On integral bounded variation

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Abstract In this paper we will investigate the concept of the q -integral p -variation introduced in 1970's by Terehin. This kind of integral variation has been mainly used, until now, to describe the regularity of functions in L^p in terms of the L^q -norm (for $q > p$). In the present paper we show that the class of functions of bounded q -integral p -variation appears a very nice tool to work with integral operators, including these corresponding to Riemann–Liouville fractional integration. We also give the acting conditions for autonomous superposition operators and prove some existence theorems for Hammerstein integral equation in this space.

Keywords Acting conditions · Bounded variation · Fractional integral · Integral variation · Nonlinear Hammerstein integral equation · Riemann–Liouville fractional integration · Superposition operator

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1 Introduction

The concept of a variation of a function dates back to the last decades of the 19th century and to the works of Jordan. Later the concept was generalized, in different directions, by Wiener (p -variation), Young (ϕ -variation), F. Riesz (Riesz variation) and Waterman (Λ -variation), to name a few (for an overview see e.g. [1]). All of the above-mentioned concepts of variation are defined pointwise and may not be directly applied in L^q -spaces. Still, because of the role that L^q -spaces play in mathematics and their numerous applications, the question how to measure the variation of a function, so it is not sensitive to a change of values in a zero-measure set, appears to be of vital importance.

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Several interesting attempts have been made. In [19] the Authors suggest the so-called *variation in the mean* that might be applied to L^1 -functions over a bounded interval. This type of a variation is constant within the equivalence classes belonging to L^1 but, as it was shown in [22], it actually represents the space of functions of bounded Jordan variation. Quite recently the concept of *lower Λ -variation* was introduced in [8] generalizing the Waterman Λ -variation to functions integrable in the Lebesgue sense. In the mentioned paper the Authors investigate the properties of the space of functions of lower Λ -variation and its elements. One of the important observations is that these spaces are actually subsets of the space L^∞ of essentially bounded functions.

In this paper we are going to deal with the idea of the integral variation that was introduced a long time before the concepts mentioned above—i.e. with the q -integral p -variation introduced by Terehin in 1972 (see [24]). This idea is different from these described above mainly because it does not imply that the functions having bounded q -integral p -variation have to be essentially bounded (cf. Example 3). In a series of papers (see [25–27]) Terehin investigated the properties of functions of bounded q -integral p -variation, including the multidimensional case. Such concept of the variation turned out to be useful in the discussion about the smoothness of functions in L^p spaces or in Sobolev spaces, described in terms of their L^q -properties (for $q > p$).

At the beginning of 1980's Borucka-Cieřlewicz introduced a more general concept of the q -integral M -variation (see [2, 3]). She generalized Terehin's concept towards the variation in the sense of Young. In these two papers this new concept was studied and a few applications were suggested (including some estimates for integral operators with the Dirichlet kernel).

We should also note that quite recently another application of the q -integral p -variation has been suggested. As one of the latest, we should mention here the papers by Donchyk concerning the problems of impulse control for the string equation—solved both in the classical space of functions of bounded p -variation and q -integral p -variation (see [12] and [13]).

Quite recently the paper [4] by Brudnyi appeared, where Author investigated the properties of functions of bounded q -integral p -variation (and its generalizations) from the point of view of the approximation theory. Author proves there that functions of bounded q -integral p -variation, for certain values of p and q , may be approximated by piecewise polynomial functions in L^q norm, with a given rate.

In the present paper we are going to investigate properties of spaces of functions of bounded q -integral p -variation and of linear integral operators defined in L^q -spaces with values being functions of bounded integral variation. We will also study the autonomous superposition operators defined in the space of functions of bounded q -integral 1-variation which will lead us to the existence theorems for certain nonlinear Hammerstein equations in such spaces. The interesting observation will follow, in the last section of the paper, that maps corresponding to the fractional Riemann–Liouville integration may be considered (under certain assumptions) as acting into the space of functions of bounded q -integral 1-variation.

2 Preliminaries

Notation. By $\mathbb{N}$ we denote the set of all positive integers. Moreover, by I we will denote the unit interval $[0, 1]$.

As usual, for $q \in [1, +\infty)$, by $L^q(J)$ we will denote the Banach space of all the equivalence classes of real-valued functions defined on a bounded interval $J \subseteq \mathbb{R}$ which are

Lebesgue integrable with q th power, endowed with the norm $\|x\|_{L^q} := (\int_I |x(t)|^q dt)^{1/q}$. Further on, we will write just L^q instead of $L^q(I)$. From now on we will always assume that $q' \in (1, +\infty)$ is a conjugate of $q \in (1, +\infty)$, that is $1/q' + 1/q = 1$. In the space of bounded functions $B(I)$ and its subspaces (like the space of continuous functions $C(I)$) we consider the norm $\|x\|_\infty = \sup_{t \in I} |x(t)|$ for $x \in B(I)$.

In the next subsection we collect basic definitions and facts which will be needed in the sequel.

2.1 Basic definitions and properties

Let us first remind the classical definition of the L^q modulus of continuity (see e.g. [28]).

Definition 1 Let $x: I \rightarrow \mathbb{R}$ be a Lebesgue measurable function and let $[a, b] \subset I$ be a fixed interval, $a < b$. The value

$$\omega_q(x; a, b) = \sup_{0 < h < b-a} \left(\int_a^{b-h} |x(t+h) - x(t)|^q dt \right)^{1/q} = \sup_{0 < h < b-a} \|x(\cdot+h) - x(\cdot)\|_{L^q(a, b-h)}$$

is called the L^q -modulus of continuity of the function x on the interval $[a, b]$.

Remark 1 A similar definition can be given in the L^∞ -case (see [24]), but we will not refer to this in the present paper.

The following example illustrates the above notion.

Example 1 Let $x: [a, b] \rightarrow \mathbb{R}$ be a step function given by

$$x(t) = \begin{cases} u, & t \in [a, c), \\ w, & t \in [c, b], \end{cases} \quad (1)$$

where u, w are distinct real numbers and $c \in (a, b)$.

We are going to show that

$$\omega_q(x; a, b) = |u - w|(\min(c - a, b - c))^{1/q}.$$

Assume first that $c - a \leq b - c$. Let us consider three cases.

(1) Let $b - a > h \geq b - c$. Then $x(t+h) - x(t) = w - u$ for $t \in [a, b-h]$. Hence

$$\left(\int_a^{b-h} |x(t+h) - x(t)|^q dt \right)^{1/q} = |u - w|(b - a - h)^{1/q}$$

and

$$\sup_{b-a > h \geq b-c} \left(\int_a^{b-h} |x(t+h) - x(t)|^q dt \right)^{1/q} = |u - w|(c - a)^{1/q}.$$

(2) Let $b - c > h \geq c - a$. Then $x(t+h) - x(t) = w - u$ for $t \in [a, c]$ and $x(t+h) - x(t) = 0$ for $t \in [c, b-h]$. Hence

$$\left(\int_a^{b-h} |x(t+h) - x(t)|^q dt \right)^{1/q} = |u - w|(c - a)^{1/q}$$

and

$$\sup_{b-c > h \geq c-a} \left(\int_a^{b-h} |x(t+h) - x(t)|^q dt \right)^{1/q} = |u - w|(c - a)^{1/q}.$$

- (3) Let $c - a > h > 0$. Then $x(t + h) - x(t) = 0$ for $t \in [a, c - h) \cup [c, b - h]$ and $x(t + h) - x(t) = w - u$ for $t \in [c - h, c)$. Hence

$$\left(\int_a^{b-h} |x(t + h) - x(t)|^q dt \right)^{1/q} = |u - w| h^{1/q}$$

and

$$\sup_{0 < h < c-a} \left(\int_a^{b-h} |x(t + h) - x(t)|^q dt \right)^{1/q} = |u - w| (c - a)^{1/q}.$$

We may arrive to similar conclusions when $c - a > b - c$.

Thus

$$\omega_q(x; a, b) = \sup_{0 < h < b-a} \left(\int_a^{b-h} |x(t + h) - x(t)|^q dt \right)^{1/q} = |u - w| (\min(c - a, b - c))^{1/q}.$$

Let us now define the integral variation of a measurable function.

Definition 2 Let $p, q \in [1, +\infty)$ and let $x: [a, b] \rightarrow \mathbb{R}$ be a Lebesgue measurable function, where $[a, b] \subset I$. The value

$$\text{ivar}_p^q(x; a, b) = \sup \left(\sum_{i=1}^N (\omega_q(x; t_{i-1}, t_i))^p \right)^{1/p},$$

where the supremum is taken over all finite partitions $a = t_0 < t_1 < \dots < t_N = b$ of the interval $[a, b]$, is called the q -integral p -variation of the function x . If $\text{ivar}_p^q(x; a, b) < +\infty$, then we say that x is a function of bounded q -integral p -variation. The set of all such functions is denoted by $IBV_p^q([a, b])$. Later, instead of $IBV_p^q(I)$ we will simply write IBV_p^q .

The concept of the q -integral p -variation was introduced by Terehin (see [24]). Following Terehin let us note that the case $q = +\infty$ recreates the well-known Wiener p -variation.

Now let us pass on to the concept of the q -integral M -variation introduced by Borucka-Cieřlewiecz (see [2,3]). For a Lebesgue measurable function $x: [0, 1] \rightarrow \mathbb{R}$ we can define

$$\text{ivar}_M^q(x; a, b) = \sup \sum_{i=1}^N M(\omega_q(x; t_{i-1}, t_i)),$$

where $M: [0, +\infty) \rightarrow \mathbb{R}$ is a continuous, non-decreasing function with $M(0) = 0$ and $M(u) > 0$ for $u > 0$, and the supremum is taken over all finite partitions $a = t_0 < t_1 < \dots < t_N = b$ of the interval $[a, b]$.

In the following proposition we have collected a few known facts concerning the q -integral p -variation.

Proposition 1 *The following properties hold:*

- (i) (cf. [24, p. 278]) $IBV_p^q \subset L^q$;
- (ii) (cf. [24, p. 278]) if $1 \leq q \leq p < +\infty$, then $L^q \subset IBV_p^q$;
- (iii) (cf. [24, Theorem 1]) if x is absolutely continuous and $x' \in L^p$ for $p > 1$, then $\text{ivar}_p^q(x) \leq \|x'\|_{L^p}$ (this is a direct consequence of Theorem 1 of [24] with $\delta = 1 - 0 = 1$ and $r = 1$);
- (iv) (cf. [2, Theorem 6]) the set IBV_p^q is a linear space;

(v) (cf. [3, Theorem 4]) the space IBV_p^q is a Banach space when endowed with a norm $\|x\|_{IBV_p^q} = \|x\|_{L^q} + \text{ivar}_p^q(x)$.

Actually the statement (iii) above shows that the Sobolev space $W^{1,p}(I) \subset IBV_p^q$ for any $q \in [1, +\infty)$. However, the results presented in [24] lead to much more sophisticated conclusions: in [24, Theorem 2] the Author gives the precise description of $W^{1,p}(I)$ in terms of the integral variation.

2.2 Basic examples

Now we are going to present a few examples that seem to be important to understand the nature of the space IBV_p^q .

Example 2 Let $x \in L^q(a, b)$ be the step function given by (1). Then $x \in IBV_p^q(x; a, b)$ and

$$\text{ivar}_p^q(x; a, b) = |u - w|(\min(c - a, b - c))^{1/q}. \quad (2)$$

To prove this let us take any partition $a = t_0 < t_1 < \dots < t_{N-1} < t_N = b$ of the interval $[a, b]$. Then $\omega_q(x; t_{i-1}, t_i) = 0$ for all such $i \in \{1, \dots, N\}$ that $c \notin (t_{i-1}, t_i)$. Hence, if $c = t_i$ for some $i = 1, \dots, N - 1$, then

$$\left(\sum_{i=1}^N (\omega_q(x; t_{i-1}, t_i))^p \right)^{1/p} = 0.$$

So we may focus on the situation when $t_{j-1} < c < t_j$ for some $j = 1, \dots, N$. Then

$$\begin{aligned} \left(\sum_{i=1}^N (\omega_q(x; t_{i-1}, t_i))^p \right)^{1/p} &= \omega_q(x; t_{j-1}, t_j) \\ &= |w - u|(\min(c - t_{j-1}, t_j - c))^{1/q} \leq |u - w|(\min(c - a, b - c))^{1/q}. \end{aligned}$$

Thus, $\text{ivar}_p^q(x; a, b) \leq |u - w|(\min(c - a, b - c))^{1/q}$. Moreover, as we can see, the inequality turns into equality when we take the partition $a = t_0 < t_1 = b$.

Another class of functions of bounded q -integral p -variation is indicated by the following

Proposition 2 Let $1 \leq p < q$. Assume that $x : [a, b] \rightarrow \mathbb{R}$ is Hölder continuous with exponent $\alpha \in [1/p - 1/q, 1]$ and $[a, b] \subset I$. Then $x \in IBV_p^q([a, b])$.

Proof Since x is Hölder continuous, there exists such $M > 0$ that $|x(t+h) - x(t)| \leq M|h|^\alpha$, if $t, t+h \in [a, b]$. Let us take any partition $a = t_0 < t_1 < \dots < t_{N-1} < t_N = b$ of the interval $[a, b]$. Then

$$\begin{aligned} \omega_q(x; t_{i-1}, t_i) &= \sup_{0 < h < t_i - t_{i-1}} \left(\int_{t_{i-1}}^{t_i - h} |x(t+h) - x(t)|^q dt \right)^{1/q} \\ &\leq M \sup_{0 < h < t_i - t_{i-1}} \left(\int_{t_{i-1}}^{t_i - h} |h|^{\alpha q} dt \right)^{1/q} \\ &\leq M \sup_{0 < h < t_i - t_{i-1}} h^\alpha \cdot (t_i - t_{i-1} - h)^{1/q} \\ &= M \left(\frac{\alpha}{\alpha + 1/q} \right)^\alpha \left(\frac{1/q}{\alpha + 1/q} \right)^{1/q} (t_i - t_{i-1})^{\alpha + 1/q}, \end{aligned}$$

since it is easy to check that the map $h \mapsto h^\alpha \cdot (t_i - t_{i-1} - h)^{1/q}$ attains its maximum in $[0, t_i - t_{i-1}]$ at $h_{\max} = \frac{\alpha}{\alpha+1/q} (t_i - t_{i-1})$. This shows that

$$\begin{aligned} \sum_{i=1}^N (\omega_q(x; t_{i-1}, t_i))^p &\leq M^p \left(\frac{\alpha}{\alpha + 1/q} \right)^{p\alpha} \left(\frac{1/q}{\alpha + 1/q} \right)^{p/q} \sum_{i=1}^N (t_i - t_{i-1}) \\ &\leq M^p \left(\frac{\alpha}{\alpha + 1/q} \right)^{p\alpha} \left(\frac{1/q}{\alpha + 1/q} \right)^{p/q} (b - a), \end{aligned}$$

because $|t_i - t_{i-1}| \leq 1$ and $|t_i - t_{i-1}|^{\alpha p + p/q} \leq |t_i - t_{i-1}|$ for $\alpha p + p/q \geq 1$. This proves that $\text{ivar}_p^q(x; a, b) \leq M \left(\frac{\alpha}{\alpha+1/q} \right)^\alpha \left(\frac{1/q}{\alpha+1/q} \right)^{1/q} (b-a)^{1/p}$. $\square$

Remark 2 If $x \in IBV_p^q([a, b])$ and $c \in (a, b)$, then

$$(\text{ivar}_p^q(x; a, c))^p + (\text{ivar}_p^q(x; c, b))^p \leq (\text{ivar}_p^q(x; a, b))^p.$$

Moreover, let us note that this inequality may be strict. Indeed, assume that $x \in IBV_p^q([a, b])$ is defined as in (1). Then we can see that $\text{ivar}_p^q(x; a, c) = \text{ivar}_p^q(x; c, b) = 0$, while $\text{ivar}_p^q(x; a, b) > 0$.

We may also prove the reversed inequality—adding to it one more term.

Proposition 3 *Let $1 \leq p \leq q$. If $x \in IBV_p^q([a, c])$ and $x \in IBV_p^q([c, b])$, then $x \in IBV_p^q([a, b])$ and*

$$\begin{aligned} (\text{ivar}_p^q(x; a, b))^p &\leq (\text{ivar}_p^q(x; a, c))^p + (\text{ivar}_p^q(x; c, b))^p \\ &\quad + \left(2 \sup_{s \in [a, b-d]} \left(\int_s^{s+d} |x(\tau)|^q d\tau \right)^{1/q} \right)^p, \end{aligned}$$

where $d = \min(c - a, b - c)$.

Proof Let us take any partition $a = t_0 < t_1 < \dots < t_{N-1} < t_N = b$ of the interval $[a, b]$. Of course, when $c = t_j$ for some $j = 1, \dots, N-1$, then we can see that

$$\begin{aligned} \sum_{i=1}^N (\omega_q(x; t_{i-1}, t_i))^p &= \sum_{i=1}^j (\omega_q(x; t_{i-1}, t_i))^p + \sum_{i=j+1}^N (\omega_q(x; t_{i-1}, t_i))^p \\ &\leq (\text{ivar}_p^q(x; a, c))^p + (\text{ivar}_p^q(x; c, b))^p. \end{aligned}$$

Let us now assume that $t_{j-1} < c < t_j$ for some $j \in \{1, \dots, N\}$. Let us look at the sum

$$\sum_{i=1}^N (\omega_q(x; t_{i-1}, t_i))^p$$

and one of its terms

$$\begin{aligned} (\omega_q(x; t_{j-1}, t_j))^p &= \left(\sup_{0 < h < t_j - t_{j-1}} \left(\int_{t_{j-1}}^{t_j - h} |x(t+h) - x(t)|^q dt \right)^{1/q} \right)^p \\ &= \sup_{0 < h < t_j - t_{j-1}} \left(\int_{t_{j-1}}^{t_j - h} |x(t+h) - x(t)|^q dt \right)^{p/q}. \end{aligned}$$

Now we will estimate the quantity $\left(\int_{t_{j-1}}^{t_j-h} |x(t+h) - x(t)|^q dt\right)^{p/q}$ for different values of h . We will use the following classical inequality

$$\left(\sum_{i=1}^m a_i\right)^{p/q} \leq \sum_{i=1}^m a_i^{p/q},$$

which holds for an arbitrary finite collection of non-negative real numbers $a_1, \dots, a_m$ (see [15]*Inequality 19). Let us consider four cases.

(1) Let $0 < h \leq \min(c - t_{j-1}, t_j - c)$. Then

$$\begin{aligned} \left(\int_{t_{j-1}}^{t_j-h} |x(t+h) - x(t)|^q dt\right)^{p/q} &= \left(\int_{t_{j-1}}^{c-h} |x(t+h) - x(t)|^q dt \right. \\ &\quad \left. + \int_{c-h}^c |x(t+h) - x(t)|^q dt + \int_c^{t_j-h} |x(t+h) - x(t)|^q dt\right)^{p/q} \\ &\leq \left(\int_{t_{j-1}}^{c-h} |x(t+h) - x(t)|^q dt\right)^{p/q} + \left(\int_{c-h}^c |x(t+h) - x(t)|^q dt\right)^{p/q} \\ &\quad + \left(\int_c^{t_j-h} |x(t+h) - x(t)|^q dt\right)^{p/q} \\ &\leq (\omega_q(x; t_{j-1}, c))^p + \left(\int_{c-h}^c |x(t+h) - x(t)|^q dt\right)^{p/q} + (\omega_q(x; c, t_j))^p. \end{aligned}$$

Let us also observe that the length of the interval in the middle term integral is $h \leq \min(c - t_{j-1}, t_j - c) \leq \min(c - a, b - c)$.

(2) When $c - t_{j-1} \leq h \leq t_j - c$, we have the following estimates

$$\begin{aligned} \left(\int_{t_{j-1}}^{t_j-h} |x(t+h) - x(t)|^q dt\right)^{p/q} &= \left(\int_{t_{j-1}}^c |x(t+h) - x(t)|^q dt \right. \\ &\quad \left. + \int_c^{t_j-h} |x(t+h) - x(t)|^q dt\right)^{p/q} \leq \left(\int_{t_{j-1}}^c |x(t+h) - x(t)|^q dt\right)^{p/q} \\ &\quad + \left(\int_c^{t_j-h} |x(t+h) - x(t)|^q dt\right)^{p/q} \leq \left(\int_{t_{j-1}}^c |x(t+h) - x(t)|^q dt\right)^{p/q} \\ &\quad + (\omega_q(x; c, t_j))^p. \end{aligned}$$

Let us observe that the length of the interval in the first integral is $c - t_{j-1} = \min(c - t_{j-1}, t_j - c) \leq \min(c - a, b - c)$.

(3) If $t_j - c \leq h \leq c - t_{j-1}$, then

$$\begin{aligned} \left(\int_{t_{j-1}}^{t_j-h} |x(t+h) - x(t)|^q dt\right)^{p/q} &= \left(\int_{t_{j-1}}^{c-h} |x(t+h) - x(t)|^q dt \right. \\ &\quad \left. + \int_{c-h}^{t_j-h} |x(t+h) - x(t)|^q dt\right)^{p/q} \leq \left(\int_{t_{j-1}}^{c-h} |x(t+h) - x(t)|^q dt\right)^{p/q} \\ &\quad + \left(\int_{c-h}^{t_j-h} |x(t+h) - x(t)|^q dt\right)^{p/q} \end{aligned}$$

$$\begin{aligned}
& + \left(\int_{c-h}^{t_j-h} |x(t+h) - x(t)|^q dt \right)^{p/q} \leq (\omega_q(x; t_{j-1}, c))^p \\
& + \left(\int_{c-h}^{t_j-h} |x(t+h) - x(t)|^q dt \right)^{p/q}.
\end{aligned}$$

The length of the interval in the last integral equals $t_j - c = \min(c - t_{j-1}, t_j - c) \leq \min(c - a, b - c)$.

- (4) Finally, let $\max(c - t_{j-1}, t_j - c) \leq h < t_j - t_{j-1}$. Here the length of the interval in the integral

$$\left(\int_{t_{j-1}}^{t_j-h} |x(t+h) - x(t)|^q dt \right)^{p/q}$$

equals to $t_j - t_{j-1} - h \leq \min(c - t_{j-1}, t_j - c) \leq \min(c - a, b - c)$.

We should also note that whatever $h \in (0, t_j - t_{j-1})$ is, the integral $\left(\int_{\alpha}^{\beta} |x(t+h) - x(t)|^q dt \right)^{p/q}$, for $[\alpha, \beta] \subset [t_{j-1}, t_j - h]$ may be estimated by

$$\begin{aligned}
\left(\int_{\alpha}^{\beta} |x(t+h) - x(t)|^q dt \right)^{p/q} & \leq \left(\left(\int_{\alpha+h}^{\beta+h} |x(t)|^q dt \right)^{1/q} + \left(\int_{\alpha}^{\beta} |x(t)|^q dt \right)^{1/q} \right)^p \\
& \leq \left(2 \sup_{s \in [t_{j-1}, t_j - (\beta - \alpha)]} \left(\int_s^{s+\beta-\alpha} |x(\tau)|^q d\tau \right)^{1/q} \right)^p, \quad (3)
\end{aligned}$$

where the last supremum should be seen as the maximum value of the integral of $|x|^q$ over the subintervals of $[t_{j-1}, t_j]$ of length $\beta - \alpha$. On the other hand, we may estimate

$$\begin{aligned}
\sup_{s \in [t_{j-1}, t_j - (\beta - \alpha)]} \left(\int_s^{s+\beta-\alpha} |x(\tau)|^q d\tau \right)^{1/q} & \leq \sup_{s \in [a, b - (\beta - \alpha)]} \left(\int_s^{s+\beta-\alpha} |x(\tau)|^q d\tau \right)^{1/q} \\
& \leq \sup_{s \in [a, b-d]} \left(\int_s^{s+d} |x(\tau)|^q d\tau \right)^{1/q},
\end{aligned}$$

since $\beta - \alpha \leq \min(c - a, b - c) = d$, and the last inequality is a consequence of the observation that for $\beta - \alpha \leq d$ the supremum of values of the integrals of the function $|x|^q$ over intervals of length $\beta - \alpha$ does not exceed the supremum of integrals of this function over the intervals of the length d . Therefore,

$$\begin{aligned}
(\omega_q(x; t_{j-1}, t_j))^p & = \sup_{0 < h < t_j - t_{j-1}} \left(\int_{t_{j-1}}^{t_j-h} |x(t+h) - x(t)|^q dt \right)^{p/q} \\
& \leq (\omega_q(x; t_{j-1}, c))^p + (\omega_q(x; c, t_j))^p + \left(2 \sup_{s \in [a, b-d]} \left(\int_s^{s+d} |x(\tau)|^q d\tau \right)^{1/q} \right)^p.
\end{aligned}$$

This shows that

$$\begin{aligned}
 \sum_{i=1}^N (\omega_q(x; t_{i-1}, t_i))^p &\leq \sum_{i=1}^{j-1} (\omega_q(x; t_{i-1}, t_i))^p + (\omega_q(x; t_{j-1}, c))^p \\
 &\quad + \sum_{i=j+1}^N (\omega_q(x; t_{i-1}, t_i))^p + (\omega_q(x; c, t_j))^p \\
 &\quad + \left(2 \sup_{s \in [a, b-d]} \left(\int_s^{s+d} |x(\tau)|^q d\tau \right)^{1/q} \right)^p \\
 &\leq (\text{ivar}_p^q(x; a, c))^p + (\text{ivar}_p^q(x; c, b))^p \\
 &\quad + \left(2 \sup_{s \in [a, b-d]} \left(\int_s^{s+d} |x(\tau)|^q d\tau \right)^{1/q} \right)^p,
 \end{aligned}$$

and consequently

$$\begin{aligned}
 (\text{ivar}_p^q(x; a, b))^p &\leq (\text{ivar}_p^q(x; a, c))^p + (\text{ivar}_p^q(x; c, b))^p \\
 &\quad + \left(2 \sup_{s \in [a, b-d]} \left(\int_s^{s+d} |x(\tau)|^q d\tau \right)^{1/q} \right)^p.
 \end{aligned}$$

This ends the proof. $\square$

Using Proposition 3 we can prove the following

Proposition 4 *Let $1 \leq p \leq q$ and let $x \in L^q(a, b)$ be a periodic step function with the basic period $T = |b - a|/M$, where $M \in \mathbb{N}$, and such that*

$$x(t) = \begin{cases} u, & t \in [a, a + \frac{T}{2}), \\ w, & t \in [a + \frac{T}{2}, a + T), \end{cases} \quad (4)$$

where u, w are distinct real numbers. Then

$$M^{1-p/q} \frac{1}{2^{p/q}} |u - w|^p |b - a|^{p/q} \leq (\text{ivar}_p^q(x; a, b))^p \leq 2M^{1-p/q} |u - w|^p |b - a|^{p/q}. \quad (5)$$

Proof First, we will slightly modify the conclusion of Proposition 3. In the case of periodic step functions we may estimate the integral $\left(\int_\alpha^\beta |x(t+h) - x(t)|^q dt \right)^{p/q}$ in a different way then in (3). Since $|x(t+h) - x(t)| \leq |u - w|$ for $t, t+h \in [a, b]$, we may observe that

$$\left(\int_\alpha^\beta |x(t+h) - x(t)|^q dt \right)^{p/q} \leq |u - w|^p |\alpha - \beta|^{p/q}.$$

Since x is periodic, it is easy to see that $\text{ivar}_p^q(x; a, a+T) = \text{ivar}_p^q(x; a+kT, a+(k+1)T)$ for $k = 0, \dots, M-1$. This lets us write:

$$\begin{aligned}
 (\text{ivar}_p^q(x; a, a+2T))^p &\leq (\text{ivar}_p^q(x; a, a+T))^p \\
 &\quad + (\text{ivar}_p^q(x; a+T, a+2T))^p + |u - w|^p |T|^{p/q}.
 \end{aligned}$$

Thus, by induction, we get

$$\begin{aligned} (\text{ivar}_p^q(x; a, a + kT))^p &\leq k(\text{ivar}_p^q(x; a, a + T))^p + (k-1)|u-w|^p|T|^{p/q} \\ &= k|u-w|^p|T/2|^{p/q} + (k-1)|u-w|^p|T|^{p/q} \leq 2k|u-w|^p|T|^{p/q} \end{aligned}$$

and

$$(\text{ivar}_p^q(x; a, a + MT))^p \leq 2M|u-w|^p \left(\frac{|b-a|}{M} \right)^{p/q} = 2M^{1-p/q}|u-w|^p|b-a|^{p/q}.$$

On the other hand,

$$\begin{aligned} (\text{ivar}_p^q(x; a, a + MT))^p &\geq \sum_{i=1}^M (\text{ivar}_p^q(x; a + (i-1)T, a + iT))^p = M(\text{ivar}_p^q(x; a, a + T))^p \\ &= M|u-w|^p|T/2|^{p/q} = M^{1-p/q} \frac{1}{2^{p/q}} |u-w|^p|b-a|^{p/q}. \end{aligned}$$

This completes the proof. $\square$

Proposition 5 Let $1 \leq p \leq q$ and let $x \in L^q$. Moreover, assume that there exists such a non-decreasing function $\varphi : [0, 1] \rightarrow [0, +\infty)$ that $(\int_a^b |x(t)|^q dt)^{1/q} \leq \varphi(|b-a|)$ for any interval $[a, b] \subset I$. If $(h_n)_{n \in \mathbb{N}}$ is a non-increasing sequence of positive real numbers such that $\sum_{n=1}^{+\infty} h_n = 1$ and $\sum_{n=1}^{+\infty} [\varphi(h_n)]^p < +\infty$, then

$$(\text{ivar}_p^q(x; 0, 1))^p \leq \sum_{n=1}^{+\infty} (\text{ivar}_p^q(x; t_{n-1}, t_n))^p + 2^p \sum_{n=1}^{+\infty} [\varphi(h_n)]^p + 2^p [\varphi(1)]^p, \quad (6)$$

where $t_0 := 0$ and $t_n := \sum_{i=1}^n h_i$.

Proof When $\sum_{n=1}^{+\infty} (\text{ivar}_p^q(x; t_{n-1}, t_n))^p = +\infty$ the inequality (6) is obvious. So we may assume that $\sum_{n=1}^{+\infty} (\text{ivar}_p^q(x; t_{n-1}, t_n))^p < +\infty$. For any fixed $n \geq 2$ by Proposition 3 we can see that

$$\begin{aligned} (\text{ivar}_p^q(x; 0, t_n))^p &\leq \sum_{i=1}^n (\text{ivar}_p^q(x; t_{i-1}, t_i))^p + 2^p \sum_{i=2}^n \left(\sup_{s \in [0, t_i - h_i]} \left(\int_s^{s+h_i} |x(\tau)|^q d\tau \right)^{1/q} \right)^p \\ &\leq \sum_{i=1}^n (\text{ivar}_p^q(x; t_{i-1}, t_i))^p + 2^p \sum_{i=2}^n [\varphi(h_i)]^p \\ &\leq \sum_{i=1}^{+\infty} (\text{ivar}_p^q(x; t_{i-1}, t_i))^p + 2^p \sum_{i=1}^{+\infty} [\varphi(h_i)]^p. \end{aligned}$$

Now, we are going to prove that not only the sequence $(\text{ivar}_p^q(x; 0, t_n))_{n \in \mathbb{N}}$ is uniformly bounded, but also the integral variation $\text{ivar}_p^q(x; 0, 1)$ in the entire interval $[0, 1]$ is bounded. To prove this let us take any partition $0 = s_0 < s_1 < \dots < s_{N-1} < s_N = 1$ of the interval I and take $n \in \mathbb{N}$ big enough that $t_n \geq s_{N-1}$. Now we can see that

$$\sum_{i=1}^{N-1} (\omega_q(x; s_{i-1}, s_i))^p \leq \left(\text{ivar}_p^q(x, 0, t_n) \right)^p.$$

Moreover,

$$\omega_q(x; s_{N-1}, s_N) \leq 2\varphi(|s_N - s_{N-1}|) \leq 2\varphi(1).$$

Thus,

$$\begin{aligned} \sum_{i=1}^N (\omega_q(x; s_{i-1}, s_i))^p &\leq \left(\text{ivar}_p^q(x, 0, t_n) \right)^p + 2^p [\varphi(1)]^p \\ &\leq \sum_{i=1}^{+\infty} (\text{ivar}_p^q(x; t_{i-1}, t_i))^p + 2^p \sum_{i=1}^{+\infty} [\varphi(h_i)]^p + 2^p [\varphi(1)]^p. \end{aligned}$$

This shows the inequality (6) and ends the proof. $\square$

Now, we are going to show that essentially unbounded functions may also belong to IBV_p^q .

Example 3 Let $x : (0, 1] \rightarrow \mathbb{R}$ be defined by $x(t) = t^{-\beta}$, where $\beta \in (0, 1/q)$ and $q > 1$. Now we are going to show that $x \in IBV_1^q$.

First, let us observe that for any $(a, b) \subset (0, 1]$ and $\alpha > 1$ we have

$$\int_a^{b-h} (t^{-\beta} - (t+h)^{-\beta})^q dt = \frac{1}{\alpha^{1-q\beta}} \int_{\alpha a}^{\alpha b - \alpha h} (s^{-\beta} - (s + \alpha h)^{-\beta})^q ds.$$

This directly leads to

$$\omega_q(x; a, b) = \frac{1}{\alpha^{1/q-\beta}} \omega_q(x; \alpha a, \alpha b).$$

We also know that since $x|_{[\frac{1}{2}, 1]}$ is a Lipschitz function, by Proposition 2, the value $\text{ivar}_1^q(x; \frac{1}{2}, 1)$ is well-defined (and finite).

Let us also observe that

$$\int_s^{s+\delta} t^{-\beta q} dt \leq \int_0^\delta t^{-\beta q} dt = \frac{1}{1-\beta q} \delta^{1-\beta q} \quad \text{for } s \in [0, 1-\delta],$$

and so

$$\sup_{s \in [0, 1-\delta]} \left(\int_s^{s+\delta} |x(t)|^q dt \right)^{1/q} = \left(\frac{1}{1-\beta q} \right)^{1/q} \delta^{1/q-\beta}.$$

This proves that for any $n \geq 2$ we have

$$\begin{aligned} \text{ivar}_1^q(x; \frac{1}{2^n}, 1) &\leq \sum_{i=1}^n \text{ivar}_1^q(x; \frac{1}{2^i}, \frac{1}{2^{i-1}}) + 2 \sum_{i=2}^n \sup_{s \in [0, 1-1/2^i]} \left(\int_s^{s+1/2^i} |x(\tau)|^q d\tau \right)^{1/q} \\ &\leq \sum_{i=1}^{+\infty} \left(\frac{1}{2^{1/q-\beta}} \right)^{i-1} \text{ivar}_1^q(x; \frac{1}{2}, 1) + 2 \left(\frac{1}{1-\beta q} \right)^{1/q} \sum_{i=0}^{+\infty} \left(\frac{1}{2^{1/q-\beta}} \right)^i \\ &= \frac{1}{1 - \frac{1}{2^{1/q-\beta}}} \text{ivar}_1^q(x; \frac{1}{2}, 1) + 2 \left(\frac{1}{1-\beta q} \right)^{1/q} \frac{1}{1 - \frac{1}{2^{1/q-\beta}}}. \end{aligned}$$

Let us now take any partition $0 = t_0 < t_1 < \dots < t_{N-1} < t_N = 1$ and such $n \in \mathbb{N}$ that $\frac{1}{2^n} < t_1$. Then, similarly as in the proof of Proposition 5,

$$\begin{aligned} \sum_{i=1}^N \omega_q(x; t_{i-1}, t_i) &= \omega_q(x; 0, t_1) + \sum_{i=2}^N \omega_q(x; t_{i-1}, t_i) \leq 2 \left(\int_0^1 t^{-\beta q} dt \right)^{1/q} \\ &\quad + \text{ivar}_1^q \left(x; \frac{1}{2^n}, 1 \right) \\ &\leq \frac{1}{1 - \frac{1}{2^{1/q-\beta}}} \text{ivar}_1^q \left(x; \frac{1}{2}, 1 \right) + 2 \left(\frac{1}{1 - \beta q} \right)^{1/q} \left(\frac{1}{1 - \frac{1}{2^{1/q-\beta}}} + 1 \right), \end{aligned}$$

which proves that all sums $\sum_{i=1}^N \omega_q(x; t_{i-1}, t_i)$ are uniformly bounded and completes the proof.

We have just shown that essentially unbounded functions may be of bounded q -integral 1-variation. The next example shows that bounded functions may be of unbounded q -integral 1-variation. The presented example is similar to the one given in [2, Theorem 3], but—contrary to that one—we will consider a bounded function.

Example 4 We are going to show that there exists a bounded function $x: [0, 1] \rightarrow \mathbb{R}$ such that $x \in L^q \setminus IBV_1^q$. Let $C > 0$ be a real number such that $C \sum_{i=1}^{+\infty} \frac{1}{i^q} = 1$ and $t_n = \sum_{i=1}^n C/i^q$ for $n = 1, 2, \dots$ and $t_0 = 0$. In every interval $[t_{n-1}, t_n]$ we define $x(t) = 1$ for $t \in [t_{n-1}, \frac{1}{2}(t_{n-1} + t_n)]$ and $x(t) = 0$ for $t \in (\frac{1}{2}(t_{n-1} + t_n), t_n)$. Let us also assume that $x(1) = 1$. In order to show that $\text{ivar}_1^q(x; 0, 1) = +\infty$, let us observe that $\text{ivar}_1^q(x; t_{n-1}, t_n) = (C/2)^{1/q} \frac{1}{n}$ and

$$\text{ivar}_1^q(x; 0, t_n) \geq (C/2)^{1/q} \sum_{i=1}^n \frac{1}{i}.$$

Thus,

$$\text{ivar}_1^q(x; 0, 1) \geq \sup_{n \in \mathbb{N}} \text{ivar}_1^q(x; 0, t_n) \geq (C/2)^{1/q} \sum_{i=1}^{+\infty} \frac{1}{i} = +\infty,$$

which proves our claim.

Remark 3 If we replace the series $\sum_{i=1}^{+\infty} \frac{1}{i^q}$ in the construction given in Example 4 by the convergent series $\sum_{i=2}^{+\infty} \frac{1}{i \ln^2 i}$, then we would get the function, for which

$$\text{ivar}_p^q(x; 0, t_n) \geq (C/2)^{1/q} \left(\sum_{i=2}^n \frac{1}{i^{p/q} \ln^{2p/q} i} \right)^{1/p}.$$

for any $p < q$ and $n \geq 2$. The right-hand sum is unbounded, so this shows that there exist L^∞ -functions that do not belong to any IBV_p^q for $p < q$.

Now we are going to ask whether the embedding $IBV_1^q \subset L^q$ is completely continuous. The answer is negative, which is shown in the next example.

Example 5 Let the sequence $(x_n)_{n \in \mathbb{N}}$ of functions belonging to L^q be given by

$$x_n(t) = \begin{cases} n^{1/q}, & t \in [0, \frac{1}{n}], \\ 0, & t \in (\frac{1}{n}, 1]. \end{cases}$$

This sequence is uniformly bounded in IBV_1^q since $\text{ivar}(x_n; 0, 1) = n^{1/q} \cdot \frac{1}{n^{1/q}} = 1$ and $\|x_n\|_{L^q} = 1$, but is not relatively compact in L^q .

By the well-known Kolmogorov-Riesz compactness condition in L^q (see e.g. [14]) we know that for a relatively compact set $(x_n)_{n \in \mathbb{N}}$ and for any $\varepsilon > 0$, there exists $\delta > 0$ that

$$\left(\int_0^{1-h} |x_n(t+h) - x_n(t)|^q dt \right)^{1/q} \leq \varepsilon, \quad (7)$$

for any $|h| \leq \delta$ and $n \in \mathbb{N}$. As we can see

$$\left(\int_0^{1-\frac{1}{n}} |x_n(t + \frac{1}{n}) - x_n(t)|^q dt \right)^{1/q} = 1 \quad \text{for } n \geq 2.$$

Hence, for $\varepsilon = 1$ there is no $\delta > 0$ for which the condition (7) holds uniformly for the entire sequence $(x_n)_{n \in \mathbb{N}}$.

3 Integral operators

In this section we are going to consider linear integral operators acting between the spaces L^q and IBV_1^q . From now on let us assume that $p = 1$ and $q \in (1, +\infty)$.

Let us take a function $k : I \times I \rightarrow \mathbb{R}$ and let us consider the integral operator

$$L^q \ni x \mapsto (Kx)(t) = \int_0^1 k(t, s)x(s)ds. \quad (8)$$

Such operators, considered on spaces of functions of bounded variation in the sense of Jordan and Wiener, have been extensively studied in [10, 11]. In particular in [11] there was presented the characterization of kernels k for which the corresponding map $K : BV(I) \rightarrow BV(I)$ is bounded. Quite recently similar necessary and sufficient conditions for the continuity of the integral operator have been also specified for ΛBV spaces (see [9]).

We are going to present now two sets of sufficient conditions for the continuity of the integral operator $K : L^q \rightarrow IBV_1^q$. Each of them is a consequence of one of the classical inequalities: the first one corresponds the generalized Minkowski inequality, while the second one corresponds to the Hölder inequality.

Theorem 1 *Let $1 < q < +\infty$ and $1/q + 1/q' = 1$. Assume that the function $k : I \times I \rightarrow \mathbb{R}$ satisfies the following conditions:*

- (A1) *k is a Lebesgue measurable function on $I \times I$;*
- (A2) *there exists such a function $m_0 \in L^{q'}$ that $\|k(\cdot, s)\|_{L^q} \leq m_0(s)$ for a.e. $s \in I$;*
- (A3) *there exists such a function $m \in L^{q'}$ that $\text{ivar}_1^q(k(\cdot, s); 0, 1) \leq m(s)$ for a.e. $s \in I$.*

Then the linear operator K , induced by k , acts from L^q to IBV_1^q and is continuous.

Proof First, we should observe that due to the generalized Minkowski inequality (see [15, Theorem 202]) for each $x \in L^q$ and a.e. $t \in I$ we have

$$\begin{aligned} \left(\int_0^1 \left| \int_0^1 k(t, s)x(s)ds \right|^q dt \right)^{1/q} &\leq \int_0^1 \left(\int_0^1 |k(t, s)x(s)|^q dt \right)^{1/q} ds \\ &\leq \int_0^1 |x(s)| \left(\int_0^1 |k(t, s)|^q dt \right)^{1/q} ds \leq \|x\|_{L^q} \|m_0\|_{L^{q'}}. \end{aligned}$$

This proves that the map K is well-defined and that $\|Kx\|_{L^q} \leq \|m_0\|_{L^{q'}} \|x\|_{L^q}$.

Let us take any partition $0 = t_0 < t_1 < \dots < t_N = 1$ of the interval I and consider

$$\sum_{i=1}^N \sup_{0 < h < t_i - t_{i-1}} \left(\int_{t_{i-1}}^{t_i-h} \left| \int_0^1 [k(t+h, s) - k(t, s)] x(s) ds \right|^q dt \right)^{1/q}.$$

Now, let us focus on one of the terms in the previous sum and apply the generalized Minkowski inequality once more:

$$\begin{aligned} & \left(\int_{t_{i-1}}^{t_i-h} \left| \int_0^1 [k(t+h, s) - k(t, s)] x(s) ds \right|^q dt \right)^{1/q} \\ & \leq \int_0^1 \left(\int_{t_{i-1}}^{t_i-h} |k(t+h, s) - k(t, s)|^q dt \right)^{1/q} ds \\ & \leq \int_0^1 |x(s)| \left(\int_{t_{i-1}}^{t_i-h} |k(t+h, s) - k(t, s)|^q dt \right)^{1/q} ds. \end{aligned}$$

This allows us to estimate

$$\begin{aligned} & \sum_{i=1}^N \sup_{0 < h < t_i - t_{i-1}} \left(\int_{t_{i-1}}^{t_i-h} \left| \int_0^1 [k(t+h, s) - k(t, s)] x(s) ds \right|^q dt \right)^{1/q} \\ & \leq \sum_{i=1}^N \sup_{0 < h < t_i - t_{i-1}} \int_0^1 |x(s)| \left(\int_{t_{i-1}}^{t_i-h} |k(t+h, s) - k(t, s)|^q dt \right)^{1/q} ds. \quad (9) \end{aligned}$$

Now, since we cannot be sure that the function

$$I \ni s \mapsto \sup_{0 < h < t_i - t_{i-1}} \left(\int_{t_{i-1}}^{t_i-h} |k(t+h, s) - k(t, s)|^q dt \right)^{1/q}$$

is measurable, let us fix positive $\varepsilon > 0$ and choose numbers $h_i \in (0, t_i - t_{i-1})$ in such a way that

$$\begin{aligned} & \sup_{0 < h < t_i - t_{i-1}} \int_0^1 |x(s)| \left(\int_{t_{i-1}}^{t_i-h} |k(t+h, s) - k(t, s)|^q dt \right)^{1/q} ds \\ & \leq \frac{\varepsilon}{N} + \int_0^1 |x(s)| \left(\int_{t_{i-1}}^{t_i-h_i} |k(t+h_i, s) - k(t, s)|^q dt \right)^{1/q} ds. \end{aligned}$$

Now we may continue our estimation (9)

$$\begin{aligned} & \sum_{i=1}^N \sup_{0 < h < t_i - t_{i-1}} \int_0^1 |x(s)| \left(\int_{t_{i-1}}^{t_i-h} |k(t+h, s) - k(t, s)|^q dt \right)^{1/q} ds \\ & \leq \int_0^1 |x(s)| \sum_{i=1}^N \left(\int_{t_{i-1}}^{t_i-h_i} |k(t+h_i, s) - k(t, s)|^q dt \right)^{1/q} ds + \varepsilon \\ & \leq \int_0^1 |x(s)| m(s) ds \leq \|m\|_{L^{q'}} \|x\|_{L^q} + \varepsilon. \end{aligned}$$

Since the inequality above holds true for any $\varepsilon > 0$, we infer that

$$\sum_{i=1}^N \omega_q(Kx; t_{i-1}, t_i) \leq \|m\|_{L^{q'}} \|x\|_{L^q},$$

which, in turn, shows that $\text{ivar}_1^q(Kx; 0, 1) \leq \|m\|_{L^{q'}} \|x\|_{L^q}$ and ends the proof. $\square$

Theorem 2 *Let $1 < q < +\infty$ and $1/q + 1/q' = 1$. Assume that the function $k: I \times I \rightarrow \mathbb{R}$ satisfies the following conditions:*

(B1) *k is a Lebesgue measurable function on $I \times I$;*

(B2) *the function $s \mapsto k(t, s)$ belongs to $L^{q'}$ for a.e. $t \in I$;*

(B3) $\mathcal{K}_0 = \left(\int_0^1 \|k(t, \cdot)\|_{L^{q'}}^q dt \right)^{1/q} < +\infty$;

(B4) $\mathcal{K} = \sup_{\mathcal{I}} \sum_{i=1}^N \sup_{0 < h < t_i - t_{i-1}} \left(\int_{t_{i-1}}^{t_i-h} \left(\int_0^1 |k(t+h, s) - k(t, s)|^{q'} ds \right)^{q/q'} dt \right)^{1/q} < +\infty$,

where the supremum is taken over the set of all partitions $\mathcal{I}$ of the interval I . Then the linear mapping K , induced by k , acts from L^q to IBV_1^q and is continuous.

Proof First, we should observe that due to Hölder's inequality and (B2) for each $x \in L^q$ and a.e. $t \in I$ the integral $\int_0^1 k(t, s)x(s)ds$ exists and is finite. Hence the map K is well-defined. Moreover,

$$\left(\int_0^1 \left| \int_0^1 k(t, s)x(s)ds \right|^q dt \right)^{1/q} \leq \left(\int_0^1 \|k(t, \cdot)\|_{L^{q'}}^q \|x\|_{L^q}^q dt \right)^{1/q}.$$

Hence $\|Kx\|_{L^q} \leq \mathcal{K}_0 \|x\|_{L^q}$.

Let us note that

$$\text{ivar}_1^q(Kx; 0, 1) = \sup_{\mathcal{I}} \sum_{i=1}^N \sup_{0 < h < t_i - t_{i-1}} \left(\int_{t_{i-1}}^{t_i-h} \left| \int_0^1 [k(t+h, s) - k(t, s)]x(s)ds \right|^q dt \right)^{1/q},$$

where the supremum is taken over the family of all finite partitions $0 = t_0 < \dots < t_N = 1$ of the interval I .

By the Hölder inequality, for a.e. $t \in I$, we have

$$\begin{aligned} \left| \int_0^1 [k(t+h, s) - k(t, s)]x(s)ds \right| &\leq \int_0^1 |k(t+h, s) - k(t, s)|x(s)ds \\ &\leq \left(\int_0^1 |k(t+h, s) - k(t, s)|^{q'} ds \right)^{1/q'} \cdot \|x\|_{L^q}. \end{aligned}$$

Let us now estimate the sum

$$\begin{aligned} &\sum_{i=1}^N \sup_{0 < h < t_i - t_{i-1}} \left(\int_{t_{i-1}}^{t_i-h} \left| \int_0^1 [k(t+h, s) - k(t, s)]x(s)ds \right|^q dt \right)^{1/q} \\ &\leq \sum_{i=1}^N \sup_{0 < h < t_i - t_{i-1}} \left(\int_{t_{i-1}}^{t_i-h} \left(\int_0^1 |k(t+h, s) - k(t, s)|^{q'} ds \right)^{q/q'} \cdot \|x\|_{L^q}^q dt \right)^{1/q} \\ &\leq \mathcal{K} \|x\|_{L^q}. \end{aligned}$$

This actually shows that

$$\text{ivar}_1^q(Kx; 0, 1) \leq \mathcal{K}\|x\|_{L^q},$$

and proves the continuity of K . $\square$

Remark 4 The concept of the variation of a function may be extended to maps with values in any Banach space (see Definition 7.3.4 in [16]). In particular, the value $\mathcal{K}$ given in (B4) may be interpreted as the *strong q -integral 1-variation of the $L^{q'}$ -valued map $\kappa: [0, 1] \rightarrow L^{q'}$* given by $\kappa(t) = k(t, \cdot) \in L^{q'}$, since

$$\mathcal{K} = \sup_{\mathcal{T}} \sum_{i=1}^N \sup_{0 < h < t_i - t_{i-1}} \left(\int_{t_{i-1}}^{t_i-h} \|\kappa(t+h) - \kappa(t)\|_{L^{q'}}^q dt \right)^{1/q}.$$

Example 6 Let $\alpha \in [1 - 1/q, 1]$ and $m \in L^{q'}$ be fixed. Assume that $k: I \times I \rightarrow \mathbb{R}$ is a Lebesgue measurable function satisfying, for all $h \in I$ and $t \in [0, 1 - h]$ the inequality

$$|k(t+h, s) - k(t, s)| \leq m(s)h^\alpha \quad \text{for a.e. } s \in I. \quad (10)$$

Then we have

$$\begin{aligned} \int_{t_{i-1}}^{t_i-h} \left(\int_0^1 |k(t+h, s) - k(t, s)|^{q'} ds \right)^{q/q'} dt &\leq \int_{t_{i-1}}^{t_i-h} h^{\alpha q} \left(\int_0^1 |m(s)|^{q'} ds \right)^{q/q'} dt \\ &\leq h^{\alpha q} (t_i - t_{i-1} - h) \|m\|_{L^{q'}}^q \leq C \|m\|_{L^{q'}}^q (t_i - t_{i-1})^{\alpha q + 1}, \end{aligned}$$

for a certain constant $C > 0$ —taking the supremum over $h \in (0, t_i - t_{i-1})$, similarly to what we did in Proposition 2. That is why

$$\begin{aligned} \sum_{i=1}^N \sup_{0 < h < t_i - t_{i-1}} \left(\int_{t_{i-1}}^{t_i-h} \left(\int_0^1 |k(t+h, s) - k(t, s)|^{q'} ds \right)^{q/q'} dt \right)^{1/q} \\ \leq C^{1/q} \|m\|_{L^{q'}} \sum_{i=1}^N (t_i - t_{i-1})^{\alpha + 1/q} \leq C^{1/q} \|m\|_{L^{q'}} \end{aligned}$$

since $(t_i - t_{i-1})^{\alpha + 1/q} \leq t_i - t_{i-1}$. Hence the condition (10) implies (B4).

4 Superposition operator

In this section we are going to answer the question concerning the acting conditions for the autonomous superposition operator defined on the space of functions of bounded q -integral 1-variation, that is we are going to find conditions which should be imposed on a function $f: \mathbb{R} \rightarrow \mathbb{R}$ so that its superposition with a function $x \in IBV_1^q$ remains in this space. The problem of acting conditions and related problems have been studied extensively for different concepts of variation (see the classical papers [17, 20], more recent results [6, 9–11, 21] and also [1] for an overview). Therefore, it seems to be important to answer this question also in this case.

The first observation in this section is a pretty obvious one.

Proposition 6 Assume that $f: \mathbb{R} \rightarrow \mathbb{R}$ is a Lipschitz function with a Lipschitz constant $L > 0$. Then for any $x \in IBV_1^q$ the superposition $f \circ x$ belongs to IBV_1^q and

$$\text{ivar}_1^q(f \circ x; 0, 1) \leq L \cdot \text{ivar}_1^q(x; 0, 1).$$

Proof The proof is the consequence of the following observation:

$$\begin{aligned} \omega_q(f \circ x; a, b) &= \sup_{0 < h < b-a} \left(\int_a^{b-h} |f(x(t+h)) - f(x(t))|^q dt \right)^{1/q} \\ &\leq L \sup_{0 < h < b-a} \left(\int_a^{b-h} |x(t+h) - x(t)|^q dt \right)^{1/q} = L \omega_q(x; a, b). \end{aligned}$$

□

Lemma 1 Assume that a Borel function $f: \mathbb{R} \rightarrow \mathbb{R}$ is not locally bounded, i.e., there exists such a bounded sequence $(b_n)_{n \in \mathbb{N}}$ that $|f(b_n)| \rightarrow +\infty$. Then there exists $x \in IBV_1^q$ such that $f \circ x \notin IBV_1^q$.

Proof Assume that $(b_n)_{n \in \mathbb{N}} \subset [-M, M] \subset \mathbb{R}$ is a sequence such that $|f(b_n) - f(0)| \geq n$. Let $(t_i)_{i \in \mathbb{N}} \subset (0, 1)$ be such that $h_i = t_i - t_{i-1} = \frac{1}{C_q} \frac{1}{i^{2q}}$ for $i \in \mathbb{N}$, where $t_0 = 0$ and $C_q = \sum_{i=1}^{+\infty} \frac{1}{i^{2q}}$. Let $x \in L^q$ be given by

$$x(t) = \begin{cases} b_i, & t \in [t_{i-1}, \frac{t_{i-1}+t_i}{2}), \\ 0, & t \in [\frac{t_{i-1}+t_i}{2}, t_i). \end{cases}$$

We are going to show that $x \in IBV_1^q$ and $f \circ x \notin IBV_1^q$. Please note that

$$\text{ivar}_1^q(x; t_{i-1}, t_i) = |b_i| \left(\frac{h_i}{2} \right)^{1/q} = |b_i| \left(\frac{1}{2C_q} \right)^{1/q} \frac{1}{i^2} \quad \text{for } i \in \mathbb{N},$$

and

$$\left(\int_a^b |x(t)|^q dt \right)^{1/q} \leq M |b - a|^{1/q} \quad \text{for any } a, b \in I \text{ with } a < b.$$

Therefore, by Proposition 5 we see that $x \in IBV_1^q$.

On the other hand,

$$\text{ivar}_1^q(f \circ x; t_{i-1}, t_i) = |f(b_i) - f(0)| \left(\frac{h_i}{2} \right)^{1/q} \geq i \left(\frac{1}{2C_q} \right)^{1/q} \frac{1}{i^2} = \left(\frac{1}{2C_q} \right)^{1/q} \frac{1}{i},$$

which proves that

$$\text{ivar}_1^q(f \circ x; 0, 1) \geq \sum_{i=1}^{+\infty} \text{ivar}_1^q(f \circ x; t_{i-1}, t_i) = +\infty.$$

□

Lemma 2 Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a Borel function which does not satisfy the local Lipschitz condition. Then there exists $x \in IBV_1^q$ such that $f \circ x \notin IBV_1^q$.

Proof Assume that $f: [-M, M] \rightarrow \mathbb{R}$ does not satisfy the Lipschitz condition for some $M > 0$. This means that for any $n \in \mathbb{N}$ there exists an interval $[a_n, b_n] \subset [-M, M]$, with non-empty interior, such that

$$|f(b_n) - f(a_n)| \geq n|a_n - b_n|.$$

By Lemma 1 we may assume that f is bounded on $[-M, M]$, which implies that $|b_n - a_n| \rightarrow 0$ as $n \rightarrow +\infty$. Let us observe that we may select a sequence of intervals $[a_n, b_n]$ in such a way that $|b_n - a_n| \leq 1$ and $|f(b_n) - f(a_n)| \geq n|a_n - b_n|$.

Now, let us observe that for each $n \in \mathbb{N}$ there exists a positive integer $M_n \geq 1$ such that

$$\frac{1}{(M_n + 1)^{1-1/q}} < |b_n - a_n| \leq \frac{1}{M_n^{1-1/q}}.$$

Hence we have

$$\frac{1}{2^{1-1/q}} \leq M_n^{1-1/q} |b_n - a_n| \leq 1.$$

Now, we will define the function x . Let $(t_i)_{i \in \mathbb{N}} \subset (0, 1)$ be such that $t_i - t_{i-1} = \frac{1}{C_q} \frac{1}{i^{2q}}$ for $i \in \mathbb{N}$, where $t_0 = 0$ and $C_q = \sum_{i=1}^{+\infty} \frac{1}{i^{2q}}$. Now, each interval $[t_{i-1}, t_i]$, $i = 1, 2, \dots$ will be divided into M_i subintervals $[s_{j-1}^i, s_j^i]$ of equal length, $t_{i-1} = s_0^i < s_1^i < \dots < s_{M_i-1}^i < s_{M_i}^i = t_i$. Hence, we have $|s_j^i - s_{j-1}^i| = \frac{|t_i - t_{i-1}|}{M_i} = \frac{1}{C_q M_i i^{2q}}$. Let x on $[s_{j-1}^i, s_j^i]$, $i \in \mathbb{N}$, $j = 1, 2, \dots, M_i$, be given by

$$x(t) = \begin{cases} b_i, & t \in [s_{j-1}^i, \frac{s_{j-1}^i + s_j^i}{2}), \\ a_i, & t \in [\frac{s_{j-1}^i + s_j^i}{2}, s_j^i]. \end{cases}$$

Clearly $x \in L^q$. Moreover by Proposition 4, we have

$$M_i^{1-1/q} |b_i - a_i| \left(\frac{1}{2C_q i^{2q}} \right)^{1/q} \leq \text{ivar}_1^q(x; t_{i-1}, t_i) \leq 2M_i^{1-1/q} |b_i - a_i| \left(\frac{1}{C_q i^{2q}} \right)^{1/q}.$$

Moreover, for any interval $[a, b] \subset [0, 1]$ we have

$$\left(\int_a^b |x(t)|^q dt \right)^{1/q} \leq M|b - a|^{1/q}.$$

By Proposition 5 it implies that

$$\begin{aligned} \text{ivar}_1^q(x; 0, 1) &\leq \sum_{i=1}^{+\infty} \text{ivar}_1^q(x; t_{i-1}, t_i) + 2M \sum_{i=1}^{+\infty} |t_i - t_{i-1}|^{1/q} + 2M \\ &\leq \sum_{i=1}^{+\infty} 2 \frac{1}{C_q^{1/q}} |b_i - a_i| M_i^{1-1/q} \frac{1}{i^2} + 2M \sum_{i=1}^{+\infty} \frac{1}{C_q^{1/q} i^2} + 2M \\ &\leq 2 \frac{1}{C_q^{1/q}} \sum_{i=1}^{+\infty} \frac{1}{i^2} + 2M \frac{1}{C_q^{1/q}} \sum_{i=1}^{+\infty} \frac{1}{i^2} + 2M < +\infty. \end{aligned}$$

Therefore, $x \in IBV_1^q$.

On the other hand, the function $f \circ x$ is also a step function but with $f(b_n)$ and $f(a_n)$ replacing b_n and a_n respectively. Hence,

$$\text{ivar}_1^q(f \circ x; t_{i-1}, t_i) \geq i M_i^{1-1/q} |b_i - a_i| \left(\frac{1}{2C_q i^{2q}} \right)^{1/q}$$

and

$$\begin{aligned} \text{ivar}_1^q(f \circ x; 0, 1) &\geq \sum_{i=1}^{+\infty} i M_i^{1-1/q} |b_i - a_i| \left(\frac{1}{2C_q i^{2q}} \right)^{1/q} \\ &\geq \frac{1}{2C_q^{1/q}} \sum_{i=1}^{+\infty} \frac{1}{i} = +\infty. \end{aligned}$$

Thus, $f \circ x \notin IBV_1^q$. □

Lemma 3 *Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a Borel function which does not satisfy the (global) Lipschitz condition. Then there exists $x \in IBV_1^q$ such that $f \circ x \notin IBV_1^q$.*

Proof Let us take two sequences $(a_n)_{n \in \mathbb{N}}, (b_n)_{n \in \mathbb{N}}$ such that $|f(b_n) - f(a_n)| > n|b_n - a_n|$. For simplicity assume that $|b_n| = \max\{|a_n|, |b_n|\}$ for $n \in \mathbb{N}$. By the previous Lemma 2 we may assume that the sequence $(b_n)_{n \in \mathbb{N}}$ is unbounded. We may also assume that the sequence $|b_n|$ is non-decreasing and $|b_n| \geq 1$. We can also easily observe that the difference $|b_n - a_n|$ may be assumed to be arbitrarily small. Indeed, if for some $\varepsilon_0 > 0$ we have $|f(s) - f(t)| \leq n|s - t|$ for all $|s - t| \leq \varepsilon_0$, then for any $a, b \in \mathbb{R}, a < b$ we have

$$\begin{aligned} |f(b) - f(a)| &\leq \sum_{i=1}^N \left| f\left(a + \frac{i}{N}(b-a)\right) - f\left(a + \frac{i-1}{N}(b-a)\right) \right| \\ &\leq n \sum_{i=1}^N \left| a + \frac{i}{N}(b-a) - \left(a + \frac{i-1}{N}(b-a)\right) \right| = n|b-a|, \end{aligned}$$

for a positive $N \in \mathbb{N}$ such that $|b-a|/N < \varepsilon_0$. This clearly implies that f is a Lipschitz function, which is not possible. Therefore, we may assume that $|b_n - a_n| \leq 1$ for $n \in \mathbb{N}$.

Let us observe that the series $\sum_{n=1}^{\infty} \frac{1}{|b_n|^q} \frac{1}{n^{2q}}$ is convergent and denote $h_n = \frac{1}{C|b_n|^q} \frac{1}{n^{2q}}$, where $C = \sum_{n=1}^{\infty} \frac{1}{|b_n|^q} \frac{1}{n^{2q}}$. We can see that not only $\sum_{n=1}^{\infty} h_n$ is convergent but $\sum_{n=1}^{\infty} h_n^{1/q}$ is convergent, too. Let us define the sequence $(t_n)_{n \in \mathbb{N}} \subset (0, 1)$ as $t_n = \sum_{i=1}^n h_i$ for $n \in \mathbb{N}$ and $t_0 = 0$.

For each $n \in \mathbb{N}$ there exists a positive integer M_n such that

$$M_n^{1-1/q} \leq \frac{|b_n|}{|b_n - a_n|} \leq (M_n + 1)^{1-1/q}.$$

This means that $M_n^{1-1/q} \leq \frac{|b_n|}{|b_n - a_n|} \leq 2^{1-1/q} M_n^{1-1/q}$ and

$$\frac{1}{2^{1-1/q}} \leq M_n^{1-1/q} \frac{|b_n - a_n|}{|b_n|} \leq 1.$$

Now, as in the proof of the Lemma 2, we divide each interval $[t_{i-1}, t_i], i = 1, 2, \dots$ into M_i subintervals $[s_{j-1}^i, s_j^i]$ of equal length, $t_{i-1} = s_0^i < s_1^i < \dots < s_{M_i-1}^i < s_{M_i}^i = t_i$. Hence,

we have $|s_j^i - s_{j-1}^i| = \frac{|t_i - t_{i-1}|}{M_i} = \frac{1}{C|b_i|^q M_i i^{2q}}$. Let x on $[s_{j-1}^i, s_j^i]$, $i \in \mathbb{N}$, $j = 1, 2, \dots, M_i$, be given by

$$x(t) = \begin{cases} b_i, & t \in [s_{j-1}^i, \frac{s_{j-1}^i + s_j^i}{2}), \\ a_i, & t \in [\frac{s_{j-1}^i + s_j^i}{2}, s_j^i). \end{cases}$$

We can see that

$$\int_0^1 |x(t)|^q dt \leq \sum_{i=1}^{+\infty} \int_{t_{i-1}}^{t_i} |b_i|^q dt = \sum_{i=1}^{+\infty} |b_i|^q \frac{1}{|b_i|^q} \frac{1}{C i^{2q}} = \frac{1}{C} \sum_{i=1}^{+\infty} \frac{1}{i^{2q}} < +\infty,$$

so $x \in L^q$.

By Proposition 4 we can see that

$$M_i^{1-1/q} |b_i - a_i| \left(\frac{1}{2C|b_i|^q i^{2q}} \right)^{1/q} \leq \text{ivar}_1^q(x; t_{i-1}, t_i) \leq 2M_i^{1-1/q} |b_i - a_i| \left(\frac{1}{C|b_i|^q i^{2q}} \right)^{1/q}$$

and

$$\text{ivar}_1^q(x; t_{i-1}, t_i) \leq \frac{2}{C^{1/q} i^2}.$$

Let us proceed to the estimation of $\text{ivar}_1^q(x; 0, 1)$. First, let us observe that for any $\delta \in (0, t_n)$ and $s \in [0, t_n - \delta]$

$$\left(\int_s^{s+\delta} |x(t)|^q dt \right)^{1/q} \leq |b_n| \delta^{1/q}.$$

We cannot apply Proposition 5 directly, but we may slightly change its proof to show that $x \in IBV_1^q$. Indeed, for any $n \geq 2$, we have

$$\begin{aligned} \text{ivar}_1^q(x; 0, t_n) &\leq \sum_{i=1}^n \text{ivar}_1^q(x; t_{i-1}, t_i) + 2 \sum_{i=2}^n \sup_{s \in [0, t_i - h_i]} \left(\int_s^{s+h_i} |x(\tau)|^q d\tau \right)^{1/q} \\ &\leq \sum_{i=1}^n \text{ivar}_1^q(x; t_{i-1}, t_i) + 2 \sum_{i=2}^n |b_i| h_i^{1/q} \\ &\leq \sum_{i=1}^{+\infty} \frac{2}{C^{1/q} i^2} + 2 \sum_{i=1}^{+\infty} \frac{|b_i|}{C^{1/q} |b_i| i^2} = \frac{4}{C^{1/q}} \sum_{i=1}^{+\infty} \frac{1}{i^2}. \end{aligned}$$

Therefore, if $0 = \sigma_0 < \sigma_1 < \dots < \sigma_{N-1} < \sigma_N = 1$ is an arbitrary partition of the interval I and $n \geq 2$ is such that $t_n \geq \sigma_{N-1}$, we see that

$$\begin{aligned} \sum_{i=1}^N \omega_q(x; \sigma_{i-1}, \sigma_i) &\leq \text{ivar}_1^q(x; 0, t_n) + 2 \left(\int_{\sigma_{N-1}}^{\sigma_N} |x(t)|^q dt \right)^{1/q} \\ &\leq \frac{4}{C^{1/q}} \sum_{i=1}^{+\infty} \frac{1}{i^2} + 2\|x\|_{L^q} < +\infty. \end{aligned}$$

This proves that $\text{ivar}_1^q(x; 0, 1) < +\infty$.

Now, we have to check that $\text{ivar}(f \circ x; 0, 1) = +\infty$. This estimation is much easier as

$$\text{ivar}_1^q(f \circ x; 0, 1) \geq \sum_{i=1}^{+\infty} \text{ivar}_1^q(f \circ x; t_{i-1}, t_i) \geq \frac{1}{2} \sum_{i=1}^{+\infty} \frac{1}{C^{1/q_i}} = +\infty.$$

□

The main result of this section is the following theorem concerning the acting conditions. It is an immediate consequence of Proposition 6 and Lemma 3, so its proof will be omitted.

Theorem 3 *The Borel function $f: \mathbb{R} \rightarrow \mathbb{R}$ generates the autonomous superposition operator $F: IBV_1^q \rightarrow IBV_1^q$ if and only if f satisfies the global Lipschitz condition.*

5 Existence theorems

In this section we will be interested in the problem of the existence of solutions to the following nonlinear Hammerstein integral equation belonging to the space IBV_1^q

$$x(t) = g(t) + \lambda \int_0^1 k(t, s) f(x(s)) ds, \quad t \in I, \quad (11)$$

where $\lambda \in \mathbb{R}$.

In what follows we will need the following lemma.

Lemma 4 *Assume that $f: \mathbb{R} \rightarrow \mathbb{R}$ is a Lipschitz function with a constant $L > 0$. Then the autonomous superposition operator $F: IBV_1^q \rightarrow L^q$, generated by f , is Lipschitz continuous with the constant $L > 0$.*

Proof First, let us note that in view of Proposition 6 the superposition operator F is well-defined, that is, $F(x) \in IBV_1^q \subseteq L^q$ for each $x \in IBV_1^q$. Moreover, for $x, y \in IBV_1^q$ we have

$$\begin{aligned} \|F(x) - F(y)\|_{L^q} &= \left(\int_0^1 |f(x(t)) - f(y(t))|^q dt \right)^{1/q} \\ &\leq L \left(\int_0^1 |x(t) - y(t)|^q dt \right)^{1/q} = L \|x - y\|_{L^q} \leq L \|x - y\|_{IBV_1^q}, \end{aligned}$$

which completes the proof. □

Theorem 4 *Let us admit the following assumptions:*

- 1° $f: \mathbb{R} \rightarrow \mathbb{R}$ is a Lipschitz continuous function with a constant $L > 0$;
- 2° the kernel $k: I \times I \rightarrow \mathbb{R}$ satisfies conditions (A1)–(A3) or (B1)–(B4);
- 3° $g \in IBV_1^q$.

Then there exists such $\delta > 0$ that for all λ with $|\lambda| < \delta$ the Eq. (11) possesses a unique solution belonging to the space IBV_1^q .

Proof The problem (11) corresponds to the operator equation

$$x = g + \lambda K(F(x)),$$

where $F: IBV_1^q \rightarrow L^q$ is an autonomous superposition operator associated to the function f , while $K: L^q \rightarrow IBV_1^q$ is a continuous linear map induced by the kernel k (see Theorem

1 and Theorem 2). By Lemma 4 the superposition operator $F : IBV_1^q \rightarrow L^q$ is a Lipschitz map. This shows that the superposition $\lambda K \circ F$ is a Lipschitz map with a constant $|\lambda| \|K\| \cdot L$. Hence, for λ small enough, it is a contraction, proving the existence and uniqueness of the solution to the problem (11). $\square$

6 Fractional Riemann–Liouville operators

In Sect. 3 we have discussed linear integral operators between spaces L^q and IBV_1^q and some sufficient conditions for the continuity of these operators have been given. Moreover, in Sect. 5, we have shown some existence theorems for nonlinear Hammerstein equations in IBV_1^q spaces. In this section we are going to show that the class of continuous operators acting from L^q to IBV_1^q , considered in Sect. 3, contains Riemann–Liouville fractional-order integrals. Hence, the results from the previous sections may be applied to some problems in which fractional derivatives are involved. To be more precise, let us now fix $\alpha \in (0, 1)$ and let us consider the linear map

$$L^q \ni x \mapsto (Kx)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{x(s)}{(t-s)^{1-\alpha}} ds, \quad (12)$$

where Γ denotes the Gamma function.

The above operator actually corresponds to the integral operator (8) with the kernel

$$k(t, s) = \begin{cases} \frac{1}{\Gamma(\alpha)}(t-s)^{-(1-\alpha)}, & \text{if } 0 \leq s < t \leq 1, \\ 0, & \text{if } 0 \leq t \leq s \leq 1. \end{cases} \quad (13)$$

This is the Riemann–Liouville left-sided fractional integral of order α , denoted in [23] by I_{0+}^α . The fractional integral operators, along with their applications, have been extensively studied through decades (see [18, 23] and the references therein). Among other properties of the map I_{0+}^α first of all we should mention that the mapping is well-defined for any $\alpha > 0$ and $q > 1$ as a mapping from L^q to L^q (see [23, Theorem 2.6]).

In [5, 7] the Authors considered the following problem

$$\begin{cases} x^{(\alpha)}(t) = f(x(t)), & t \in (0, 1], \\ x^{(\alpha-1)}(0) = x_0, \end{cases} \quad (14)$$

where $x^{(\alpha)}$ denotes the fractional derivative of the function x of the order $\alpha \in (0, 1)$.

The problem (14) corresponds to the operator equation (cf. [7, Proposition 4])

$$x = y_0 + I_{0+}^\alpha F(x),$$

where $y_0(t) = \frac{x_0}{\Gamma(\alpha)} t^{\alpha-1}$. Let us observe that, by Example 3, the function y_0 belongs to IBV_1^q for $\alpha \in (1 - \frac{1}{q}, 1)$. As we will see below, Theorem 4 may be applied also to the problems of the form (14).

Theorem 5 *If K is given by (12) and $\alpha \in (1 - \frac{1}{q}, 1)$, then for each $x \in L^q$, $Kx \in IBV_1^q$ and the mapping $K : L^q \rightarrow IBV_1^q$ is continuous.*

Proof We are going to show that $k : I \times I \rightarrow \mathbb{R}$ given by (13) satisfies the assumptions of Theorem 1. It is—obviously—measurable and the function $t \mapsto k(t, s)$ is in L^q for a.e. $s \in I$. In order to check the assumption (A2) we should fix $s \in I$ and calculate

$$\begin{aligned}\|k(\cdot, s)\|_{L^q} &= \frac{1}{\Gamma(\alpha)} \left(\int_s^1 (t-s)^{-(1-\alpha)q} dt \right)^{1/q} \\ &= \frac{1}{\Gamma(\alpha)} \cdot \left(\frac{1}{1-(1-\alpha)q} \right)^{1/q} (1-s)^{1/q-(1-\alpha)}.\end{aligned}$$

As we can see, because $q'/q - (1-\alpha)q' > -1$ the integral

$$\int_0^1 |m_0(s)|^{q'} ds = \left(\frac{1}{\Gamma(\alpha)} \cdot \left(\frac{1}{1-(1-\alpha)q} \right)^{1/q} \right)^{q'} \int_0^1 (1-s)^{q'/q-(1-\alpha)q'} ds$$

is finite, which shows that (A2) is satisfied with $m_0(s) = \frac{1}{\Gamma(\alpha)} \cdot \left(\frac{1}{1-(1-\alpha)q} \right)^{1/q} (1-s)^{1/q-(1-\alpha)}$.

Let us now observe that

$$\text{ivar}_1^q(k(\cdot, s); s, 1) \leq \frac{1}{\Gamma(\alpha)} \text{ivar}_1^q(t^{-(1-\alpha)}; 0, 1) \text{ for a.e. } s \in I$$

and by Proposition 3

$$\begin{aligned}\text{ivar}_1^q(k(\cdot, s); 0, 1) &\leq \text{ivar}_1^q(k(\cdot, s); 0, s) + \text{ivar}_1^q(k(\cdot, s); s, 1) + 2 \frac{1}{\Gamma(\alpha)} \left(\int_0^1 t^{-(1-\alpha)q} dt \right)^{1/q} \\ &\leq \frac{1}{\Gamma(\alpha)} \text{ivar}(t^{-(1-\alpha)}; 0, 1) + 2 \frac{1}{\Gamma(\alpha)} \left(\frac{1}{1-(1-\alpha)q} \right)^{1/q} \leq C\end{aligned}$$

for a certain constant C (cf. Example 3). So k satisfies the assumption (A3) of Theorem 1 with a constant (and thus $L^{q'}$ -integrable) function $m(s) = C$. $\square$

Remark 5 Let $\alpha > 1/q$. According to Theorem 3.6 of [23] the image $I_{0+}^\alpha(L^q) \subset H^{\alpha-\frac{1}{q}}$, where H^s denotes the space of Hölder continuous functions of order $s \in (0, 1]$. If $1 - 1/q \geq 1/q$, i.e. if $q \geq 2$, then $\alpha > 1 - 1/q \geq 1/q$ implies that $I_{0+}^\alpha(L^q) \subset H^{\alpha-\frac{1}{q}} \cap IBV_1^q$. Please note that Proposition 2 does not allow us to conclude that a functions belonging to the Hölder class $H^{\alpha-\frac{1}{q}}$ belong to IBV_1^q .

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