ORIGINAL ARTICLE

Students' agency, creative reasoning, and collaboration in mathematical problem solving

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Abstract

This paper aims to give detailed insights of interactional aspects of students' agency, reasoning, and collaboration, in their attempt to solve a linear function problem together. Four student pairs from a Norwegian upper secondary school suggested and explained ideas, tested it out, and evaluated their solution methods. The student–student interactions were studied by characterizing students' individual mathematical reasoning, collaborative processes, and exercised agency. In the analysis, two interaction patterns emerged from the roles in how a student engaged or refrained from engaging in the collaborative work. Students' engagement reveals aspects of how collaborative processes and mathematical reasoning co-exist with their agencies, through two ways of interacting: bi-directional interaction and one-directional interaction. Four student pairs illuminate how different roles in their collaboration are connected to shared agency or individual agency for merging knowledge together in shared understanding. In one-directional interactions, students engaged with different agencies as a primary agent, leading the conversation, making suggestions and explanations sometimes anchored in mathematical properties, or, as a secondary agent, listening and attempting to understand ideas are expressed by a peer. A secondary agent rarely reasoned mathematically. Both students attempted to collaborate, but rarely or never disagreed. The interactional pattern in bi-directional interactions highlights a mutual attempt to collaborate where both students were the driving forces of the problem-solving process. Students acted with similar roles where both were exercising a shared agency, building the final argument together by suggesting, accepting, listening, and negotiating mathematical properties. A critical variable for such a successful interaction was the collaborative process of repairing their shared understanding and reasoning anchored in mathematical properties of linear functions.

Keywords Agency · Collaboration · Reasoning · Shared understanding

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Introduction

Students' mathematical communication, reasoning, and problem-solving are highlighted in the research literature as important aspects for fostering students' learning of mathematics (Pijls & Dekker, 2011; Seidouvy & Schindler, 2019; Sidenvall, 2019). Therefore, promoting students' mathematical communication such as making reasoning meaningful to oneself and peers, listening to one another, and solving mathematics together (Mueller et al., 2012) are central issues for progress in their mathematical understanding. Students' collaboration and reasoning are key interactional aspects in math-talk and small group work. It is a well-researched area but still highlighted as a topic where more knowledge is needed, because when students are given the opportunity they can construct their own solution procedures, important for their mathematical understanding (cf., Lithner, 2017; Mueller et al., 2012; Stockero et al., 2019).

Collaboration in classrooms has often been studied as an outcome or as a process (Dillenbourg et al., 1996). A research approach on the outcome is often seen as an individual's activity with focus on learning (Child & Shaw, 2018; Dillenbourg et al., 1996), whereas a research approach on collaborative processes emphasize the whole group and in particular the participant interaction (Seidouvy and Schindler, 2019). With the latter view on collaboration, this study defines collaboration as "a coordinated, synchronous activity that is the result of a continued attempt to construct and maintain a shared conception of a problem" (Roschelle & Teasley, 1995, p. 70). With this focus, the study views productive collaboration enacted when students build and maintain a shared conception of a mathematical problem, which is their shared understanding of the problem at hand (Roschelle & Teasley, 1995). Moreover, for students to pool their knowledge together for a shared understanding, the study sees the processes of *building* by introducing and accepting knowledge, *monitoring* ongoing activity, and *repairing* conflicting interpretations, as central activities for such interaction.

Opportunities for studying situations of collaborative processes are in this study viewed through a problem-solving session, emphasizing active engagement in a learning process (Lithner, 2017). An active learning process is not common in so-called easier learning processes where solution procedures are imitated but rather found in students' problem solving when they attempt to construct their own solution procedures through reasoning (Lithner, 2017). Mathematical reasoning is a central interactional aspect of learning mathematics where arguments are important for the learning process and not only as an outcome of learning mathematics (Yackel, 2001). Therefore, this study views mathematical argumentation and reasoning as an interactional accomplishment and what students "take as acceptable, individually and collectively, and not whether an argument might be considered mathematically valid" (Yackel, 2001, p. 6). In line with this view, including all students at any competence level in mathematics, is Lithner's framework of mathematical reasoning (Lithner, 2008, 2017). From this framework, a student's reasoning is explained as "the line of thought adopted to produce assertions and reach conclusions in task solving" (Lithner, 2017, p. 939).

In situations of mathematical communication, a student may become a producer of mathematics through their own and joint processes of grappling and exploring mathematics, sometimes hitting the wrong trail, and sometimes making sense of solutions, rather than being a reproducer (Freudenthal, 1991; Schoenfeld, 2013). Schoenfeld (2013) says: "A major issue is, when if ever do students get to develop a mathematical voice? That is, when do they get to propose ideas and answers, defend them, and become recognized as producers of mathematics themselves?" (p. 613). If these situations of mathematical communication include mathematical talk between students, they are in a student–student interaction. How well students interact in pairs and small groups is important for students' progress in becoming a mathematical producer, rather than an imitator who reproduces mathematics without understanding the conceptual parts (Lithner, 2017). Varhol et al. (2020) researched student interactions and found that some specific types of interactions were important for making mathematical progress in algebraic generalization. That study, and other studies on students' group work or pairwise collaboration, contends that quality student interactions and the dynamics of the processes in students' collaborations need to be further explored (Seidouvy & Schindler, 2019; van de Pol et al., 2018; Varhol et al., 2020).

Students engage with one another and take on different roles while interacting. Sometimes a student leads the conversation, other times he or she might listen or withdraw from the conversation. Thus, roles in collaborative work can "change from moment to moment" (Child & Shaw, 2018, p. 1). Hence, another central interactional aspect in group work, when considering students' actions and engagement in reasoning and collaboration, is students' *exercised agency* (Mueller et al., 2012). This study see agency as Gresalfi et al. (2009) define it; "the way in which he or she acts, or refrains from acting, and the way in which her or his action contributes to the joint action of the group in which he or she is participating" (p. 53).

The field needs a better understanding of peer interaction patterns in collaborative mathematical activity. Little is known about the dynamics of students' collaborative interaction in mathematics classroom, and few studies have taken the process view of collaboration (Seidouvy & Schindler, 2019). This study focuses on three particular aspects of students' interactions: collaborative processes, mathematical reasoning, and exercised agency. To investigate conditions for fruitful collaboration, Kuhn (2015) states that "it is essential to understand the underlying mechanisms" (p. 47). Studying interactional aspects separately and seen in interplay in students' interaction patterns may therefore give a better understanding of underlying processes of collaboration, which can furthermore enable collaborative aspects to become teachable. Research on collaboration (Child & Shaw, 2018; Kuhn, 2015) and reasoning (Lithner, 2017) argues that learning is enacted in both instances and that there is a need for insights of the underlying processes causing this learning opportunity. Therefore, unpacking students' social interactions found in the interplay of specific interactional aspects can make collaboration and reasoning more manageable for teachers and students, and consequentially promote productive quality interactions for learning opportunities in mathematics classrooms.

The aim of this study is to give detailed insights of interactional aspects with upper secondary students' roles in their collaboration and reasoning, when

attempting to solve a linear function task together. With this aim, the research question is *What are the patterns of interaction for creating a shared understanding through the interplay between students' creative reasoning, collaboration, and exercised agency in a mathematical problem-solving session?*

Theory

Mathematical reasoning

Mathematical reasoning can be defined as "the explicit act of justifying choices and conclusions by mathematical arguments" (Boesen et al., 2014, p. 75). In line with this statement is the framework *Creative mathematically founded reasoning*[1] (CMR, Lithner, 2008), which identifies two major types of reasoning: *creative reasoning* and *imitative reasoning.* The latter reasoning type is seen in students' use of remembered facts and memorized algorithms without considering their meaning. Such path of reasoning has its strength in quickly solving tasks in school mathematics. However, without the conceptual part it may lead to rote learning (Lithner, 2017). Creative reasoning, on the other hand, has the strength of promoting deeper understanding of mathematical procedures and concepts (Lithner, 2008). If engaged with creative reasoning, students are considering mathematical properties with the task they are solving or discussing, which makes it likely to develop an understanding (Lithner, 2017). Creative reasoning is characterized by three aspects: *creativity, plausibility, and anchoring.* These three aspects are interconnected in reasoning as follows: A student may create a new idea or recreate a forgotten one (creativity) using arguments which are meaningful and logical to the student who is employing them (plausibility) and that are based on mathematical properties (anchoring). Creativity is therefore a student's attempt to create or recreate a reasoning sequence that, to some extent, is new to them. A student's reasoning, expressed as arguments, is creative when supported by plausible arguments. Plausible arguments are explanations of strategy choices, implementations of the strategies, and explanation of why a strategy or solution will work or not (Olsson, 2018). The arguments are creative when explanations and suggestions are mathematically anchored justifications (Granberg and Olsson, 2015). Lithner (2008) explains the difference in mathematical property as superficial or intrinsic: "In deciding if 9/15 or 2/3 is larger, the size of the numbers (9, 15, 2 and 3) is a surface property that is insufficient to consider while the quotient captures the intrinsic property" (p. 261). Central CMR-components are presented in Table 1.

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[1] In line with Lithner (2008, 2017) and his colleagues studying creative mathematically founded reasoning, this study uses the wording creative reasoning or acronym CMR for linguistically simplicity.

Table 1 Overview of central elements of creative reasoning

Creative mathematically founded reasoning (CMR)		
Creativity	Plausibility	Anchoring
New idea Recreating a forgotten idea	Explanation of strategy choice Explanation of strategy implementation Explanation of why something is true	Ideas connected to mathematical properties and concepts Intrinsic: mathematical concepts and properties

Collaboration

When students work together and attempt to do thinking together, their "emergent interplay of ideas" (Martin & Towers, 2015) promotes a *shared understanding* of the problem. A shared understanding is students' collaborative outcome from engaging in collaborative processes where two or more students work together to solve a problem and attempt to produce a joint outcome (Roschelle and Teasley, 1995). In line with a view on students' merging of ideas into a shared conception for solving a mathematical problem, Roschelle and Teasley (1995) define *collaboration* as a "coordinated, synchronous activity that is the result of a continued attempt to construct and maintain a shared conception of a problem" (p. 70).

The way students engage in collaborative processes relates to how well collaboration is maintained and fostered (Child & Shaw, 2016), and collaborative processes are important to study in students' interactions for the dynamics of joint mathematical problem solving. Students' attempts to create and uphold a shared understanding, through coordination of language and actions (e.g., Baker, 2015; Roschelle & Teasley, 1995; Sarmiento & Stahl, 2008), entail what Roschelle & Teasley (1995) call collaborative processes of building, monitoring, and repairing. The collaborative process *building* means to suggest ideas to initiate collaboration or it could be a continuation or ending of collaborative work (Alrø & Skovsmose, 2004; Child & Shaw, 2018; Roschelle & Teasley, 1995). For instance, if a peer accepts the suggested idea, such as a problem-solving strategy or implementation of an algorithm, a student contributes to build a shared understanding. A student could also read out loud and point at the problem to be solved. Sometimes, a peer listens to a suggestion or asks questions about an idea, which is important for *monitoring* a groups' shared understanding (Roschelle & Teasley, 1995). A question about an idea might result in a monitoring action, such as an explanation. If an explanation does not make sense or a suggested idea seems wrong to a peer, then students might experience a discrepancy between viewpoints (Dillenbourg, 1999). But if students try to restore their shared understanding about the problem, they are in the collaborative process of *repairing* (Roschelle & Teasley, 1995). Therefore, important actions for repairing a shared understanding are negotiations and corrections of conflicting interpretations, such as paraphrasing or repeating an utterance in one's own words (Alrø & Skovsmose, 2004).

Through these processes, coordination is seen as mutual exchange of utterances and taken actions, e.g., hand gestures, inputs to a dynamic software program, and

explanations and justification of mathematical ideas. Students need to engage in a mutual exchange of ideas and actions to access each other's thinking. In conversations, students can take turns suggesting, questioning, and negotiate ideas. Therefore, a central interactional practice in conversations is *turn-takings* (Roschelle & Teasley, 1995; Sidnell, 2010). Turn-takings may therefore promote students' co-construction of a shared understanding through processes where students build, monitor, and repair the meaning or a strategy for solving a problem (Roschelle & Teasley, 1995).

Trying to fix the differences in opinions (repairing), understanding an explanation (monitoring), or introducing ideas and suggestions as well as accepting them (building) are important social interactions co-existing with students' mathematical reasoning (Granberg & Olsson, 2015). Components of the collaborative processes are presented in Table 2.

Agency

A student can participate in group work by contributing with ideas or by listening to peers. The participation may include actively seeking to solve the task at hand, by making their own attempts, which involves guesses, trials and wrong paths, and investigations that may lead to evaluating their own mathematical production (Freudenthal, 1991). However, another student or the same student, in a different situation or in a different group, could refrain from making suggestions and actions, or active listening. A student may, for various reasons, resist to attend to a collaborating peer. Therefore, the nature of a student's exercised agency will vary within different interactions and situations (Gresalfi et al., 2009).

There are several possible perspectives to study when focusing on students' exercised agency in mathematics. Carlsen, Erfjord, Hundeland, and Monaghan (2016) point to how different cultures, people, and artifacts shape students' actions and decisions, and thus, their exercised agency. Gresalfi et al. (2009) are concerned with students' engagement in classroom activities. This focus concerns students' act of complying or refraining (Sengupta-Irving, 2016), as well as about the given opportunities to act, either from a peer or from a teacher (Langer-Osuna, 2018). The latter aspect concerns distribution of agency (Gresalfi et al., 2009). A teacher may distribute agency through his or her authority to the given group (Engle & Conant, 2002), or there could be a social conflict between group members, consequentially inhibiting students' talk and activities (Langer-Osuna, 2018).

Table 2 Overview of central elements of collaborative processes

Collaborative processes for building and maintaining a shared understanding		
Building	Monitoring	Repairing
Accepting ideas	Asking questions	Negotiations
Making suggestions	Explaining an idea	Correcting conflicting interpretations
Stating a problem	Observing and responding to one another's interpretations and ideas	Counter-suggestions
Pointing out mathematical properties		Reformulations

The mentioned aspects of teacher authority, social conflicts, or different cultures and artifacts are relevant for studying students' collaboration and mathematical reasoning. However, to focus the agency aspect on student—student interaction, this study views students' exercised agency as their engagement in participation for making mathematical arguments or refraining from making mathematical arguments (Gresalfi et al., 2009).

To investigate students' argumentation when collaboratively solving mathematical problems, a framework on how students' agency is expressed in discursive practices (Mueller et al., 2012) is adopted. This framework is based on the definition on agency from Gresalfi et al. (2009) and suggests that students may exercise different agencies, such as *shared agency, primary agency,* or *secondary agency* (Mueller et al., 2012). See Table 3 for an overview of central elements of agency.

A shared agency is students' co-construction of arguments, where all participants contribute with their ideas. Therefore, ideas, suggestions, and actions are all important elements to building an argument from the ground, and only existing because of all the participants' contributions (Mueller et al., 2012). Contrasting shared agency is students' individual agency, where students are either a primary agent or a secondary agent. A student may act with primary agency when he or she makes the final argument based on correction from a peer or assimilates a peer's argument, or by making sense of a peer's faulty or flawed idea. A secondary agent makes inputs influencing the original argument. These inputs are either corrections or extended or flawed arguments, formed by the primary agent to a final argument (Mueller et al., 2012).

Methods

The study is characterized as an instrumental case study (Stake, 2003)—an in-depth study of the particular case of four student pairs, to advance the understanding of an interplay between students' creative reasoning, collaborative processes, and exercised agency.

Data collection and participants

This article reports on four pairs of students, age 15–16, who collaboratively worked on a function task (Fig. 1). The students were enrolled in their first year of a

Table 3 Overview of central elements of agency

Agency		
Shared agency	Primary agent	Secondary agent
Co-construction of arguments	Makes the final argument Assimilates another student's argument Makes sense of a peer's faulty or flawed argument	Makes input for the main argument

theoretical mathematics program. Data was collected in a Norwegian upper secondary school in 2017, where three mathematics classes (69 students altogether) and their three teachers participated.

In the design process of the larger study, which this case is a part of, the three teachers and the researcher (author) planned and evaluated lessons together emphasizing collaborative work and math-talk. Both prior to and after the planned lessons, the teachers and researcher discussed how to assist and interact with the students in order to promote mathematical reasoning and collaborative work. Students worked together in the same pairs over five consecutive months in their regular classroom setting. Based on the teacher–researcher conversations, the teachers attempted to ask open-ended questions with minimum guiding intervention, aiming to provide students with opportunities to make connections between function representations for understanding the function concept together. The three teachers were considered ordinary and engaged teachers, but not particularly used to organize classrooms for collaborative interactions. Hence, they were previous to the study not particularly aware of their teacher approach to support student pairs or whole classroom discussion for collaborative inquiries. This was a deliberate choice for the study, which this case study is a part of.

The students were organized into 33 pairs based on the following criteria: (1) reasoning competence; (2) understanding of functions; and (3) likeliness to engage in math-talk with one another. The two first criteria were based on the students' scores on a mathematics test. The last criterion was based on conversations with the teachers when considering point 1 and 2.

Due to practicality of observing students through video recordings, two pairs in each of the three classrooms were set as a condition. For an in-depth analysis of interactional aspects in students' collaboration, six pairs were chosen. The six pairs were chosen based on the three criteria above, where two aspects particularly stood out: (1) students should express a high level of reasoning competence, which meant that they attempted to explain their thinking and anchored it in mathematics and (2) the likeliness of student pairs to be verbal and share thoughts with one another. These aspects were discussed with their teachers when making the pairs.

Two out of six student pairs did not exercise turn-takings, a criterion considered important for creating a shared understanding of the problem. These two pairs did not have successful conversations, since they were not engaged in sharing thoughts with one another. The four other student pairs exercised interactions with reasoning, collaborative processes, and different agencies. From the analysis ("Data analysis"), two distinct ways of interacting were seen, two pairs within each interaction pattern.

- Create a straight line $y = mx + c$
- Create another straight line in a way that the corresponding graphs are perpendicular.
- Formulate a rule for when two straight lines are perpendicular.
- Test the rule for other straight lines.

Fig. 1 The function task (reformulated from Olsson (2018))

In this article, four student pairs will illuminate the two typical interaction-patterns. These pairs' interactions are presented in "Results."

The linear function problem

During the time of the study, three main tasks were given. The presented task (Fig. 1) facilitated the richest conversations. In these conversations, students more often anchored their arguments in mathematical properties (Lithner, 2017), such as the slope number being a varying parameter, and students more frequently engaged in all of the collaborative processes of building, monitoring, and repairing of their shared understanding (Roschelle & Teasley, 1995).

The function tasks that were planned emphasized mathematical reasoning and non-routine solving of tasks, where the struggle ought to be more like a challenge to solve, rather than an obstacle (Hiebert & Grouws, 2007; Lithner, 2017; Stein et al., 2008). In order to facilitate a challenge easy to discuss, however not too difficult, tasks without a known procedure for the students to follow were emphasized. Therefore, students were likely not to withdraw from mathematical conversations due to differences in their level of competence in mathematics. The task presented in Fig. 1 had previously and successfully been tested for similar purpose at another school (Olsson, 2018).

An aim with this task was also to facilitate opportunities for the students to connect different function representations to construct their own solutions to the linear function problem. In connecting different function representations, such as graphical and algebraic representations, students can build a more comprehensive function concept (Best & Bikner-Ahsbahs, 2017), rather than view functions as "topics" to be learned in isolation of the others" (Thompson, 1994, p. 24). The function concept is regarded in school mathematics as difficult for students to learn, but very important to understand, since it has a central role in organizing and connecting many mathematical ideas (Michelsen, 2006).

When students solved the function tasks, they were encouraged to use the dynamic software program GeoGebra, which may promote active investigation of different function representations (Olsson, 2019; Preiner, 2008). The strength of the program, to easily adjust representations in the algebraic field or the graphical field, gave students rapid feedback on well-justified suggestions or simple guesses. Students' engagement with GeoGebra for solving a function task may contribute to students' mathematical reasoning (Granberg & Olsson, 2015) and, thus, their way of interacting. However, GeoGebra does not interpret the meaning. Students need to make their own meanings of their findings, which is important for students in producing their own mathematics. In this case study, the unit of analysis is the student–student interaction, unlike Olsson (2018) where GeoGebra additionally was included in the unit of analysis.

Data analysis

Video recordings of the eight students' talk and actions were viewed multiple times, and the first step of analysis comprised denoting longer sequences where students

often made justifications and explanations anchored in mathematical properties as sequences of *creative reasoning* (CMR) (Lithner, 2017). These sequences were thus coded as CMR-sequences (creating a solution with plausible arguments anchored in mathematics) and further transcribed.

Then, a second analytical step comprised the coding of each student's utterance in the CMR-sequences with respect to *collaborative processes* (Roschelle & Teasley, 1995). How a CMR sequence was coded for collaborative processes (see Table 2 for details) and for aspects of creative reasoning (see Table 1 for details) is exemplified in "Bi-directional interaction."

The third step in the analytical process was to provide a thick description (Powell et al., 2003) of the four pairs' collaboration and reasoning within their CMR sequences. This step enabled a description of students' participation in the collaborating dyads, which made it possible to characterize students' agency in their conversations as primary, secondary, or shared (Mueller et al., 2012). First, a description was made on the students' interactions when engaged in reasoning, concerning how they attempted to engage with each other. If students constructed a solution sequence where they connected ideas and thoughts making a shared understanding of the current situation, it was recognized as *shared agency*. If students engaged individually, clearly having different roles when suggesting ideas or explaining thoughts, their agency was recognized as *primary or secondary*. After characterizing students' interactions, it was possible to describe students' typical roles in their engagement connected to how they reasoned about linear functions and the ways they collaborated to solve the given task.

Tables 1, 2, and 3 present an overview of characterization of codes for CMR, collaborative processes, and agency, respectively. The headings in the tables are the interactional categories, and the bullet points describe the interactional aspects within each category and were used as codes in the analysis. Excerpts in "Results" outline typical interactions found between the student pairs in their CMR sequences. The excerpts are chosen because of their clear interaction patterns.

Results

Studying students' interactional aspects of reasoning, collaborative processes, and agencies contributed to rich descriptions of students' participation, which further grew into two distinct ways of interacting in the conversation, here named *bi-directional interaction* and *one-directional interaction*. Characteristics for the bi-directional interaction were students who engaged with similar roles; mutually attempting to understand each other's ideas; making suggestions, listening, and negotiating mathematical properties; and mutually driving the problem solving process forward. Such interaction was found in the student pairs: Philip and Noah, and Emma and Hannah. Characteristics for the one-directional interaction were students who engaged with different roles and one student who led the conversation by being the primary reasoner and suggestion-maker for solving the problem, which a peer attempted to understand and occasionally contributed with input to the final outcome of a reasoning sequence. Such interaction was found in the student pairs: Olivia and Oscar, and Leah and Isaac. In the following excerpts, interactional aspects of

reasoning, collaborative processes, and exercised agency are italicized followed by a number from the students' utterances in the conversations.

Bi-directional interaction

Philip and Noah had found instances for when two linear functions were perpendicular onto each other. In the challenge that followed, they attempted to make a rule explaining when the functions were perpendicular to each other.

Excerpt 1

1	Noah	I don't know how we should write it [the rule], because…
2	Philip	Okay. It's slope number. What is it called? It's m… (writing on the laptop)
3	Noah	Yes
4	Philip	Divided by 1, right. Then we'll have…
5	Noah	No. One divided by m is right

Noah *stated* a problem for formulating the rule, thus initiating a new focus in a process characterized as *building* (1). Philip responded by *monitoring* their problem when he *asked* what the slope number was called, which was answered by himself (2). In Philip's turn, he also *built* their shared understanding by *suggesting* a focus *anchored* in the slope number (2). Noah continued to *build* when he *accepted* the initiation of formulating the rule (3). Again, Philip continued to *build* when he *suggested* it was m divided by 1 (4). This was not logical to Noah, and he *disagreed* with Philip's suggestion of the rule. Noah made a *counter-suggestion*: 1 divided by m (5). Thus, Noah's countersuggestion can be seen as a *repairing* of their shared understanding. Together, Noah and Philip created a new reasoning sequence for making a rule based on the generalized slope number m.

   To specify the coding procedure for this excerpt: The sequence was characterized as CMR (Lithner, 2017), because in the conversation students' reasoning was *new* to them when they created an expression for the rule. Their explanation involved a connection between the perpendicular functions which was *anchored* in the mathematical property of the slope number m. Their solution made sense to the student pair. Thus, it was *plausible* to them. Concerning their collaborative processes (Roschelle & Teasley, 1995); students were, for instance, *building* a shared understanding when *suggesting* a focus of the rule *anchored* in the slope number, and *repairing* when making a *countersuggestion* to the rule.

   In the further unfolding problem-solving path, the student pair continued in a turn-taking conversation.

Excerpt 2

6	Philip	Yeah, but is it a rule?
7	Noah	Yes
8	Philip	How we should say something, in a way
9	Noah	A rule. Oh yeah. The slope number on one line is something in connection to the other line. I don't know how to formulate, explain it. (Pause) You have the slope number for one [line] and slope number for the other one. To figure out the second one, I use that formula (pointing at the formula they have written down). So …
10	Philip	No, right. We need to figure out something better to say than the first and the second. Maybe like *a* and *b*, or something
11	Noah	Yes. I can try (writing on the laptop). Slope number for line b is $-m/1$. Wait. -1 divided by the slope number for line a. We say that. At line *a*
12	Philip	Yeah, but slope number (pointing at the screen). We can use *m*, as $mx + c$. Instead of "slope number." To add some subject content, concepts, with numbers and letters

Philip and Noah *agreed* upon the rule $-1/m$. However, Philip was initially not sure whether it was a proper mathematical rule (6, 8). Noah *explained* his take on the meaning of a mathematical rule and attempted an *explanation* of the rule they had found (9). Noah was building their shared understanding by *anchoring* his reasoning on how two linear function's slope numbers were connected (9). Together both contributed to generalize their findings (10, 11) into a rule constructed from observations of pairs of perpendicular lines, which they emphasized as "subject content, concepts, with numbers and letters" (12). Their attempt to create a rule was *co-constructed* through their turn-taking conversation.

Thus, Noah and Philip were both actively participating in solving the problem by *making suggestions, observing suggested strategies, taking initiatives, and making counter-suggestions.* The students' line of thought in the excerpts is *collaboratively built*, and their reasoning is *co-constructed* and does not exist without the peer's input. Hence, Noah and Philip exercised *shared agency* creating arguments together through collaborative processes and CMR.

Emma and Hannah began their problem solving by testing different linear functions to be perpendicular to the line $y = 4x + 2$. Their dialogue was focused on varying parameters (slope number and constant), and an anticipation where the lines would appear in the coordinate system. Excerpt 3 shows Emma and Hannah's continuing conversation and struggle about the connection between linear functions for being perpendicular.

Excerpt 3

13	Hannah	y equals. I have Caps Lock on. Eh, what should we try?
14	Emma	x. 1 divided by 4…. plus 2
15	Hannah	No, it's not the same. I thought, if it were the same [the rule could be] that we only put minus sign in the front. But that doesn't work
16	Emma	I don't know what we have been doing
17	Hannah	Wait. If we take 4 divided by minus… (writing on a paper). No We have. What we actually have been doing… We put number 4 as the denominator in the fraction, in a way. But I don't know

Emma and Hannah struggled to make sense of their own suggestions (15, 16, 17). Hannah *initiated* a new sequence of guess and check, thus collaboratively *building* (13). Emma responded with further *building* when *suggesting* a new linear function: $y = 1/4x + 2$ (14). Her suggestion conflicted with Hanna's expectation of perpendicularity of two linear functions (15). She thought that the pair of slope numbers should have been 4 and -4. Hannah's *correction*, at least for herself, was about the conflicting interpretation of the connection between the linear functions (15). Thus, Hannah *observed* that what she had previously thought did not make sense, which is characterized as *repairing* their progress (15). Emma *observed* that their input did not result in a desired outcome, and she *monitored* their problem situation (16). Hannah continued *monitoring* by attempting a further *explanation* of the algebraic representation, stating indirectly that the slope number of one line was found as a denominator in the slope number of a perpendicular function (17). She vaguely *suggested* that it ought to be a negative slope number but rejected her own suggestion (17).

Emma and Hannah struggled actively together attempting to find a pair of perpendicular lines. Both students addressed the slope numbers of the perpendicular pair of linear functions, and their reasoning sequence was *anchored* in the mathematical property of the slope number, mainly referred to in the algebraic expressions. The students attempted in a joint effort to solve the linear problem with *co-constructed reasoning*. Therefore, their participation in the interaction, seen in the excerpt and before and after, is characterized as a *shared agency*.

The two pairs, Philip and Noah, and Emma and Hannah, typically attempted to make sense of each other's ideas and thoughts. Similar for both pairs were a joint effort and engagement in different aspects of working together: making suggestions, listening to one another, and expressing disagreement with actions or suggestions made by each other or oneself. Although the two collaborating pairs often were engaged in similar ways, Philip and Noah stood out when it came to working together through reasoning and engagement in all the collaborative processes. This pair more often entered processes of repairing of their shared understanding of the function problem, compared to Emma and Hanna.

One-directional interaction

Olivia and Oscar were not engaged in a mutual exchange of ideas and actions for solving the function problem. In the following conversation Olivia and Oscar began their problem solving.

Excerpt 4

1	Oscar	Okay. $y = x + 1$ (Olivia begins to use the laptop). And then you can take $-x + 1$. It should be perpendicular
2	Olivia	Now?
3	Oscar	Yeah. There. It's perpendicular. Um. Rule. Um. You put… They have the same value, but minus in front. I don't know. Constant x has the same value, wait…
4	Olivia	It's not the same value, or?
5	Oscar	No. When they have the same constant, but one is positive and one is negative, I don't know. It's kind of a rule. Wait. We should have a rule (looking at the given task). Yes, this works. Right?
6	Olivia	Yes, it's not a formula, so it's okay
7	Oscar	But if the slope values are the same, because the constant can be different. It'll only go higher or lower on the first line-thing (pointing at the laptop screen). The straight line, as it was called

Oscar was the driving force for solving the problem. He made *suggestions*, such as *y* equals $x + 1$ and $-x + 1$, and *stated* problems "They have same value, but minus in front… Constant x has the same value" (1, 3). Oscar attempted an *explanation* about the observation of the linear functions, *justified* in the "constant x," which was *plausible* to him and *anchored* in mathematical property of the slope number (3, 5). Olivia *accepted* Oscar's explanations and suggestions, by putting her own words to Oscar's ideas: "Yes, it is not a formula, so it's okay" (6) and later she said "Yes, the constant can be different, and it doesn't have to be the same." Thus, the students were *suggesting* and *accepting*, in the process of *building*.

Oscar and Olivia demonstrated the process of *monitoring* when Olivia *asked* if her input looked correct (2), and when she *questioned* the meaning of the numbers having same value (4). Oscar *monitored* when he attempted an *explanation* trying to say that the slope numbers should be the same numbers only with different signs (5). In his last statement, he differentiated between the slope number and the constant (7). Thus, Oscar expressed more details to their rule: the perpendicular lines had to have the same slope numbers with opposite signs and the constant could be arbitrary numbers.

From their interaction and way of participating, Oscar and Olivia had different roles in the *building* and *monitoring* process, which characterized their interactions throughout their problem-solving path. They did not have instances of *repairing*. Oscar exercised agency when *suggesting* and *explaining* his ideas. Thus, he was the initiator and *primary agent* of reasoning in the conversation. Olivia *accepted* Oscar's explanation of the rule, which she distinguished from a formula (6). Oscar further

explained his thoughts, after Olivia's acknowledgement of his suggestions (7). Olivia was a *secondary agent* who contributed with trying to understand how Oscar was thinking and what she should do to execute his ideas into action in GeoGebra. Their student–student interaction did not demonstrate a mutual and synchronous way of making and maintaining a shared understanding of the problem. Nor did they exercise a shared agency for solving the problem.

Leah and Isaac found a set of linear functions making a perpendicular pair: $y = x + 3$ and $y = -x + 3$. They expressed a relation between the two linear functions: two opposite slope numbers, which they further referred to as either $-x$ and x, or -1 and $+1$. They agreed upon this relation as the rule and Leah initiated testing of their rule for other linear functions.

Excerpt 5

8	Isaac	Okay. Then we create a line. We choose the line y equals, we choose $2x + 3$. Okay. It's probably a bad number. Okay, and then for every x there is 2. Then the negative version of this should be $y = -2x$, and then it should intersect… (writes on the laptop) But it isn't going to… It's always minus… The line should always be $-x$. Like that (writes on the laptop)
9	Leah	Always?
10	Isaac	Yes, because if I use $-2x$, it doesn't work. $y = -x$ and it should intersect…
11	Leah	3
12	Isaac	Are you sure?
13	Leah	Yes. Try (Isaac writes on the laptop)
14	Isaac	Hmm. No. 4
15	Leah	Oh. It's… (Leah uses the laptop). It's because this is f***ed
16	Isaac	Okay, but wait. It doesn't work for every number. Okay, we'll figure it out
17	Leah	I think it's wrong to say it's always $-x$. Try minus… What did you use here? 2?
18	Isaac	This is $2x + 3$
19	Leah	Then you should try $-2x$.

Isaac contended for their new pair of linear functions that always one of the slope numbers had to be $-x$ (8). Leah *questioned* his statement (9), which Isaac replied to by *pointing out* the different variables $-2x$ and $-x$ (10), which is likely his way of *anchoring* the reasoning in the mathematical property of the slope numbers being -2 and -1. They further attempted to *evaluate* their input in GeoGebra (11–15). Towards the end of the turn-taking sequence, Isaac *interpreted* their observations with guessing and checking with GeoGebra and said that it did not work for every case (16). Leah *repeated* his observation and specified that a perpendicular line not always was $-x$ (17), meaning that for a pair of linear functions one line did not have to have a slope number equal to -1. She continued *building* by *suggesting* $-2x$ for a perpendicular line (19), which Isaac previously had observed would not result in a perpendicular line (10).

In Leah and Isaac's conversation, there were mainly collaborative processes characterized as *building* by making *suggestions* for pairwise linear functions (8, 10, 13, 18, 19), which was *monitored* by *observations, evaluations*, and *questions* (9, 12, 14, 16). They did not engage in processes of *repairing*. Both students contributed to the problem-solving process; however, it was primarily Isaac that *suggested* and *made arguments* connected to mathematical properties *anchored* in the algebraic representation of the slope number. Isaac acted as the *primary agent* leading the conversation and making the final arguments. Leah, on the other hand, either *built* by *accepting* Isaac's ideas or *monitored* by *questioning, repeating*, and *observing* how Isaac's ideas and utterances played out in GeoGebra. Therefore, she exercised *secondary agency*.

In the interaction of the two pairs, Olivia and Oscar, and Leah and Isaac, only one student attempted to make sense of the collaborating peer's suggestions. Similar for both pairs were a mutual attempt to solve the function problem where they exercised different roles in the problem-solving session. Moreover, it was Oscar and Isaac who led the conversations, whereas Olivia and Leah attempted to understand their peers' thoughts and actions. The student pair Olivia and Oscar stood out compared to Leah and Isaac in the way that their roles in the interaction were more clearly divided in primary and secondary agency, particularly how Oscar made the reasoning sequences which Olivia tried to understand. The interplay in the interaction patterns found in their attempt of collaboration is in a clear contrast to, particularly, Philip and Noah's well-functioning collaboration.

Discussion

The study's research question is *What are the patterns of interaction for creating a shared understanding through the interplay between students' creative reasoning, collaboration, and exercised agency in a mathematical problem-solving session?*

The case study has analyzed videos of students' attempt to collaborate and engage in mathematical talk to solve a linear function problem. The findings reveal an interplay between exercised agency, reasoning, and collaborative processes, affecting how student–student interactions are expressed. Based on the students' roles in pairwise collaboration, considering interactional aspects, a pattern for their interaction evolved in the analysis process into bi-directional interaction and one-directional interaction.

If students exercise mutual attempts to understand one another (e.g., Mueller et al., 2012; Roschelle & Teasley, 1995), using arguments logically, and anchored in mathematical properties of the reasoning sequence (Lithner, 2017), where both are the driving force of the problem-solving process, they are in a *bi-directional interaction*. Philip and Noah, and Hannah and Emma revealed such an interactional pattern. Students typically negotiated mathematical properties (Lithner, 2017) to make a pair of perpendicular lines, and they often represented functions dynamically. Students' active investigations of mathematics using GeoGebra is seen as support for students' reasoning and activity (e.g., Granberg & Olsson, 2015; Olsson, 2019; Preiner, 2008), and observed in the two student pairs' engagement. Students expressed functions algebraically and graphically, thus,

actively engaged with GeoGebra's feedback. Therefore, functions were not operated on as separate topics to be learned (Thompson, 1994), important for a deeper understanding of the function concept (Best & Bikner-Ahsbahs, 2017).

Both student-pairs participated in turn-taking conversations (Sidnell, 2010) building on suggested and explained ideas (Alrø & Skovsmose, 2004). Students exercised coordination and synchronicity in their interaction which characterizes "true" collaboration, where it is likely to achieve a shared understanding (Baker, 2015; Roschelle & Teasley, 1995; Sarmiento & Stahl, 2008).

Philip and Noah's interaction, viewed holistically, showed an interaction pattern where both students engaged in collaborative processes of building, monitoring, and repairing. When Philip and Noah engaged in a repairing process, they more often anchored their reasoning in mathematical properties of linear functions, compared to the other pairs. Moreover, Philip and Noah willingly shared ideas and entered situations with conflicting ideas (Dillenbourg, 1999), such as the formulation of the rule using the connection between an algebraic expression and a graphical representation of the function concept. In the mutual and synchronous interaction, Philip and Noah co-constructed reasoning sequences and participated with shared agency. Therefore, their agency co-existed with collaborative processes and creative reasoning important for shared understanding.

When students exercise different roles in the problem-solving process where the final outcome is expressed repeatedly by one of the students (Mueller et al., 2012), they are in a *one-directional interaction*. In such an interaction, this study suggests that, a primary agent utters creative reasoning, and a secondary agent listens or tries to understand a peer's argumentation. Thus, a student with a secondary agency rarely or never engages in creative reasoning. Moreover, their collaborative processes are characterized by building and monitoring instances, missing the important repairing instances valuable for a possible evolving of students' understanding.

Such a pattern was demonstrated in the student pairs Olivia and Oscar, and Leah and Isaac. In both pairs, one student led the process of solving the task. The co-working student often attempted to understand suggestions or explanations made by the primary agent. Thus, students engaged with different agencies. Here, Oscar and Isaac participated as the primary agents, and Olivia and Leah as the secondary agents. The secondary agents exercised their agency through expressed ideas and questions about the primary agents' ideas. Their input was either assimilated into the final outcome (Mueller et al., 2012) of the reasoning, or considered by the peer who refined the input or neglected it. Concerning collaborative processes for making a shared understanding (Roschelle & Teasley, 1995): Olivia and Oscar or Isaac and Leah did not experience conflicting ideas. Even though they attempted to collaborate, they did not seem to have a shared understanding, and they rarely experienced discrepancies in their ideas, which they would have to repair.

Oscar expressed more details to the solution strategies, and anchored ideas in mathematics (Lithner, 2017) concerning aspects of the rule for making two linear functions perpendicular than the other primary agent Isaac, who often led the conversation and made suggestions for solving the problem. Both Olivia and Leah tried to understand how their peers were thinking. Consequentially, their student–student

interaction was not a mutual and synchronous way of creating a shared understanding of the function problem.

Students who engage in a bi-directional or one-directional manner may change roles in a problem solving process (Child & Shaw, 2016). A student might assimilate a peer's input to construct a final argument, which is in this study recognized as a primary agent (Mueller et al., 2012) in a one-directional interaction and as shared agency (Mueller et al., 2012) in a bi-directional interaction. Therefore, characteristics of "assimilating a peer's construct into a final argument" do not indicate what kind of agency a student has exercised. Therefore, to categorize the interactional aspect agency, it is important to study the collaborating pair's turn-takings and other interactional aspects at play, such as collaborative processes and mathematical reasoning.

Concluding thoughts

This study illuminates two interactional patterns from four student pairs: Noah and Philip, Hannah and Emma, Oscar and Olivia, and Leah and Isaac. The limitation of the small sample is acknowledged; however, rich descriptions and the detailed analysis have provided important insights in students' interactions when collaborating and reasoning about functions. More student pairs' interactions should be studied, and their engagement in a broader specter of mathematical problems, to further reveal nuances to interactional patterns found in students' attempt to collaborate and reason mathematically. Researching several instances of students' interactions in different settings may contribute to more nuances in how students, of different age, learning different mathematical topics, in different environments, develop their own mathematical voice (Schoenfeld, 2013; Sengupta-Irving, 2016).

If students interact in a dynamic way, as described above, where both participate in equal roles and show authority over mathematical ideas during the problem-solving process, they might construct a shared understanding from merging an interplay of ideas. Such dynamic structure of pairwise collaboration reveals important components in students' mathematical communication (Sidenvall, 2019), to better understand underlying mechanisms for fruitful collaboration (Child & Shaw, 2018; Kuhn, 2015; Seidouvy & Schindler, 2019). These identified conditions are both being promoters in a problem-solving process, both making reasoning anchored in mathematical properties, and both being engaged in different collaborative processes. Such an interactional pattern in a bi-directional interaction promotes learning of mathematics through quality interactions (Pijls and Dekker, 2011; Varhol et al., 2020).

The contrast to such a dynamic interplay is a monotonic one-directional interplay, where only one student evolves his or her individual problem space. In such instances of one-directional interaction, teacher involvement should be suggested for supporting both students to build (accept and suggest) and monitor (explain and ask question) emphasizing plausible and mathematically founded argumentation.

Reasons for the two patterns of interactions that occurred might have several explanations. One reason might be students' individual experiences and personalities.

Comparing the student pairs in the study, the student pair Philip and Noah stood out and demonstrated a productive interaction, as described above and in results. It is likely that their individual personalities were a good match. Thus, affective patterns or individual personalities come into play and influence students' conversations (Cobb et al., 2009). Other influencing aspects are students' background and disposition for learning, self-confidence and past success in mathematics, and beliefs about their roles and roles of others (Mueller et al., 2012).

A second reason is the given task. Although the given task presents an opportunity for creative reasoning (Granberg and Olsson, 2015), it does not automatically initiate a productive engagement in the other interactional aspects of this study. To promote interactional aspects such as collaboration and agency, a mathematical problem might entail other features than explicitly discussing the slope number and constant in a linear function found in an algebraic and graphical expression. If an individual student has experienced constraints with mathematics, it might prevent them for further engaging in a productive social interaction with a peer. Therefore, it might be worthwhile considering a relatable or meaningful context for students to reason about, and for translating between function representations.

A third reason to consider is classroom norms and teacher involvement in students' interactions. Regarding the first aspect, classroom norms, Yackel and Cobb (1996) differentiate between a social norm as an expected explanation to a given task, whereas a sociomathematical norm is an acceptable mathematical explanation. Both aspects are important for individual and collective learning, and students' engagement in both would influence their pattern of interaction. The second aspect, teacher involvement, has the potential to provide students with necessary resources for social norms and sociomathematical norms. A study found that important teacher guidance for collaborative inquiry happens through supporting student contributions with well-defined structure for mathematical work (Staples, 2007).

An agenda for further research on the three mentioned reasons for the two interactional patterns could be studies of different tasks promoting CMR, collaborative processes, and exercising of shared agency. Another future study could focus on teacher guidance: opportunities and limitations with teacher actions for students' productive interactional pattern. A third study could include interviews with students focusing on individual variables, such as beliefs about one-self as a mathematics learner, self-confidence, and the role of the collaborating peer. A broader understanding of these influential aspects might give a more complete understanding of the underlying mechanisms for productive interactions through the interplay of collaborative processes, mathematical reasoning, and exercising of agency.

If a teacher observes such an interaction pattern, the teacher could facilitate a change of roles: a secondary agent attempts to suggest strategies and explain outcomes or connection anchored in mathematical properties. A teacher's awareness of interactional patterns concerning collaboration processes, reasoning, and agency, has the potential to contribute to students' fruitful interactional dynamic. Moreover, such teacher actions has the potential to model important aspects to facilitate students' synchronicity and coordination pooling knowledge together to construct their own mathematical knowledge for understanding mathematical ideas (Stockero et al., 2019). Therefore, further research on teacher's actions facilitating dynamic

interactions between the peers, where students are exercising shared agency, might provide additional insights to foster students' learning of mathematics through quality interactions.

Disclaimer

The manuscript has not been submitted to more than one journal for simultaneous consideration.

The submitted work is original and is not published elsewhere in any form or language (partially or in full). This manuscript has not been split up into several parts. Results are presented clearly, honestly, and without fabrication, falsification, or inappropriate data manipulation.

No data, text, or theories by others are presented as if they were the author's own ("plagiarism"). Proper acknowledgements to other works have been given. The author had permissions for the use of software. Research articles have been cited appropriate and relevant literature in support of the claims made. Author has avoided untrue statements.

Compliance with ethical standards

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