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FOREWORD

Professor María Teresa Lozano and universal links

Enrique Artal[1] · **Antonio F. Costa**[2] ·
Milagros Izquierdo[3]

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María Teresa, Maite, Lozano is a great person and mathematician, in these pages we can only give a very small account of her results trying to resemble her personality. We will focus our attention only on a few of the facets of her work, mainly in collaboration with Mike Hilden and José María Montesinos because as Maite Lozano pointed out in an international conference in Umeå University in June 2017, where she was a plenary speaker:

> *I am specially proud of been part of the team Hilden-Lozano-Montesinos (H-L-M), and of our mathematical achievements*

1 Historical motivation of universal links and knots

A first step in the study of 3-dimensional manifolds, before trying to classify them, is to find a good system of representation and to have a list. There are several classical ways to do that: using Heegaard diagrams, surgery on links, crystallizations. These proceedings contain a good account in several articles. There is another very important and classical way: using branched coverings of the sphere.

The use of branched coverings of the sphere to represent manifolds is inspired in the great importance of these objects in the study of Riemann surfaces. In fact, Riemann surfaces

✉ Enrique Artal
artal@unizar.es

Antonio F. Costa
acosta@mat.uned.es

Milagros Izquierdo
miizq@mai.liu.se; milagros.izquierdo@liu.se

1 Departamento de Matemáticas, IUMA, Universidad de Zaragoza, C. Pedro Cerbuna 12, 50009 Zaragoza, Spain

2 Departamento de Matemáticas Fundamentales, UNED, Paseo Senda del Rey 9, 28040 Madrid, Spain

3 Matematiska Institutionen Linköpings Universitet, 581 83 Linköping, Sweden

were defined originally as the right objects associated to meromorphic multi-valued functions which are no more than branched coverings of the Riemann sphere.

The most important advance in the use of branched coverings of the sphere in order to represent manifolds is a celebrated theorem by one of the founders of the topology, James W. Alexander[1] who wrote in 1920:

Every closed orientable triangulable n-manifold M is a branched covering of the n-dimensional sphere

The title of Alexander paper's, *Note on Riemann spaces*, makes evident the inspiration in Riemann surfaces for this result. The Alexander Theorem reduces the possibility to obtain a list of manifolds of dimension n to the consideration of subcomplexes of codimension two of the n-sphere and monodromy representations of the fundamental groups of the complement of such subcomplexes on symmetric groups. We get in that way a reduction of dimension.

In the 70's there was an important improvement in dimension three of Alexander's result, independently produced by H. Hilden[2] and J.M. Montesinos[3]:

Every 3-dimensional, closed, orientable manifold is a 3-fold irregular covering of the 3-sphere branched on knots (connected 1-submanifolds of the sphere).

If links are allowed as branching locus, the result was also obtained by U. Hirsch[4]. This result reduced the listing of 3-manifolds to the list of 3-coloured knots.

Looking carefully at the proof Alexander's paper (as the Mexican mathematician A. Ramírez[5] remarks) the branched subcomplex may be fixed, more concretely: *every closed, orientable triangulable n-manifold is a covering of the n-sphere branched on an (n-2)-skeleton of an n-simplex*. For instance in dimension two all closed, orientable surfaces are coverings of the 2-sphere branched on three points. This type of coverings of the 2-sphere are of very special importance: these surfaces have a representation as complex algebraic curves with coefficients in an algebraic field (Belyi curves). Then Alexander theorem tells us the existence of Belyi curves of all genera. In dimension three and greater, the $(n-2)$-skeleton of an n-simplex is not a submanifold of the n-sphere. Considering all the coverings of the n-sphere branched on such subcomplex we obtain a list of polyhedra, a lot of them are not manifolds. It is of great interest then to obtain a submanifold S of the sphere such that all manifolds are coverings of the sphere with singular set the submanifold S.

For dimension three the problem consists in to find a link L such that every 3-manifold is a covering of the sphere branched on L, following Thurston terminology, *a universal link*.

2 The dream team versus universal knot

In January 1982 Bill Thurston sent a letter to J.M. Montesinos, where he described a first universal link: a link of six connected components. But Thurston, making use of his great

1 *Note on Riemann spaces*, Bull. Am. Math. Soc. **26** (1920), no. 8, 370–372.

2 *Every closed orientable 3-manifold is a 3-fold branched covering space of* $\mathbb{S}^3$, Bull. Am. Math. Soc. **80** (1974), 1243–1244.

3 *A representation of closed orientable 3-manifolds as 3-fold branched coverings of* $\mathbb{S}^3$, Bull. Am. Math. Soc. **80** (1974), 845–846.

4 *Über offene Abbildungen auf die 3-Sphäre*, Math. Z. **140** (1974), 203–230.

5 *On a theorem of Alexander*, An. Inst. Mat. Univ. Nac. Autónoma México **15** (1975), no. 1, 77–81.

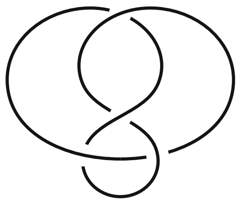

Fig. 1 Figure eight knot

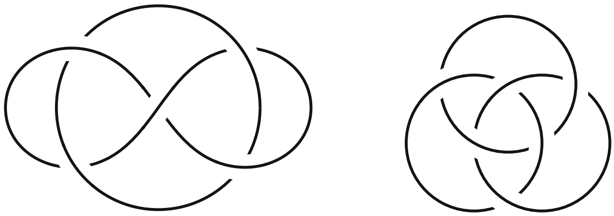

Fig. 2 Whitehead and Borromean links

intuition, asked if such a universal link may be reduced to a knot and furthermore if some simple knots (as Fig. 1) or links are universal.

Hilden-Lozano-Montesinos proved that, in fact, he was right. Note that at this time Thurston[6] was studying the hyperbolic structure of figure eight knot. Observe that the figure eight knot is the simplest knot that can be universal (the trefoil knot is a fibered torus knot and the branched coverings on this knot belong to a restrictive family of 3-manifolds: graph or Waldhausen manifolds).

Montesinos and Maite Lozano were at that time in Zaragoza and M. Hilden visited them. The team Hilden-Lozano-Montesinos started to work in this problem. And very soon they produced a universal knot announced in 1983[7] in a Conference in Vancouver[8]. This paper is a true *tour de force*: some of the figures of this paper have several pages and the first universal knot appears in a projection with more than 250 crossings!!

Later they improve very much their methods, and in an article published in 1983 in *Collectanea Mathematica*[9] they prove that the Whitehead link, the Borromean rings and the knot 9_{46} are universal. These are simple and well-known links (Fig. 2).

Some years later, the dream team answered positively the Thurston's question on the universality of the figure eight knot, in a 1985-article in *Topology*[10]. In this paper, Hilden-Lozano-Montesinos proved as well that all the non-toroidal 2-bridge knots are universal.

The team Hilden-Lozano-Montesinos starts here a fructiferous life of collaboration, and as Montesinos and Hilden confess, Maite Lozano plays a very essential rôle in this machinery

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[6] *Three-dimensional geometry and topology. Vol. 1*, Princeton Mathematical Series, vol. 35, Princeton University Press, Princeton, NJ, 1997, Edited by Silvio Levy.

[7] *Universal knots*, Bull. Am. Math. Soc. (N.S.) **8** (1983), no. 3, 449–450.

[8] *Universal knots*, Knot theory and manifolds (Vancouver, B.C., 1983), Lecture Notes in Math., vol. 1144, Springer, Berlin, 1985, pp. 25–59.

[9] *The Whitehead link, the Borromean rings and the knot* 9_{46} *are universal*, Collect. Math. **34** (1983), no. 1, 19–28.

[10] *On knots that are universal*, Topology **24** (1985), no. 4, 499–504.

and probably without her we should not have so wonderful and historical pages of the topology of 3-dimensional manifolds.

In 1987 Hilden, Lozano, Montesinos and W. Witten (the father of the Fields medallist Edward Witten) came out with an amazing (following È. Vinberg and O. Shvartsman[11]) and very important result regarding the geometrization of 3-manifolds: *every closed, orientable 3-manifold underlies a hyperbolic orbifold*. They reached this result by proving that every closed, orientable 3-manifold is an orbifold covering of finite degree of the hyperbolic orbifold $O = \mathbb{H}^3/U$ consisting of the 3-sphere with singular set the Borromean rings with conic singularities of orders 4, 4, 4. With other words, each closed, orientable 3-manifold is uniformized as $M = \mathbb{H}^3/H$ with H a finite index subgroup of the finitely generated group U of orientation-preserving isometries of $\mathbb{H}^3$ (a fundamental region for U is a pyritohedron). The group U is so a *universal group*. The result appeared in *Inventiones Mathematicae*[12]. In the same article it is proved that the group U is arithmetic. The key ingredient of the proof of the result is that every closed, orientable 3-manifold is a branched covering of the 3-sphere with branching set the Borromean rings with branching indices 1, 2 and 4.

The team Hilden-Lozano-Montesinos has found many infinite families of universal orbifolds with singular sets non-toroidal 2-bridge knots or links, or the three-parametric family with singular set the Borromean rings with branching indices m, $m \geq 3$; $2p$, $p \geq 2$; $2q$, $q \geq 2$, in a work in *Hiroshima Mathematical Journal*[13].

In a 1993-paper[14] the dream team showed that the orbifold lying on the 3-sphere with singular set the figure eight knot with cyclic isotropy group of order 12 is a universal orbifold, and so its orbifold fundamental group is a universal group. This universal group is also arithmetic as it is the case for the figure eight knot[15].

These arithmeticity properties are closely related to another framework where the team H-L-M has obtained remarkable progress, namely the study of character varieties. Given a group G (say finitely presented) the traces of characters in $\mathrm{SL}(2;\mathbb{C})$ form an algebraic variety. The structure of Wirtinger presentation makes easier the presentation of this *character variety*, specially for groups of knot complements, but more important, the presence of geometric structures (specially hyperbolic), provide interesting additional properties. More precisely, since $\mathrm{PSL}(2;\mathbb{C})$ is the orientation-preserving isometry group of the hyperbolic space and the holonomy of a hyperbolic structure can be lifted to $\mathrm{SL}(2;\mathbb{C})$, such a structure provide special characters; the irreducible components containing these characters are quite special since they are curves, and they are called the *excellent* components of the character variety. The list of progress of the dream team in this subject is remarkable, and covers a long list of papers from the nineties till today. We would like to point out two main features of their work on character varieties. They have been able to compute these excellent curves for a long list of knots with a strong computational cost, adding new techniques to their excellent background in mathematics. And also, these excellent curves are not only nice invariants of the knots (and links), but the character values for the meridians provide information on the study of cone manifolds and transitions of geometries, developed in the next section.

[11] *Discrete groups of motions of spaces of constant curvature*, Geometry, II, Encyclopaedia Math. Sci., vol. 29, Springer, Berlin, 1993, pp. 139–248.

[12] *On universal groups and three-manifolds*, Invent. Math. **87** (1987), no. 3, 441–456.

[13] *On universal hyperbolic orbifold structures in $\mathbb{S}^3$ with the Borromean rings as singularity*, Hiroshima Math. J. **40** (2010), no. 3, 357–370.

[14] *Universal 2-bridge knot and link orbifolds*, J. Knot Theory Ramifications **2** (1993), no. 2, 141–148.

[15] *The arithmeticity of the figure eight knot orbifolds*, Topology '90 (Columbus, OH, 1990), Ohio State Univ. Math. Res. Inst. Publ., vol. 1, de Gruyter, Berlin, 1992, pp. 169–183.

3 Works on cone manifolds and transitions of Thurston's geometries

The study of deformations, degenerations and transitions among different geometries is not only a very interesting and intricate piece of the work of María Teresa Lozano, but it is also elegant and beautiful. The framework where these transitions are understood is the concept of Seifert fibered cone manifold. A (Seifert fibered) cone manifold is a generalization of a (Seifert fibered) orbifold by allowing any singular angle α around singular points. Orbifolds are containied in the family of cone manifolds.

In 1995[16] the team Hilden-Lozano-Montesinos provided one uniparametric family of cone manifolds on the 3-sphere singular along the figure eight knot such that these cone manifolds have hyperbolic structure for cone angles ranging from 0 to $2\pi/3$, turning to spherical structures when the cone angle ranges from $2\pi/3$ to π. The limit case, for cone angle $2\pi/3$, gives Euclidean structure. In fact, as the title of the article says, H-L-M provided the uniparametric family of fundamental polyhedra for the (cone manifold-) fundamental groups of the cone manifolds.

Jumping to very recent times, in 2015[17] M.T. Lozano together with J.M. Montesinos studied the degeneration of some of Thurston's 3-manifold geometries in the framework of the two-parametric family $M(R, S)$ of real quaternion subalgebras of the algebra $M(2, \mathbb{C})$ of 2×2 complex matrices, with R and S two non-zero real parameters. The group $X(R, S)$ of unit quaternions of $M(R, S)$ is both a Seifert fibered 3-manifold and a 3-dimensional real Lie group. For different values of R and S the Lie group $X(R, S)$ is endowed with a Riemannian structure. In this way, Lozano-Montesinos create a suitable framework to study concrete deformations of geometric structures. For instance, along the line $R = S$, one passes from the spherical geometry for $X_{(-1,-1)}$ to $SL(2, R)$ for $X_{(1,1)}$ through Heisenberg geometry. One should remark as well that the group $E(2)$ of Euclidean transformations appears as the limit when R tends to 0 and S keeps constant.

To end this section we want to highlight the article of Lozano-Montesinos appeared in 2016[18] where they studied continuous families of geometric Seifert cone-manifold structures. In this paper they study Seifert fibered cone manifolds lying on Seifert manifolds with orbit space $\mathbb{S}^2$, with no incompressible fiberwise torus such that the singular set is a link with no more than three components which can include exceptional or general fibers. This family includes some interesting subfamilies, as the Seifert manifolds with orbit space $\mathbb{S}^2$ and finite fundamental group, and also the Seifert fibered cone manifolds lying on manifolds obtained by Dehn surgery on a torus knot $K(r, s)$ with singularity the core of the surgery. As a consequence, L-M obtain the holonomy of the Thurston geometry possessed by any given Seifert fibered orbifold obtained by surgery on a torus knot.

4 The volume

This volume is based in the special session Geometric Topology, in honor to Professor María Teresa Lozano, that took place at University of Zaragoza, January 30th–February 2017 3rd, within the Congress of Real Sociedad Matemática Española. That special session and this

[16] *On a remarkable polyhedron geometrizing the figure eight knot cone manifolds*, J. Math. Sci. Univ. Tokyo **2** (1995), no. 3, 501–561.

[17] *On the degeneration of some 3-manifold geometries via unit groups of quaternion algebras*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM **109** (2015), no. 2, 669–715.

[18] *On continuous families of geometric Seifert conemanifold structures*, J. Knot Theory Ramifications **25** (2016), no. 14, 1650083, 40.

volume are devoted to the mathematics that Professor Lozano and her colleagues, students and friends have worked on: representation of 3-manifolds by graphs and crystallizations; orbifold and branched coverings and dynamics of branched coverings; manifolds with singularities: stratifolds and orbifolds; lens spaces; singularities; flows with singularities; Riemann and Klein surfaces: moduli spaces, families of Riemann surfaces, automorphism groups.

This volume is a good example of the wide range of fields of interest in low dimensional topology. It is also an example of the considerable current activity in this extense area of mathematics. We give thanks to the authors who contributed to this Special Issue, to the referees of the papers for their great work and to the Editor in Chief of RACSAM, M. López Pellicer for his help and patience. Also we want to thank Red Española de Topología [19] and IUMA (University of Zaragoza) for finantial support to the Special Session in the Congress of RSME at University of Zaragoza.

Last but not least, this volume is a connected collection of mathematical works; the connecting thread of the volume consists of the concepts of branched covering, universal coverings and covering transformation groups.
Enrique Artal, Antonio F. Costa, Milagros Izquierdo

[19] Acción complementaria MTM201569385REDT, Ministerio de Economía y Competitividad