

On symmetric representations of groups of automorphism of bordered Klein surfaces

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Abstract An automorphism of a bordered compact Klein surface induces a permutation of its boundary components and we study the corresponding representations of groups of automorphisms of such surfaces in the corresponding finite symmetric groups. The principal result gives a characterization of those abstract representations of a finite group G in symmetric groups which, up to the natural equivalence, proceed from compact Klein surfaces on which G acts as a group of dianalytic automorphisms. We deal with such representations for actions of G given by a so called smooth epimorphisms $\Lambda \rightarrow G$, where Λ are so called non-Euclidean crystallographic groups (NEC-groups). For such data we calculate the kernels of our representations which allow, as an application, to get some results on the minimal degree of such representations for finite abstract groups.

Keywords Klein surfaces · Automorphisms of Klein surfaces · Boundary of Klein surfaces · NEC-groups · Real forms of complex algebraic curves · Symmetric (permutation) representation of finite groups

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1 Introduction

The importance of compact connected Riemann surfaces with nontrivial groups of analytic automorphisms comes from the fact that they represent singular points in the moduli spaces of

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such surfaces of given genus, and the groups themselves describe the nature of such singularities in some way. On the other hand there is a well known functorial equivalence between such surfaces and smooth, irreducible, complex, projective, algebraic curves and particular role play so called symmetric Riemann surfaces which correspond to complex curves having real forms. More precisely, under this equivalence, a Riemann surface X admits a symmetry σ , by which we understand an antiholomorphic involution, if and only if the corresponding curve \mathcal{C}_X has a real form $\mathcal{C}_X(\sigma)$. Furthermore, two such symmetries σ and τ define real forms $\mathcal{C}_X(\sigma)$ and $\mathcal{C}_X(\tau)$, birationally isomorphic over the field \mathbb{R} of real numbers, if and only if they are conjugate in the group $\text{Aut}^\pm(X)$ of all, including antiholomorphic, automorphisms of X and

$$\text{Aut}^\mathbb{R}(\mathcal{C}_X(\sigma)) \cong \text{Cent}_{\text{Aut}^\pm(X)}(\sigma),$$

where Cent stands for the centralizer.

Now for a Riemann surface X with a symmetry σ having nonempty set of fixed points, the orbit space X/σ is a bordered topological surface with, inherited from X , the natural dianalytic structure of so called Klein surface. Notice also, that each Klein surface arise from a symmetric Riemann surface in this way and a corresponding symmetry is unique up to conjugation in $\text{Aut}^\pm(X)$. A study of groups of dianalytic automorphisms of such surfaces is equivalent to a study of groups of birational automorphisms of real algebraic curves due to isomorphism

$$\text{Aut}(X/\sigma) \cong \text{Aut}^\mathbb{R}(\mathcal{C}_X(\sigma))$$

and the advantage of the former is made by a counterpart of the Riemann uniformization theorem, which together with some principal facts from covering theory allows their study through a well developed combinatorial theory of non-Euclidean crystallographic groups.

Next an automorphism of a bordered Klein surface induces a permutation of its boundary components and we study the associated representations of groups of automorphisms of such surfaces in the corresponding finite symmetric groups. The principal result gives a characterization of those abstract representations of a finite group G in symmetric groups which, up to the natural equivalence of symmetric representations, proceed from bordered Klein surfaces on which G acts as a group of dianalytic automorphisms. We deal with such representations for actions of finite groups G given by a so called smooth epimorphism $\Lambda \rightarrow G$, where Λ are so called non-Euclidean crystallographic groups. For them, we calculate the kernels of the corresponding representations which allow, as an application, to get some illustrative examples and general results on the minimal degree of such representations as well. Our results describe in such a way, a nature of groups of automorphisms of real algebraic curves.

2 Preliminaries

We shall use a combinatorial approach based on non-Euclidean crystallographic groups (NEC-groups in short); we send the reader to the monographs [1] and [3] for detailed exposition of the whole theory.

2.1 Non-Euclidean crystallographic groups

An *NEC-group* is a discrete and cocompact subgroup of the group \mathcal{G} of isometries of the hyperbolic plane \mathcal{H} including those which reverse orientation. If such a subgroup contains only orientation preserving isometries, it is called a Fuchsian group.

Macbeath [9] and Wilkie [13] associated to every NEC-group Λ a signature $s = s(\Lambda)$, which determines its algebraic structure. It has the form

$$(g; \pm; [m_1, \dots, m_r]; \{C_1, \dots, C_k\}). \tag{1}$$

Particular role, in our considerations, will play the brackets $C_i = (n_{i1}, \dots, n_{is_i})$, called the *period cycles* with the numbers $n_{ij} \geq 2$, called the *link periods*.

A group Λ with signature (1) has the presentation with the following generators, called *canonical generators* :

$x_1, \dots, x_r, e_i, c_{ij}, 1 \leq i \leq k, 0 \leq j \leq s_i$ and $a_1, b_1, \dots, a_h, b_h$ if the sign is + or d_1, \dots, d_h otherwise,

and relators:

$$x_i^{m_i}, i = 1, \dots, r, c_{ij-1}^2, c_{ij}^2, (c_{ij-1}c_{ij})^{n_{ij}}, c_{i0}e_i^{-1}c_{is_i}e_i, i = 1, \dots, k, j = 1, \dots, s_i \text{ and}$$

$$x_1 \dots x_r e_1 \dots e_k a_1 b_1 a_1^{-1} b_1^{-1} \dots a_h b_h a_h^{-1} b_h^{-1} \text{ or } x_1 \dots x_r e_1 \dots e_k d_1^2 \dots d_h^2,$$

according to whether the sign is + or -.

Generators $c_{i,j}$ are hyperbolic reflections and reflections $c_{i,j-1}, c_{i,j}$ corresponding to a link period $n_{i,j}$ are said to be consecutive or neighbouring.

Every NEC-group has associated a fundamental region, whose hyperbolic area $\mu(\Lambda)$ for an NEC-group Λ with signature (1) is given by

$$2\pi \left(\varepsilon g + k - 2 + \sum_{i=1}^r \left(1 - \frac{1}{m_i} \right) + \frac{1}{2} \sum_{i=1}^k \sum_{j=1}^{s_i} \left(1 - \frac{1}{n_{ij}} \right) \right), \tag{2}$$

where $\varepsilon = 2$ if the sign is + and $\varepsilon = 1$ otherwise. Finally, it is known that an abstract group with the above presentation can be realized as an NEC-group with the signature (1) if and only if (2) is positive and if Λ' is a subgroup of finite index in an NEC-group Λ then it is an NEC-group itself and there is a Hurwitz–Riemann formula, which says that

$$[\Lambda : \Lambda'] = \mu(\Lambda')/\mu(\Lambda). \tag{3}$$

2.2 Riemann and Klein surfaces and their group of automorphisms

A Klein surface is a compact topological surface with a dianalytic structure which, in contrast to the classical analytic structure, involve the conjugation $z \rightarrow \bar{z}$ for transition maps between charts and for the local forms of automorphisms between such surfaces. Similarly, as for Riemann surfaces, a bordered Klein surface of algebraic genus $g \geq 2$ can be represented as the orbit space \mathcal{H}/Γ with an NEC-group Γ having a signature $(g'; \pm; [-]; \{(-), \cdot^k, (-)\})$, called *bordered surface NEC-group*, where k is the number of boundary components of X and where $g = \varepsilon g' + k - 1$, with $\varepsilon = 2$ if X is orientable (the sign is +) or $\varepsilon = 1$ if X is nonorientable (the sign is -). Furthermore a finite group G is a group of automorphisms of a surface so represented if and only if $G \cong \Lambda/\Gamma$ for some NEC group Λ and, generally, an epimorphism $\Lambda \rightarrow G$ with the kernel being bordered surface NEC-group will be called *smooth*. Finally two dianalytic actions given by smooth epimorphisms $\theta : \Lambda \rightarrow G$ and $\theta' : \Lambda' \rightarrow G'$ are *topologically equivalent* if and only if the diagram

$$\begin{array}{ccc} \Lambda & \xrightarrow{\varphi} & \Lambda' \\ \theta \downarrow & & \downarrow \theta' \\ G & \xrightarrow{\psi} & G' \end{array}$$

commutes for some isomorphisms $\varphi : \Lambda \rightarrow \Lambda'$ and $\psi : G \rightarrow G'$.

3 Representations of groups of automorphisms of Klein surfaces on their boundary

Let G be a group of automorphisms of a compact Klein surface X with the boundary consisting of the connected components $\mathcal{O}_1, \dots, \mathcal{O}_n$, each of which turns out to be homeomorphic to a circle. We define $\rho_X : G \rightarrow S_n$ by

$$\rho_X(g)(i) = j \Leftrightarrow g(\mathcal{O}_i) = \mathcal{O}_j$$

This representation will be called *geometric*. With this definition we have the following

Lemma 3.1 *The mapping ρ_X is a homomorphism whose kernel is either cyclic or dihedral.*

Proof If $\rho_X(g)(i) = i$, then g induces an isometry of the circle \mathcal{O}_i . So the assertion follows from the fact that a finite group of isometries of a circle is either cyclic or dihedral, from [5, 10] on the extension of automorphisms of bordered Klein surfaces and from the fact that an orientation preserving automorphism of a compact Riemann surface having a non-discrete set of fixed points is trivial. □

As an immediate consequence we obtain the following result of Bujalance [2] about groups of automorphisms of bordered Klein surfaces with a small number of boundary components.

Corollary 3.2 *The group of automorphisms of a Klein surface having $1 \leq k \leq 4$ boundary components is solvable.* □

Here and throughout the remainder of the paper we shall use left hand side notation for conjugations and commutators i.e. $a^x = xax^{-1}$ and $[a, x] = axa^{-1}x^{-1}$.

Theorem 3.3 *Let $\theta : \Lambda \rightarrow G$ be an epimorphism defining a group of automorphism G of a Klein surface $X = \mathcal{H}/\Gamma$, where $\Gamma = \ker \theta$. Given a set of canonical generators of Λ , let \mathcal{C} be the set of its canonical reflections. Let $\{c_i : i \in I\}$ be a maximal subset of \mathcal{C} consisting of the pairwise non-conjugate reflections, mapped by θ into the unit. Then*

$$\ker \rho_X = \bigcap_{g \in G} \bigcap_{i \in I} g (\theta (C(\Lambda, c_i))) g^{-1}$$

Proof It is well known how to calculate the number of boundary components of a compact Klein surface X , in terms of θ and the signature of Λ [3]. But here we shall need not only such quantitative results but also a qualitative description of boundary components. We shall use an approach developed in [6] and already successfully applied in [4] and [7] for example. The connected components of the boundary of X are in the bijective correspondence with the reflections of Λ which are in Γ and which are not conjugated in Γ . On the other hand, each reflection of Γ is conjugate to some canonical reflection from \mathcal{C} . So we have to look for representatives of the conjugacy classes with respect to the conjugations by the elements of Γ in the set which, in principle comes from

$$\{c^\lambda : c \in \mathcal{C}, \theta(c) = 1, \lambda \in \Lambda\}. \tag{4}$$

Observe however that if $c_i \stackrel{\Lambda}{\sim} c_{i'}$, say $c_{i'} = c_i^\lambda$ and $c_i^{\lambda'}, c_i^{\lambda''} \in \Gamma$, then $c_i^{\lambda''} = c_i^{\lambda''\lambda}$. It follows that actually the canonical reflections in (4) can be runed over the set $\{c_i : i \in I\}$ from the statement.

Now, given $i \in I$ we have

$$\lambda c_i \lambda^{-1} \stackrel{\Gamma}{\sim} \lambda' c_i \lambda'^{-1} \Leftrightarrow C(\Lambda, c_i) \ni \lambda^{-1} \gamma \lambda' = \lambda^{-1} \lambda' (\lambda'^{-1} \gamma \lambda')$$

for some $\gamma \in \Gamma$, which is equivalent to saying that $\lambda^{-1} \lambda' \in C(\Lambda, c_i) \Gamma$, as Γ is a normal subgroup of Λ . Therefore conjugates of c_i give rise to

$$[\Lambda : C(\Lambda, c_i) \Gamma] = [\Lambda / \Gamma : C(\Lambda, c_i) \Gamma / \Gamma] = [G : \theta(C(\Lambda, c_i))] \tag{5}$$

boundary components in X .

Now given $h = \theta(\lambda_h) \in G$ and the boundary component \mathcal{O} corresponding to a reflection c , $h(\mathcal{O})$ is the boundary component corresponding to c^{λ_h} . Hence $h \in \ker \rho_X$ if and only if for given $i \in I$ and $\lambda \in \Lambda$, we have $\lambda_h \lambda c_i \lambda^{-1} \lambda_h^{-1} \stackrel{\Gamma}{\sim} \lambda c_i \lambda^{-1}$. This means that $C(\Lambda, c_i) \ni \lambda^{-1} \gamma \lambda_h \lambda = \lambda^{-1} \lambda_h \lambda (\lambda_h \lambda)^{-1} \gamma (\lambda_h \lambda)$ or equivalently $\lambda^{-1} \lambda_h \lambda \in C(\Lambda, c_i) \Gamma$, which means that $g h g^{-1} \in \theta(C(\Lambda, c_i))$ for arbitrary $i \in I$, $g \in G$ and in turn $h \in g^{-1} \theta(C(\Lambda, c_i)) g$.

Conversely, let

$$h \in \bigcap_{g \in G} \bigcap_{i \in I} g (\theta(C(\Lambda, c_i))) g^{-1}$$

and let \mathcal{O}_i be a boundary component corresponding to c_i^λ for some $\lambda \in \Lambda$. Then $h = \theta(\lambda) \theta(\lambda_i) \theta(\lambda)^{-1} = \theta(\lambda \lambda_i \lambda^{-1})$ for some $\lambda_i \in \Lambda$ commuting with c_i . So $h(\mathcal{O}_i)$ corresponds to $(c_i^\lambda)^{\lambda \lambda_i \lambda^{-1}} = c_i^\lambda$ and therefore $h(\mathcal{O}_i) = \mathcal{O}_i$. Thus $h \in \ker \rho_X$, which completes the proof. \square

Having calculated the kernel of the representation ρ_X for the group G of dianalytic automorphisms of a bordered Klein surface X given by the smooth epimorphism $\theta : \Lambda \rightarrow G$, we obtain at once the following, well known and obvious.

Corollary 3.4 *Given a finite group G , there exists a bordered Klein surface X such that $G \subseteq \text{Aut}(X)$ and ρ_X is faithful.*

Proof Let g_1, \dots, g_k be an arbitrary set of generators of G ; clearly we can assume that $k \geq 2$. Let Λ be an NEC-group with signature $(k; +; [-]; \{(-)\})$ and consider an epimorphism $\theta : \Lambda \rightarrow G$ for which $\theta(a_i) = \theta(b_i) = g_i$ and $\theta(e_1) = \theta(c_1) = 1$. Then for $X = \mathcal{H} / \Gamma$, where $\Gamma = \ker \theta$, the representation ρ_X is faithful by Theorem 3.3, since $C(\Lambda, c_1) = \langle c_1, e_1 \rangle$. \square

Remark 3.5 There is a bijection between irreducible G -invariant subsets of the set of boundary components of X and the conjugacy classes in Λ of canonical reflections of Λ which are in Γ . Furthermore, the subset corresponding to the class of a reflection c has $[G : \theta(C(\Lambda, c))]$ elements. Indeed, (4) is the set of all reflections of Γ . As we already observed, reflections c in (4) actually run over the set \mathcal{C}' of representatives of the conjugacy classes in Λ of all canonical reflections of Λ . Observe also that given $c, c' \in \mathcal{C}'$ which are in Γ , their conjugations give boundary components in two disjoint G -invariant subsets. Finally, it is trivial that all conjugations of given $c \in \mathcal{C}'$ are conjugate in Λ and so the subset corresponding to c is irreducible, its cardinality is found in (5). \square

4 The characterization of geometric representations

We already have shown that given a finite group there is a bordered Klein surface X for which G is a group of dianalytic automorphisms and ρ_X is faithful. It should be interesting to find

for given G the minimum $n = n(G)$ so that $\rho(G) \subset S_n$. Here we shall prove our principal result which gives necessary and sufficient conditions for an abstract permutational representation $\rho : G \rightarrow S_n$ to be equivalent to a representation ρ_X proceeding from a bordered Klein surface X with G as a group of dianalytic automorphisms.

Remark 4.1 We already know that the boundary components of X correspond to the conjugacy classes in Γ of reflections in Γ . Furthermore recall that given $g = \theta(\lambda) \in G$ and a boundary component \mathcal{O} , say corresponding to the reflection c , we have

$$\rho_X(g)(\mathcal{O}) = \mathcal{O}', \tag{6}$$

where \mathcal{O}' is a boundary component corresponding to c^λ . □

A representation $\rho : G \rightarrow S_n = S(\Omega)$ of a finite group G in the symmetric group on $\Omega = \{1, \dots, n\}$ is said to be irreducible if there are no G -invariant proper subsets of Ω . Now, given an arbitrary representation ρ , there is a partition $\Omega = \Omega_1 \cup \dots \cup \Omega_k$, where Ω_i is G -invariant of cardinality n_i and it does not contain G -invariant subsets. In this way we obtain the decomposition $\rho = \rho_1 \oplus \dots \oplus \rho_k$ into irreducible representations ρ_i of G in $S_{n_i} = S(\Omega_i)$. Observe however that ρ_i may not be faithful even if ρ is faithful.

Now two representations $\rho : G \rightarrow S(\Omega), \rho' : G' \rightarrow S(\Omega')$ are said to be *equivalent* if there is an isomorphism $\alpha : G \rightarrow G'$ and a bijection $\beta : \Omega \rightarrow \Omega'$ such that the diagram

$$\begin{array}{ccc} G & \xrightarrow{\alpha} & G' \\ \rho \downarrow & & \downarrow \rho' \\ S(\Omega) & \xrightarrow{\tilde{\beta}} & S(\Omega') \end{array}$$

where $\tilde{\beta}(f) = \beta f \beta^{-1}$ commutes.

Observe that topologically equivalent actions give rise to equivalent representations. Finally, a representation of G , equivalent to the canonical representation of G on the set of left cosets G/H for some subgroup H of G , is said to be regular with respect to H . With these definitions we have the following

Theorem 4.2 *An abstract representation $\rho : G \rightarrow S_n$ is equivalent to the representation of a group of automorphisms of some Klein surface on its set of boundary components if and only if ρ is the product of regular representations with respect to cyclic or dihedral subgroups.*

Proof Let $\rho_X : G \rightarrow S(\Omega)$ be a representation induced by the action of G on the set Ω of boundary components of a Klein surface X defined by a smooth epimorphism $\theta : \Lambda \rightarrow G$. Then by the Remark 3.5 we know that irreducible invariant subsets correspond to conjugacy classes of reflections of Λ which are in Γ . Let Ω_i be such a subset, say of cardinality n_i , and let

$$c_i^{\lambda_1}, \dots, c_i^{\lambda_{n_i}} \tag{7}$$

be the non-conjugate in Γ reflections corresponding to the elements $\mathcal{O}_1, \dots, \mathcal{O}_{n_i}$ of Ω_i . Then, given $g = \theta(\lambda) \in G$, we have

$$\begin{aligned}
 g(\mathcal{O}_j) = \mathcal{O}_k &\Leftrightarrow c_i^{\lambda\lambda_j} \overset{\Gamma}{\sim} c_i^{\lambda_k} \\
 &\Leftrightarrow \lambda_k^{-1} \gamma \lambda \lambda_j \in C(\Lambda, c_i) \quad \text{for some } \gamma \in \Gamma \\
 &\Leftrightarrow \lambda_k^{-1} \lambda \lambda_j \in C(\Lambda, c_i) \Gamma \\
 &\Leftrightarrow \theta(\lambda_k)^{-1} g \theta(\lambda_j) \in \theta(C(\Lambda, c_i)) \\
 &\Leftrightarrow g(\theta(\lambda_j) \theta(C(\Lambda, c_i))) = \theta(\lambda_k) \theta(C(\Lambda, c_i))
 \end{aligned}$$

and therefore, since, by (5), $\theta(\lambda_1), \dots, \theta(\lambda_{n_i})$ are representatives of all cosets in $G/\theta(C(\Lambda, c_i))$, we see that the action of ρ_X on Ω_i is regular with respect to $\theta(C(\Lambda, c_i))$, which by [11, 12], is either cyclic or dihedral.

Conversely, assume that $\rho = \rho_1 \oplus \dots \oplus \rho_k$ is a decomposition of a representation ρ into irreducible representations ρ_i of G in the symmetric group S_{m_i} on the sets Ω_i and suppose that ρ_1, \dots, ρ_s are regular with respect to dihedral subgroups $D_{n_i} = \langle a_i, b_i \mid a_i^2, b_i^2, (a_i b_i)^{n_i} \rangle$ and $\rho_{s+1}, \dots, \rho_k$ are regular with respect to cyclic subgroups $Z_{n_i} = \langle a_i \rangle$. Let m be the order of $a = a_{s+1} \dots a_k$, let g_1, \dots, g_l be the set of generators of G , let Λ be an NEC group with signature

$$\left(l; +; [m]; \left\{ (2, 2, n_1), \dots, (2, 2, n_s), (-, \overset{k-s}{\dots}, (-) \right\} \right)$$

and let $\theta : \Lambda \rightarrow G$ be an epimorphism defined by the assignment

$$\theta(a_i) = \theta(b_i) = g_i \quad \text{for } i = 1, \dots, l,$$

$$\theta(c_{i0}) = \theta(c_{i3}) = a_i, \theta(c_{i1}) = 1, \theta(c_{i2}) = b_i, \theta(e_i) = 1 \quad \text{for } i = 1, \dots, s,$$

$$\theta(c_{i0}) = 1, \theta(e_i) = a_i \quad \text{for } i = s + 1, \dots, k, \text{ and } \theta(x_1) = a^{-1}.$$

Then for $X = \mathcal{H}/\Gamma$, where $\Gamma = \ker \theta$ we have a representation $\rho_X : G \rightarrow S_n$, where $n = n_1 + \dots + n_k$, equivalent to ρ . □

5 On the degree of geometrical representations of finite groups

Let $G = \Lambda/\Gamma$, where Λ is an NEC group, Γ is a surface NEC-group and let $\theta : \Lambda \rightarrow G$ be the corresponding smooth epimorphism. We know that given a finite group G there is a bordered Klein surface X with k boundary components for which $\rho_X : G \rightarrow S_k$ is faithful and a natural problem considered here is to find the minimal k in such situation.

5.1 A general theorem

Theorem 5.1 *The minimum degree $k(G)$ of a geometrical representation of a finite group G is a minimal value of*

$$\frac{1}{2} \left(\frac{N}{|d_1 d'_1|} + \dots + \frac{N}{|d_s d'_s|} \right) + \frac{N}{|a_{s+1}|} + \dots + \frac{N}{|a_k|}, \tag{8}$$

where $|x|$ denotes the order of x , $s \leq k$ run over all nonnegative integers, a_i are arbitrary elements and d'_i, d_i are arbitrary involutions for which

$$\left(\bigcap_{g \in G} \bigcap_{j=1}^s g \langle d_j, d'_j \rangle g^{-1} \right) \cap \left(\bigcap_{g \in G} \bigcap_{i=s+1}^k g \langle a_i \rangle g^{-1} \right) = 1$$

Proof Let $G = \Lambda / \Gamma$. Observe, that for a reflection c corresponding to an empty period cycle we have $C(\Lambda, c) = \langle e, c \rangle$ while for a triple consecutive reflections c', c, c'' corresponding to two consecutive link periods equal to 2, $C(\Lambda, c) = \langle c \rangle \oplus \langle c', c'' \rangle$. On the other hand, each boundary component of $X = \mathcal{H} / \Gamma$ correspond to a conjugate of a canonical reflection c of Λ which is in Γ . Furthermore, if such a reflection corresponds to an empty period cycle with connecting generator e , then it gives rise to $|G|/|\theta(e)|$ boundary components of X by (5), while if it corresponds to a nonempty period cycle, then for the neighbouring reflections c', c'' , the products $c'c$ and cc'' have orders 2 and c giving rise to $|G|/2|\theta(c')\theta(c'')|$ boundary components in X by (5) again. There are no boundary components of X which arise in any other way and so the minimal value of (8) give a lower bound for $k(G)$ in virtue of Theorem 3.3.

On the other hand having arbitrary elements a_i and arbitrary involutions d'_i, d''_i for which (8) is minimal, let g_1, \dots, g_l be an arbitrary set of generators for G . Then an epimorphism constructed in the second part of Theorem 4.2 gives a Klein surface X with a group of automorphisms G for which ρ_X is a geometrical permutation representation of G of the minimal degree. □

5.2 Groups for which $k(G) = |G|$

It is obvious that $k(G) \leq |G|$ and here we shall look for the groups for which $k(G) = |G|$. We start with rather obvious

Example 5.2 Let $G = \langle g \rangle = \mathbb{Z}_N$, be a cyclic group of order N , where $N = p_1^{\alpha_1} \dots p_r^{\alpha_r}$ for different primes p_1, \dots, p_r and $r > 1$. Let $N_i = N/p_i^{\alpha_i}, i = 1, \dots, r$. The subgroup $G_i = \langle g^{p_i^{\alpha_i}} \rangle$ has index $p_i^{\alpha_i}$ in G . Moreover

$$\bigcap_{1 \leq i \leq r} G_i = 1.$$

Hence, there exists a geometrical representation of degree $p_1^{\alpha_1} + \dots + p_r^{\alpha_r}$ which is the minimum degree of a faithful permutation representation for a cyclic group of order N . If $r = 1$ then obviously the minimum degree of a geometrical representation is equal to N . □

Example 5.3 Let D_N be a dihedral group of order $2N$, where $N = p_1^{\alpha_1} \dots p_r^{\alpha_r}, p_1, \dots, p_r$ are different primes and $r > 1$. Let G be a cyclic subgroup of D_N of order N . Every element of G is a product of two involutions, so for a subgroup $G_i, i = 1, \dots, r$, from the previous example, there exist involutions d_i, d'_i such that $|\langle d_i, d'_i \rangle : G_i| = 2$. Similarly as in Example 5.2 we see that

$$G \cap \bigcap_{1 \leq i \leq r} \langle d_i, d'_i \rangle = 1.$$

The maximal subgroup of D_N which has trivial intersection with G has order 2 and so it is generated by an involution. No element outside G is central, so

$$\bigcap_{g \in G} \bigcap_{1 \leq i \leq r} g^{-1} \langle d_i, d'_i \rangle g = 1.$$

Now suppose that N is a power of a prime, $N > 2$. Then for a noncentral involution $d \in D_N$ we have

$$\bigcap_{g \in G} g^{-1} \langle d \rangle g = 1.$$

Hence the minimum degree of a faithful geometrical representation is equal to $|D_N|/2 = N$ and is the same as the minimum degree of a faithful permutation representation. \square

Example 5.4 Let Q_{2^n} be a generalized quaternion group of order 2^{n+1} :

$$Q_{2^n} = \langle x, y \mid x^{2^n} = 1, y^2 = x^{2^{n-1}} y^{-1} x y = x^{-1} \rangle.$$

It is well known that every nontrivial subgroup of Q_{2^n} contains a subgroup $\langle x^{2^{n-1}} \rangle$. Therefore we obtain the minimal value of (8) for $r = 1$ and $a_1 = 1$. Hence the minimum degree of a faithful geometrical representation is equal to 2^{n+1} . \square

Proposition 5.5 *If G is a finite group then $k(G) = |G|$ if and only if G is one of the following groups:*

- (a) a cyclic group of order a power of a prime,
- (b) a generalized quaternion group,
- (c) an elementary abelian group of order 2^n .

Proof The implication \Leftarrow follows from the previous examples and we shall prove the converse. If $|G|$ is divisible by two different primes p and q then it contains two cyclic subgroups of orders p and q respectively. Since they have trivial intersection, $k(G) \leq |G|/p + |G|/q < |G|$. Thus G is a p -group. If every two nontrivial subgroups of $|G|$ have nontrivial intersection then either G is cyclic or $p = 2$ and G is a generalized quaternion group (see for instance [8], Satz 5.3.7). Suppose G has two subgroups $\langle a_1 \rangle$ and $\langle a_2 \rangle$ of order p^m and p^k respectively. Then $k(G) \leq |G| (1/p^m + 1/p^k)$. If the right hand side of this inequality is equal to $|G|$, then $p = 2$ and $m = k = 1$, which means that G is elementary abelian of order 2^n , for some n . \square

It is known that if we put $n = 2$ in the last item of the above proposition we get the description of groups G for which the minimum degree of a faithful permutation representation is equal to $|G|$.

5.3 On $k(G)$ for finite simple groups at large

If G is a finite simple group then the sum from Theorem 5.1 has obviously only one component coming from a cyclic or dihedral subgroup of maximal order.

Lemma 5.6 *An arbitrary maximal subgroup of a finite simple group is not cyclic.*

Proof Suppose that a maximal subgroup M of a finite simple group G is cyclic. Then obviously $N_G(M) = M$ and for every $g \in G \setminus M$ we have $g^{-1}Mg \cap M = 1$. Otherwise $g^{-1}Mg \cap M$ would be a nontrivial central subgroup in $\langle g^{-1}Mg, M \rangle = G$ which contradicts the simplicity of G . Hence, by Frobenius theorem, M has a nontrivial normal complement which again gives a contradiction. \square

On the other hand there exist infinitely many finite simple groups containing maximal subgroups which are dihedral. But also in this case $k(G)$ does not need to be equal to the minimal degree of a permutation representation of G .

Lemma 5.7 *Let p be a prime, $p > 3$, and let $G = \text{PSL}(2, p)$. Every maximal subgroup M of G is isomorphic to one of the following groups:*

- (a) a dihedral group of order $p - 1$ or $p + 1$,
- (b) a semidirect product of a Sylow p -subgroup of G and a cyclic group of order $(p - 1)/2$,
- (c) A_4 , A_5 or S_4 – the alternating group of degree 4 or 5 or the symmetric group of degree 4. \square

Example 5.8 If p is a prime, $p > 3$, then $k(\text{PSL}(2, p))$ is bigger than the minimum degree of a permutation representation of $\text{PSL}(2, p)$ because maximal subgroups of the biggest order are those described in the items (b) and (c). It is immediately seen that the minimal degree of a faithful permutation representation of $\text{PSL}(2, p)$ ($p > 11$) is equal to $p + 1$, when $k(\text{PSL}(2, p)) = p(p - 1)/2$. \square

The minimum degree of faithful permutation representations of finite simple groups were determined in a series of papers by several authors. For many reasons it would be interesting to describe $k(G)$ for these groups, which however is not an easy matter and perhaps is a good task for further and more detailed study and a sequel is planned by the authors. We finish this paper with an example of alternating groups which indicates some difficulties in determining the precise value of $k(G)$.

Example 5.9 We start with the symmetric group S_n of degree n , where the minimum degree of a faithful permutation representation is obviously equal to n . Let s be the maximal order of elements of S_n , which is the maximum of the least common multiple of numbers n_1, n_2, \dots, n_k such that $n_1 + \dots + n_k \leq n$. Since each cyclic subgroup of S_n is contained in a subgroup which is dihedral we have $k(G) = n!/2s$ which, for instance, give $k(S_5) = 10$ and $k(S_8) = 8!/30 = 1344$. The case of alternating groups A_n is a little bit more involved. Here also the minimum degree of a permutation representation is equal to n but now a cyclic subgroup of the maximal order may not be contained in a dihedral subgroup which makes the problem nontrivial. For instance the maximal order of elements of A_7 is 7, but the normalizer of a subgroup of order 7 in A_7 , has order 21 (there is not a subgroup of order 14) and it is easily seen that $k(A_7) = (7!/2)/(2 \cdot 6) = 210$ as A_7 contains a dihedral subgroup of order 12. The maximal order of elements of A_8 is 15, that is the same as for S_8 , but there is not a dihedral subgroup of order ≥ 15 . Hence $k(A_8) = k(S_8)$. In the general case A_n , $n > 5$ one can write that $k(G) \geq n!/4s$, where s is the maximal order of an element of A_n . \square

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