

Existence of periodic solutions of fourth-order nonlinear difference equations

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Abstract By making use of the critical point theory, the existence of periodic solutions for fourth-order nonlinear difference equations is obtained. The proof is based on the Saddle Point Theorem in combination with variational technique. The problem is to solve the existence of periodic solutions of fourth-order nonlinear difference equations. Results obtained complement the existing one.

Keywords Existence · Periodic solutions · Fourth-order · Nonlinear difference equations · Discrete variational theory

Mathematics Subject Classification (2000) 39A11

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1 Introduction

Let \mathbf{N} , \mathbf{Z} and \mathbf{R} denote the sets of all natural numbers, integers and real numbers respectively. For any $a, b \in \mathbf{Z}$, define $\mathbf{Z}(a) = \{a, a + 1, \dots\}$, $\mathbf{Z}(a, b) = \{a, a + 1, \dots, b\}$ when $a \leq b$. Let the symbol $*$ denote the transpose of a vector.

The present paper considers the following fourth-order nonlinear difference equation

$$\Delta^2 (r_{n-2} \Delta^2 u_{n-2}) = f(n, u_{n+1}, u_n, u_{n-1}), \quad n \in \mathbf{Z}, \tag{1.1}$$

where Δ is the forward difference operator $\Delta u_n = u_{n+1} - u_n$, $\Delta^2 u_n = \Delta(\Delta u_n)$, $r_n > 0$ is real valued for each $n \in \mathbf{Z}$, $f \in C(\mathbf{Z} \times \mathbf{R}^3, \mathbf{R})$, r_n and $f(n, v_1, v_2, v_3)$ are T -periodic in n for a given positive integer T .

We may think of (1.1) as a discrete analogue of the following fourth-order functional differential equation

$$[r(t)u''(t)]'' = f(t, u(t + 1), u(t), u(t - 1)), \quad t \in \mathbf{R}. \tag{1.2}$$

Equation (1.2) includes the following equation

$$u^{(4)}(t) = f(t, u(t)), \quad t \in \mathbf{R}, \tag{1.3}$$

which is used to model deformations of elastic beams [9,29]. Equations similar in structure to (1.2) arise in the study of the existence of solitary waves of lattice differential equations, see Smets and Willem [31].

Recently, the theory of nonlinear difference equations has been widely used to study discrete models appearing in many fields such as computer science, economics, neural network, ecology, cybernetics, etc. For the general background of difference equations, one can refer to monographs [1,21,25]. Since the last decade, there has been much progress on the qualitative properties of difference equations, which included results on stability and attractivity [13,23,39] and results on oscillation and other topics, see [1–3,5,10,16–19,22,34–38].

In 1995, Peterson and Ridenhour [27] considered the disconjugacy of the following equation

$$\Delta^4 u_{n-2} + q_n u_n = 0, \quad n \in \mathbf{Z}.$$

Yan, Liu [34] in 1997 and Thandapani, Arockiasamy [32] in 2001 studied the following fourth-order difference equation of form,

$$\Delta^2 (r_n \Delta^2 u_n) + f(n, u_n) = 0, \quad n \in \mathbf{Z}. \tag{1.4}$$

the authors obtain criteria for the oscillation and nonoscillation of solutions for Eq. (1.4).

When $\beta > 2$, in Theorem 1.1, Cai et al. [7] have obtained some criteria for the existence of periodic solutions of the following fourth-order difference Eq.

$$\Delta^2 (r_{n-2} \Delta^2 u_{n-2}) + f(n, u_n) = 0, \quad n \in \mathbf{Z}. \tag{1.5}$$

Furthermore, [7] is the only paper we found which deals with the problem of periodic solutions to fourth-order difference Eq. (1.5). When $\beta < 2$, can we still find the periodic solutions of (1.5)?

By using various methods and techniques, such as Schauder fixed point theorem, the cone theoretic fixed point theorem, the method of upper and lower solutions, coincidence degree theory, a series of existence results of nontrivial solutions for differential equations have been obtained in [4,6,8,20]. Critical point theory is also an important tool to deal with problems

on differential equations [9, 11, 12, 24, 29, 33]. Because of applications in many areas for difference equations [1, 21, 25], recently, a few authors have gradually paid attention to applying critical point theory to deal with periodic solutions on discrete systems, see [16–18, 30, 36, 38]. Particularly, Guo and Yu [16–18] and Shi et al. [30] studied the existence of periodic solutions of second-order nonlinear difference equations by using the critical point theory. Compared to one-order or second-order difference equations, the study of higher-order equations, and in particular, fourth-order equations, has received considerably less attention (see, for example, [1, 7, 10, 14, 21, 27, 28, 32, 34] and the references contained therein). However, to the best of our knowledge, results obtained in the literature on the periodic solutions of (1.1) are very scarce. Since f in (1.1) depends on u_{n+1} and u_{n-1} , the traditional ways of establishing the functional in [16–18, 36, 38] are inapplicable to our case. The main purpose of this paper is to give some sufficient conditions for the existence of periodic solutions to fourth-order nonlinear difference equations. The main approaches used in our paper are variational techniques and the Saddle Point Theorem. In particular, our results complement the result in the literature [7]. In fact, one can see the following Remark 1.4 for details.

For basic knowledge on variational methods, we refer the reader to [15, 24, 26, 29].
Let

$$\underline{r} = \min_{n \in \mathbf{Z}(1, T)} \{r_n\}, \quad \bar{r} = \max_{n \in \mathbf{Z}(1, T)} \{r_n\}.$$

Now we state the main results of this paper.

Theorem 1.1 *Assume that the following hypotheses are satisfied: (F₁) there exists a function $F(n, v_1, v_2) \in C^1(\mathbf{Z} \times \mathbf{R}^2, \mathbf{R})$ such that*

$$\begin{aligned} F(n+T, v_1, v_2) &= F(n, v_1, v_2), \\ \frac{\partial F(n-1, v_2, v_3)}{\partial v_2} + \frac{\partial F(n, v_1, v_2)}{\partial v_2} &= f(n, v_1, v_2, v_3); \end{aligned}$$

(F₂) *there exists a constant $M_0 > 0$ for all $(n, v_1, v_2) \in \mathbf{Z} \times \mathbf{R}^2$ such that*

$$\left| \frac{\partial F(n, v_1, v_2)}{\partial v_1} \right| \leq M_0, \quad \left| \frac{\partial F(n, v_1, v_2)}{\partial v_2} \right| \leq M_0;$$

(F₃) $F(n, v_1, v_2) \rightarrow +\infty$ *uniformly for $n \in \mathbf{Z}$ as $\sqrt{v_1^2 + v_2^2} \rightarrow +\infty$.*

Then for any given positive integer $m > 0$, (1.1) has at least one mT -periodic solution.

Remark 1.1 Assumption (F₂) implies that there exists a constant $M_1 > 0$ such that (F'₂) $|F(n, v_1, v_2)| \leq M_1 + M_0(|v_1| + |v_2|)$, $\forall (n, v_1, v_2) \in \mathbf{Z} \times \mathbf{R}^2$.

Theorem 1.2 *Assume that (F₁) holds; further (F₄) there exist constants $R_1 > 0$ and α , $1 < \alpha < 2$ such that for $n \in \mathbf{Z}$ and $\sqrt{v_1^2 + v_2^2} \geq R_1$,*

$$0 < \frac{\partial F(n, v_1, v_2)}{\partial v_1} v_1 + \frac{\partial F(n, v_1, v_2)}{\partial v_2} v_2 \leq \alpha F(n, v_1, v_2);$$

(F₅) *there exist constants $a_1 > 0$, $a_2 > 0$ and γ , $1 < \gamma \leq \alpha$ such that*

$$F(n, v_1, v_2) \geq a_1 \left(\sqrt{v_1^2 + v_2^2} \right)^\gamma - a_2, \quad \forall (n, v_1, v_2) \in \mathbf{Z} \times \mathbf{R}^2.$$

Then for any given positive integer $m > 0$, (1.1) has at least one mT -periodic solution.

Remark 1.2 Assumption (F_4) implies that for each $n \in \mathbf{Z}$ there exist constants $a_3 > 0$ and $a_4 > 0$ such that $(F'_4) F(n, v_1, v_2) \leq a_3 \left(\sqrt{v_1^2 + v_2^2}\right)^\alpha + a_4, \forall(n, v_1, v_2) \in \mathbf{Z} \times \mathbf{R}^2$.

Remark 1.3 The results of Theorems 1.1 and 1.2 ensure that (1.1) has at least one mT -periodic solution. However, in some cases, we are interested in the existence of nontrivial periodic solutions for (1.1).

In this case, we have

Theorem 1.3 Assume that (F_1) holds; further $(F_6) F(n, 0) = 0, f(n, v_1, v_2, v_3) = 0$ if and only if $v_2 = 0$, for all $n \in \mathbf{Z}$; (F_7) there exists a constant $\alpha, 1 < \alpha < 2$ such that for $n \in \mathbf{Z}$,

$$0 < \frac{\partial F(n, v_1, v_2)}{\partial v_1} v_1 + \frac{\partial F(n, v_1, v_2)}{\partial v_2} v_2 \leq \alpha F(n, v_1, v_2), \quad \forall(v_1, v_2) \neq 0;$$

(F_8) there exist constants $a_5 > 0$ and $\gamma, 1 < \gamma \leq \alpha$ such that

$$F(n, v_1, v_2) \geq a_5 \left(\sqrt{v_1^2 + v_2^2}\right)^\gamma, \quad \forall(n, v_1, v_2) \in \mathbf{Z} \times \mathbf{R}^2.$$

Then for any given positive integer $m > 0$, (1.1) has at least one nontrivial mT -periodic solution.

Theorem 1.4 Assume that $(F_1) - (F_3)$ and (F_6) hold; further (F_9) there exist constants $a_6 > 0$ and $\theta, 0 < \theta < 2$ such that

$$F(n, v_1, v_2) \geq a_6 \left(\sqrt{v_1^2 + v_2^2}\right)^\theta, \quad \forall(n, v_1, v_2) \in \mathbf{Z} \times \mathbf{R}^2.$$

Then for any given positive integer $m > 0$, (1.1) has at least one nontrivial mT -periodic solution.

If $f(n, u_{n+1}, u_n, u_{n-1}) = -f(n, u_n)$, (1.1) reduces to (1.5). Then, we have the following results.

Theorem 1.5 Assume that the following hypotheses are satisfied: (F_{10}) there exists a functional $F(n, v) \in C^1(\mathbf{Z} \times \mathbf{R}, \mathbf{R}), F(n + T, v) = F(n, v)$ such that

$$\frac{\partial F(n, v)}{\partial v} = f(n, v);$$

$(F_{11}) F(n, 0) = 0$, for all $n \in \mathbf{Z}$;

(F_{12}) there exists a constant $\alpha, 1 < \alpha < 2$ such that for $n \in \mathbf{Z}$,

$$\alpha F(n, v) \leq v f(n, v) < 0, \quad |v| \neq 0;$$

(F_{13}) there exist constants $a_7 > 0$ and $\gamma, 1 < \gamma \leq \alpha$ such that

$$F(n, v) \leq -a_7 |v|^\gamma, \quad \forall(n, v) \in \mathbf{Z} \times \mathbf{R}.$$

Then for any given positive integer $m > 0$, (1.5) has at least one nontrivial mT -periodic solution.

Theorem 1.6 Assume that (F_{10}) holds; further (F_{14}) there exists a constant $M_0 > 0$ for all $(n, v) \in \mathbf{Z} \times \mathbf{R}$ such that $|f(n, v)| \leq M_0$; (F_{15}) $F(n, v) \rightarrow -\infty$ uniformly for $n \in \mathbf{Z}$ as $v \rightarrow +\infty$; (F_{16}) $F(n, 0) = 0$, $f(n, v) = 0$ if and only if $v = 0$, for all $n \in \mathbf{Z}$; (F_{17}) there exist constants $a_8 > 0$ and θ , $0 < \theta < 2$ such that

$$F(n, v) \leq -a_8|v|^\theta, \quad \forall (n, v) \in \mathbf{Z} \times \mathbf{R}.$$

Then for any given positive integer $m > 0$, (1.5) has at least one nontrivial mT -periodic solution.

Remark 1.4 When $\beta > 2$, in Theorem 1.1, Cai et al. [7] have obtained some criteria for the existence of periodic solutions of (1.5). When $\beta < 2$, we can still find the periodic solutions of (1.5). Hence, Theorems 1.5 and 1.6 complement the existing one.

The rest of the paper is organized as follows. First, in Sect. 2, we shall establish the variational framework associated with (1.1) and transfer the problem of the existence of periodic solutions of (1.1) into that of the existence of critical points of the corresponding functional. Some related fundamental results will also be recalled. Then, in Sect. 3, we shall complete the proof of the results by using the critical point method. Finally, in Sect. 4, we shall give two examples to illustrate the main results.

2 Variational structure and some lemmas

In order to apply the critical point theory, we shall establish the corresponding variational framework for (1.1) and give some lemmas which will be of fundamental importance in proving our main results. We start by some basic notations.

Let S be the set of sequences $u = (\dots, u_{-n}, \dots, u_{-1}, u_0, u_1, \dots, u_n, \dots) = \{u_n\}_{n=-\infty}^{+\infty}$, that is

$$S = \{\{u_n\} | u_n \in \mathbf{R}, n \in \mathbf{Z}\}.$$

For any $u, v \in S$, $a, b \in \mathbf{R}$, $au + bv$ is defined by

$$au + bv = \{au_n + bv_n\}_{n=-\infty}^{+\infty}.$$

Then S is a vector space.

For any given positive integers m and T , E_{mT} is defined as a subspace of S by

$$E_{mT} = \{u \in S | u_{n+mT} = u_n, \forall n \in \mathbf{Z}\}.$$

Clearly, E_{mT} is isomorphic to \mathbf{R}^{mT} . E_{mT} can be equipped with the inner product

$$\langle u, v \rangle = \sum_{j=1}^{mT} u_j v_j, \quad \forall u, v \in E_{mT}, \quad (2.1)$$

by which the norm $\|\cdot\|$ can be induced by

$$\|u\| = \left(\sum_{j=1}^{mT} u_j^2 \right)^{\frac{1}{2}}, \quad \forall u \in E_{mT}. \quad (2.2)$$

It is obvious that E_{mT} with the inner product (2.1) is a finite dimensional Hilbert space and linearly homeomorphic to \mathbf{R}^{mT} .

On the other hand, we define the norm $\|\cdot\|_s$ on E_{mT} as follows:

$$\|u\|_s = \left(\sum_{j=1}^{mT} |u_j|^s \right)^{\frac{1}{s}}, \tag{2.3}$$

for all $u \in E_{mT}$ and $s > 1$.

Since $\|u\|_s$ and $\|u\|_2$ are equivalent, there exist constants c_1, c_2 such that $c_2 \geq c_1 > 0$, and

$$c_1 \|u\|_2 \leq \|u\|_s \leq c_2 \|u\|_2, \quad \forall u \in E_{mT}. \tag{2.4}$$

Clearly, $\|u\| = \|u\|_2$. For all $u \in E_{mT}$, define the functional J on E_{mT} as follows:

$$\begin{aligned} J(u) &= -\frac{1}{2} \sum_{n=1}^{mT} r_{n-1} (\Delta^2 u_{n-1})^2 + \sum_{n=1}^{mT} F(n, u_{n+1}, u_n) \\ &:= -H(u) + \sum_{n=1}^{mT} F(n, u_{n+1}, u_n), \end{aligned} \tag{2.5}$$

where

$$H(u) = \frac{1}{2} \sum_{n=1}^{mT} r_{n-1} (\Delta^2 u_{n-1})^2, \quad \frac{\partial F(n-1, v_2, v_3)}{\partial v_2} + \frac{\partial F(n, v_1, v_2)}{\partial v_2} = f(n, v_1, v_2, v_3).$$

Clearly, $J \in C^1(E_{mT}, \mathbf{R})$ and for any $u = \{u_n\}_{n \in \mathbf{Z}} \in E_{mT}$, by using $u_0 = u_{mT}, u_1 = u_{mT+1}$, we can compute the partial derivative as

$$\frac{\partial J}{\partial u_n} = -\Delta^2 (r_{n-2} \Delta^2 u_{n-2}) + f(n, u_{n+1}, u_n, u_{n-1}).$$

Thus, u is a critical point of J on E_{mT} if and only if

$$\Delta^2 (r_{n-2} \Delta^2 u_{n-2}) = f(n, u_{n+1}, u_n, u_{n-1}), \quad \forall n \in \mathbf{Z}(1, mT).$$

Due to the periodicity of $u = \{u_n\}_{n \in \mathbf{Z}} \in E_{mT}$ and $f(n, v_1, v_2, v_3)$ in the first variable n , we reduce the existence of periodic solutions of (1.1) to the existence of critical points of J on E_{mT} . That is, the functional J is just the variational framework of (1.1).

Let

$$P = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & -1 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ -1 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}$$

be a $mT \times mT$ matrix. By matrix theory, we see that the eigenvalues of P are

$$\lambda_k = 2 \left(1 - \cos \frac{2k}{mT} \pi \right), \quad k = 0, 1, 2, \dots, mT - 1. \tag{2.6}$$

Thus, $\lambda_0 = 0, \lambda_1 > 0, \lambda_2 > 0, \dots, \lambda_{mT-1} > 0$. Therefore,

$$\begin{cases} \lambda_{\min} = \min\{\lambda_1, \lambda_2, \dots, \lambda_{mT-1}\} = 2 \left(1 - \cos \frac{2}{mT} \pi \right), \\ \lambda_{\max} = \max\{\lambda_1, \lambda_2, \dots, \lambda_{mT-1}\} = \begin{cases} 4, & \text{when } mT \text{ is even,} \\ 2 \left(1 + \cos \frac{1}{mT} \pi \right), & \text{when } mT \text{ is odd.} \end{cases} \end{cases} \tag{2.7}$$

Let

$$W = \ker P = \{u \in E_{mT} \mid Pu = 0 \in \mathbf{R}^{mT}\}.$$

Then

$$W = \{u \in E_{mT} \mid u = \{c\}, c \in \mathbf{R}\}.$$

Let V be the direct orthogonal complement of E_{mT} to W , i.e., $E_{mT} = V \oplus W$. For convenience, we identify $u \in E_{mT}$ with $u = (u_1, u_2, \dots, u_{mT})^*$.

Let E be a real Banach space, $J \in C^1(E, \mathbf{R})$, i.e., J is a continuously Fréchet-differentiable functional defined on E . J is said to satisfy the Palais-Smale condition (P.S. condition for short) if any sequence $\{u^{(k)}\} \subset E$ for which $\{J(u^{(k)})\}$ is bounded and $J'(u^{(k)}) \rightarrow 0 (k \rightarrow \infty)$ possesses a convergent subsequence in E .

Let B_ρ denote the open ball in E about 0 of radius ρ and let ∂B_ρ denote its boundary.

Lemma 2.1 (Saddle Point Theorem [24,29]) *Let E be a real Banach space, $E = E_1 \oplus E_2$, where $E_1 \neq \{0\}$ and is finite dimensional. Suppose that $J \in C^1(E, \mathbf{R})$ satisfies the P.S. condition and (J_1) there exist constants $\sigma, \rho > 0$ such that $J|_{\partial B_\rho \cap E_1} \leq \sigma$; (J_2) there exists $e \in B_\rho \cap E_1$ and a constant $\omega \geq \sigma$ such that $J_{e+E_2} \geq \omega$.*

Then J possesses a critical value $c \geq \omega$, where

$$c = \inf_{h \in \Gamma} \max_{u \in B_\rho \cap E_1} J(h(u)), \quad \Gamma = \{h \in C(\bar{B}_\rho \cap E_1, E) \mid h|_{\partial B_\rho \cap E_1} = id\}$$

and id denotes the identity operator.

Lemma 2.2 *Assume that $(F_1) - (F_3)$ are satisfied. Then J satisfies the P.S. condition.*

Proof Let $\{u^{(k)}\} \subset E_{mT}$ be such that $\{J(u^{(k)})\}$ is bounded and $J'(u^{(k)}) \rightarrow 0$ as $k \rightarrow \infty$. Then there exists a positive constant M_2 such that $|J(u^{(k)})| \leq M_2$.

Let $u^{(k)} = v^{(k)} + w^{(k)} \in V + W$. For k large enough, since

$$-\|u\|_2 \leq \langle J'(u^{(k)}), u \rangle = -\langle H'(u^{(k)}), u \rangle + \sum_{n=1}^{mT} f(n, u_{n+1}^{(k)}, u_n^{(k)}, u_{n-1}^{(k)}) u_n,$$

combining with (F_2) and (F_3) , we have

$$\begin{aligned} \langle H'(u^{(k)}), v^{(k)} \rangle &\leq \sum_{n=1}^{mT} f(n, u_{n+1}^{(k)}, u_n^{(k)}, u_{n-1}^{(k)}) v_n^{(k)} + \|v^{(k)}\|_2 \\ &\leq 2M_0 \sum_{n=1}^{mT} |v_n^{(k)}| + \|v^{(k)}\|_2 \\ &\leq (2M_0\sqrt{mT} + 1) \|v^{(k)}\|_2. \end{aligned}$$

On the other hand, we know that

$$\langle H'(u^{(k)}), v^{(k)} \rangle = \sum_{n=1}^{mT} r_{n-1} (\Delta^2 v_{n-1}^{(k)}, \Delta^2 v_{n-1}^{(k)}) = \sum_{n=1}^{mT} r_n (\Delta^2 v_n^{(k)}, \Delta^2 v_n^{(k)}) = 2H(v^{(k)}).$$

Since

$$\frac{r}{2} \lambda_{\min} \|x^{(k)}\|_2^2 \leq \frac{r}{2} (x^{(k)})^* P(x^{(k)}) \leq H(v^{(k)}) \leq \frac{\bar{r}}{2} (x^{(k)})^* P(x^{(k)}) \leq \frac{\bar{r}}{2} \lambda_{\max} \|x^{(k)}\|_2^2,$$

and

$$\begin{aligned} \lambda_{\min} \left\| v^{(k)} \right\|_2^2 &\leq \left\| x^{(k)} \right\|_2^2 = \sum_{n=1}^{mT} \left(v_{n+1}^{(k)} - v_n^{(k)}, v_{n+1}^{(k)} - v_n^{(k)} \right) = \left(v^{(k)} \right)^* P \left(v^{(k)} \right) \\ &\leq \lambda_{\max} \left\| v^{(k)} \right\|_2^2, \end{aligned}$$

where $x^{(k)} = \left(\Delta v_1^{(k)}, \Delta v_2^{(k)}, \dots, \Delta v_{mT}^{(k)} \right)^*$, we get

$$\frac{\underline{r}}{2} \lambda_{\min}^2 \left\| v^{(k)} \right\|_2^2 \leq H \left(v^{(k)} \right) \leq \frac{\bar{r}}{2} \lambda_{\max}^2 \left\| v^{(k)} \right\|_2^2. \quad (2.8)$$

Thus, we have

$$\underline{r} \lambda_{\min}^2 \left\| v^{(k)} \right\|_2^2 \leq \left(2M_0 \sqrt{mT} + 1 \right) \left\| v^{(k)} \right\|_2.$$

The above inequality implies that $\{v^{(k)}\}$ is bounded.

Next, we shall prove that $\{w^{(k)}\}$ is bounded. Since

$$\begin{aligned} M_2 &\geq J \left(u^{(k)} \right) = -H \left(u^{(k)} \right) + \sum_{n=1}^{mT} F \left(n, u_{n+1}^{(k)}, u_n^{(k)} \right) \\ &= -H \left(v^{(k)} \right) + \sum_{n=1}^{mT} \left[F \left(n, u_{n+1}^{(k)}, u_n^{(k)} \right) - F \left(n, w_{n+1}^{(k)}, w_n^{(k)} \right) \right] + \sum_{n=1}^{mT} F \left(n, w_{n+1}^{(k)}, w_n^{(k)} \right), \end{aligned}$$

combining with (2.8), we get

$$\begin{aligned} &\sum_{n=1}^{mT} F \left(n, w_{n+1}^{(k)}, w_n^{(k)} \right) \\ &\leq M_2 + H \left(v^{(k)} \right) - \sum_{n=1}^{mT} \left[F \left(n, u_{n+1}^{(k)}, u_n^{(k)} \right) - F \left(n, w_{n+1}^{(k)}, w_n^{(k)} \right) \right] \\ &\leq M_2 + \frac{\bar{r}}{2} \lambda_{\max}^2 \left\| v^{(k)} \right\|_2^2 \\ &\quad + \sum_{n=1}^{mT} \left| \frac{\partial F \left(n, \theta v_{n+1}^{(k)} + w_{n+1}^{(k)}, u_n^{(k)} \right)}{\partial v_1} v_{n+1}^{(k)} + \frac{\partial F \left(n, w_{n+1}^{(k)}, \theta v_n^{(k)} + w_n^{(k)} \right)}{\partial v_2} v_n^{(k)} \right| \\ &\leq M_2 + \frac{\bar{r}}{2} \lambda_{\max}^2 \left\| v^{(k)} \right\|_2^2 + 2M_0 \sqrt{mT} \left\| v^{(k)} \right\|_2. \end{aligned}$$

where $\theta \in (0, 1)$. It is not difficult to see that $\left\{ \sum_{n=1}^{mT} F \left(n, w_{n+1}^{(k)}, w_n^{(k)} \right) \right\}$ is bounded.

By (F_3) , $\{w^{(k)}\}$ is bounded. Otherwise, assume that $\left\| w^{(k)} \right\|_2 \rightarrow +\infty$ as $k \rightarrow \infty$. Since there exist $z^{(k)} \in \mathbf{R}$, $k \in \mathbf{N}$, such that $w^{(k)} = \left(z^{(k)}, z^{(k)}, \dots, z^{(k)} \right)^* \in E_{mT}$, then

$$\left\| w^{(k)} \right\|_2 = \left(\sum_{n=1}^{mT} \left| w_n^{(k)} \right|^2 \right)^{\frac{1}{2}} = \left(\sum_{n=1}^{mT} \left| z^{(k)} \right|^2 \right)^{\frac{1}{2}} = \sqrt{mT} \left| z^{(k)} \right| \rightarrow +\infty$$

as $k \rightarrow \infty$. Since $F(n, w_{n+1}^{(k)}, w_n^{(k)}) = F(n, z^{(k)}, z^{(k)})$, then $F(n, w_{n+1}^{(k)}, w_n^{(k)}) \rightarrow +\infty$ as $k \rightarrow \infty$. This contradicts the fact that $\left\{ \sum_{n=1}^{mT} F(n, w_{n+1}^{(k)}, w_n^{(k)}) \right\}$ is bounded. Thus the P.S. condition is verified. \square

Lemma 2.3 Assume that (F_1) , (F_4) and (F_5) are satisfied. Then J satisfies the P.S. condition.

Proof Let $\{u^{(k)}\} \subset E_{mT}$ be such that $\{J(u^{(k)})\}$ is bounded and $J'(u^{(k)}) \rightarrow 0$ as $k \rightarrow \infty$. Then there exists a positive constant M_3 such that $|J(u^{(k)})| \leq M_3$.

For k large enough, we have

$$\left| \langle J'(u^{(k)}), u^{(k)} \rangle \right| \leq \|u^{(k)}\|_2.$$

So

$$\begin{aligned} & M_3 + \frac{1}{2} \|u^{(k)}\|_2^2 \\ & \geq J(u^{(k)}) - \frac{1}{2} \langle J'(u^{(k)}), u^{(k)} \rangle \\ & = \sum_{n=1}^{mT} \left[F(n, u_{n+1}^{(k)}, u_n^{(k)}) - \frac{1}{2} \left(\frac{\partial F(n-1, u_n^{(k)}, u_{n-1}^{(k)})}{\partial v_2} \cdot u_n^{(k)} \right. \right. \\ & \quad \left. \left. + \frac{\partial F(n, u_{n+1}^{(k)}, u_n^{(k)})}{\partial v_2} \cdot u_n^{(k)} \right) \right] \\ & = \sum_{n=1}^{mT} \left[F(n, u_{n+1}^{(k)}, u_n^{(k)}) - \frac{1}{2} \left(\frac{\partial F(n, u_{n+1}^{(k)}, u_n^{(k)})}{\partial v_1} \cdot u_{n+1}^{(k)} \right. \right. \\ & \quad \left. \left. + \frac{\partial F(n, u_{n+1}^{(k)}, u_n^{(k)})}{\partial v_2} \cdot u_n^{(k)} \right) \right]. \end{aligned}$$

Take

$$\begin{aligned} I_1 &= \left\{ n \in \mathbf{Z}(1, mT) \mid \sqrt{(u_{n+1}^{(k)})^2 + (u_n^{(k)})^2} \geq R_1 \right\}, \\ I_2 &= \left\{ n \in \mathbf{Z}(1, mT) \mid \sqrt{(u_{n+1}^{(k)})^2 + (u_n^{(k)})^2} < R_1 \right\}. \end{aligned}$$

By (F_4) , we have

$$\begin{aligned} & M_3 + \frac{1}{2} \|u^{(k)}\|_2^2 \\ & \geq \sum_{n=1}^{mT} F(n, u_{n+1}^{(k)}, u_n^{(k)}) - \frac{1}{2} \sum_{n \in I_1} \left[\frac{\partial F(n, u_{n+1}^{(k)}, u_n^{(k)})}{\partial v_1} \cdot u_{n+1}^{(k)} + \frac{\partial F(n, u_{n+1}^{(k)}, u_n^{(k)})}{\partial v_2} \cdot u_n^{(k)} \right] \\ & \quad - \frac{1}{2} \sum_{n \in I_2} \left[\frac{\partial F(n, u_{n+1}^{(k)}, u_n^{(k)})}{\partial v_1} \cdot u_{n+1}^{(k)} + \frac{\partial F(n, u_{n+1}^{(k)}, u_n^{(k)})}{\partial v_2} \cdot u_n^{(k)} \right] \end{aligned}$$

$$\begin{aligned} &\geq \sum_{n=1}^{mT} F\left(n, u_{n+1}^{(k)}, u_n^{(k)}\right) - \frac{\alpha}{2} \sum_{n \in I_1} F\left(n, u_{n+1}^{(k)}, u_n^{(k)}\right) \\ &\quad - \frac{1}{2} \sum_{n \in I_2} \left[\frac{\partial F\left(n, u_{n+1}^{(k)}, u_n^{(k)}\right)}{\partial v_1} \cdot u_{n+1}^{(k)} + \frac{\partial F\left(n, u_{n+1}^{(k)}, u_n^{(k)}\right)}{\partial v_2} \cdot u_n^{(k)} \right] \\ &= \left(1 - \frac{\alpha}{2}\right) \sum_{n=1}^{mT} F\left(n, u_{n+1}^{(k)}, u_n^{(k)}\right) \\ &\quad + \frac{1}{2} \sum_{n \in I_2} \left[\alpha F\left(n, u_{n+1}^{(k)}, u_n^{(k)}\right) - \frac{\partial F\left(n, u_{n+1}^{(k)}, u_n^{(k)}\right)}{\partial v_1} \cdot u_{n+1}^{(k)} - \frac{\partial F\left(n, u_{n+1}^{(k)}, u_n^{(k)}\right)}{\partial v_2} \cdot u_n^{(k)} \right]. \end{aligned}$$

The continuity of $\alpha F(n, v_1, v_2) - \frac{\partial F(n, v_1, v_2)}{\partial v_1} v_1 - \frac{\partial F(n, v_1, v_2)}{\partial v_2} v_2$ with respect to the second and third variables implies that there exists a constant $M_4 > 0$ such that

$$\alpha F(n, v_1, v_2) - \frac{\partial F(n, v_1, v_2)}{\partial v_1} v_1 - \frac{\partial F(n, v_1, v_2)}{\partial v_2} v_2 \geq -M_4,$$

for $n \in \mathbf{Z}(1, mT)$ and $\sqrt{v_1^2 + v_2^2} \leq R_1$. Therefore,

$$M_3 + \frac{1}{2} \|u^{(k)}\|_2 \geq \left(1 - \frac{\alpha}{2}\right) \sum_{n=1}^{mT} F\left(n, u_{n+1}^{(k)}, u_n^{(k)}\right) - \frac{1}{2} mT M_4.$$

By (F5), we get

$$\begin{aligned} M_3 + \frac{1}{2} \|u^{(k)}\|_2 &\geq \left(1 - \frac{\alpha}{2}\right) a_1 \sum_{n=1}^{mT} \left[\sqrt{\left(u_{n+1}^{(k)}\right)^2 + \left(u_n^{(k)}\right)^2} \right]^\gamma - \left(1 - \frac{\alpha}{2}\right) a_2 mT - \frac{1}{2} mT M_4 \\ &> \left(1 - \frac{\alpha}{2}\right) a_1 |u_n^{(k)}|^\gamma - M_5, \end{aligned}$$

where $M_5 = \left(1 - \frac{\alpha}{2}\right) a_2 mT + \frac{1}{2} mT M_4$.

Combining with (2.4), we have

$$M_3 + \frac{1}{2} \|u^{(k)}\|_2 \geq \left(1 - \frac{\alpha}{2}\right) a_1 c_1^\gamma \|u^{(k)}\|_2^\gamma - M_5.$$

Thus,

$$\left(1 - \frac{\alpha}{2}\right) a_1 c_1^\gamma \|u^{(k)}\|_2^\gamma - \frac{1}{2} \|u^{(k)}\|_2 \leq M_3 + M_5.$$

This implies that $\{\|u^{(k)}\|_2\}$ is bounded on the finite dimensional space E_{mT} . As a consequence, it has a convergent subsequence. □

3 Proof of the main results

In this Section, we shall prove our main results by using the critical point method.

Proof of Theorem 1.1 By Lemma 2.2, we know that J satisfies the P.S. condition. In order to prove Theorem 1.1 by using the Saddle Theorem, we shall prove the conditions (J_1) and (J_2) .

From (2.8) and (F'_2) , for any $v \in V$,

$$\begin{aligned} J(v) &= -H(v) + \sum_{n=1}^{mT} F(n, v_{n+1}, v_n) \\ &\leq -\frac{r}{2}\lambda_{\min}^2 \|v\|_2^2 + mTM_1 + M_0 \sum_{n=1}^{mT} (|v_{n+1}| + |v_n|) \\ &\leq -\frac{r}{2}\lambda_{\min}^2 \|v\|_2^2 + mTM_1 + 2M_0\sqrt{mT} \|v\|_2 \rightarrow -\infty \text{ as } \|v\|_2 \rightarrow +\infty. \end{aligned}$$

Therefore, it is easy to see that the condition (J_1) is satisfied.

In the following, we shall verify the condition (J_2) . For any $w \in W$, $w = (w_1, w_2, \dots, w_{mT})^*$, there exists $z \in \mathbf{R}$ such that $w_n = z$, for all $n \in \mathbf{Z}(1, mT)$. By (F_3) , we know that there exists a constant $R_0 > 0$ such that $F(n, z, z) > 0$ for $n \in \mathbf{Z}$ and $|z| > \frac{R_0}{\sqrt{2}}$. Let $M_6 = \min_{n \in \mathbf{Z}, |z| \leq R_0/\sqrt{2}} F(n, z, z)$, $M_7 = \min\{0, M_6\}$. Then

$$F(n, z, z) \geq M_7, \quad \forall (n, z, z) \in \mathbf{Z} \times \mathbf{R}^2.$$

So we have

$$J(w) = \sum_{n=1}^{mT} F(n, w_{n+1}, w_n) = \sum_{n=1}^{mT} F(n, z, z) \geq mTM_7, \quad \forall w \in W.$$

The conditions of (J_1) and (J_2) are satisfied. □

Proof of Theorem 1.2 By Lemma 2.3, J satisfies the P.S. condition. To apply the Saddle Point Theorem, it suffices to prove that J satisfies the conditions (J_1) and (J_2) .

For any $w \in W$, since $H(w) = 0$, we have

$$J(w) = \sum_{n=1}^{mT} F(n, w_{n+1}, w_n).$$

By (F_5) ,

$$J(w) \geq a_1 \sum_{n=1}^{mT} \left(\sqrt{w_{n+1}^2 + w_n^2} \right)^\gamma - a_2 mT \geq -a_2 mT.$$

Combining with (F'_4) , (2.4) and (2.8), for any $v \in V$, we get, like before,

$$\begin{aligned} J(v) &\leq -\frac{r}{2}\lambda_{\min}^2 \|v\|_2^2 + a_3 \sum_{n=1}^{mT} \left(\sqrt{v_{n+1}^2 + v_n^2} \right)^\alpha + a_4 mT \\ &\leq -\frac{r}{2}\lambda_{\min}^2 \|v\|_2^2 + a_3 c_2^\alpha \left[\sum_{n=1}^{mT} (v_{n+1}^2 + v_n^2) \right]^{\frac{\alpha}{2}} + a_4 mT \\ &\leq 7 - \frac{r}{2}\lambda_{\min}^2 \|v\|_2^2 + 2^{\frac{\alpha}{2}} a_3 c_2^\alpha \|v\|_2^\alpha + a_4 mT. \end{aligned}$$

Let $\mu = -a_2 mT$, since $1 < \alpha < 2$, there exists a constant $\rho > 0$ large enough such that

$$J(v) \leq \mu - 1 < \mu, \quad \forall v \in V, \|v\|_2 = \rho.$$

Thus, by Lemma 2.1, (1.1) has at least one mT -periodic solution. □

Proof of Theorem 1.3 Similarly to the proof of Lemma 2.3, we can prove that J satisfies the P.S. condition. We shall prove this theorem by the Saddle Point Theorem. Firstly, we verify the condition (J_1) .

In fact, (F_4) clearly implies (F'_4) . For any $v \in V$, by (F'_4) and (2.4), we have again $J(v) \rightarrow -\infty$ as $\|v\|_2 \rightarrow +\infty$.

Next, we show that J satisfies the condition (J_2) . For any given $v_0 \in V$ and $w \in W$. Let $u = v_0 + w$. So

$$\begin{aligned} J(u) &= -H(u) + \sum_{n=1}^{mT} F(n, u_{n+1}, u_n) \\ &= -H(v_0) + \sum_{n=1}^{mT} F(n, (v_0)_{n+1} + w_{n+1}, (v_0)_n + w_n) \\ &\geq -\frac{\bar{r}}{2} \lambda_{\max}^2 \|v_0\|_2^2 + a_5 \sum_{n=1}^{mT} \left[\sqrt{((v_0)_{n+1} + w_{n+1})^2 + ((v_0)_n + w_n)^2} \right]^\gamma \\ &\geq -\frac{\bar{r}}{2} \lambda_{\max}^2 \|v_0\|_2^2 + a_5 \sum_{n=1}^{mT} |(v_0)_n + w_n|^\gamma \\ &\geq -\frac{\bar{r}}{2} \lambda_{\max}^2 \|v_0\|_2^2 + a_5 c_1^\gamma \left[\sum_{n=1}^{mT} |(v_0)_n + w_n|^2 \right]^{\frac{\gamma}{2}} \\ &= -\frac{\bar{r}}{2} \lambda_{\max}^2 \|v_0\|_2^2 + a_5 c_1^\gamma [\|v_0\|_2^2 + \|w\|_2^2]^{\frac{\gamma}{2}} \\ &\geq -\frac{\bar{r}}{2} \lambda_{\max}^2 \|v_0\|_2^2 + a_5 c_1^\gamma \|v_0\|_2^\gamma + a_5 c_1^\gamma \|w\|_2^\gamma. \end{aligned}$$

Since $1 < \gamma < 2$, there exists a constant $\delta > 0$ small enough such that

$$J(v_0 + w) \geq \delta^\gamma \left(a_5 c_1^\gamma - \frac{\bar{r}}{2} \lambda_{\max}^2 \delta^{2-\gamma} \right) > 0,$$

for $v_0 \in V$, $\|v_0\|_2 = \delta$ and for any $w \in W$.

Take $v = \delta^\gamma \left(a_5 c_1^\gamma - \frac{\bar{r}}{2} \lambda_{\max}^2 \delta^{2-\gamma} \right)$. Then for $v_0 \in V$ and for any $w \in W$, we get $\|v_0\|_2 = \delta$ and $J(v_0 + w) \geq v > 0$.

By the Saddle Point Theorem, there exists a critical point $\bar{u} \in E_{mT}$, which corresponds to a mT -periodic solution of (1.1).

In the following, we shall prove that \bar{u} is nontrivial, i.e., $\bar{u} \notin W$. Otherwise, $\bar{u} \in W$. Since $J'(\bar{u}) = 0$, then

$$\Delta^2 (r_{n-2} \Delta^2 \bar{u}_{n-2}) = f(n, \bar{u}_{n+1}, \bar{u}_n, \bar{u}_{n-1}).$$

On the other hand, $\bar{u} \in W$ implies that there is a point $z \in \mathbf{R}$ such that $\bar{u}_n = z$, for all $n \in \mathbf{Z}(1, mT)$. That is, $\bar{u}_1 = \bar{u}_2 = \dots = \bar{u}_n = \dots = z$. Thus, $f(n, \bar{u}_{n+1}, \bar{u}_n, \bar{u}_{n-1}) = f(n, z, z, z) = 0$, for all $n \in \mathbf{Z}(1, mT)$. From (F_6) , we know that $z = 0$. Therefore, by (F_6) , we have

$$J(\bar{u}) = \sum_{n=1}^{mT} F(n, \bar{u}_{n+1}, \bar{u}_n) = \sum_{n=1}^{mT} F(n, 0) = 0.$$

This contradicts $J(\bar{u}) \geq v > 0$. The proof of Theorem 1.3 is finished. □

Remark 3.1 The techniques of the proof of the Theorem 1.4 are just the same as those carried out in the proof of Theorem 1.3. We do not repeat them here.

Remark 3.2 Due to Theorems 1.3 and 1.4, the conclusion of Theorems 1.5 and 1.6 is obviously true.

4 Examples

As an application of the main theorems, finally, we give two examples to illustrate our results.

Example 1 For all $n \in \mathbf{Z}$, assume that

$$\Delta^2 (r_{n-2} \Delta^2 u_{n-2}) = 2\alpha u_n \left[\varphi(n) (u_{n+1}^2 + u_n^2)^{\alpha-1} + \varphi(n-1) (u_n^2 + u_{n-1}^2)^{\alpha-1} \right], \quad (4.1)$$

where $r_n > 0$ is real valued for each $n \in \mathbf{Z}$, φ is continuously differentiable and $\varphi(n) > 0$, T is a given positive integer, $r_{n+T} = r_n$, $\varphi(n+T) = \varphi(n)$, $1 < \alpha < 2$. We have

$$f(n, v_1, v_2, v_3) = 2\alpha v_2 \left[\varphi(n) (v_1^2 + v_2^2)^{\alpha-1} + \varphi(n-1) (v_2^2 + v_3^2)^{\alpha-1} \right]$$

and

$$F(n, v_1, v_2) = \varphi(n) (v_1^2 + v_2^2)^\alpha.$$

Then

$$\frac{\partial F(n-1, v_2, v_3)}{\partial v_2} + \frac{\partial F(n, v_1, v_2)}{\partial v_2} = 2\alpha v_2 \left[\varphi(n) (v_1^2 + v_2^2)^{\alpha-1} + \varphi(n-1) (v_2^2 + v_3^2)^{\alpha-1} \right].$$

It is easy to verify all the assumptions of Theorem 1.3 are satisfied. Consequently, for any given positive integer $m > 0$, (4.1) has at least one nontrivial mT -periodic solution.

Example 2 For all $n \in \mathbf{Z}$, assume that

$$\Delta^2 (r_{n-2} \Delta^2 u_{n-2}) = 2\theta u_n \left[\left(1 + \cos^2 \frac{n\pi}{T} \right) (u_{n+1}^2 + u_n^2)^{\theta-1} + \left(1 + \cos^2 \frac{(n-1)\pi}{T} \right) (u_n^2 + u_{n-1}^2)^{\theta-1} \right], \quad (4.2)$$

where $r_n > 0$ is real valued for each $n \in \mathbf{Z}$, T is a given positive integer, $r_{n+T} = r_n$, $0 < \theta < 2$. We have

$$f(n, v_1, v_2, v_3) = 2\theta v_2 \left[\left(1 + \cos^2 \frac{n\pi}{T} \right) (v_1^2 + v_2^2)^{\theta-1} + \left(1 + \cos^2 \frac{(n-1)\pi}{T} \right) (v_2^2 + v_3^2)^{\theta-1} \right]$$

and

$$F(n, v_1, v_2) = \left(1 + \cos^2 \frac{n\pi}{T} \right) (v_1^2 + v_2^2)^\theta.$$

Then

$$\begin{aligned} & \frac{\partial F(n-1, v_2, v_3)}{\partial v_2} + \frac{\partial F(n, v_1, v_2)}{\partial v_2} \\ &= 2\theta v_2 \left[\left(1 + \cos^2 \frac{n\pi}{T}\right) (v_1^2 + v_2^2)^{\theta-1} + \left(1 + \cos^2 \frac{(n-1)\pi}{T}\right) (v_2^2 + v_3^2)^{\theta-1} \right]. \end{aligned}$$

It is easy to verify all the assumptions of Theorem 1.4 are satisfied. Consequently, for any given positive integer $m > 0$, (4.2) has at least one nontrivial mT -periodic solution.

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