

Parovičenko spaces with structures

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Received: 5 June 2013 / Accepted: 23 November 2013 / Published online: 6 December 2013
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Abstract We study an analogue of the Parovičenko property in categories of compact spaces with additional structures. In particular, we present an internal characterization of this property in the class of compact median spaces.

Keywords Structured compact space · Normally supercompact space · Median space

Mathematics Subject Classification (2010) Primary: 54D30; Secondary: 54B25, 54B30, 18B30

1 Introduction

A compact space K is said to be *Parovičenko* if for every continuous surjections $f: A \rightarrow B$ and $g: K \rightarrow B$, where A, B are compact metrizable spaces, there exists a continuous surjection $h: K \rightarrow A$ satisfying $f \circ h = g$. This definition naturally generalizes to higher weights, leading to the notion of a κ -Parovičenko space, where κ is a cardinal bounding the weights of the spaces A, B . It is also natural to consider the same property in other categories. We shall concentrate on categories of *structured compact spaces*, defined below. We shall present a general (and rather standard) result on the existence of structured Parovičenko

W. Kubiś research was supported by the GAČR Grant P 201/12/0290 (Czech Republic).

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spaces. Next, we discuss one particular example: compact median spaces. The main result is an internal characterization of Parovičenko objects in this category.

The concept of a Parovičenko space was introduced by van Douwen and van Mill [14], where the authors actually used its internal properties, instead of the external category-theoretic condition stated above. Originally, Parovičenko [10] showed that every compact space of weight $\leq \aleph_1$ is a continuous image of the Čech–Stone remainder \mathbb{N}^* of the natural numbers; he also proved that, assuming the Continuum Hypothesis, this is the unique Parovičenko compact space of weight \aleph_1 .

Later on, Negreponis [9] generalized Parovičenko’s results to higher cardinals; some of his questions were several years later answered by Dow [2]. Finally, van Douwen and van Mill [13] showed that the Continuum Hypothesis is equivalent to the existence of a unique Parovičenko space of weight \aleph_1 . At the same time, Błaszczyk and Szymański [1] gave a simpler proof of Parovičenko theorem, also finding another characterization of \mathbb{N}^* , under the Continuum Hypothesis. Some further simplifications were subsequently found by Engelking [4]. Nowadays, Parovičenko spaces (dually, Parovičenko Boolean algebras) belong to the folklore of topology and set theory.

The aim of this note is to extend this concept to categories of compact spaces with structures, emphasizing one particular case: compact median spaces. This class was studied, in the context of Parovičenko theorem, by van Mill [15], who showed that none of the obvious subspaces of the superextension of the natural numbers (that plays the role of the Čech–Stone compactification of \mathbb{N} in the category of compact median spaces) has the Parovičenko property. Our main results are: the existence of Parovičenko median spaces and an internal characterization of such spaces. We finally show that the Continuum Hypothesis is equivalent to the uniqueness of a Parovičenko median space of weight 2^{\aleph_0} , an analogue of the result of van Douwen and van Mill [14].

All topological spaces under consideration are assumed to be at least Hausdorff. We shall use standard terminology concerning general topology and category theory.

2 Structured compact spaces

In this section we present category-theoretic approach to compact spaces with additional structures. We first explain this concept informally, the precise definition is given below. Roughly speaking, compact spaces with structures are objects of the form $\langle K, a \rangle$, where a is a *structure* on K , so far with no particular meaning. Given two compact spaces with structures $\langle K, a \rangle$ and $\langle L, b \rangle$, we would like to consider continuous mappings $f: K \rightarrow L$ that preserve the structures. If the structures are not clearly defined, the only thing we can say is that structure-preserving maps are *some* particular continuous maps. It is natural to require that identities should be structure-preserving and compositions of structure-preserving maps should be structure-preserving. Here we arrive at the proper concept: Structured compact spaces can be defined by a faithful functor from some (possibly very abstract) category \mathfrak{K} into the category of compact spaces $\mathcal{C}\text{omp}$.

To be more precise, by $\mathcal{C}\text{omp}$ we shall understand the category whose objects are non-empty compact spaces and arrows are continuous surjections (i.e. quotient maps). Restricting attention to surjective maps is convenient for our applications, especially when dealing with inverse sequences, although there is no problem to consider arbitrary continuous maps.

Recall that a functor $F: \mathfrak{K} \rightarrow \mathcal{C}$ is *faithful* if it is one-to-one on hom-sets, namely, given morphisms $f: a \rightarrow b$, $g: a \rightarrow b$, where a, b are objects of \mathfrak{K} , the equality $F(f) = F(g)$ implies $f = g$.

Fix a faithful covariant functor $\Phi: \mathfrak{K} \rightarrow \mathfrak{C}omp$. We can look at the graph of Φ as a category whose objects are pairs of the form $\langle K, a \rangle$, where a is a \mathfrak{K} -object such that $K = \Phi(a)$ and an arrow from $\langle K, a \rangle$ to $\langle L, b \rangle$ is a \mathfrak{K} -arrow $f: a \rightarrow b$, identified by Φ with a continuous surjection $\Phi(f): K \rightarrow L$. The fact that Φ is faithful simply means that f is uniquely determined by $\Phi(f)$. In other words, we can call a and b *structures* on K and L , respectively, and say that $\Phi(f)$ is a *structure-preserving* continuous map. We shall talk about Φ -structured compact spaces, having in mind objects of the form $\langle K, a \rangle$, where a is a \mathfrak{K} -object such that $K = \Phi(a)$. For simplicity, we shall say that $g: \langle K, a \rangle \rightarrow \langle L, b \rangle$ is Φ -preserving if $g = \Phi(f)$ for a (necessarily unique) \mathfrak{K} -arrow $f: a \rightarrow b$.

Summarizing, a category of structured compact spaces is given just by a faithful functor into the category of compact spaces. This clearly covers all algebraic structures as well as many more abstract ones.

For example, consider the category \mathfrak{K} of all Banach spaces with linear isometric embeddings. Given a Banach space X , let $\Phi(X)$ be the dual unit ball of X endowed with the weak-star topology. It is well-known that this is a faithful contravariant¹ functor into $\mathfrak{C}omp$. Thus, dual balls of Banach spaces can be viewed as compact spaces with structures, where the structure is actually a Banach space, even though all the information is “contained” in its dual unit ball.

All the considerations above belong to the topic of concrete categories, the only formal difference is that we deal with functors into the category of compact spaces instead of the category of sets.

2.1 Parovičenko spaces

From now on we fix a faithful covariant functor $\Phi: \mathfrak{K} \rightarrow \mathfrak{C}omp$. We shall work in the category of Φ -structured compact spaces. It is natural to define the *weight* of a Φ -structured compact space $\langle K, a \rangle$ to be just the weight of K .

From now on, we fix a regular cardinal $\kappa \geq \aleph_0$. We shall say that a Φ -structured compact space $\langle P, \pi \rangle$ is κ -Parovičenko if for every Φ -preserving surjection $f: \langle L, b \rangle \rightarrow \langle K, a \rangle$ of spaces of weight $< \kappa$, for every Φ -preserving surjective map $p: \langle P, \pi \rangle \rightarrow \langle K, a \rangle$ there exists a Φ -preserving surjection $q: \langle P, \pi \rangle \rightarrow \langle L, b \rangle$ such that $p = f \circ q$.

In order to show the existence and good properties of Parovičenko spaces it is necessary to make some natural assumptions on the functor Φ .

Namely, from now on we shall assume that Φ satisfies the following conditions:

- (A) Given Φ -preserving surjections $f: \langle X, a \rangle \rightarrow \langle Z, c \rangle, g: \langle Y, b \rangle \rightarrow \langle Z, c \rangle$ with $w(X) < \kappa$ and $w(Y) < \kappa$, there exist Φ -preserving surjective maps $f': \langle W, d \rangle \rightarrow \langle X, a \rangle$ and $g': \langle W, d \rangle \rightarrow \langle Y, b \rangle$ such that $w(W) < \kappa$ and $f \circ f' = g \circ g'$.
- (B) There exist, up to isomorphism, at most $\kappa^{<\kappa}$ many Φ -structured compact spaces of weight $< \kappa$.
- (C) Given an infinite cardinal λ , every inverse sequence of Φ -preserving surjections between spaces of weight $< \lambda$ has a limit in the category of Φ -preserving surjections; if the length of this sequence is $< cf(\lambda)$ then the weight of the limit is $< \lambda$.
- (D) Given an uncountable cardinal λ , every Φ -structured compact space of weight $\lambda \geq \kappa$ is the limit of a continuous inverse sequence of length $\leq \lambda$ consisting of Φ -structured compact spaces of weight $< \lambda$ and with Φ -preserving surjections.
- (F) Given a regular cardinal $\lambda > \aleph_0$, given an inverse sequence $\mathbf{X} = \langle X_\alpha, p_\alpha^\beta, \lambda \rangle$ of Φ -preserving surjective maps between spaces of weight $< \lambda$, given a Φ -preserving surjec-

¹ This is not an obstacle, because one can always replace the category \mathfrak{K} by the opposite one.

tion $f: \lim \mathbf{X} \rightarrow Y$, there exists $\alpha < \lambda$ and a Φ -preserving surjection $f': X_\alpha \rightarrow Y$ such that $f = f' \circ p_\alpha$, where p_α is the canonical projection onto X_α .

- (T) There exists a Φ -structured compact space $\langle E, e \rangle$ such that every Φ -structured compact space has a Φ -preserving surjection onto E .

It should be clear that the identity functor of $\mathcal{C}omp$ satisfies conditions (A)–(T). Other natural functors, coming from algebraic structures on compact spaces, are described in Sect. 2.2 below. In conditions (C), (D) and (F) λ is an arbitrary infinite cardinal. Condition (A) will be sometimes called the *reversed amalgamation property*. Condition (F) will be called the *factorization property*. Condition (T) (the existence of a weakly terminal object) is assumed for the sake of convenience only, it could be replaced by a weaker one saying that for every Φ -structured compact spaces X, Y there exists a Φ -structured compact Z and Φ -preserving surjective maps $f: Z \rightarrow X, g: Z \rightarrow Y$.

The next fact belongs to the folklore, we sketch its proof only for completeness.

Lemma 2.1 *Let K be a compact space of weight $\leq \kappa^{<\kappa}$ and let L be a compact space of weight $< \kappa$. Then the set of all surjective maps from K onto L has cardinality $\leq \kappa^{<\kappa}$.*

Proof Let $Z(K)$ denote the lattice of all closed G_δ subsets of K . Then $|Z(K)| \leq (\kappa^{<\kappa})^{\aleph_0} = \kappa^{<\kappa}$. Fix a closed basis \mathcal{B} in L of cardinality $\mu < \kappa$ that consists of closed G_δ sets. Every surjective map $f: K \rightarrow L$ is uniquely determined by its inverse restricted to \mathcal{B} , which is an embedding of \mathcal{B} into the lattice $Z(K)$. Finally, $Z(K)$ has at most $(\kappa^{<\kappa})^\mu = \kappa^{<\kappa}$ subsets of cardinality $\leq \mu$. □

Lemma 2.2 *Let $\langle K, a \rangle$ be a Φ -structured compact space of weight $\leq \kappa^{<\kappa}$. Then there exists a Φ -structured compact space $\langle L, b \rangle$ of weight $\leq \kappa^{<\kappa}$, together with a Φ -preserving surjective map $h: \langle L, b \rangle \rightarrow \langle K, a \rangle$, satisfying the following condition:*

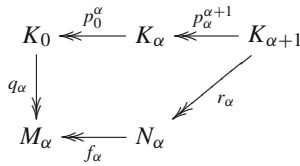
- (L) *Given Φ -preserving surjections $p: \langle K, a \rangle \rightarrow \langle M, c \rangle$ and $f: \langle N, d \rangle \rightarrow \langle M, c \rangle$ such that $w(N) < \kappa$, there exists a Φ -preserving surjection $q: \langle L, b \rangle \rightarrow \langle N, d \rangle$ such that $p \circ h = f \circ q$. In other words, the diagram*

$$\begin{array}{ccc}
 L & \xrightarrow{q} & N \\
 h \downarrow & & \downarrow f \\
 K & \xrightarrow{p} & M
 \end{array}$$

is commutative.

Proof Let us consider the family \mathcal{F} consisting of all (M, N, q, f) where M, N are Φ -structured compact spaces, $w(N) < \kappa$ and $q: K \twoheadrightarrow M, f: N \twoheadrightarrow M$ are Φ -preserving surjections. As $w(K) \leq \kappa^{<\kappa}$ and $w(N) < \kappa$ for each $(M, N, q, f) \in \mathcal{F}$, by condition (B) and Lemma 2.1 we conclude that $|\mathcal{F}| \leq (\kappa^{<\kappa})^{<\kappa} = \kappa^{<\kappa}$. Fix an enumeration $\mathcal{F} = \{(M_\alpha, N_\alpha, q_\alpha, f_\alpha): \alpha < \kappa^{<\kappa}\}$.

We define inductively an inverse sequence $\mathcal{S} = \langle K_\alpha, p_\alpha^\beta; \alpha < \kappa^{<\kappa} \rangle$ of Φ -structured compact spaces whose bonding maps are Φ -preserving surjections. We start with $K_0 = K$ and at each limit stage we use condition (C). Given $\alpha < \kappa^{<\kappa}$, we require that there exists a Φ -preserving surjection $r_\alpha: K_{\alpha+1} \twoheadrightarrow N_\alpha$ such that the following diagram



commutes. This is possible, because of the reversed amalgamation (condition (A)) for the mappings $q_\alpha \circ p_0^\alpha, f_\alpha$.

Finally, we let L to be the limit of this sequence in the category of Φ -structured compact spaces, which exists by condition (C). The same condition applied to $\lambda = (\kappa^{<\kappa})^+$ says that $w(L) \leq \kappa^{<\kappa}$. Finally, by the construction it is clear that condition (L) holds. \square

Theorem 2.3 *There exists a κ -Parovičenko Φ -structured compact space of weight $\leq \kappa^{<\kappa}$.*

Proof We construct a Φ -continuous inverse sequence of length κ^+ , starting from a weakly terminal object given by condition (T). At each successor step we use Lemma 2.2, while at limit steps we use condition (C). Finally, the factorization property of Φ (condition (F) applied to $\lambda = \kappa^+$) shows that the limit of this sequence is κ -Parovičenko. \square

The following statement is an adaptation of well-known results in Fraïssé-Jónsson theory, see e.g. [3] or [7].

Theorem 2.4 *Let P be a κ -Parovičenko Φ -structured compact space, where $\kappa > \aleph_0$. Then:*

- (1) *For every Φ -structured compact space K of weight $\leq \kappa$ there exists a Φ -preserving surjection $p: P \twoheadrightarrow K$.*
- (2) *If $\kappa = w(P)$ is regular then for every Φ -preserving surjections $f, g: P \twoheadrightarrow K$, where $w(K) < \kappa$, there is a Φ -isomorphism $h: P \twoheadrightarrow P$ such that $f = g \circ h$.*
- (3) *If κ is regular then there exists at most one (up to isomorphism) κ -Parovičenko Φ -structured compact space of weight κ .*

Proof (1): First, suppose that $w(K) < \kappa$. By (T), there are Φ -preserving surjections $f: P \twoheadrightarrow E$ and $g: K \twoheadrightarrow E$, where E is a weakly terminal Φ -object specified in condition (T). By the definition of a κ -Parovičenko space, we get a Φ -preserving surjection $q: P \twoheadrightarrow K$ additionally satisfying $g \circ p = f$.

Assume now that $w(K) = \kappa$. By (D), $K = \varprojlim \langle K_\alpha, q_\alpha^\beta, \kappa \rangle$, where the sequence is continuous, all bonding maps are Φ -preserving surjections and $w(K_\alpha) < \kappa$ for each $\alpha < \kappa$. We may assume that $K_0 = E$. Suppose we have constructed Φ -preserving surjections $p_\alpha: P \twoheadrightarrow K_\alpha$ for $\alpha < \delta$, where $\delta < \kappa$ is a fixed ordinal, such that $q_\alpha^\beta \circ p_\beta = p_\alpha$ for every $\alpha < \beta < \delta$. If δ is a limit ordinal, we use the continuity of the sequence to construct p_δ . If $\delta = \beta + 1$ then we find p_δ just using the definition of a κ -Parovičenko Φ -structured compact space. Finally, the surjective map $p: P \twoheadrightarrow K$ is the limit of the sequence $\{p_\alpha\}_{\alpha < \kappa}$. This completes the proof of (1).

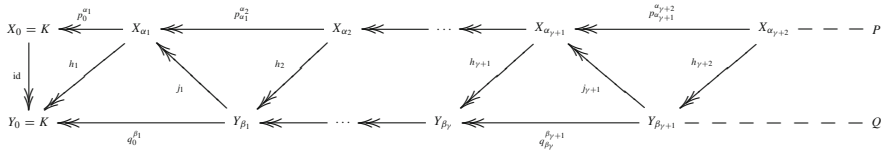
(2) and (3): We shall prove both statements simultaneously. Namely, assume P and Q are two Φ -structured κ -Parovičenko compact spaces of weight κ . Assume $P = \varprojlim \langle X_\alpha, p_\alpha^\beta, \kappa \rangle$ and $Q = \varprojlim \langle Y_\alpha, q_\alpha^\beta, \kappa \rangle$, where both sequences are continuous, all bonding maps are Φ -preserving surjections and all spaces X_α, Y_α have weight $< \kappa$. Now observe that the inverse sequence $\langle X_\alpha, p_\alpha^\beta, \kappa \rangle$ has the following property:

- (P) Given $\alpha < \kappa$, given a Φ -preserving surjection $f: K \twoheadrightarrow X_\alpha$ with $w(K) < \kappa$, there exist $\beta > \alpha$ and a Φ -preserving surjection $g: X_\beta \twoheadrightarrow K$ such that

$$f \circ g = p_\alpha^\beta.$$

This is easily proved by applying the κ -Parovičenko property and the factorization property (F). Obviously, the sequence $\langle Y_\alpha, q_\alpha^\beta, \kappa \rangle$ satisfies the same condition.

Now suppose for the moment that $X_0 = Y_0 = K$. A standard back-and-forth argument gives the required Φ -isomorphism from P to Q . This is visualized in the following commutative diagram:



The successor steps are made by applying condition (P) and the limit stages are taken care by condition (C).

Now, assume that $K = E$ (a weakly terminal object from condition (T)). Then the back-and-forth argument above shows that P is isomorphic to Q , proving (3).

Finally, in order to show (2), let us assume that $P = Q$, $X_\alpha = Y_\alpha$ and $p_\alpha^\beta = q_\alpha^\beta$ in the diagram above. Using the factorization property (F), we may assume that there are $\alpha_1, \beta_1 < \kappa$ such that $f = p_{\alpha_1} \circ p_0^{\alpha_1}$ and $g = q_{\beta_1} \circ q_0^{\beta_1}$. The same back-and-forth argument sketched in the diagram above gives the required isomorphism h satisfying $f = g \circ h$. \square

2.2 Topological algebras

We now make a brief discussion of possible applications of the concepts of this section. Namely, fix a countable first-order language L consisting of function symbols (i.e. algebraic operations) and fix a class \mathfrak{K} of compact L -algebras, that is, if $K \in \mathfrak{K}$ then K is a compact Hausdorff space which is at the same time an L -model such that all L -operations in K are continuous. Let us assume that \mathfrak{K} is closed under standard products, that is, a product of a family $\mathcal{F} \subseteq \mathfrak{K}$ is the usual product $\prod \mathcal{F}$ with the Tikhonov topology, and all L -operations are defined coordinate-wise. Let us also assume that \mathfrak{K} is stable under closed subalgebras. In other words, if $K \in \mathfrak{K}$ and $K' \subseteq K$ is a closed subspace that is also closed under all L -operations, then $K' \in \mathfrak{K}$.

These two assumptions ensure that \mathfrak{K} has inverse limits, that are actually inverse limits in the category of compact spaces.

In order to describe everything in the language of Φ -structured compact spaces, we consider the natural forgetful functor Φ from \mathfrak{K} into $\mathcal{C}omp$, where \mathfrak{K} is viewed as the category of all continuous surjections that preserve all L -operations.

It follows that \mathfrak{K} (or formally the just-defined functor Φ) satisfies conditions (A), (B) and (C). In fact, (C) follows from the remarks above; (B) is a standard “counting” argument (using the fact that the language is countable); (A) can be deduced from the fact that \mathfrak{K} has products and is closed under substructures. More precisely, given continuous epimorphisms of L -structures $f : X \rightarrow Z, g : Y \rightarrow Z$, their pullback belongs to the category \mathfrak{K} . Recall that the pullback of $\langle f, g \rangle$ is a pair of maps $\langle f', g' \rangle$, where $f' : W \rightarrow X, g' : W \rightarrow Y$ are defined as follows:

$$W = \{ \langle x, y \rangle \in X \times Y : f(x) = g(y) \},$$

$$f'(x, y) = x \quad \text{and} \quad g'(x, y) = y.$$

Condition (T) is satisfied as long as there exists an L -algebra E in \mathfrak{K} such that every other $K \in \mathfrak{K}$ has a continuous homomorphism onto E . Finally, condition (D) requires an addi-

tional assumption. Namely, we need to know that \mathfrak{K} is closed under continuous homomorphic images. More precisely, given $K \in \mathfrak{K}$, given a compact L -algebra L , if there exists a continuous epimorphism $f: K \rightarrow L$, then $L \in \mathfrak{K}$. Summarizing, we have:

Theorem 2.5 *Let L be a countable first-order language consisting of function symbols only and let \mathfrak{K} be a class of compact L -algebras with the following properties:*

- (1) \mathfrak{K} is stable under products and closed subalgebras,
- (2) \mathfrak{K} is stable under continuous epimorphisms.

Then \mathfrak{K} satisfies conditions (A), (B), (C), (D) and (F).

Proof In view of the remarks above, only condition (D) requires a proof. This is a rather standard closing-off argument, although we were unable to find it explicitly written in the literature, therefore we sketch it below. We use the method of elementary submodels and we refer to [5, Chapter 17] for more details and explanations.

Namely, given a compact space K and given an elementary submodel M of a big enough $H(\theta)$ (the family of sets of hereditary cardinality $< \theta$), there is a natural equivalence relation \sim_M on K , defined by $x \sim_M y$ iff M contains no continuous function $f: K \rightarrow \mathbb{R}$ with $f(x) \neq f(y)$. The quotient space with respect to this relation is denoted by K/M and the quotient map is denoted by $q_M: K \rightarrow K/M$. The crucial fact is that K/M is an L -algebra and q_M is a homomorphism, whenever M “knows” the L -operations of M . This is satisfied whenever $L \in M$ is countable.

Note that the weight of K/M is $\leq |M|$ and the Skolem–Löwenheim theorem says that there exists a countable elementary submodel M of $H(\theta)$ such that K and any fixed countable set is in M .

Finally, we need to know that if θ is big enough so that $K \in H(\theta)$, then the family of all elementary substructures of $H(\theta)$ that contain K and all the L -operations on K is directed and closed under unions of arbitrary chains. A suitable chain of elementary submodels of $H(\theta)$, starting with a countable one, provides an inverse sequence witnessing (D). \square

One should admit that condition (T) is usually trivial to check, therefore it has been ignored in the statement above.

3 Compact median spaces

A topological space is *supercompact* if it has a subbase \mathcal{B} for its closed sets (called a *binary subbase*) such that every linked family $\mathcal{F} \subseteq \mathcal{B}$ has nonempty intersection. A family \mathcal{F} is *linked* if $A \cap B \neq \emptyset$ for every $A, B \in \mathcal{F}$. By Alexander’s subbase lemma, every supercompact space is compact. A nontrivial result of Strok and Szymański [11] says that every compact metric space is supercompact. By the result of van Douwen and van Mill [13], every infinite supercompact space contains non-trivial convergent sequences, therefore not all compact spaces are supercompact. We say that two sets $A, B \subseteq X$ are *screened* with C, D if $A \cap D = \emptyset = B \cap C$ and $C \cup D = X$. A closed subbase \mathcal{F} is a *normal subbase* if any $S, T \in \mathcal{F}$ with $S \cap T = \emptyset$ are screened with some $S', T' \in \mathcal{F}$.

A topological space X which possesses a normal binary subbase is called *normally supercompact*. For more information about supercompactness we refer to van Mill’s book [15]. We now discuss briefly convexity structures in median spaces. For more details we refer to van de Vel’s monograph [12].

Supercompact spaces, especially normally supercompact spaces, have some nice geometric properties, in the sense of general convexity structures. Recall that an *interval convexity* (convexity for short) on the set X is the family \mathcal{G} of subsets of X which satisfies the following conditions:

- (1) $\emptyset, X \in \mathcal{G}$;
- (2) if $\mathcal{G}' \subseteq \mathcal{G}$ then $\bigcap \mathcal{G}' \in \mathcal{G}$;
- (3) if $A \subseteq X$ and for each $a, b \in A$ there is $C \in \mathcal{G}$ with $a, b \in C \subseteq A$ then $A \in \mathcal{G}$.

Elements of \mathcal{G} are called *convex sets*. A convex set whose complement is convex is called a *halfspace*. By (2) we can define the *convex hull* $\text{conv } A$ of any set $A \subseteq X$ as the set $\bigcap \{F \in \mathcal{G} : A \subseteq F\}$. We write $[a, b]$ instead of $\text{conv}\{a, b\}$ and call it the *segment* joining a and b .

Let X be a normally supercompact space with a fixed normal binary subbase \mathcal{B} . The interval map $I_{\mathcal{B}} : X \times X \rightarrow \mathcal{P}(X)$ is defined by the formula:

$$I_{\mathcal{B}}(a, b) = \bigcap \{B \in \mathcal{B} : a, b \in B\}.$$

It is not hard to see that the family $\mathcal{G}_{I_{\mathcal{B}}} = \{A \subseteq X : I_{\mathcal{B}}(a, b) \subseteq A \text{ for each } a, b \in A\}$ is a convexity containing \mathcal{B} and $I_{\mathcal{B}}(a, b) = [a, b]$, where the segment $[a, b]$ is considered with respect to the convexity $\mathcal{G}_{I_{\mathcal{B}}}$. Since \mathcal{B} is binary, $[x, y] \cap [x, z] \cap [y, z] \neq \emptyset$ for each $x, y, z \in X$. Moreover, this intersection contains exactly one point $m(x, y, z)$, called the *median* of x, y, z . This follows from the fact that each two distinct points can be screened with two sets from \mathcal{B} . Since each element of \mathcal{B} is convex, we see that every normally supercompact space satisfies the condition:

CC_2 : each two distinct points can be screened with closed convex sets.

Let us note that the convexity $\mathcal{G}_{I_{\mathcal{B}}}$ is binary in the sense that each finite linked subfamily of $\mathcal{G}_{I_{\mathcal{B}}}$ has nonempty intersection (see [15, Theorem 1.3.3]). So, each normally supercompact space is a compact space with a binary convexity satisfying CC_2 .

It turns out that the converse is true as well. If X is a compact space with a binary convexity \mathcal{G} satisfying CC_2 then, by compactness, the collection of all closed convex sets is a closed subbase for the topology of X . Moreover, the collection of all closed convex sets is normal (see [6, Proposition V.1.2]). In conclusion, the following two classes topological spaces are equal:

- compact spaces with a binary convexity satisfying CC_2
- normally supercompact spaces.

Such spaces will be called *compact median spaces*.

3.1 Basic properties of compact median spaces

In the class of compact median spaces the following counterpart of Urysohn lemma is true (see [16, Theorem 3.3]):

Proposition 3.1 *If X is a compact median space and x, y are distinct points of X then there is a continuous map $f : X \rightarrow [0, 1]$ such that $f(x) = 0, f(y) = 1$ and $f^{-1}(A)$ is convex for each interval $A \subseteq [0, 1]$.*

The map f from the last theorem is a special case of a more general concept of convexity preserving map. Namely $f : X \rightarrow Y$, is called a *convexity preserving map*, if $f^{-1}(G) \in \mathcal{F}$

for each $G \in \mathcal{G}$, where X and Y are spaces with convexities \mathcal{F} and \mathcal{G} , respectively. In case of binary convexities, this is equivalent to

$$f([a, b]) \subseteq [f(a), f(b)] \text{ for every } a, b \in X. \tag{cp}$$

For median spaces, convexity preserving maps are precisely the median preserving maps (with the obvious meaning). Indeed, a convexity preserving map f necessarily preserves the median, because of (cp). On the other hand, if f is median preserving and $x \in [a, b]$ then $m(x, a, b) = x$ and hence $f(x) = m(f(x), f(a), f(b)) \in [f(a), f(b)]$, showing that f satisfies (cp).

From now on, we will consider the category \mathfrak{CM} consisting of all compact median spaces and surjective continuous median preserving maps. Such maps will be called *epimorphisms* and denoted $f : X \twoheadrightarrow Y$ for $X, Y \in \mathfrak{CM}$. Recall that a subset X of a median space M is *median-closed* (or *median-stable*) if $m(x, y, z) \in X$ whenever $x, y, z \in X$. It follows from Proposition 3.1 that:

Theorem 3.2 (cf. [16, Thm. 3.4]) *Every compact median space X is isomorphic to a median-closed subset of the Tikhonov cube $[0, 1]^\kappa$, where $\kappa = w(X)$. In the case of zero-dimensional spaces, Tikhonov cube can be replaced by the Cantor cube $\{0, 1\}^\kappa$.*

Let us mention the following consequence (cf. [8, Theorem 2.2]):

Theorem 3.3 *Each compact median space X of weight $\kappa \geq \aleph_0$ can be represented as the limit of an inverse sequence of compact median spaces $\langle X_\alpha; p_\alpha^\beta; \alpha \leq \beta < \kappa \rangle$, where:*

1. $|X_0| = 1$.
2. for a limit ordinal $\lambda < \kappa; X_\lambda = \varprojlim \langle X_\alpha : \alpha < \lambda \rangle$.
3. $w(X_\alpha) < w(X)$ for each $\alpha < \kappa$.
4. Each bonding map p_α^β is convexity preserving.
5. If X is zero-dimensional then so is each X_α and moreover for each $\alpha < \kappa$ there exist closed convex sets $A_\alpha, B_\alpha \subseteq X_\alpha$ such that $A_\alpha \cup B_\alpha = X_\alpha, X_{\alpha+1} = (A_\alpha \times \{0\}) \cup (B_\alpha \times \{1\})$ and $p_\alpha^{\alpha+1}$ is the projection.

Lemma 3.4 *Let K, L be compact median spaces, where K is zero-dimensional of weight $\kappa \geq \aleph_0$, and let $f : K \rightarrow L$ be an epimorphism. There exists an inverse sequence of compact median spaces $\mathcal{S} = \langle Y_\alpha, f_\alpha^\beta, \alpha < \beta < \kappa \rangle$ such that $K = \varprojlim \mathcal{S}$ and*

1. $Y_0 = L, f_0 = f$;
2. for every limit ordinal $\lambda < \kappa, Y_\lambda = \varprojlim \langle Y_\alpha : \alpha < \lambda \rangle$;
3. for each $\alpha < \kappa$ there exist closed convex sets $E_\alpha, F_\alpha \subseteq X_\alpha$ such that $E_\alpha \cup F_\alpha = Y_\alpha, Y_{\alpha+1} = (E_\alpha \times \{0\}) \cup (F_\alpha \times \{1\})$, and $f_\alpha^{\alpha+1}$ is the canonical projection.

Proof Let $K = \varprojlim \mathcal{S}$ where $\mathcal{S} = \langle X_\alpha, p_\alpha^\beta, \alpha \leq \beta < \kappa \rangle$ is an inverse sequence of compact median zero-dimensional spaces such as in the Theorem 3.3.

Inductively we define a map $h_{\alpha+1} : K \rightarrow Y_{\alpha+1}$ by

$$h_{\alpha+1}(x) = \begin{cases} (h_\alpha(x), 0) & x \in p_{\alpha+1}^{-1}(A_\alpha \times \{0\}) \\ (h_\alpha(x), 1) & \text{otherwise.} \end{cases}$$

and $Y_{\alpha+1} = (E_\alpha \times \{0\}) \cup (F_\alpha \times \{1\})$ and $q_\alpha^{\alpha+1} : Y_{\alpha+1} \rightarrow Y_\alpha$ is the projection, where $E_\alpha = h_\alpha[p_{\alpha+1}^{-1}(A_\alpha \times \{0\})]$ and $F_\alpha = h_\alpha[p_{\alpha+1}^{-1}(B_\alpha \times \{1\})]$. Let $Y_0 = L$ and $h_0 = f$. If $\alpha < \kappa$ is a limit ordinal then let h_α be the map induced by $\{h_\beta : \beta < \alpha\}$ and let

$Y_\beta = \varprojlim(Y_\alpha : \alpha < \beta)$. Let $Y = \varprojlim(Y_\alpha, q_\alpha^\beta, \kappa)$. Since all h_α are epimorphisms, the induced map $h : K \rightarrow Y$ is an epimorphism. We claim that h is one-to-one, i.e. it is an isomorphism. Indeed, if $x, y \in K$ are different points, then there is $\alpha < \kappa$ and clopen sets $p_{\alpha+1}^{-1}(A_\alpha \times \{0\})$, $p_{\alpha+1}^{-1}(B_\alpha \times \{1\})$ such that $x \in p_{\alpha+1}^{-1}(A_\alpha \times \{0\})$ and $y \in p_{\alpha+1}^{-1}(B_\alpha \times \{1\})$. Therefore $h(x) \in q_{\alpha+1}^{-1}(E_\alpha \times \{0\})$ and $h(y) \in q_{\alpha+1}^{-1}(F_\alpha \times \{1\})$. \square

Lemma 3.5 ([6, Lemma 1.1, p. 45]) *Let A, B be two disjoint closed convex subset of a zero-dimensional compact median space. Then there exists a clopen halfspace $H \subseteq X$ such that $A \cap H = \emptyset$ and $B \subseteq H$.*

Lemma 3.6 *If X is a compact median space and \mathcal{R} is a normal family which is a subbase both for the closed sets and for the convexity, then any normal subbase $\mathcal{B} \subseteq \mathcal{R}$ for the closed sets is a subbase for the convexity. Moreover any $x \in X$ and closed convex $C \subseteq X \setminus \{x\}$ can be screened with some $A, B \in \mathcal{B}$.*

Proof Let \mathcal{R} be a normal subbase both for the closed sets and for the convexity and let $\mathcal{B} \subseteq \mathcal{R}$ be a subbase for the closed sets. The convexity \mathcal{C}' generated by the family \mathcal{B} satisfies the condition CC_2 . We check that $\mathcal{C} \subseteq \mathcal{C}'$. Let $C \in \mathcal{C}$. Define $I : X^2 \rightarrow X$ and $I' : X^2 \rightarrow X$ by $I(x, y) = \bigcap \{A \in \mathcal{C} : x, y \in A\}$ and $I'(x, y) = \bigcap \{A \in \mathcal{C}' : x, y \in A\}$. It suffices to check that if $a, b \in C$ then $I'(a, b) \subseteq C$. Fix $a, b \in C$ and $x \in I'(a, b)$. For every $c, d, g \in X$ the set $I(c, d) \cap I(d, g) \cap I(c, g)$ is a singleton by CC_2 . Hence $\{m(a, b, x)\} = I(a, b) \cap I(a, x) \cap I(b, x) \subseteq I'(a, b) \cap I'(a, x) \cap I'(b, x) = \{x\}$. Thus $x = m(a, b, x) \in I(a, b) \subseteq C$.

Let $x \in X$ and let $C \subseteq X \setminus \{x\}$ be a closed convex set. By CC_2 , the intersection $\bigcap \{[x, c] : c \in C\} \cap C$ is a singleton, say $\{c_0\}$. Again by CC_2 , there are $A, B \in \mathcal{B}$ such that $x \notin B$ and $c_0 \notin A$ and $A \cup B = X$. Finally, $C \cap A = \emptyset$, because if $c \in C \cap A$ then $c_0 \in [x, c] \subseteq A$, which would be a contradiction. \square

4 Parovičenko median spaces

Let κ be a fixed infinite regular cardinal. We shall now discuss κ -Parovičenko median spaces, aiming at their internal characterization.

Lemma 4.1 *For every compact median space K there exists a zero-dimensional compact median space K_0 with $w(K) = w(K_0)$ and an epimorphism $f : K_0 \rightarrow K$.*

Proof By Theorem 3.2, we may assume that $K \subseteq [0, 1]^\kappa$, where the cube is considered with the product median structure. Consider the Cantor set $C = 2^\omega$ with the median operation induced from its standard linear ordering (not the product structure). Let $h : C \rightarrow [0, 1]$ be the standard order preserving continuous surjection. Then h is an epimorphism of compact median spaces. Furthermore, the power $h^\kappa : C^\kappa \rightarrow [0, 1]^\kappa$ is an epimorphism. Finally, we may set $K_0 = (h^\kappa)^{-1}[K]$. \square

Lemma 4.2 *Every κ -Parovičenko median space is zero-dimensional.*

Proof Let P be a κ -Parovičenko median space. It is sufficient to show that two distinct elements of P can be separated by clopen sets. Fix $a \neq b$ in P . By Proposition 3.1, there exists a continuous median preserving map $q : P \rightarrow [0, 1]$ such that $q(a) = 0$ and $q(b) = 1$. By Lemma 4.1, we can find a 0-dimensional compact median space L and an epimorphism $f : L \rightarrow [0, 1]$. Let H be a clopen halfspace in L that separates f^0 from $f^{-1}(1)$. As P

is κ -Parovičenko, there exists an epimorphism $g : P \rightarrow L$ such that $f \circ g = q$. The set $W = g^{-1}(H)$ is clopen in P and it separates a from b . \square

Theorem 4.3 *For every infinite regular cardinal κ , there exists a κ -Parovičenko compact median space of weight $\leq \kappa^{<\kappa}$.*

Proof Let Φ denote the forgetful functor from the category of compact median spaces to \mathbf{Comp} . In view of Theorem 2.5, conditions (A)–(D) and (F), (T) are satisfied, therefore the theorem follows from Theorem 2.3. \square

By the theorem above combined with Theorems 2.3 and 2.4(3), without any extra set-theoretic assumptions there exists a unique ω -Parovičenko compact median space P_ω of countable weight. It is an easy exercise to check that P_ω carries the topology of the Cantor set. On the other hand, its median structure seems to be rather complicated and not definable by any simple formula.

4.1 Main result

Let P be a compact median space. The collection of all clopen halfspaces in P will be denoted by $H(P)$ and let $H(P)^+ = H(P) \setminus \{\emptyset\}$. Recall that $H(P)^+$ separates the points of P . Given a family $\mathcal{A} \subseteq H(P)$, it is natural to consider the following quotient space P/\mathcal{A} : Given $x, y \in P$, we define $x \sim y$ iff no member of \mathcal{A} separates x from y . The space P/\mathcal{A} is formally the quotient P/\sim , endowed with the quotient topology. It is actually a median space, where the median is well-defined by the formula $m([x]_\sim, [y]_\sim, [z]_\sim) = m(x, y, z)$, $x, y, z \in P$. The canonical quotient map is obviously median preserving. We shall use this construction in the proof below. Note that the topological quotient space P/\mathcal{A} can be defined as long as \mathcal{A} is a family of clopen subsets of P . The fact that \mathcal{A} consists of halfspaces allows us to get a median structure on the quotient. Note also that $P/H(P) = P$.

Theorem 4.4 *Given a compact median space P , the following are equivalent:*

- (a) P is a κ -Parovičenko space.
- (b) If $f : P \rightarrow K$ is an epimorphism, $w(K) < \kappa$ and $E, F \subseteq K$ are closed convex sets such that $E \cup F = K$ then there exists $H \in H(P)$ such that $f[H] = E$ and $f[P \setminus H] = F$.
- (c) P is zero-dimensional and satisfies conditions:
 - (M1) If $\mathcal{A}, \mathcal{B} \subseteq H(P)$, $|\mathcal{A} \cup \mathcal{B}| < \kappa$ and any $A \in \mathcal{A}$ and $B \in \mathcal{B}$ are disjoint then there exists $C \in H(P)$ with $\bigcup \mathcal{A} \subseteq C$ and $\bigcup \mathcal{B} \subseteq P \setminus C$,
 - (M2) If $\mathcal{A} \subseteq H(P)$ is linked and $|\mathcal{A}| < \kappa$, then there exists $C \in H(P)^+$ such that $C \subseteq \bigcap \mathcal{A}$,
 - (M3) for each $A \in H(P)^+$ there exists $B_0, B_1 \in H(P)^+$ such that $B_0 \cap B_1 = \emptyset$ and $B_0 \cup B_1 \subseteq A$.

Proof (b) \Rightarrow (a) Let $f : K \rightarrow L$ and $q : P \rightarrow L$ be epimorphisms, where K, L are compact median spaces of weight $< \kappa$. By Lemma 4.1 we can assume that K is zero-dimensional. Using Lemma 3.4, we find an inverse sequence of compact median spaces $\mathcal{S} = \langle Y_\alpha, f_\alpha^\beta, \alpha < \beta < \kappa \rangle$ such that $K = \varprojlim \mathcal{S}$ and

1. $Y_0 = L$;
2. for a limit ordinal $\lambda < \kappa$, $Y_\lambda = \varprojlim \langle Y_\alpha : \alpha < \lambda \rangle$;
3. For each $\alpha < \kappa$ there exist closed convex sets $A_\alpha, B_\alpha \subseteq X_\alpha$ such that $A_\alpha \cup B_\alpha = Y_\alpha$, and $Y_{\alpha+1} = (A_\alpha \times \{0\}) \cup (B_\alpha \times \{1\})$, and $f_\alpha^{\alpha+1}$ is the canonical projection.

Assume that we have already defined an epimorphism $g_\alpha : P \rightarrow X_\alpha$. By condition (b), there is $H \in H(P)$ such that $f[H] = A_\alpha$ and $f[P \setminus H] = B_\alpha$. Now, we define an epimorphism $g_{\alpha+1} : P \rightarrow Y_{\alpha+1}$ by the formula:

$$g_{\alpha+1}(x) = \begin{cases} (g_\alpha(x), 0), & \text{if } x \in H \\ (g_\alpha(x), 1), & \text{otherwise.} \end{cases}$$

If $\alpha < \kappa$ is a limit ordinal, let g_α be induced by $\{g_\beta : \beta < \alpha\}$. Finally, the map $g : P \rightarrow K$ induced by $\{g_\beta : \beta < \kappa\}$ is an epimorphism and $f \circ g = q$.

(c) \Rightarrow (b) Let $f : P \rightarrow K$ be an epimorphism with $w(K) < \kappa$ and let $E, F \subseteq K$ be closed convex sets with $E \cup F = K$. Since $w(K) = \tau < \kappa$, there exists a dense subset $D \subseteq E \cap F$ such that $|D| \leq \tau$. □

Claim 4.5 *Given an open set $W \subseteq K$ whose complement is convex, there exists a family $\{H_\alpha : \alpha < \tau\} \subseteq H(P)$ such that $f^{-1}(W) = \bigcup_{\alpha < \tau} H_\alpha$, where $\tau = w(K)$.*

Proof of Claim 4.5 By [6, Corollary V.1.3], the collection \mathcal{R} of closed halfspaces is a normal subbase both for the closed sets and for the convexity. Let $\mathcal{B} \subseteq \mathcal{R}$ be a normal subbase for the closed sets such that $|\mathcal{B}| = \tau$. By Lemma 3.6, \mathcal{B} generates the convexity structure of K . Moreover, any $x \in X$ and closed convex set $C \subseteq X \setminus \{x\}$ can be screened with some $A, B \in \mathcal{B}$. Given $x \in W$, let $A, B \in \mathcal{B}$ be such that $x \in A \subseteq W$ and $A \cup B = K$. Hence, there exists a collection of closed halfspaces $\{U_\alpha : \alpha \in \tau\}$ such that $W = \bigcup \{U_\alpha : \alpha \in \tau\}$. By Lemma 3.5 applied to the closed convex disjoint sets $P \setminus f^{-1}(W), f^{-1}(U_\alpha)$, there exists $H_\alpha \in H(P)$ such that $f^{-1}(U_\alpha) \subseteq H_\alpha \subseteq f^{-1}(W)$. Finally we get $f^{-1}(W) = \bigcup \{H_\alpha : \alpha \in \tau\}$. □

Claim 4.6 *Given $p \in K$, there are $H_0, H_1 \in H^+(P)$ such that $H_0 \cap H_1 = \emptyset$ and $H_0 \cup H_1 \subseteq f^{-1}(p)$.*

Proof of Claim 4.6 Let $p \in P$. By Claim 4.5, there is a collection of clopen halfspaces $\{H_\alpha : \alpha \in \tau\} \subseteq H(P)$ such that $f^{-1}(p) = \bigcap \{H_\alpha : \alpha \in \tau\}$. By condition (M2), there is $G \in H^+(P)$ such that $G \subseteq \bigcap \{H_\alpha : \alpha \in \tau\}$. Using condition (M3), we find $H_1, H_0 \in H^+(P)$ such that $H_0 \cap H_1 = \emptyset$ and $H_0 \cup H_1 \subseteq G \subseteq f^{-1}(p)$. □

By Claim 4.5 there are $\{H_\alpha^E : \alpha < \tau\} \subseteq H(P)$ and $\{H_\alpha^F : \alpha < \tau\} \subseteq H(P)$ such that $f^{-1}(K \setminus E) = \bigcup \{H_\alpha^F : \alpha < \tau\}$ and $f^{-1}(K \setminus F) = \bigcup \{H_\alpha^E : \alpha < \tau\}$. By Claim 4.6 for each $d \in D$ there are disjoint $H_d^1, H_d^0 \in H(P)$ such that $H_d^0 \cup H_d^1 \subseteq f^{-1}(d)$. Define $\mathcal{A} = \{H_\alpha^F : \alpha < \tau\} \cup \{H_d^0 : d \in D\}$ and $\mathcal{C} = \{H_\alpha^E : \alpha < \tau\} \cup \{H_d^1 : d \in D\}$. Notice that $|\mathcal{A} \cup \mathcal{C}| < \kappa$ and $A \cap C = \emptyset$ for every $A \in \mathcal{A}$ and $C \in \mathcal{C}$. By condition (M1), there is $H \in H(P)$ such that $\bigcup \mathcal{A} \subseteq H$ and $\bigcup \mathcal{C} \subseteq P \setminus H$. Since $f[\bigcup \mathcal{A}], f[\bigcup \mathcal{C}]$ are dense subsets of F and E , respectively, we get $f[H] = F$ and $f[P \setminus H] = E$.

(a) \Rightarrow (c) We first show (M1).

Let $\mathcal{A}, \mathcal{B} \subseteq H(P)$ be such that $|\mathcal{A} \cup \mathcal{B}| < \kappa$ and $A \cap B = \emptyset$ for every $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Let $L = P / (\mathcal{A} \cup \mathcal{B} \cup \mathcal{A}' \cup \mathcal{B}')$, where $\mathcal{A}' = \{P \setminus A : A \in \mathcal{A}\}$ and $\mathcal{B}' = \{P \setminus B : B \in \mathcal{B}\}$. Since $\mathcal{A} \cup \mathcal{B} \cup \mathcal{A}' \cup \mathcal{B}' \subseteq H(P)$ is a normal binary family, the canonical quotient mapping $q : P \rightarrow L$ is an epimorphism (i.e. it is median preserving). Define $E = \bigcap \{L \setminus A^+ : A \in \mathcal{A}\}$ and $F = \bigcap \{L \setminus B^+ : B \in \mathcal{B}\}$. The sets E, F are closed convex and $E \cup F = L$. Put $K = E \times \{0\} \cup F \times \{1\}$ and define an epimorphism $f : K \rightarrow L$ by $f(x, i) = x$. By condition (a), there is an epimorphism $g : P \rightarrow K$ such that $q = f \circ g$. Now we shall show that $\bigcup \mathcal{A} \subseteq g^{-1}(F \times \{0\})$ and $\bigcup \mathcal{B} \subseteq g^{-1}(E \times \{1\})$. Let $x \in A \in \mathcal{A}$. Since $q(x) \in A^+ \subseteq F \setminus E$, we get $g(x) \in F \times \{0\}$.

Next we show (M2).

Let $\mathcal{A} \subseteq H(P)$ be a linked family and $|\mathcal{A}| < \kappa$. Put $L = P/(\mathcal{A} \cup \mathcal{A}')$, where $\mathcal{A}' = \{P \setminus A : A \in \mathcal{A}\}$ and $\mathcal{A} \cup \mathcal{A}'$ is a normal binary family. Let $q : P \rightarrow L$ be the canonical epimorphism. Since \mathcal{A} is linked and $H(P)$ is binary we have $\bigcap \{A^+ : A \in \mathcal{A}\} \neq \emptyset$. Let $x \in \bigcap \{A^+ : A \in \mathcal{A}\}$ and $K = (L \times \{0\}) \cup (\{x\} \times \{1\})$. Define $f : K \rightarrow L$ by $f(x, i) = x$. Then f is an epimorphism. By condition (a) there is an epimorphism $g : P \rightarrow K$ such that $q = f \circ g$. We shall prove that $g^{-1}(\{x\} \times \{1\}) \subseteq \bigcap \mathcal{A}$. Let $a \in g^{-1}(\{x\} \times \{1\})$ and suppose that $a \notin A$ for some $A \in \mathcal{A}$. Then $a \in P \setminus A$ and hence $q(a) \in (P \setminus A)^+$. This is a contradiction with $q(a) = f(g(a)) = f(x, 1) = x \in \bigcap \{A^+ : A \in \mathcal{A}\}$.

We now show (M3).

Let $A \in H(P)^+$. Define an epimorphism $q : P \rightarrow \{0, 1\}$ by

$$q(x) = \begin{cases} 0 & \text{if } x \in A, \\ 1 & \text{otherwise.} \end{cases}$$

Consider the set

$$\lambda 3 = \{\mathcal{R} : \mathcal{R} \text{ is a maximal linked system in } \mathcal{P}(3)\}$$

equipped with the topology generated by the family $\{A^+ : A \subseteq 3\}$, where $A^+ = \{\xi \in \lambda 3 : A \in \xi\}$ (see [15, Chap. II] for details). It is easy to see that $\lambda 3 = \{\xi_0, \xi_1, \xi_2, \xi_3\}$, where $\xi_i = \{A \subseteq \{0, 1, 2\} : i \in A\}$ for $i \in \{0, 1, 2\}$ and $\xi_3 = \{\{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}$. The space $\lambda 3$ is a compact median space with the median operation $m(x, y, z) = (x \cap y) \cup (x \cap z) \cup (y \cap z)$. Let us define a map $f : \lambda 3 \rightarrow \{0, 1\}$ by the formula:

$$f(\xi_i) = \begin{cases} 1 & \text{if } i = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\{2\}^+ = \{\xi_2\}$ is a clopen halfspace in $\lambda 3$, the map f is an epimorphism. By condition (a), there is an epimorphism $g : P \rightarrow K$ such that $q = f \circ g$. Now, we prove that $g^{-1}(\{0, 1\}^+) \subseteq A$. Let $a \in g^{-1}(\{0, 1\}^+)$. Then $g(a) \in \{0, 1\}^+ = \{\xi_0, \xi_1, \xi_3\}$. So $f(g(a)) = 0$, which implies that $q(a) = f(g(a)) = 0$. Therefore $a \in A$, by the definition of q . It remains to note that $g^{-1}(\{0\}^+)$ and $g^{-1}(\{1\}^+)$ are clopen disjoint halfspaces contained in A , which completes the proof. \square

5 A characterization of the Continuum Hypothesis

By Theorem 2.4(3), if $\tau^+ = 2^\tau$ then there is a unique τ^+ -Parovičenko space of weight 2^τ . We prove the converse implication by constructing two concrete examples of τ^+ -Parovičenko spaces of weight $\leq 2^\tau$ which are not homeomorphic whenever $\tau^+ \neq 2^\tau$.

Lemma 5.1 *For every regular cardinal κ such that $\tau^+ \leq \kappa \leq 2^\tau$ there exists a τ^+ -Parovičenko compact median space P of weight $\leq 2^\tau$ such that*

$$P = \varprojlim \langle K_\alpha; p_\alpha^\beta; \alpha \leq \beta < \kappa \rangle,$$

where

1. K_α is a compact median space of weight $\leq 2^\tau$ for every $\alpha < \kappa$,
2. $K_\alpha = \varprojlim \langle K_\beta, p_\gamma^\beta; \gamma \leq \beta < \alpha \rangle$ and p_β^α is a projection from K_α onto K_β for a limit ordinal $\alpha < \kappa$, and $\beta < \alpha$,

3. $K_\alpha = K_\beta \times D^{2^\tau}$ and p_β^α is the projection on the second coordinate, for every even ordinal $\alpha = \beta + 1$.

The same proof as in Lemma 2.2 works.

Proof We define an inverse sequence $\langle K_\alpha; p_\alpha^\beta; \alpha \leq \beta < \kappa \rangle$, where the spaces K_α are compact median of weight $\leq 2^\tau$ and the maps p_α^β are epimorphisms, in following way. Let $K_0 = D^{2^\tau}$. Assume that $\alpha < \kappa$ and spaces K_β for $\beta < \alpha$ and p_γ^β for $\gamma \leq \beta < \alpha$ are already defined. If α is a limit ordinal let us define $K_\alpha = \varprojlim \langle K_\beta, p_\gamma^\beta; \gamma \leq \beta < \alpha \rangle$ and p_β^α as a projection from K_α onto K_β for $\beta < \alpha$. If $\alpha = \beta + 1$ and α is an odd ordinal then we define K_α and p_β^α as in Lemma 2.2. If $\alpha = \beta + 1$ and α is an even ordinal then we define $K_\alpha = K_\beta \times D^{2^\tau}$ and p_β^α is a projection on second coordinate.

Let $P = \varprojlim \langle K_\alpha; p_\alpha^\beta; \alpha \leq \beta < 2^\tau \rangle$. The space P is compact median and each projection $p_\alpha: P \rightarrow K_\alpha$ is an epimorphism. Obviously $w(P) = 2^\tau$. Let $q: P \rightarrow L$ and $f: K \rightarrow L$ be epimorphisms, where $w(K) \leq \tau$. Since $w(L) \leq \tau$ and κ is regular, there is an even ordinal $\alpha < \kappa$ and an epimorphism $q': K_\alpha \rightarrow L$ such that $q = q' \circ p_\alpha$. By the construction, there is an epimorphism $p: K_{\alpha+1} \rightarrow K$ such that $q' \circ p_{\alpha+1}^\alpha = f \circ p$. It remains to note that

$$f \circ (p \circ p_{\alpha+1}) = q' \circ p_{\alpha+1}^{\alpha+1} \circ p_{\alpha+1} = q' \circ p_\alpha = q,$$

which completes the proof that P is a τ^+ -Parovičenko space. □

Theorem 5.2 *For every regular cardinal κ such that $\tau^+ \leq \kappa \leq 2^\tau$ there is a τ^+ -Parovičenko compact median space P of weight $\leq 2^\tau$ such that $\chi(x, P) \geq \kappa$ for all $x \in P$.*

Proof By Lemma 5.1, there is τ^+ -Parovičenko compact median space P with properties (1)–(3).

Assume that there is $x \in P$ such that $\chi(x, P) = \lambda < \kappa$. Without loss of the generality, we can assume that $\mathcal{B}_x \subseteq H(P)^+$ is a subbase of size λ at the point x . As $\text{cf}(\kappa) = \kappa$, there is an odd ordinal $\alpha < \lambda$ such that $\mathcal{B}_x = \{(p_\alpha)^\gamma(U_\gamma) : \gamma < \lambda \text{ and } U_\gamma \subseteq K_\alpha \text{ is clopen}\}$. Therefore $\{x\} = \bigcap \{(p_\alpha)^\gamma(U_\gamma) : \gamma < \lambda \text{ and } U_\gamma \subseteq K_\alpha \text{ is clopen}\} = (p_\alpha)^{-1}(\bigcap \{U_\gamma : \gamma < \lambda \text{ and } U_\gamma \subseteq K_\alpha \text{ is clopen}\})$. Setting $p_\alpha(x) = x_\alpha$, we get a contradiction with $(p_\alpha)^{-1}(x_\alpha) = (p_{\alpha+1})^{-1}((p_\alpha^{\alpha+1})^{-1}(x_\alpha))$ and $|(p_\alpha^{\alpha+1})^{-1}(x_\alpha)| = 2^{2^\tau}$. □

Theorem 5.3 *There exists a τ^+ -Parovičenko compact median space P of weight 2^τ and $\chi(x, P) = \tau^+$ for some $x \in P$.*

Proof By Lemma 5.1 there is a τ^+ -Parovičenko compact median space P_0 with properties (1)–(3) for $\kappa = \tau^+$.

We define a sequence $\{W_\alpha : \alpha < \tau^+\}$ of G_δ closed convex sets of P_0 . Let $A = \bigcap \{p_\alpha^{-1}(1) : \alpha < \omega\}$. Suppose we have already defined $\{W_\beta\}_{\beta < \alpha}$ so that if β is a limit ordinal or an odd ordinal then $W_\beta \subsetneq \bigcap \{W_\gamma : \gamma < \beta\}$ and if $\beta = \gamma + 1$ is an even ordinal then $W_{\gamma+1} = (p_{\gamma+1})^{-1}(K_\gamma \times A)$. If $\alpha < \tau^+$ is a limit ordinal or an odd ordinal then by Theorem 4.4 (M2),(M3) there is $W_\alpha \subsetneq \bigcap \{W_\beta : \beta < \alpha\}$ and $W_\alpha \in H(P)^+$. Let $D = \bigcap \{U_\alpha : \alpha < \tau^+ \text{ and } \alpha \text{ is an odd ordinal}\}$ and $P = P_0/D$. Note that $\text{int } D = \emptyset$. Suppose that $\text{int } D \neq \emptyset$. There exists an odd ordinal $\alpha < \tau^+$ and a clopen set $U \subset K_\alpha$ such that $(p_\alpha)^{-1}(U) \subseteq D \subset (p_{\alpha+1})^{-1}(K_\alpha \times A)$. Hence $(p_\alpha^{\alpha+1})^{-1}(U) \subseteq K_\alpha \times A$, but this is a contradiction with $\emptyset = \text{int } A \subseteq D^{2^\tau}$.

We shall show that P is a 0-dimensional normally supercompact space. Let $q: P_0 \rightarrow P_0/D$ be a quotient map. Let $\mathcal{B} = \{q[H] : D \subseteq H \text{ or } D \cap H = \emptyset \text{ or } H \in H(P)^+\}$. It is

easy to show that \mathcal{B} consists of clopen sets in P and separates points, hence \mathcal{B} is a subbase for P . If $\mathcal{P} \subseteq \mathcal{B}$ is a linked family then $\{q^{-1}(H) : H \in \mathcal{P}\}$ is the linked family. Thus P is a supercompact space. Since \mathcal{B} consists of clopen sets, P is a normally supercompact space.

Now, we shall show that P satisfies conditions (M1)–(M3).

(M1): Assume that $\mathcal{A}, \mathcal{P} \subseteq H(P)$, $|\mathcal{A} \cup \mathcal{P}| < \tau^+$ and any $A \in \mathcal{A}$ and $B \in \mathcal{P}$ are disjoint. If there is $C \in \mathcal{A} \cup \mathcal{P}$ such that $q[D] \in C$ then it easy to see that there is $H \in H(P)^+$ such that $D \subseteq H$ and $\bigcup \mathcal{A} \subseteq q[H]$ and $\bigcup \mathcal{P} \subseteq P \setminus q[H]$. If $q[D] \notin \mathcal{A} \cup \mathcal{P}$ then $D \cap V = \emptyset$ for $V \in \{q^{-1}(W) : W \in \mathcal{A} \cup \mathcal{P}\}$. There exists an odd ordinal $\alpha < \tau^+$ such that $W_\alpha \cap V = \emptyset$ for $V \in \{q^{-1}(W) : W \in \mathcal{A} \cup \mathcal{P}\}$. By the condition (M1) for P there exists $H \in H(P)^+$ such that $\bigcup \{q^{-1}(W) : W \in \mathcal{A}\} \cup \{W_\alpha\} \subseteq H$ and $\bigcup \{q^{-1}(W) : W \in \mathcal{P}\} \subseteq P \setminus H$. Thus $D \subset H$.

(M2): Assume that $\mathcal{A} \subseteq H(P)$ is a linked family and $|\mathcal{A}| < \tau^+$, then $\{q^{-1}(W) : W \in \mathcal{A}\}$ is a linked family. Hence there exists $C \in H(P_0)^+$ such that $C \subseteq \bigcap \{q^{-1}(W) : W \in \mathcal{A}\}$. If $D \cap \bigcap \{q^{-1}(W) : W \in \mathcal{A}\} = \emptyset$ then $q[C] \in H(P)^+$, otherwise there exists an odd ordinal $\alpha < \tau^+$ such that $W_\alpha \subsetneq \bigcap \{q^{-1}(W) : W \in \mathcal{A}\}$. Therefore $\bigcap \{q^{-1}(W) : W \in \mathcal{A}\} \cap (P_0 \setminus W_\alpha) \neq \emptyset$ and we can find $C \in H(P_0)^+$ such that $C \subseteq \bigcap \{q^{-1}(W) : W \in \mathcal{A}\} \cap (P_0 \setminus W_\alpha) \neq \emptyset$, so $q[C] \in H(P)^+$ and $q[C] \subseteq \bigcap \mathcal{A}$.

(M3): Let $A \in H(P)^+$. Since $\text{int } D = \emptyset$, we have $q^{-1}(A) \neq D$. There exists an odd ordinal $\alpha < \tau^+$ such that $q^{-1}(A) \setminus W_\alpha \neq \emptyset$. Since $W_\alpha \in H(P_0)^+$, there exists $B_0, B_1 \in H(P_0)^+$ such that $B_0 \cap B_1 = \emptyset$ and $B_0 \cup B_1 \subseteq q^{-1}(A) \setminus W_\alpha \neq \emptyset$. It easy to see that $\chi(q[D], P) \leq \tau^+$. Since the sequence $\{W_\alpha : \alpha < \tau^+ \text{ and } \alpha \text{ is an odd ordinal}\}$ is strictly decreasing we have $\chi(q[D], P) = \tau^+$. This completes the proof. \square

Corollary 5.4 *If every two τ^+ -Parovičenko spaces of weight $\leq 2^\tau$ are homeomorphic then $2^\tau = \tau^+$.*

Proof Suppose that $2^\tau > \tau^+$, then by Theorem 5.2 for $(\tau^+)^+ \leq 2^\tau$ there exists a τ^+ -Parovičenko compact median space P of weight $\leq 2^\tau$ such that $\chi(x, P) \geq (\tau^+)^+$ for all $x \in P$. By Theorem 5.3, there exists a τ^+ -Parovičenko compact median space P_1 of weight 2^τ and $\chi(x, P_1) = \tau^+$ for some $x \in P_1$; hence P and P_1 are not homeomorphic, a contradiction. \square

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