sequence (μ_n) of sign-measures $\mu_n = \frac{1}{2}(\delta_{x_n} - \delta_x)$, $n \in \mathbb{N}$.
- (2) The space $C_p(\beta\mathbb{N})$ does not satisfy the JNP. This follows directly from the Grothendieck theorem, see [5, Corollary 4.5.8].
- (3) There exists a compact space K containing a copy of $\beta\mathbb{N}$ but without non-trivial convergent sequences such that $C_p(K)$ satisfies the JNP, see Example 1 below.

If a compact space K contains a closed subset Z that is metrizable, then $C_p(K)$ has a complemented subspace isomorphic to $C_p(Z)$. Consequently, if compact K contains a non-trivial convergent sequence, then $C_p(K)$ has a complemented subspace isomorphic to c_0 . However for every infinite compact K the space $C_p(K)$ contains a subspace isomorphic to c_0 but not necessary complemented in $C_p(K)$. Nevertheless, there exists a compact space K without infinite convergent sequences and such that $C_p(K)$ enjoys the JNP and so contains a complemented subspace isomorphic to c_0 , as follows from Theorem 1 below.

It turns out that the Josefson–Nissenzweig property characterizes an interesting case related with Problem 1 and provides a complete solution to Problem 2.

Theorem 1 *For a Tychonoff space X the following conditions are equivalent:*

- (1) $C_p(X)$ satisfies the JNP;
 - (2) $C_p(X)$ contains a complemented subspace isomorphic to c_0 ;
 - (3) $C_p(X)$ has a quotient isomorphic to c_0 ;
 - (4) $C_p(X)$ admits a linear continuous map onto c_0 .
- If the space X is pseudocompact, then the conditions (1)–(4) are equivalent to*
- (5) $C_p(X)$ contains a complemented infinite-dimensional metrizable subspace;
 - (6) $C_p(X)$ contains a complemented infinite-dimensional separable subspace;
 - (7) $C_p(X)$ has an infinite-dimensional Polishable quotient.

We recall that a locally convex space X is *Polishable* if X admits a stronger Polish locally convex topology. Equivalently, Polishable locally convex spaces can be defined as images of separable Fréchet spaces under continuous linear maps. Clearly, the subspace c_0 of $\mathbb{R}^{\mathbb{N}}$ is Polishable.

A topological space X is *pseudocompact* if it is Tychonoff and each continuous real-valued function on X is bounded. It is known (see [3]) that a Tychonoff space X is not pseudocompact if and only if $C_p(X)$ contains a complemented copy of $\mathbb{R}^{\mathbb{N}}$. Combining this characterization with Theorem 1, we obtain another characterization related to Problem 1.

Corollary 1 *For a Tychonoff space X the following conditions are equivalent:*

- (1) $C_p(X)$ has an infinite-dimensional Polishable quotient;
- (2) $C_p(X)$ contains a complemented infinite-dimensional Polishable subspace;
- (3) $C_p(X)$ contains a complemented subspace isomorphic to $\mathbb{R}^{\mathbb{N}}$ or c_0 ;

Corollary 2 *The space $C_p(\beta\mathbb{N})$*

- (1) *has a quotient isomorphic to ℓ_{∞} ;*
- (2) *contains a subspace isomorphic to c_0 ;*
- (3) *does not admit a continuous linear map onto c_0 ;*
- (4) *has no Polishable infinite-dimensional quotients;*
- (4) *contains no complemented separable infinite-dimensional subspaces.*

Indeed, the first claim follows from [3, Proposition], the others follow from Theorem 1 and the statement (2) after Definition 1.

In the final Sect. 5 we shall characterize Tychonoff spaces X whose function space $C_p(X)$ is Polishable and prove the following theorem.

Theorem 2 *For a Tychonoff space X the following conditions are equivalent:*

- (1) $C_p(X)$ is Polishable;
- (2) $C_k(X)$ is Polishable;
- (3) $C_k(X)$ is Polish;
- (4) X is a submetrizable hemicompact k -space.

In this theorem $C_k(X)$ denotes the space of continuous real-valued functions on X , endowed with the compact-open topology. It should be mentioned that a locally convex space is Polish if and only if it is a separable Fréchet space, by using, for example, the Birkhoff–Kakutani theorem [20, Theorem 9.1].

3 Proof of Theorem 1

We start with the following

Lemma 1 *Let a Tychonoff space X be continuously embedded into a compact Hausdorff space K (i.e. there exists a continuous injection from X into K .) Let (μ_n) be a sequence of finitely supported sign-measures on X (and so, on K) such that*

- (1) $\|\mu_n\| = 1$ for all $n \in \mathbb{N}$, and
- (2) $\mu_n(f) \rightarrow_n 0$ for all $f \in C(K)$.

Then there exists an infinite subset Ω of $\mathbb{N}$ such that

- (a) *the closed subspace $Z = \bigcap_{k \in \Omega} \{f \in C_p(X) : \mu_k(f) = 0\}$ of $C_p(X)$ is complemented in the subspace $L = \{f \in C_p(X) : \lim_{k \in \Omega} \mu_k(f) = 0\}$ of $C_p(X)$;*
- (b) *the quotient space L/Z is isomorphic to the subspace c_0 of $\mathbb{R}^{\mathbb{N}}$;*
- (c) *L contains a complemented subspace isomorphic to c_0 ;*
- (d) *the quotient space $C_p(X)/Z$ is infinite-dimensional and metrizable (and so, separable).*

Proof (I) First we show that the set $M = \{\mu_n : n \in \mathbb{N}\}$ is not relatively weakly compact in the dual of the Banach space $C(K)$. Indeed, assume on the contrary that the closure $\overline{M}$ of M in the weak topology of $C(K)^*$ is weakly compact. Applying the Eberlein–Šmulian theorem [1, Theorem 1.6.3], we conclude that $\overline{M}$ is weakly sequentially compact. Thus (μ_n) has a subsequence (μ_{k_n}) that weakly converges to some element $\mu_0 \in C(K)^*$. Taking into account that the sequence (μ_n) converges to zero in the weak* topology of $C(K)^*$, we conclude that $\mu_0 = 0$ and hence (μ_{k_n}) is weakly convergent to zero in $C(K)^*$. Denote by W the countable set $\bigcup_{n \in \mathbb{N}} \text{supp}(\mu_n)$. The measures $\mu_n, n \in \mathbb{N}$, can be considered as elements of the unit sphere of the Banach space $\ell_1(W) \subset C(K)^*$. By the Schur theorem [1, Theorem 2.3.6], the weakly convergent sequence (μ_{k_n}) is convergent to zero in the norm topology of $\ell_1(W)$, which is not possible as $\|\mu_n\| = 1$ for all $n \in \mathbb{N}$. Thus the set M is not relatively weakly compact in $C(K)^*$.

- (II) By the Grothendieck theorem [1, Theorem 5.3.2] there exist a number $\epsilon > 0$, a sequence $(m_n) \subset \mathbb{N}$ and a sequence (U_n) of pairwise disjoint open sets in K such that $|\mu_{m_n}(U_n)| > \epsilon$ for any $n \in \mathbb{N}$. Clearly, $\lim_{n \rightarrow \infty} \mu_k(U_n) = 0$ for any $k \in \mathbb{N}$, since

$$\sum_{n \in \mathbb{N}} |\mu_k|(U_n) = |\mu_k| \left(\bigcup_{n \in \mathbb{N}} U_n \right) \leq |\mu_k|(K) = 1.$$

Thus we can assume that the sequence (m_n) is strictly increasing.

For some strictly increasing sequence $(n_k) \subset \mathbb{N}$ we have $U_{n_k} \cap \text{supp}(\mu_{m_{n_i}}) = \emptyset$ for all $k, i \in \mathbb{N}$ with $k > i$.

Put $v_k = \mu_{m_{n_k}}$ and $W_k = U_{n_k}$ for all $k \in \mathbb{N}$. Then

- (A1) $v_k(f) \rightarrow_k 0$ for every $f \in C(K)$;
- (A2) $|v_k(W_k)| > \epsilon$ for every $k \in \mathbb{N}$;
- (A3) $|v_k(W_n)| = 0$ for all $k, n \in \mathbb{N}$ with $k < n$.

- (III) By induction we shall construct a decreasing sequence (N_k) of infinite subsets of $\mathbb{N}$ with $\min N_k < \min N_{k+1}$ for $k \in \mathbb{N}$ such that $|v_n|(W_m) \leq \epsilon/3^k$ for every $k \in \mathbb{N}, m = \min N_k, n \in N_k$ and $n > m$. Let $N_0 = \mathbb{N}$. Assume that for some $k \in \mathbb{N}$ an infinite subset N_{k-1} of $\mathbb{N}$ has been constructed. Let F be a finite subset of N_{k-1} with $|F| > 3^k/\epsilon$ and $\min F > \min N_{k-1}$. For every $i \in F$ consider the set

$$\Lambda_i = \{n \in N_{k-1} : |v_n|(W_i) \leq \epsilon/3^k\}.$$

For every $n \in N_{k-1}$ we get $|v_n|(X) \geq \sum_{i \in F} |v_n|(W_i)$. Hence there exists $i \in F$ such that

$$|v_n|(W_i) \leq 1/|F| \leq \epsilon/3^k.$$

Thus $N_{k-1} = \bigcup_{i \in F} \Lambda_i$, so for some $m \in F$ the set Λ_m is infinite. Put

$$N_k = \{n \in \Lambda_m : n > m\} \cup \{m\}.$$

Then $\min N_{k-1} < \min F \leq m = \min N_k$ and $|v_n|(W_m) \leq \epsilon/3^k$ for $n \in N_k$ with $n > m$.

(IV) Let $i_k = \min N_k$, $\lambda_k = v_{i_k}$ and $V_k = W_{i_k}$ for $k \in \mathbb{N}$. Then

(B1) $\lambda_k(f) \rightarrow_k 0$ for every $f \in C(K)$;

(B2) $|\lambda_k(V_k)| > \epsilon$ for every $k \in \mathbb{N}$;

(B3) $|\lambda_k(V_l) = 0$ and $|\lambda_l(V_k) \leq \epsilon/3^k$ for all $k, l \in \mathbb{N}$ with $k < l$.

Clearly, the set

$$\Omega = \{n \in \mathbb{N} : \mu_n = \lambda_k \text{ for some } k \in \mathbb{N}\}$$

is infinite. Put

$$Z = \bigcap_{n \in \mathbb{N}} \{f \in C_p(X) : \lambda_n(f) = 0\}$$

and $L = \{f \in C_p(X) : \lambda_n(f) \rightarrow_n 0\}$. Clearly, Z and L are subspaces of $C_p(X)$ and Z is closed in L and in $C_p(X)$. The linear operator

$$S : L \rightarrow c_0, \quad S : f \mapsto (\lambda_n(f))_n,$$

is continuous and $\ker S = Z$.

We shall construct a linear continuous map $P : c_0 \rightarrow L$ such that $S \circ P$ is the identity map on c_0 . For every $k \in \mathbb{N}$ there exists a continuous function $\varphi_k : K \rightarrow [-1, 1]$ such that

$$\varphi_k(s) = \lambda_k(V_k)/|\lambda_k(V_k)|$$

for $s \in V_k \cap \text{supp}(\lambda_k)$ and $\varphi_k(s) = 0$ for $s \in (K \setminus V_k)$. Then

$$\lambda_k(\varphi_k) = |\lambda_k(V_k)| > \epsilon,$$

$\lambda_n(\varphi_k) = 0$ for all $n, k \in \mathbb{N}$ with $n < k$ and

$$|\lambda_n(\varphi_k)| \leq |\lambda_n|(V_k) \leq \epsilon/3^k$$

for all $n, k \in \mathbb{N}$ with $n > k$.

(V) Let $(x_n) \in c_0$. Define a sequence $(x'_n) \in \mathbb{R}^{\mathbb{N}}$ by the recursive formula

$$x'_n := \left[x_n - \sum_{1 \leq k < n} x'_k \lambda_n(\varphi_k) \right] / \lambda_n(\varphi_n) \text{ for } n \in \mathbb{N}.$$

We shall prove that $(x'_n) \in c_0$.

First we show that $\sup_n |x'_n| < \infty$. Since $(x_n) \in c_0$, there exists $m \in \mathbb{N}$ such that $\sup_{n \geq m} |x_n| < \epsilon$. Put

$$M_n = \max \left\{ 2, \max_{1 \leq k < n} |x'_k| \right\}$$

for $n \geq 2$. For every $n > m$ we get

$$|x'_n| = |x_n - \sum_{1 \leq k < n} x'_k \lambda_n(\varphi_k)| / \lambda_n(\varphi_n) \leq \left[\epsilon + M_n \sum_{1 \leq k < n} \epsilon / 3^k \right] / \epsilon \leq 1 + M_n / 2 \leq M_n.$$

Hence $M_{n+1} = \max\{M_n, |x'_n|\} \leq M_n$ for $n > m$. Thus $d := \sup_n |x'_n| \leq M_{m+1} < \infty$.

Now we show that $x'_n \rightarrow_n 0$. Given any $\delta > 0$, find $v \in \mathbb{N}$ such that $d < 3^v \delta$. Since $(x_n) \in c_0$ and $\lambda_n(f) \rightarrow_n 0$ for any $f \in C(K)$, there exists $m > v$ such that for every $n \geq m$

$$|x_n| < \delta \epsilon \text{ and } d \sum_{1 \leq k \leq v} |\lambda_n(\varphi_k)| < \delta \epsilon.$$

Then for $n \geq m$ we obtain

$$\begin{aligned} |x'_n| &\leq \left[|x_n| + \sum_{1 \leq k \leq v} |x'_k| \cdot |\lambda_n(\varphi_k)| + \sum_{v < k < n} |x'_k| \cdot |\lambda_n(\varphi_k)| \right] / \lambda_n(\varphi_n) \\ &\leq \left[\delta \epsilon + \sum_{1 \leq k \leq v} d \cdot |\lambda_n(\varphi_k)| + \sum_{v < k < n} d \cdot |\lambda_n(\varphi_k)| \right] / \epsilon < \delta + \delta \\ &\quad + \sum_{v < k < n} d / 3^k < 2\delta + d / 3^v < 3\delta. \end{aligned}$$

Thus $(x'_n) \in c_0$.

Clearly, the operator

$$\Theta : c_0 \rightarrow c_0, \quad \Theta : (x_n) \mapsto (x'_n),$$

is linear and continuous. We prove that Θ is surjective. Let $(y_n) \in c_0$. Set $t = \sup_n |y_n|$. Let

$$x_n = \sum_{k=1}^n \lambda_n(\varphi_k) y_k \text{ for } n \in \mathbb{N}.$$

First we show that $(x_n) \in c_0$. Given any $\delta > 0$, find $v \in \mathbb{N}$ with $\epsilon t < \delta 3^v$. Clearly, there exists $m > v + 2$ such that $|y_n| < \delta$ and $\sum_{k=1}^v t |\lambda_n(\varphi_k)| < \delta$ for $n \geq m$. Then for every $n \geq m$ we obtain

$$\begin{aligned} |x_n| &\leq \sum_{k=1}^n |\lambda_n(\varphi_k)| \cdot |y_k| \leq \sum_{k=1}^v t \cdot |\lambda_n(\varphi_k)| + \sum_{v < k < n} t \cdot |\lambda_n(\varphi_k)| + |\lambda_n(\varphi_n)| \cdot |y_n| \\ &< \delta + \sum_{v < k < n} t \epsilon / 3^k + ||\lambda_n|| \cdot |y_n| < \delta + t \epsilon / 3^v + \delta < 3\delta. \end{aligned}$$

Thus $(x_n) \in c_0$. Clearly, $\Theta((x_n)_{n \in \mathbb{N}}) = (y_n)_{n \in \mathbb{N}}$; so Θ is surjective.

(VI) The operator

$$T : c_0 \rightarrow C_p(X), \quad T : (x_n) \mapsto \sum_{n=1}^{\infty} x_n \cdot \varphi_n|X,$$

is well-defined, linear and continuous, since the functions φ_n , $n \in \mathbb{N}$, have pairwise disjoint supports and $\varphi_n(X) \subset [-1, 1]$, $n \in \mathbb{N}$. Thus the linear operator

$$\Phi = T \circ \Theta : c_0 \rightarrow C_p(X)$$

is continuous.

Let $x = (x_k) \in c_0$ and $x' = (x'_k) = \Theta(x)$. Then

$$\Phi(x) = T(x') = \sum_{k=1}^{\infty} x'_k \varphi_k|X.$$

Using (B3) and the definition of Θ , we get for every $n \in \mathbb{N}$

$$\begin{aligned} \lambda_n(\Phi(x)) &= \sum_{k=1}^{\infty} x'_k \lambda_n(\varphi_k) = x'_n \lambda_n(\varphi_n) + \sum_{1 \leq k < n} x'_k \lambda_n(\varphi_k) \\ &= \left(x_n - \sum_{1 \leq k < n} x'_k \lambda_n(\varphi_k) \right) + \sum_{1 \leq k < n} x'_k \lambda_n(\varphi_k) = x_n; \end{aligned}$$

so $\lambda_n(\Phi(x)) \rightarrow_n 0$. This implies that $\Phi(x) \in L$ and $S \circ \Phi(x) = x$ for every $x \in c_0$. Therefore the operator $P := \Phi \circ S: L \rightarrow L$ is a continuous linear projection with $\ker P = \ker S = Z$. Thus the subspace Z is complemented in L . Since $S \circ \Phi$ is the identity map on c_0 , the map $S: L \rightarrow c_0$ is open. Indeed, let U be a neighbourhood of zero in L ; then $V = \Phi^{-1}(U)$ is a neighbourhood of zero in c_0 and

$$V = S \circ \Phi(V) \subset S(U).$$

Thus the quotient space L/Z is topologically isomorphic to c_0 and $\Phi(c_0)$ is a complemented subspace of L , isomorphic to c_0 . In particular, Z has infinite codimension in L and in $C_p(X)$.

(VII) Finally we prove that the quotient space $C_p(X)/Z$ is first countable and hence metrizable. Let

$$U_n = \{f \in C_p(X) : |f(x)| < 1/n \text{ for every } x \in \bigcup_{k=1}^n \text{supp}(\lambda_k)\}, n \in \mathbb{N}.$$

The first countability of the quotient space $C_p(X)/Z$ will follow as soon as for every neighbourhood U of zero in $C_p(X)$ we find $n \in \mathbb{N}$ with $Z + U_n \subset Z + U$. Clearly we can assume that

$$U = \bigcap_{x \in F} \{f \in C_p(X) : |f(x)| < \delta\}$$

for some finite subset F of X and some $\delta > 0$.

By the continuity of the operator $\Phi: c_0 \rightarrow C_p(X)$, there exists $n \in \mathbb{N}$ such that for any $y = (y_k) \in c_0$ with

$$\max_{1 \leq k \leq n} |y_k| \leq 1/n$$

we get $\Phi(y) \in \frac{1}{2}U$. Replacing n by a larger number, we can assume that $\frac{1}{n} < \frac{1}{2}\delta$ and

$$F \cap \bigcup_{k=1}^{\infty} \text{supp}(\lambda_k) \subset \bigcup_{k=1}^n \text{supp}(\lambda_k).$$

Let $f \in U_n$. Choose a function $h \in C_p(K)$ such that $h(x) = f(x)$ for every

$$x \in F \setminus \bigcup_{k=1}^{\infty} \text{supp}(\lambda_k)$$

and $h(x) = 0$ for every $x \in \bigcup_{k=1}^n \text{supp}(\lambda_k)$. Put $g = h|_X$. Then $g \in L$, since $\lambda_k(g) = \lambda_k(h) \rightarrow_k 0$. Put $y = S(g)$ and $\xi = \Phi(y)$. Since $g(x) = 0$ for

$$x \in \bigcup_{k=1}^n \text{supp} \lambda_k,$$

we have $|\lambda_k(g)| = 0 < \frac{1}{n}$ for $1 \leq k \leq n$, so $\max_{1 \leq k \leq n} |y_k| < \frac{1}{n}$. Hence $\xi = \Phi(y) \in \frac{1}{2}U$, so $\max_{x \in F} |\xi(x)| < \frac{1}{2}\delta$. For $\varsigma = g - \xi$ we obtain

$$S(\varsigma) = S(g) - S \circ \Phi \circ S(g) = S(g) - S(g) = 0,$$

so $\varsigma \in Z$. Moreover $f - \varsigma \in U$. Indeed, we have

$$|f(x) - \varsigma(x)| = |f(x) - g(x) + \xi(x)| = |\xi(x)| < \delta$$

for $x \in F \setminus \bigcup_{k=1}^\infty \text{supp}(\lambda_k)$ and

$$|f(x) - \varsigma(x)| = |f(x) - g(x) + \xi(x)| \leq |f(x)| + |g(x)| + |\xi(x)| < \frac{1}{n} + 0 + \frac{1}{2}\delta < \delta$$

for every $x \in \bigcup_{k=1}^n \text{supp}(\lambda_k)$. Thus $f = \varsigma + (f - \varsigma) \in Z + U$, so $U_n \subset Z + U$. Hence $Z + U_n \subset Z + U$. $\square$

Lemma 2 *Let X be a Tychonoff space. Each metrizable continuous image of $C_p(X)$ is separable.*

Proof It is well-known [11, 2.3.18] that the Tychonoff product $\mathbb{R}^X$ has *countable cellularity*, which means that $\mathbb{R}^X$ contains no uncountable family of pairwise disjoint non-empty open sets. Then the dense subspace $C_p(X)$ of $\mathbb{R}^X$ also has countable cellularity and so does any continuous image Y of $C_p(X)$. If Y is metrizable, then Y is separable according to Theorem 4.1.15 in [11]. $\square$

Lemma 3 *Let X be a pseudocompact space. A closed linear subspace S of $C_p(X)$ is separable if and only if S is Polishable.*

Proof If S is Polishable, then S is separable, being a continuous image of a separable Fréchet locally convex space. Now assume that S is separable. Fix a countable dense subset $\{f_n\}_{n \in \mathbb{N}}$ in S and consider the continuous map

$$f : X \rightarrow \mathbb{R}^{\mathbb{N}}, \quad f : x \mapsto (f_n(x))_{n \in \mathbb{N}}.$$

By the pseudocompactness of X and the metrizability of $\mathbb{R}^{\mathbb{N}}$, the image $M := f(X)$ is a compact metrizable space. The continuous surjective map $f : X \rightarrow M$ induces an isomorphic embedding

$$C_p f : C_p(M) \rightarrow C_p(X), \quad C_p f : \varphi \mapsto \varphi \circ f.$$

So, we can identify the space $C_p(M)$ with its image $C_p f(C_p(M))$ in $C_p(X)$. We claim that $C_p(M)$ is closed in $C_p(X)$. Given any function $\varphi \in C_p(X) \setminus C_p(M)$, we should find a neighborhood $O_\varphi \subset C_p(X)$ of φ , which is disjoint with $C_p(M)$.

We claim that there exist points $x, y \in X$ such that $f(x) = f(y)$ and $\varphi(x) \neq \varphi(y)$. In the opposite case, $\varphi = \psi \circ f$ for some bounded function $\psi : M \rightarrow \mathbb{R}$. Let us show that the function ψ is continuous. Consider the continuous map

$$h : X \rightarrow M \times \mathbb{R}, \quad h : x \mapsto (f(x), \varphi(x)).$$

The pseudocompactness of X implies that the image $h(X) \subset M \times \mathbb{R}$ is a compact closed subset of $M \times \mathbb{R}$. Let $\text{pr}_M : h(X) \rightarrow M$ and $\text{pr}_{\mathbb{R}} : h(X) \rightarrow \mathbb{R}$ be the coordinate projections. It follows that

$$\text{pr}_{\mathbb{R}} \circ h = \varphi = \psi \circ f = \psi \circ \text{pr}_M \circ h,$$

which implies that $\text{pr}_{\mathbb{R}} = \psi \circ \text{pr}_M$. The map $\text{pr}_M : h(X) \rightarrow M$ between the compact metrizable spaces $h(X)$ and M is closed and hence is quotient. Then the continuity of the map $\text{pr}_{\mathbb{R}} = \psi \circ \text{pr}_M$ implies the continuity of ψ . Now we see that the function $\varphi = \psi \circ f$ belongs to the subspace $C_p(M) \subset C_p(X)$, which contradicts the choice of φ . This contradiction shows that $\varphi(x) \neq \varphi(y)$ for some points $x, y \in X$ with $f(x) = f(y)$. Then

$$O_\varphi := \{\phi \in C_p(X) : \phi(x) \neq \phi(y)\}$$

is a required neighborhood of φ , disjoint with $C_p(M)$.

Therefore the subspace $C_p(M)$ of $C_p(X)$ is closed and hence $C_p(M)$ contains the closure S of the dense set $\{f_n\}_{n \in \mathbb{N}}$ in S . Since the space $C_p(M)$ is Polishable, so is its closed subspace S . $\square$

Now we are at the position to prove the main Theorem 1:

Proof of Theorem 1 First, for a Tychonoff space X we prove the equivalence of the conditions:

- (1) $C_p(X)$ satisfies the JNP;
- (2) $C_p(X)$ contains a complemented subspace isomorphic to c_0 ;
- (3) $C_p(X)$ has a quotient isomorphic to c_0 ;
- (4) $C_p(X)$ admits a continuous linear map onto c_0 .

The implication (1) $\Rightarrow$ (2) follows from Lemma 1, applied to the Stone-Ćech compactification $K = \beta X$ of X . The implications (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are trivial.

To prove the implication (4) $\Rightarrow$ (1) assume that there exists a continuous linear map T from $C_p(X)$ onto c_0 . Let

$$\{e_n^*\}_{n \in \mathbb{N}} \subset c_0^*$$

be the sequence of coordinate functionals. By definition of c_0 , $e_n^*(y) \rightarrow_n 0$ for every $y \in c_0$. For every $n \in \mathbb{N}$ consider the linear continuous functional

$$\lambda_n \in C_p(X)^*, \quad \lambda_n(f) = e_n^*(Tf),$$

which can be thought as a finitely supported sign-measure on X . It follows that $\lambda_n(f) = e_n^*(Tf) \rightarrow_n 0$ for every $f \in C_p(X)$.

We shall show that the union $S = \bigcup_{n=1}^{\infty} S_n$ of supports $S_n = \text{supp}(\lambda_n)$ of the sign-measures λ_n is bounded in X in the sense that for any $\varphi \in C_p(X)$ the image $\varphi(S)$ is bounded in $\mathbb{R}$, since in the opposite case we get a function $\psi \in C_p(X)$ with $\lambda_n(\psi) \not\rightarrow 0$. Indeed, suppose that for some $\varphi \in C_p(X)$ the image $\varphi(S)$ is unbounded in $\mathbb{R}$; without loss of generality we can assume that φ is non-negative.

We can find inductively an increasing sequence $\{n_k\}_{k \in \mathbb{N}} \subset \mathbb{N}$ such that

$$\max \varphi(S_{n_k}) > 3 + \max \varphi(S_{n_i})$$

for all $k, i \in \mathbb{N}$ with $k > i$. For every $k \in \mathbb{N}$ choose $x_k \in S_{n_k}$ with $\varphi(x_k) = \max \varphi(S_{n_k})$. Then $\varphi(x_k) - \varphi(x_i) > 3$ for all $k, i \in \mathbb{N}$ with $k > i$. Since the space X is Tychonoff, for every $k \in \mathbb{N}$ we can find an open neighborhood $U_k \subset \{x \in X : |\varphi(x) - \varphi(x_k)| < 1\}$ of x_k such that $U_k \cap \text{supp}(\lambda_{n_k}) = \{x_k\}$.

Observe that for any $k > i$ and any points $x \in U_k$ and $y \in U_i \cup \text{supp}(\lambda_{n_i})$ we get

$$\varphi(x) - \varphi(y) \geq \varphi(x_k) - 1 - (\max \varphi(S_{n_i}) + 1) > 3 - 2 = 1,$$

which implies $U_k \cap \text{supp}(\lambda_{n_i}) = \emptyset$ and that the family $(\varphi(U_k))_{k \in \mathbb{N}}$ is discrete in $\mathbb{R}$ and consequently $(U_k)_{k \in \mathbb{N}}$ is discrete in X .

Taking into account that $U_k \cap \text{supp}(\lambda_{n_k}) = \{x_k\}$, we can inductively construct a sequence $(\psi_k)_{k \in \mathbb{N}}$ of functions $\psi_k : X \rightarrow \mathbb{R}$ such that $\text{supp}(\psi_k) := \{x \in X : \psi_k(x) \neq 0\} \subset U_k$ and

$$\lambda_{n_k}(\psi_k) > 1 + \sum_{i < k} |\lambda_{n_k}(\psi_i)|$$

for all $k \in \mathbb{N}$.

Since the family $(U_k)_{k \in \mathbb{N}}$ is discrete, the function

$$\psi : X \rightarrow \mathbb{R}, \quad \psi : x \mapsto \sum_{k=1}^{\infty} \psi_k(x)$$

is well-defined and continuous. For every $k \in \mathbb{N}$ and $i > k$ we have $U_i \cap \text{supp}(\lambda_{n_k}) = \emptyset$ and hence

$$|\lambda_{n_k}(\psi)| = \left| \sum_{i \leq k} \lambda_{n_k}(\psi_i) \right| \geq |\lambda_{n_k}(\psi_k)| - \sum_{i < k} |\lambda_{n_k}(\psi_i)| > 1.$$

But this contradicts $\lambda_n(\psi) \rightarrow_n 0$. This contradiction shows that the set $S = \bigcup_{k \in \mathbb{N}} \text{supp}(\lambda_k)$ is bounded in X .

Next, we show that the set $B = \{(x_n) \in c_0 : \forall n \in \mathbb{N} \ |x_n| \leq \|\lambda_n\|\}$ is absorbing. Given any sequence $(x_n) \in c_0$, find $f \in C_p(X)$ with $Tf = (x_n)$. Put $\|f\|_S := \sup\{|f(x)| : x \in S\}$ and observe that $|x_n| = |e_n^*(Tf)| = |\lambda_n(f)| \leq \|\lambda_n\| \cdot \|f\|_S$, so $(x_n) \in \|f\|_S \cdot B$. Thus B is absorbing. Clearly, B is absolutely convex and closed in c_0 , so it is a barrel in the Banach space $(c_0, \|\cdot\|_{\infty})$. Thus B is a neighbourhood of zero in $(c_0, \|\cdot\|_{\infty})$ and consequently, $\inf_n \|\lambda_n\| > 0$. For every $n \in \mathbb{N}$ put

$$\mu_n = \frac{\lambda_n}{\|\lambda_n\|} \in C_p(X)^*$$

and observe that the sequence (μ_n) witnesses that the function space $C_p(X)$ has the JNP.

Now assuming that the space X is pseudocompact, we shall prove that the conditions (1)–(4) are equivalent to

- (5) $C_p(X)$ contains a complemented infinite-dimensional metrizable subspace;
- (6) $C_p(X)$ contains a complemented infinite-dimensional separable subspace;
- (7) $C_p(X)$ has an infinite-dimensional Polishable quotient.

It suffices to prove the implications $(2) \Rightarrow (5) \Rightarrow (6) \Rightarrow (7) \Rightarrow (1)$. The implication $(2) \Rightarrow (5)$ is trivial and $(5) \Rightarrow (6) \Rightarrow (7)$ follow from Lemmas 2 and 3, respectively.

$(7) \Rightarrow (1)$ Assume that the space $C_p(X)$ contains a closed subspace Z of infinite codimension such that the quotient space $E := C_p(X)/Z$ is Polishable. Denote by τ_p the quotient topology of $C_p(X)/Z$ and by $\tau_0 \supset \tau_p$ a stronger separable Fréchet locally convex topology on E . Denote by τ_{∞} the topology of the quotient Banach space $C(X)/Z$. Here $C(X)$ is endowed with the sup-norm $\|f\|_{\infty} := \sup_{x \in X} |f(x)|$ (which is well-defined as X is pseudocompact).

The identity maps between (E, τ_0) and (E, τ_{∞}) have closed graphs, since $\tau_p \subset \tau_0 \cap \tau_{\infty}$. Using the Closed Graph Theorem we infer that the topologies τ_0 and τ_{∞} are equal. Let G be

a countable subset of $C(X)$ such that the set $\{g + Z : g \in G\}$ is dense in the Banach space $C(X)/Z$. Then the set

$$G + Z = \{g + z : g \in G, z \in Z\}$$

is dense in $C(X)$. Let $(g_n)_{n \in \mathbb{N}}$ be a linearly independent sequence in G such that its linear span G_0 has $G_0 \cap Z = \{0\}$ and $G_0 + Z = G + Z$. Let $f_1 = g_1$ and $v_1 \in C_p(X)^*$ with $v_1|Z = 0$ such that $v_1(f_1) = 1$.

Assume that for some $n \in \mathbb{N}$ we have chosen

$$(f_1, v_1), \dots, (f_n, v_n) \in C_p(X) \times C_p(X)^*$$

such that

$$\text{lin}\{f_1, \dots, f_n\} = \text{lin}\{g_1, \dots, g_n\}$$

and

$$v_j|Z = 0, \quad v_j(f_i) = \delta_{j,i} \quad \text{for all } i, j \in \{1, \dots, n\}.$$

Put

$$f_{n+1} = g_{n+1} - \sum_{i=1}^n v_i(g_{n+1})f_i.$$

Then

$$\text{lin}\{f_1, \dots, f_{n+1}\} = \text{lin}\{g_1, \dots, g_{n+1}\}$$

and $v_j(f_{n+1}) = 0$ for $1 \leq j \leq n$. Let $v_{n+1} \in C_p(X)^*$ with $v_{n+1}|Z = 0$ such that $v_{n+1}(f_i) = 0$ for $1 \leq i \leq n$ and $v_{n+1}(f_{n+1}) = 1$.

Continuing on this way we can construct inductively a biorthogonal sequence $((f_n, v_n))_{n \in \mathbb{N}}$ in $C_p(X) \times C_p(X)^*$ such that $\text{lin}\{f_n : n \in \mathbb{N}\} = \text{lin}\{g_n : n \in \mathbb{N}\}$ and $v_n|Z = 0$, $v_n(f_m) = \delta_{n,m}$ for all $n, m \in \mathbb{N}$. Then $\text{lin}\{f_n : n \in \mathbb{N}\} + Z$ is dense in $(C(X), \|\cdot\|_\infty)$. Let $\mu_n = v_n/\|v_n\|$ for $n \in \mathbb{N}$. Then $\|\mu_n\| = 1$ and $\mu_n(f_m) = 0$ for all $n, m \in \mathbb{N}$ with $n \neq m$.

We prove that $\mu_n(f) \rightarrow_n 0$ for every $f \in C_p(X)$. Given any $f \in C(X)$ and $\varepsilon > 0$, find $m \in \mathbb{N}$ and $g \in \text{lin}\{f_1, \dots, f_m\} + Z$ with $d(f, g) < \varepsilon$; clearly $d(f, g) = \|f - g\|_\infty$. Then $\mu_n(g) = 0$ for $n > m$, so

$$|\mu_n(f)| = |\mu_n(f - g)| \leq \|\mu_n\| \cdot \|f - g\|_\infty < \varepsilon$$

for $n > m$. Thus $\mu_n(f) \rightarrow_n 0$, which means that the space $C_p(X)$ has the JNP. $\square$

4 An example of Plebanek

In this section we describe the following example suggested to the authors by Grzegorz Plebanek [26].

Example 1 (Plebanek) There exists a compact Hausdorff space K such that

- (1) K contain no nontrivial converging sequences but contains a copy of $\beta\mathbb{N}$;
- (2) the function space $C_p(K)$ has the JNP.

We need some facts to present the construction of the space K . By definition, the *asymptotic density* of a subset $A \subset \mathbb{N}$ is the limit

$$d(A) := \lim_{n \rightarrow \infty} \frac{|A \cap [1, n]|}{n}$$

if this limit exists. The family $\mathcal{Z} = \{A \subset \mathbb{N} : d(A) = 0\}$ of sets of asymptotic density zero in $\mathbb{N}$ is an ideal on $\mathbb{N}$. Recall the following standard fact (here $A \subset^* B$ means that $A \setminus B$ is finite).

Fact 1 *For any countable subfamily $\mathcal{C} \subset \mathcal{Z}$ there is a set $B \in \mathcal{Z}$ such that $C \subset^* B$ for all $C \in \mathcal{C}$.*

Let $\mathfrak{A} = \{A \subset \mathbb{N} : d(A) \in \{0, 1\}\}$ be the algebra of subsets of $\mathbb{N}$ generated by $\mathcal{Z}$. We now let K be the Stone space of the algebra $\mathfrak{A}$ so we treat elements of K as ultrafilters on $\mathfrak{A}$. There are three types of such $x \in K$:

- (1) $\{n\} \in x$ for some $n \in \mathbb{N}$; then $x = \{A \in \mathfrak{A} : n \in A\}$ is identified with n ;
- (2) x contains no finite subsets of $\mathbb{N}$ but $Z \in x$ for some $Z \in \mathcal{Z}$;
- (3) $Z \notin x$ for every $Z \in \mathcal{Z}$; this defines the unique

$$p = \{A \in \mathfrak{A} : d(A) = 1\} \in K.$$

To see that K is the required space it is enough to check the following two facts.

Fact 2 *The space K contains no nontrivial converging sequence.*

Proof In fact we check that every infinite $X \subset K$ contains an infinite set Y such that $\overline{Y}$ is homeomorphic to $\beta\mathbb{N}$. Note first that for every $Z \in \mathcal{Z}$, the corresponding clopen set

$$\widehat{Z} = \{x \in K : Z \in x\},$$

is homeomorphic to $\beta\mathbb{N}$ because $\{A \in \mathfrak{A} : A \subset Z\} = 2^Z$.

For an infinite set $X \subset K$, we have two cases:

Case 1, $X \cap \mathbb{N}$ is infinite. There is an infinite $Z \subset X \cap \mathbb{N}$ having density zero. Then every subset of Z is in $\mathfrak{A}$, which implies that $\overline{Z} \cong \beta\mathbb{N}$.

Case 2, $X \cap (K \setminus \mathbb{N})$ is infinite. Let us fix a sequence of different $x_n \in X \cap (K \setminus \mathbb{N})$ such that $x_n \neq p$ for every n . Then for every n we have $Z_n \in x_n$ for some $Z_n \in \mathcal{Z}$. Take $B \in \mathcal{Z}$ as in Fact 1. Then $B \in x_n$ because x_n is a nonprincipal ultrafilter on $\mathfrak{A}$ so $A_n \setminus B \notin x_n$. Again, we conclude that $\{x_n : n \in \mathbb{N}\}$ is $\beta\mathbb{N}$. $\square$

Fact 3 *If $v_n = \frac{1}{n} \sum_{k \leq n} \delta_k$ and $\mu_n = \frac{1}{2}(v_n - \delta_p)$ for $n \in \mathbb{N}$, then $v_n(f) \rightarrow_n \delta_p(f)$ and $\mu_n(f) \rightarrow_n 0$ for every $f \in C(K)$.*

Proof Observe $v_n(A) \rightarrow_n d(A)$ for every $A \in \mathfrak{A}$ since elements of $\mathfrak{A}$ have asymptotic density either 0 or 1. This means that, when we treat v_n as measures on K then $v_n(V)$ converges to $\delta_p(V)$ for every clopen set $V \subset K$. This implies the assertion since every continuous function on K can be uniformly approximated by simple functions built from clopens. $\square$

5 Proof of Theorem 2

Let us recall that a topological space X is called

- *submetrizable* if X admits a continuous metric;

- *hemicompact* if X has a countable family $\mathcal{K}$ of compact sets such that each compact subset of X is contained in some compact set $K \in \mathcal{K}$;
- a *k-space* if a subset $F \subset X$ is closed if and only if for every compact subset $K \subset X$ the intersection $F \cap K$ is closed in K .

In order to prove Theorem 2, we should check the equivalence of the following conditions for every Tychonoff space X :

- (1) $C_k(X)$ is Polishable;
- (2) $C_p(X)$ is Polishable;
- (3) $C_k(X)$ is Polish;
- (4) X is a submetrizable hemicompact k -space.

The implication (1) $\Rightarrow$ (2) follows from the continuity of the identity map $C_k(X) \rightarrow C_p(X)$.

(2) $\Rightarrow$ (3) Assume that the space $C_p(X)$ is Polishable and fix a stronger Polish locally convex topology τ on $C_p(X)$. Let $C_\tau(X)$ denote the separable Fréchet space $(C_p(X), \tau)$. By τ_k denote the compact open topology of $C_k(X)$. Taking into account that the space $C_p(X)$ is a continuous image of the Polish space $C_\tau(X)$, we conclude that $C_p(X)$ has countable network and by [2, I.1.3], the space X has countable network and hence is Lindelöf. By the normality (and the Lindelöf property) of X , each closed bounded set in X is countably compact (and hence compact). So X is a μ -space. By Theorem 10.1.20 in [25, Theorem 10.1.20] the function space $C_k(X)$ is barrelled. The continuity of the identity maps $C_k(X) \rightarrow C_p(X)$ and $C_\tau(X) \rightarrow C_p(X)$ implies that the identity map $C_k(X) \rightarrow C_\tau(X)$ has closed graph. Since $C_k(X)$ is barrelled and $C_\tau(X)$ is Fréchet, we can apply the Closed Graph Theorem 4.1.10 in [25] and conclude that the identity map $C_k(X) \rightarrow C_\tau(X)$ is continuous.

Next, we show that the identity map $C_\tau(X) \rightarrow C_k(X)$ is continuous. Given any compact set $K \subset X$ and any $\varepsilon > 0$ we have to find a neighborhood $U \subset C_\tau(X)$ of zero such that

$$U \subset \{f \in C(X) : f(K) \subset (-\varepsilon, \varepsilon)\}.$$

The continuity of the restriction operator $R : C_p(X) \rightarrow C_p(K)$, $R : f \mapsto f|_K$, and the continuity of the identity map $C_\tau(X) \rightarrow C_p(X)$ imply that the restriction operator $R : C_\tau(X) \rightarrow C_p(K)$ is continuous and hence has closed graph. The continuity of the identity map $C_k(K) \rightarrow C_p(K)$ implies that R seen as an operator $R : C_\tau(X) \rightarrow C_k(K)$ still has closed graph. Since the spaces $C_\tau(X)$ and $C_k(K)$ are Fréchet, the Closed Graph Theorem 1.2.19 in [25] implies that the restriction operator $R : C_\tau(X) \rightarrow C_k(K)$ is continuous. So, there exists a neighborhood $U \subset C_\tau(X)$ of zero such that

$$R(U) \subset \{f \in C_k(K) : f(K) \subset (-\varepsilon, \varepsilon)\}.$$

Then $U \subset \{f \in C(X) : f(K) \subset (-\varepsilon, \varepsilon)\}$ and we are done. Hence $\tau = \tau_k$ is a Polish locally convex topology as claimed.

The implication (3) $\Rightarrow$ (1) is trivial.

(3) $\Rightarrow$ (4) If the function space $C_k(X)$ is Polish, then by Theorem 4.2 in [24], X is a hemicompact k -space. Taking into account that the space $C_p(X)$ is a continuous image of the space $C_k(X)$, we conclude that $C_p(X)$ has countable network and by [2, I.1.3], the space X has countable network. By [17, 2.9], the space X is submetrizable.

(4) $\Rightarrow$ (3) If X is a submetrizable hemicompact k -space, then $X = \bigcup_{n \in \omega} X_n$ for some increasing sequence $(X_n)_{n \in \omega}$ of compact metrizable spaces such that each compact subset of X is contained in some compact set X_n . Then the function space $C_k(X)$ is Polish, being topologically isomorphic to the closed subspace

$$\left\{ (f_n)_{n \in \omega} \in \prod_{n \in \omega} C_k(X_n) : \forall n \in \omega \quad f_{n+1} \upharpoonright X_n = f_n \right\}$$

of the countable product $\prod_{n=1}^{\infty} C_k(X_n)$ of separable Banach spaces. $\square$

The authors thank to the referee for his/her valuable comments and remarks.

Open Access This article is distributed under the terms of the Creative Commons Attribution 4.0 International License (<http://creativecommons.org/licenses/by/4.0/>), which permits unrestricted use, distribution, and reproduction in any medium, provided you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons license, and indicate if changes were made.

References

- Albiac, F., Kalton, N.J.: Topics in Banach Space Theory. Springer, Berlin (2006)
- Arhangel'skii, A.V.: Topological Function Spaces, Mathematics Application, vol. 78. Kluwer Academic Publishers, Dordrecht (1992)
- Banakh, T., Kąkol, J., Śliwa, W.: Metrizable quotients of C_p -spaces. *Topol. Appl.* **249**, 95–102 (2018)
- Cembranos, P., Mendoza, J.: Banach Spaces of Vector-Valued Functions, Lecture Notes in Mathematics, vol. 1676. Springer, Berlin (1997)
- Dales, H.D., Dashiell Jr., F.H., Lau, A.T.M., Strauss, D.: Banach Spaces of Continuous Functions as Dual Spaces. Springer, Berlin (2016)
- Diestel, J.: Sequences and Series in Banach spaces. Springer, Berlin (1984)
- Dow, A.: Efimov spaces and the splitting number. *Topol. Proc.* **29**, 105–113 (2005)
- Dow, A.: Compact sets without converging sequences in the random real model. *Acta Math. Univ. Comenianae* **76**, 161–171 (2007)
- Dow, A., Shelah, S.: An Efimov space from Martin's Axiom. *Houston J. Math.* **39**, 1423–1435 (2013)
- Efimov, B.: Subspaces of dyadic bicomacta. *Doklady Akademii Nauk USSR* **185**, 987–990 (1969). ((Russian); English transl., *Soviet Mathematics. Doklady* **10**, 453–456 (1969))
- Engelking, R.: General Topology. Heldermann, Berlin (1989)
- Fedorchuk, V.V.: A bicomactum whose infinite closed subsets are all n -dimensional. *Math. USSR Sbornik* **25**, 37–57 (1976)
- Fedorchuk, V.V.: Completely closed mappings, and the consistency of certain general topology theorems with the axioms of set theory. *Math. USSR Sbornik* **28**, 3–33 (1976)
- Ferrando, J.C., Kąkol, J., Lopez-Pellicer, M., Śliwa, W.: On the separable quotient problem for Banach spaces. *Funct. Approx.* **59**, 153–173 (2018)
- Ferrando, J.C., Kąkol, J., Saxon, S.A.: The dual of of the locally convex space $C_p(X)$. *Funct. Approx.* **50**, 1–11 (2014)
- Geschke, S.: The coinitalities of Efimov spaces, set theory and its applications, Babinkostova et al., Editors. *Contemp. Math.* **533**, 259–265 (2011)
- Gruenhage, G.: Generalized metric spaces. *Handbook of Set-Theoretic Topology*, pp. 423–501. North-Holland, Amsterdam (1984)
- Hart, P.: Efimov's problem. In: Pearl, E. (ed.) *Problems in Topology II*, pp. 171–177. Elsevier, Oxford (2007)
- Johnson, W.B., Rosenthal, H.P.: On weak*-basic sequences and their applications to the study of Banach spaces. *Stud. Math.* **43**, 166–168 (1975)
- Kechris, A.S.: Classical descriptive set theory. *Grad. Texts Math.* **156** (1995)
- Kąkol, J., Saxon, S.A.: Separable quotients in $C_c(X)$, $C_p(X)$ and their duals. *Proc. Am. Math. Soc.* **145**, 3829–3841 (2017)
- Kąkol, J., Śliwa, W.: Efimov spaces and the separable quotient problem for spaces $C_p(K)$. *J. Math. Anal. App.* **457**, 104–113 (2018)
- Lacey, E.: Separable quotients of Banach spaces. *An. Acad. Brasil. Ciênc.* **44**, 185–189 (1972)
- McCoy, R.A., Ntantu, I.: Completeness properties of function spaces. *Topol. Appl.* **22**(2), 191–206 (1986)
- Pérez Carreras, P., Bonnet, J.: *Barrelled Locally Convex Spaces*, North-Holland Mathematics Studies, vol. 131. North-Holland, Amsterdam (1987)
- Plebanek, G.: On Some Extremely Disconnected Compact Space. *Private Commun.* (2018)

27. Rosenthal, H.P.: On quasi-complemented subspaces of Banach spaces, with an appendix on compactness of operators from $L_p(\mu)$ to $L_r(v)$. J. Funct. Anal. **4**, 176–214 (1969)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.