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LETTER

Profit and loss attribution: an empirical study

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Abstract
The profit and loss (P&L) attribution for each business year into different risk factors (e.g., interest rates, credit spreads, foreign exchange rate etc.) is a regulatory requirement, e.g., under Solvency 2. Three different decomposition principles are prevalent: one-at-a-time (OAT), sequential updating (SU) and average sequential updating (ASU) decompositions. In this research, using financial market data from 2003 to 2022, we demonstrate that the OAT decomposition can generate significant unexplained P&L and that the SU decompositions depends significantly on the order or labeling of the risk factors. On the basis of an investment in a foreign stock, we further explain that the SU decomposition is not able to identify all relevant risk factors. This potentially effects the hedging strategy of the portfolio manager. In conclusion, we suggest to use the ASU decomposition in practice.

Keywords Profit and loss attribution · Change analysis · Sequential decompositions · Shapley value · Solvency 2

JEL Classification D53 · C58 · G22

1 Introduction

The analysis of the profits and losses (P&L) between two reporting dates is a common task in risk management. The *risk factors*, must be identified and their contribution to the P&L has to be quantified. The current regulation for insurance companies in Europe, Solvency 2, requires that when an internal model is used, a *profit and loss attri-*

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bution or *change analysis* must be performed in sufficient detail according to the risk categorization selected in the internal model, see Article 240 in Solvency 2. Typically, for the market risk module, the investment P&L is allocated to the Solvency Capital Requirement (SCR) sub-modules: interest rate, credit spread, equities, real estate and foreign exchange. There are at least two basic requirements: a P&L attribution should be able to fully explain the P&L, i.e., there should not be any unexplained P&L and a change analysis should identify *all* relevant risk factors.

The P&L attribution can also have an economic impact when trading decisions, e.g., the amount of the foreign exchange hedge, are based on this analysis.

In practice, there are many well-known decompositions that can be used to identify the contribution of different risk factors: There is the *sequential updating* (SU) decomposition, which is also known as *waterfall*, see Cadoni [1]. The SU decomposition depends on the order or labeling of the risk factors: if there are d risk factors, there are $d!$ SU decompositions.

The average of the $d!$ possible SU decompositions is called the *average sequential updating* (ASU) decomposition. The ASU decomposition is also known as the *Shapley value* or *Shapley-Shubik* decomposition and has many desirable properties, see Friedman and Moulin [4] and references therein.

The *one-at-a-time* (OAT) decomposition is also known as *bump and reset*, see Cadoni [1]. In general, the OAT decomposition is not able to fully explain the P&L.

All decompositions are linear; therefore, to decompose a portfolio it is possible to decompose the instruments of the portfolio individually. Frei [3], Jetses and Christiansen [5] and Christiansen [2] applied the OAT, SU and ASU decompositions recursively to multiple sub-intervals. In practice, one would apply a decomposition on annual, quarterly, monthly or more granular sub-intervals.

For example, a risk manager working in an insurance company has to make many decisions to implement a change analysis: which of the $d! + 2$ decompositions (OAT, $d!$ SU or ASU) is the most appropriate? How to choose the size of the sub-intervals?

In Sect. 2, we apply these decompositions to a typical investment of an insurance company; a corporate bond. We show empirically that the unexplained P&L of the OAT decomposition is significant and that the P&L attribution differs significantly for the different SU decompositions. A priori, it is unclear, which of the $d!$ SU decomposition works best.

Further, in Sect. 3, we provide an example of a partially hedged portfolio with significant foreign exchange exposure, but the OAT and some SU decompositions assign zero P&L to the risk factor describing the foreign exchange rate. This potentially effects the hedging strategy of the portfolio manager: a naive SU or OAT decomposition may lead to wrong trading and hedging decisions. The ASU decomposition does not have this drawback. Therefore, both the OAT and the SU decompositions are unsuitable for performing a change analysis.

From a theoretical point of view, the ASU decomposition is preferred because it is able to fully explain the P&L and does not depend on the order or labeling of the risk factors. Moreover, it seems reasonable to keep the sub-intervals as small as possible in order to take into account the whole paths of the risk factors, see Mai [6], and to prevent inconsistencies when using conflicting sub-intervals for different purposes. However, the available computational power limits the number of the sub-intervals.

Our empirical experiments based on a corporate bond indicate that the ASU decomposition changes significantly using annual versus daily data. However, the difference between the ASU decomposition based on monthly and daily data is less than 0.2% across twenty business years and all risk factors.

In conclusion, we suggest to use the ASU decomposition and monthly or more granular sub-intervals to obtain a change analysis of a (corporate) bond portfolio.

2 Change analysis of a bond

Let us look at the evolution of the risk factors relevant for pricing a USD A-rated corporate bond with a maturity of ten years. We consider a European investor, so the P&L has to be converted into EUR and is expressed in percentage points of the nominal in EUR. We take into account the time period $2002/12/31 - 2022/12/31$, i.e., twenty business years. The most granular available data is daily data.

The bond is modeled as a constant maturity bond, i.e., at *every* day the maturity of the bond is assumed to be $\mathcal{T} = 10$ years. This is the usual assumption if the insurance portfolio is grouped into benchmark portfolios that are represented by a bond with an average rating and duration. The risk factors are: interest rates (IR), credit spread (CS) and foreign exchange rate (FX).

The following Bloomberg tickers are used: USSW10 Index, C40410Y Index and USDEUR Curncy, which describe the US 10-year interest rates, the US corporate credit spreads for an A-rated class and a maturity of ten years and the USDEUR exchange rate, respectively. The price of the bond at time $t \geq 0$ with maturity $\mathcal{T}$ at time t is given by

$$P(r(t), s(t), x(t)) = \frac{x(t)}{(1 + r(t) + s(t))^{\mathcal{T}}}, \tag{1}$$

where $r(t)$ and $s(t)$ denote a realization of IR and CS at time t for time horizon $\mathcal{T}$. The term $x(t)$ denotes a realization of the foreign exchange rate between the foreign currency and the domestic currency at time t. In this section, we consider only a bond. Other instruments could be treated similarly by defining P appropriately.

Under Solvency 2, an insurance company has to explain the P&L

$$\Delta P = P(r(1), s(1), x(1)) - P(r(0), s(0), x(0))$$

for every business year, i.e., between two reporting dates 0 and 1.

The OAT decomposition decomposes the P&L ΔP by

$$\begin{aligned} \Delta P = &\{P(r(0), s(1), x(0)) - P(r(0), s(0), x(0))\} \\ &+ \{P(r(1), s(0), x(0)) - P(r(0), s(0), x(0))\} \\ &+ \{P(r(0), s(0), x(1)) - P(r(0), s(0), x(0))\} + R, \end{aligned} \tag{2}$$

where $R \in \mathbb{R}$ is the *unexplained* P&L. The first, second and third term in (2) corresponds to the contribution of CS, IR and FX to the P&L, respectively. That is, to

Table 1 Definition of the six possible update orders of the risk factors IR, CS and FX to obtain six SU decompositions

Update order	1	2	3	4	5	6
	CS IR FX	IR CS FX	IR FX CS	CS FX IR	FX IR CS	FX CS IR

obtain the contribution of a risk factor, fix all other risk factors at the origin, i.e., at time 0 and allow only the risk factor of interest to move from time 0 to time 1.

The SU decomposition is similarly defined but updates the risk factors sequentially: after updating one risk factor, it is not reset to time 0 but kept at time 1, see Cadoni [1]. If there are three risk factors, there are $3! = 6$ SU decompositions. For example, the three risk factors IR, CS and FX can be ordered according to Table 1. For example, the update order (CS, IR, FX) results in to the following SU decomposition:

$$\begin{aligned}\Delta P = & \{P(r(0), s(1), x(0)) - P(r(0), s(0), x(0))\} \\ & + \{P(r(1), s(1), x(0)) - P(r(0), s(1), x(0))\} \\ & + \{P(r(1), s(1), x(1)) - P(r(1), s(1), x(0))\} . \end{aligned} \tag{3}$$

The first, second and third term in Eq. (3) is interpreted as the contributions of the risk factors CS, IR and FX, respectively. The SU decomposition fully explains the P&L, i.e., there is no unexplained P&L, because the right-hand side of Eq. (3) is actually a telescoping series. The ASU decomposition is defined as the average of all possible SU decompositions.

One may also divide the business year into m sub-intervals and apply a static decomposition recursively along the sub-intervals to define a decomposition in a multi-period setting, see Frei [3], Jetses and Christiansen [5] and Christiansen [2]. For example, these sub-intervals can be chosen annual, quarterly, monthly, weekly or daily.

In the business year 2020 (2003), the OAT decomposition has an unexplained P&L of -2.4% (-0.8%) using quarterly (daily) sub-intervals. The absolute average unexplained P&L over all twenty business years are 0.4% for annual and quarterly sub-intervals and 0.3% for monthly and daily sub-intervals. No matter which sub-intervals we choose, the unexplained P&L can be significant.

Figure 1 shows the six SU decompositions for the update orders defined in Table 1 and the business year 2022. The OAT decomposition is shown on the left. The ASU decomposition is shown on the right. It can be seen that the SU decomposition clearly depends on the updating order of the risk factors, a pattern that can also be observed in other years. The range (i.e., the difference between the largest and smallest value) for the risk factor FX (IR) in 2022 (2003) across the six SU decompositions using quarterly (daily) sub-intervals is 1.9% (0.9%).

On average over all twenty business years, the range across the six possible permutations of the FX, IR, CS risk factors for monthly sub-intervals are 0.4%, 0.3% and 0.2%, respectively. No matter which sub-intervals we choose, the SU decomposition may depend significantly on the order or labeling of the risk factors.

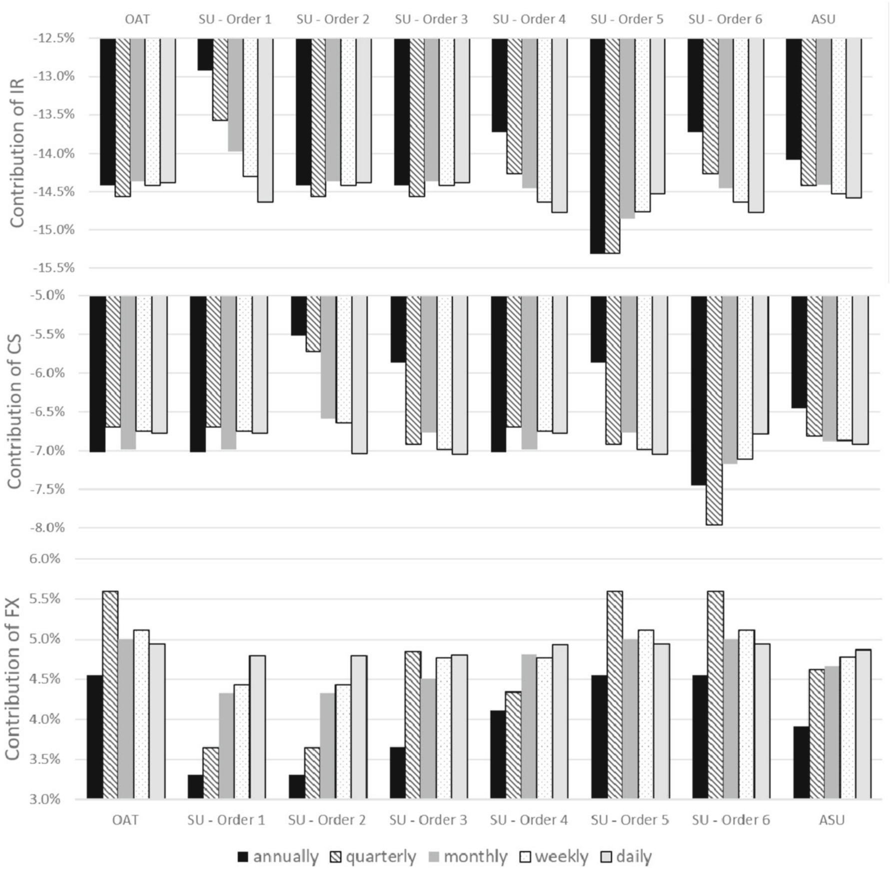

Fig. 1 OAT, SU and ASU decompositions for business year 2022 of a US corporate bond

The ASU decomposition is defined by the average of the six SU decompositions. The ASU decomposition is independent of the order of the risk factors and is able to fully explain the P&L. Like the SU or the OAT decomposition, the ASU decomposition depends on the size of the sub-intervals. In 2022, the range across annual, quarterly, monthly, weekly and daily sub-intervals of the ASU decomposition for the contribution of the risk factor FX is 1.0%.

While it seems economically reasonable to use daily sub-intervals to obtain the ASU decomposition in order to take into account the whole paths of the risk factors, for the corporate bond we considered in this section, it may be sufficient to use monthly sub-intervals since the difference of the ASU decomposition based on daily versus monthly sub-intervals is less than 0.2% across all twenty business years and the three risk factors FX, IR and CS.

3 SU decomposition of a US stock

In this section, we provide an example to illustrate that a SU decomposition may assign zero P&L to the FX risk factor even though the portfolio has significant FX exposure. In particular, we consider an investment by a European investor in the US stock market such that the FX risk is hedged by an FX-future. The FX-hedge is not perfect; some FX-risk remains. However, there is a SU decomposition which indicates that the P&L is only driven by the movements of the stock market. We will also work with real market data.

Consider a static setting and a partially hedged portfolio: Suppose there are two risk factors, say a foreign exchange rate X and foreign stock Y, and some portfolio P such that $X(t), Y(t) \in \mathbb{R}$ for $t \in \{0, 1\}$ and

$$P(X(1), Y(1)) \neq P(X(0), Y(1)), \tag{4}$$

$$P(X(1), Y(0)) = P(X(0), Y(0)). \tag{5}$$

Equation (4) says that the P&L of P depends on $X(1)$. Equation (5) says that the portfolio is partially hedged with respect to X: if only X changes and Y remains constant, the value of the portfolio does not change.

We apply the decompositions OAT, SU and ASU to obtain the contribution of the risk factor X to a portfolio that satisfy Equations (4, 5): Equation (6) is the contribution of X to the P&L according to the OAT and the SU decomposition updating the risk factor X first. Equation (7) is the contribution of X according to the SU decomposition updating the risk factor Y first. Equation (8) is the contribution of X according to the ASU decomposition:

$$P(X(1), Y(0)) - P(X(0), Y(0)) = 0, \tag{6}$$

$$P(X(1), Y(1)) - P(X(0), Y(1)) \neq 0, \tag{7}$$

$$\frac{1}{2}\left(P(X(1), Y(0)) - P(X(0), Y(0)) + P(X(1), Y(1)) - P(X(0), Y(1))\right) \neq 0. \tag{8}$$

Example 1 We consider a portfolio P in EUR consisting in a long position in the S&P 500, Y for short, and a short position in $Y(0)$ FX-forwards with strike $X(0)$, where X models the USDEUR exchange rate, i.e.,

$$P(X(t), Y(t)) = X(t)Y(t) + Y(0)(X(0) - X(t)), \quad t \in \{0, 1\}. \tag{9}$$

P satisfies Eqs. (4, 5). The S&P 500 changed in the business year 2003 from $Y(0) = 880$ to $Y(1) = 1110$ points. The USDEUR exchange rate changed from $X(0) = 0.95$ to $X(1) = 0.79$. The portfolio in Eq. (9) changed from $P(0) = 836.0$ EUR to $P(1) = 1017.7$ EUR, i.e., by 22%, while the S&P 500 increased by 26%. Therefore, the movements of X must also have a significant impact on the P&L of P. However, if we apply the SU decomposition, which updates X first, or the OAT decomposition, the contribution of X is defined as zero, see Eq. (6). According to the ASU decomposition, the contribution of X is given by 20.45 EUR and the contribution of Y is given by 161.25 EUR, see Eq. (8).

In summary, the OAT decomposition and some SU decompositions may assign a zero contribution to the risk factor X, even though X may have significant impact on the P&L. The ASU decomposition assigns a non-zero contribution to X. This is potentially relevant to an insurance company because the decomposition of the P&L usually has some impact on the portfolio manager's trading and hedging strategy.

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Declarations

Conflict of interest The authors report no Conflict of interest to declare.

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