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Operators preserving ℓ_∞

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Abstract Let Y be a Banach space, let the space ℓ_∞ be real, let W denote the Banach space ℓ_∞/c_0, and let Q denote the quotient map $\ell_\infty \to W$. In 1981, Partington proved there is a topological embedding J of ℓ_∞ into W such that the composition QJ is an isometry; in particular, Q preserves ℓ_∞. In this paper we prove that *if the kernel of an operator* $T : \ell_\infty/c_0 \to Y$ *does not contain an isometric copy of* c_0 (in particular, if T is injective), *then* T *preserves* ℓ_∞, *and hence* T *is non-weakly compact*. This, in turn, allows us to extend Partington's theorem: we show that natural quotient mappings of some real function spaces preserve ℓ_∞. We also remark that our results apply to some quotients of both Orlicz and Marcinkiewicz spaces.

Keywords Banach space · Banach lattice · Operator preserving ℓ_∞ · Operator weakly compact · Orlicz space · Marcinkiewicz space

Mathematics Subject Classification (2000) 47B37 · 46B03 · 46B42

1 Introduction

In what follows we use notations from the abstract. For notions and notations undefined here we refer the reader to the monographs [2,3]. All operators are linear and continuous, spaces are of infinite dimension, and Γ denotes an infinite set endowed with the discrete topology.

The present paper deals with operators preserving the real Banach space ℓ_∞ or, what comes to the same thing, real $C(\beta\mathbf{N})$, where $\beta\mathbf{N}$ is the Čech–Stone compactification of the discrete space of positive integers $\mathbf{N}$.

Let X, Y, V be Banach spaces, and let K be a compact Hausdorff space. An operator $T : X \to Y$ preserves [isometrically] V if there is a subspace U of X, isomorphic [isometric,

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resp.] to V, such that the restricted operator $T_{|U}$ is an isomorphism [isometry, resp.]. The first theorem on such operators is due to Pełczyński [5]. In 1962, he proved that

(*) Every non-weakly compact operator $T : C(K) \to Y$ preserves c_0.

In 1968 Rosenthal strengthened partially the above-cited Pełczyński's theorem by showing that

(**) Every non-weakly compact operator $T : C(\beta\Gamma) = \ell_\infty(\Gamma) \to Y$ preserves ℓ_∞; in particular, the quotient map $Q : \ell_\infty \to \ell_\infty/c_0$ preserves ℓ_∞

(see [7]; cf. [2, Proposition 2.f.4]). In 1970 he published the classical by now result [8, Proposition 1.2 and Remark 1 on p. 30]:

(R) If an operator $T : \ell_\infty(\Gamma) \to Y$ is such that $\inf\{\|Te_\gamma\| : \gamma \in \Gamma\} > 0$, where $\{e_\gamma : \gamma \in \Gamma\}$ is the unit vector basis of $c_0(\Gamma)$, then there exists $\Gamma' \subset \Gamma$ with $\mathrm{card}(\Gamma') = \mathrm{card}(\Gamma)$ such that $T_{|\ell_\infty(\Gamma')}$ is an isomorphism.

Here $\ell_\infty(\Gamma')$ denotes the closed subspace of $\ell_\infty(\Gamma)$ consisting of the elements with support included in Γ'.

Remark 1 It is easy to see that if, in the hypothesis of (R), the unit vector basis $\{e_\gamma : \gamma \in \Gamma\}$ is replaced by any norm-bounded subset $\{u_\gamma : \gamma \in \Gamma\}$ of $\ell_\infty(\Gamma)$ whose elements are pairwise disjoint, then T is an isomorphism on a subspace U of $\ell_\infty(\Gamma)$, isomorphic to $\ell_\infty(\Gamma)$, spanned on the set $\{u_\gamma : \gamma \in \Gamma'\}$ as in (P) below.

In 1981, Partington added a new information to the Rosenthal result (**): he proved that every automorphism of the quotient space ℓ_∞/c_0 preserves ℓ_∞ in a particular way [4, Theorem 1]:

(P) For the real Banach space ℓ_∞ and every norm $|||\,|||$ on $W = \ell_\infty/c_0$, equivalent to the usual (quotient) norm, there exist pairwise disjoint and norm-bounded elements $\{u_n : n \in \mathbf{N}\}$ in ℓ_∞ such that for every $(a_n) \in \ell_\infty$ we have

$$\left|\left|\left| Q\left((p)\sum_{n=1}^{\infty} a_n u_n\right)\right|\right|\right| = \sup_{n\geq 1} |a_n|, \tag{1}$$

where $(p)\sum_{n=1}^{\infty} a_n u_n$ denotes the formal pointwise sum of the $a_n u_n$ in ℓ_∞.

The following result of the present author extends partially Partington's theorem (P) to the Banach lattice-case (see [11, Proof of Theorem 1.1 and Remark on p. 3006]; cf. [12, Proof of Proposition 1]):

(W) Let E be a Dedekind complete Banach lattice, and let M be its closed ideal that does not contain a copy of ℓ_∞. If E contains a lattice-topological [lattice-isometric, resp.] copy of $\ell_\infty(\Gamma)$, then the quotient map $E \to E/M$ preserves [lattice-isometrically, resp.] $\ell_\infty(\Gamma)$.
In particular, the quotient map $Q : \ell_\infty(\Gamma) \to \ell_\infty(\Gamma)/c_0(\Gamma)$ preserves $\ell_\infty(\Gamma)$ lattice-isometrically.

(The examples of such pairs of E and M can be found both in the class of Orlicz and in the class of Marcinkiewicz spaces [6, Lemma 1 and remarks before Corollary 7]).

For the results on operators on Banach lattices/spaces preserving c_0, ℓ_1, or ℓ_∞ see [3, pp. 196–199, 324, 341–343]; the case of operators preserving some $C(K)$-spaces is addressed in the survey article [9, pp. 1579–1593].

In the present paper, we extend and supplement the above-cited theorems on operators preserving ℓ_∞. Our main results are included in Theorems 1 and 2, and their applications to concrete cases are given in Example 1, and Corollaries 4 and 5.

2 The results

Our basic theorem is given below. In what follows, for the set Γ fixed, the letter Q denotes the natural quotient map from $\ell_\infty(\Gamma)$ onto $\ell_\infty(\Gamma)/c_0(\Gamma)$, and $\|\,\|_W$ denotes the natural (quotient and lattice) norm on $W := \ell_\infty(\Gamma)/c_0(\Gamma)$.

Theorem 1 *Let W denote* [*an isomorphic copy of*] *the real space $\ell_\infty(\Gamma)/c_0(\Gamma)$. If the kernel of an operator $T : W \to Y$ does not contain an isometric* [*isomorphic, resp.*] *copy of c_0 then T preserves ℓ_∞. In particular, T is non-weakly compact.*

Proof Since, by Partington's result (P), the isomorphic case follows from the isometric case, we shall assume without loss of generality that W is endowed with the norm $\|\,\|_W$. Let us fix Γ.

Let us consider first the case Γ countable. Without loss of generality we set $W = \ell_\infty/c_0$. Our goal is to show that *there is a pairwise disjoint and norm-bounded sequence (u_n) in W such that* $\inf_{n\geq 1} \|T(Qu_n)\|_Y > 0$; then we shall apply Rosenthal's theorem (R).

We define a new norm $|||\,|||$ on W by the formula

$$|||w||| := \|w\|_W + \|Tw\|_Y, \tag{2}$$

where $\|\,\|_Y$ is a norm on Y. Now we apply (P) and choose a sequence (u_n) in ℓ_∞ as in (1), and we set $x_n := Qu_n$, $n = 1, 2, \ldots$. Hence

$$1 = |||x_n||| = \|x_n\|_W + \|Tx_n\|_Y, \quad n = 1, 2, \ldots. \tag{3}$$

By (P), the sequence (u_n) consists of pairwise disjoint elements and "spans" a copy U of ℓ_∞ in ℓ_∞ in the same way as in (P); we thus have

$$\left|\left|\left| Q\left((p)\sum_{n=1}^{\infty} a_n u_n\right)\right|\right|\right| = \sup_{n\geq 1} |a_n|, \tag{4}$$

for every $(a_n) \in \ell_\infty$. So that $Q(U)$ is an isometric copy of ℓ_∞ in W. Now we claim that

$$\limsup_{n\to\infty} \|Tx_n\|_Y = \limsup_{n\to\infty} \|TQ(u_n)\|_Y > 0. \tag{5}$$

Let us suppose the contrary, i.e.,

$$\lim_{n\to\infty} \|TQ(u_n)\|_Y = 0. \tag{6}$$

Let (N_k) be a sequence of infinite and pairwise disjoint subsets of $\mathbf{N}$ such that $\mathbf{N} = \bigcup_{k=1}^{\infty} N_k$, and let us set

$$y_k := Q\left((p)\sum_{n\in N_k} u_n\right), \quad k = 1, 2, \ldots. \tag{7}$$

By (4), the elements y_k are well defined (in W) and $|||y_k||| = 1$ for all k. Since Q is a lattice homomorphism and the elements (u_n) are pairwise disjoint, we have

$$|y_k| = \left|Q\left((p)\sum_{n\in N_k} u_n\right)\right| = Q\left(\left|(p)\sum_{n\in N_k} u_n\right|\right)$$
$$= Q\left((p)\sum_{n\in N_k} |u_n|\right) \geq Q(|u_n|) = |Qu_n| = |x_n|,$$

for all $n \in N_k$ (the latter inequality $\geq$ follows from the inequality $|a+b| \geq |a|$ for the elements a and b disjoint).

Hence, by (2) (and since $\|\|_W$ is a lattice norm), we obtain

$$1 = \||y_k|\| \geq \|y_k\|_W \geq \|x_n\|_W, \quad \text{for all } n \in N_k. \tag{8}$$

By (2), (3), (6) and (8), we obtain that $1 \geq \|y_k\|_W \geq \lim_{n\to\infty} \|x_n\|_W = 1$, i.e.,

$$\|y_k\|_W = 1, \quad \text{for all } k = 1, 2, \ldots, \tag{9}$$

whence, by (2) again, $Ty_k = 0$ for all k. But, by (7) and (9), the elements (y_k) are pairwise disjoint in $W = C(\beta\mathbf{N}\backslash\mathbf{N})$ and are of norm one, thus they span an isometric copy of c_0. Hence the kernel of T contains an isometric copy of c_0, but this contradicts the hypothesis of our theorem. Thus our claim (6) is false, so (5) must be true.

Now we apply the general case of Rosenthal's result (R) (see Remark 1) to the space U_1 isomorphic to ℓ_∞ and spanned—as in (P)—by an infinite subsequence (u_{n_j}) of (u_n) such that $\inf_{j\to\infty} \|TQ(u_{n_j})\|_Y > 0$; and without loss of generality we may assume that

$$\text{the operator } TQ \text{ acts on } U_1 \text{ as an isomorphism.}$$

Set $Q_1 := Q_{|U_1}$, $W_1 := Q_1(U_1)$, $T_1 := T_{|W_1}$, and let S denote the composition $T_1 \circ Q_1$. We thus have that U_1 and W_1 are isomorphic copies of ℓ_∞, and that S *is an isomorphism from* U_1 *onto* $Y_1 := T_1(W_1)$. Moreover, by (4), Q_1 *is an isomorphism from* U_1 *onto* W_1.

From all this follows that $T_1 = T_{|W_1} = S \circ Q_1^{-1}$ is an isomorphism from W_1 (=an isomorphic copy of ℓ_∞) into Y. In other words, T preserves ℓ_∞.

If the set Γ is uncountable, then the space $\ell_\infty(\Gamma)/c_0(\Gamma)$ contains an isometric copy of ℓ_∞/c_0 (see [13, Corollary 2]). Now we apply the just proved result for Γ countable.

Now let us consider the case when the operator $T : \ell_\infty(\Gamma)/c_0(\Gamma) \to Y$ is injective. If Γ is countable, Theorem 1 implies that T preserves ℓ_∞. Moreover, if Γ is uncountable, from the above-cited results (W) and (R) we obtain immediately that T preserves $\ell_\infty(\Gamma)$. More exactly: from the proof of Theorem 1 it follows that, in each of the either cases, there is a set $\{u_\gamma : \gamma \in \Gamma\}$ of pairwise disjoint and norm bounded elements of $\ell_\infty(\Gamma)$ such that $\inf_{\gamma\in\Gamma} \|TQ(u_\gamma)\|_Y > 0$.

Hence, by Theorem 1, we have the following result.

Corollary 1 *Every injective operator* $T : \ell_\infty(\Gamma)/c_0(\Gamma) \to Y$ *preserves* $\ell_\infty(\Gamma)$*: we have*

$$\inf_{\gamma\in\Gamma} \|TQ(u_\gamma)\|_Y > 0,$$

where $(u_\gamma)_{\gamma\in\Gamma}$ *are pairwise disjoint and norm bounded elements of* $\ell_\infty(\Gamma)$*; hence Rosenthal's result* (R) *applies to the operator* T *and the family* $(Qu_\gamma)_{\gamma\in\Gamma}$.

Moreover, if Y *is a WCG-space, then the kernel of every operator* $T : \ell_\infty/c_0 \to Y$ *contains an isometric copy of* c_0*; in particular,* T *is not injective.*

(The second part of Corollary 1 follows from the well known fact that a weakly compactly generated (WCG) Banach space cannot contain an isomorphic copy of ℓ_∞).

In the next corollary we supplement Rosenthal's result (R) in a particular case of the condition $\inf_{\gamma\in\Gamma} \|Te_\gamma\| = 0$.

Corollary 2 *Let* T *be an operator from* $\ell_\infty(\Gamma) \to Y$ *such that* $\ker T = c_0(\Gamma)$*. Then* T *preserves* $\ell_\infty(\Gamma)$.

Proof The operator $\overline{T} : \ell_\infty(\Gamma)/c_0(\Gamma) \to Y$ of the form $\overline{T}(Qx) := Tx$, is injective. By Corollary 1, from the form of $\overline{T}$ we obtain $\inf_{\gamma\in\Gamma} \|Tu_\gamma\|_Y > 0$, where $\{u_\gamma : \gamma \in \Gamma\}$ is a norm-bounded subset of $\ell_\infty(\Gamma)$ whose elements are pairwise disjoint. By Remark 1, the operator T preserves $\ell_\infty(\Gamma)$.

The following example illustrates Corollary 2 (for another application of this corollary see the proof of Theorem 2).

Example 1 Let c denote the Banach space of all real convergent sequences, let y^* be the "lim" functional on c, and let x^* be any fixed continuous extension of y^* to ℓ_∞. Further, let $\mathcal{F}$ denote the set (with card$(\mathcal{F}) = 2^{\aleph_0}$) of all strictly increasing functions $f : \mathbf{N} \to \mathbf{N}$, and let ξ be any function $\mathcal{F} \to (0, 1]$. By x^*_f we denote the element of ℓ^*_∞ defined by the formula $x^*_f(x) := x^*(x \circ f)$; here $x \in \ell_\infty$, and $(x \circ f)(n) = x(f(n))$, $n \geq 1$. The operator $T_\xi : \ell_\infty \to \ell_\infty(\mathcal{F})$ defined by the formula

$$T_\xi(x) = (\xi(f) \cdot x^*_f(x))_{f\in\mathcal{F}}$$

is obviously continuous, and it is easy to check that $\ker T_\xi = c_0$. From Corollary 2 we obtain that T_ξ preserves ℓ_∞.

The next two corollaries also follow from Corollary 1.

Let Y be a closed subspace of a Banach space X, and set $W := \ell_\infty(\Gamma)/c_0(\Gamma)$. Let $S : \ell_\infty(\Gamma)$ be an operator such that $S(c_0(\Gamma)) = Y \cap \mathrm{Im}S$, and let q and Q denote the natural quotient maps $\ell_\infty(\Gamma) \to W$ and $X \to X/Y$, respectively. By [6, Theorem 2], the induced operator $R : W \to X/Y$ defined by the rule $R \circ q = Q \circ S$ is injective. From Corollary 1 we thus obtain:

Corollary 3 *With the notations as above, and with the hypothesis*

$$S(c_0(\Gamma)) = Y \cap ImS,$$

the quotient space X/Y contains an isomorphic copy of ℓ_∞.

Now let us consider the quotient space $X^{**}/\iota(X)$, where ι denotes the canonical embedding of X into X^{**}. Assume there is an isomorphic embedding S_0 of c_0 into X. Then $S := S_0^{**}$ embeds ℓ_∞ into X^{**} with

$$\iota(X) \cap \mathrm{Im}S = \iota(S_0(c_0)) = S(c_0).$$

By [6, Corollary 2], for every separable subspace V of ℓ_∞ containing a copy of c_0, the quotient space $X = \ell_\infty/V$ contains a copy of $c_0(\Gamma)$, where card$(\Gamma) = 2^{\aleph_0}$. Hence, by Corollary 3, we obtain:

Corollary 4 *If X contains a copy of c_0, then the quotient space $X^{**}/\iota(X)$ contains an isomorphic copy of ℓ_∞.*

In particular, for the Banach space $X = \ell_\infty/C[0, 1]$, the quotient space X^{**}/X contains an isomorphic copy of ℓ_∞.

The next theorem has an application to some quotient mappings.

Theorem 2 *Let U be a closed subspace of a Banach space X. If U is isomorphic to ℓ_∞ and $T : X \to Y$ is an operator such that the subspace $U_0 := U \cap \ker T$ is isomorphic to c_0, then T preserves ℓ_∞.*

Proof Let R and S be two isomorphisms from ℓ_∞ onto U and from c_0 onto U_0, respectively. Then the subspace $X_0 := R^{-1}(U_0)$ of ℓ_∞ is isomorphic to c_0, and the operator $\tau_0 := R^{-1}S$ maps c_0 onto X_0. By [2, Theorem 2.f.12(i)], there is an automorphism τ of ℓ_∞ extending τ_0. It follows that

$$\text{the operator } \widetilde{S} := R\tau \text{ is an extension of } S, \text{ and } \widetilde{S} \text{ maps } \ell_\infty \text{ onto } U. \tag{10}$$

Now let us consider the restricted operator $T_{|U}$. By (10), the composition $T_{|U}\widetilde{S}$ maps ℓ_∞ into Y, and $\ker(T_{|U}\widetilde{S}) = c_0$ because $\ker(T_{|U}) = U_0$. By Corollary 2, the space ℓ_∞ contains an isomorphic copy V of ℓ_∞ such that $T_{|U}\widetilde{S}$ acts on V as an isomorphism. But since $\widetilde{S}$ is an isomorphism and the subspace $U_1 := \widetilde{S}(V)$ of U is also a copy of ℓ_∞, the operator T acts on U_1 as an isomorphism; that is, T preserves ℓ_∞, as claimed.

In the corollary below, we show how Theorem 2 works in concrete cases.

Let X denote one of the *real* Banach spaces, endowed with the "sup"-norm and built on the interval [0, 1]: $X = \mathcal{L}^b_\infty[0, 1]$—of all bounded and Lebesgue-measurable functions, or $X = \mathcal{B}^b_1[0, 1]$—of all bounded functions of Baire class one. In [6, Corollary 5] it has been proved that the quotient space $X/C[0, 1]$ contains a complemented copy of ℓ_∞/c_0. From Theorem 2 we obtain additional information about the quotient map $q : X \to X/C[0, 1]$.

Corollary 5 *With the notations as above, q preserves ℓ_∞.*

Proof Let (x_n) be a sequence of positive, pairwise disjoint and norm-one elements of $C[0, 1]$. Then the closed linear span $U_0 := [x_n]$ of (x_n) is an isometric copy of c_0, and its pointwise "span" U, defined in X as in (P), is isometric to ℓ_∞. Moreover, by Dini's theorem, the pointwise sum $(p)\sum_{n=1}^\infty t_n x_n$, where $t_n \geq 0, n = 1, 2, \ldots,$ lies in $C[0, 1]$ if and only if $t_n \to 0$ as $n \to \infty$. It follows that $U \cap \ker q = U_0$, and hence, by Theorem 2, q preserves ℓ_∞.

3 Further applications of Theorem 1

Let E denote a real Banach lattice and let E_a be the order continuous part of E (for unexplained in this section notions and fundamental facts concerning the theory of Banach lattices and Orlicz and Marcinkiewicz spaces we refer the reader to [1–3,10]; cf. [6, Section 4]). In [13, Theorem 1 and Corollary 3], it has been proved that if E is Dedekind complete with $E \neq E_a$ then the quotient Banach lattice E/E_a contains an isomorphic (or isometric) copy of $W = \ell_\infty/c_0$ (cf. [6, Corollary 6]).

For example, if ℓ_ϕ denotes a sequence Orlicz space such that the Orlicz function ϕ does not fulfil the Δ_2-condition at 0, then the quotient space ℓ_ϕ/h_ϕ contains an isometric copy of W (see [6, Corollary 7]).

Similarly, if Ψ denotes a Marcinkiewicz function, then the quotient Marcinkiewicz space $M(\Psi)/M_0(\Psi)$ contains a copy of W whenever $M(\Psi) \neq M_0(\Psi)$ (see [6, Corollary 9]).

Hence, by Theorem 1 and Corollary 1, we obtain:

Corollary 6 *Let E be a Dedekind complete Banach lattice such that $E \neq E_a$ (i.e., the norm on E is not order continuous). If the kernel of an operator $T : E/E_a \to Y$ does not contain an isomorphic copy of c_0 (e.g., if T is injective), then T preserves ℓ_∞.*

In particular, every injective operator $T : E/E_a \to Y$ is non-weakly compact.

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