Correction to: Yield curve shapes of Vasicek interest rate models, measure transformations and an application for the simulation of pension products

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Published online: 2 June 2021
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Correction to: European Actuarial Journal

<https://doi.org/10.1007/s13385-019-00214-0>

Due to an error in the proof of Lemma 1 in the appendix of [1], the proof of the assertions (c) and (d) of Theorem 4 on the existence of humped and dipped yield curves in the two-factor Vasicek model needs a modification. However, the assertions stay valid. It can directly be realized that the proof of Lemma 1 holds for $h := f - g$ and not for $h := f + g$ as we falsely claimed. Consequently, Lemma 1 can not be applied. Using the notation of Section 3 in [1], we will reformulate Theorem 4 slightly and then give the corrected proof for humped and dipped yield curves.

Theorem 4 *In a two-factor Vasicek model defined as in Eq. (13), the following assertions hold under the assumption of $a \neq b$:*

- (a) *There are normal yield curves but only for $r(t) < y_{\text{asympt}}$*
- (b) *there are inverse yield curves but only for $r(t) > y_{\text{asympt}}$*
- (c) *there are humped yield curves for $r(t) \geq y_{\text{asympt}}$*
- (d) *there are dipped yield curves for $r(t) \leq y_{\text{asympt}}$*

Before we prove the theorem, we introduce two useful lemmas.

The original article can be found online at <https://doi.org/10.1007/s13385-019-00214-0>.

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Lemma 1 Let $D \subseteq \mathbb{R}$ be a convex set. Let $f, g : D \rightarrow \mathbb{R}$ be strictly concave (convex) functions and $f(x) \neq g(x)$. Then, both functions have a maximum of two intersections.

Proof We prove this lemma for f and g being strictly concave. For f and g being strictly convex, the proof follows analogously.

We assume that f and g have (at least) three intersections $x_1 < x_2 < x_3$. As f and g are strictly concave, for $\alpha \in (0;1)$ we have

$$\begin{aligned} f(\alpha x_1 + (1 - \alpha)x_3) &> \alpha f(x_1) + (1 - \alpha)f(x_3), \\ g(\alpha x_1 + (1 - \alpha)x_3) &> \alpha g(x_1) + (1 - \alpha)g(x_3). \end{aligned}$$

Taking the difference between the two inequalities, we obtain for $\alpha \in (0;1)$

$$f(\alpha x_1 + (1 - \alpha)x_3) - g(\alpha x_1 + (1 - \alpha)x_3) > 0$$

because $f(x_1) = g(x_1)$ and $f(x_3) = g(x_3)$. As $x_1 < x_2 < x_3$, there is an $\bar{\alpha} \in (0;1)$ so that $x_2 = \bar{\alpha}x_1 + (1 - \bar{\alpha})x_3$ and the previous inequality holds for x_2

$$f(x_2) - g(x_2) = f(\bar{\alpha}x_1 + (1 - \bar{\alpha})x_3) - g(\bar{\alpha}x_1 + (1 - \bar{\alpha})x_3) > 0,$$

which contradicts the assumption of $f(x_2) = g(x_2)$. Consequently, f and g can have at most two intersections. $\square$

Lemma 2 Let $a, \sigma, x > 0$ and $K_c(x)$ be defined as

$$K_c(x) := -\frac{\sigma^2}{4a^2} \frac{B_a(x)}{x} (c - (1 - e^{-ax})).$$

The first derivative of $K_c(x)$ with respect to x is

- strictly convex for $c \geq 1$,
- strictly concave for $c \leq -14$.

Proof First, we will prove that $\frac{\partial K_c(x)}{\partial x}$ is strictly convex for $c \geq 1$. The third derivative of $K_c(x)$ with respect to x is

$$\begin{aligned} \frac{\partial^3 K_c(x)}{\partial x^3} &= -\frac{\sigma^2}{4a^2} \left(\frac{\partial^3 B_a(x)/x}{\partial x^3} (c - (1 - e^{-ax})) - 3a \frac{\partial^2 B_a(x)/x}{\partial x^2} e^{-ax} \right. \\ &\quad \left. + 3a^2 \frac{\partial B_a(x)/x}{\partial x} e^{-ax} - a^3 \frac{B_a(x)}{x} e^{-ax} \right). \end{aligned}$$

Inspection of $\frac{B_a(x)}{x}$ delivers the following properties for $a, x > 0$:

$$\begin{aligned}
\frac{B_a(x)}{x} &= \frac{1 - e^{-ax}}{ax} > 0, \\
\frac{\partial B_a(x)/x}{\partial x} &= \frac{(1 + ax)e^{-ax} - 1}{ax^2} < 0, \\
\frac{\partial^2 B_a(x)/x}{\partial x^2} &= \frac{2(1 - (1 + ax)e^{-ax}) - a^2 x^2 e^{-ax}}{ax^3} > 0, \\
\frac{\partial^3 B_a(x)/x}{\partial x^3} &= \frac{6((1 + ax)e^{-ax} - 1) + 3a^2 x^2 e^{-ax} + a^3 x^3 e^{-ax}}{ax^4} < 0.
\end{aligned}$$

These inequalities directly imply

$$\frac{\partial^3 K_c(x)}{\partial x^3} > 0$$

for $c \geq 1$. Therefore, the first derivative of $K_c(x)$ is strictly convex for $c \geq 1$.

Next, we will prove that $\frac{\partial K_c(x)}{\partial x}$ is strictly concave for $c \leq -14$. The third derivative of $K_c(x)$ can be rewritten as

$$\begin{aligned}
\frac{\partial^3 K_c(x)}{\partial x^3} &= \frac{\sigma^2}{4a^2} \left(\frac{6(c-1) + (2-c)e^{-ax}(a^3 x^3 + 3a^2 x^2 + 6ax + 6)}{ax^4} \right. \\
&\quad \left. - \frac{e^{-2ax}(8a^3 x^3 + 12a^2 x^2 + 12ax + 6)}{ax^4} \right) \\
&=: \frac{\sigma^2}{4a^2} \frac{6(c-1) + (2-c)z(x) - z(2x)}{ax^4} =: \frac{\sigma^2}{4a^2} \frac{k_c(x)}{ax^4}.
\end{aligned}$$

Hence, $\frac{\partial^3 K_c(x)}{\partial x^3}$ has the same sign as $k_c(x)$. Therefore, we consider $k_c(x)$. It attains the following limits

$$\lim_{x \downarrow 0} k_c(x) = 0, \quad \lim_{x \uparrow \infty} k_c(x) = 6(c-1)$$

and has the first derivative of

$$\frac{\partial k_c(x)}{\partial x} = a^4 x^3 e^{-ax} (16e^{-ax} - (2-c)).$$

As $a^4 x^3 e^{-ax} > 0$ as well as e^{-ax} is strictly monotonically decreasing from 1 to 0 for $a, x > 0$, $\frac{\partial k_c(x)}{\partial x}$ has at most one zero in $x > 0$ depending on c and it is given by

$$\bar{x}_c = -\frac{1}{a} \ln \left(\frac{2-c}{16} \right).$$

$\frac{\partial k_c(x)}{\partial x}$ does not have any zero in $x > 0$ for $c \leq -14$ and we have

$$\frac{\partial k_c(x)}{\partial x} \leq 16a^4 x^3 e^{-ax} (e^{-ax} - 1) < 0.$$

Consequently, $k_c(x)$ is negative since it strictly monotonically decreases from zero on and thus $\frac{\partial^3 K_c(x)}{\partial x^3}$ is negative for $x > 0$. Therefore, the first derivative of $K_c(x)$ is strictly concave if $c \leq -14$. $\square$

We are now ready to give the corrected proof of the assertions (c) and (d) in Theorem 4:

Proof of Theorem 4 (c) We consider a short rate $r(t) \geq y_{\text{asympt}}$ and Transformation 1

$$\begin{aligned} y(t, x) &= -\frac{\sigma^2}{4a^2} \frac{B_a(x)}{x} \left(c_A - (1 - e^{-ax}) \right) - \frac{\eta^2}{4b^2} \frac{B_b(x)}{x} \left(c_B - (1 - e^{-bx}) \right) \\ &\quad - \rho \frac{\sigma\eta}{ab} \frac{B_{a+b}(x)}{x} + \theta - \frac{\sigma^2}{2a^2} - \frac{\eta^2}{2b^2} - \rho \frac{\sigma\eta}{ab} \\ &=: K_{c_A}(x) + K_{c_B}(x) + R_1(x) + y_{\text{asympt}} \end{aligned}$$

where

$$\begin{aligned} c_A &:= -\frac{4a^2}{\sigma^2} \left(X(t) + \psi(t) - \theta + \frac{\sigma^2}{2a^2} + \rho \frac{\sigma\eta}{ab} \right), \\ c_B &:= -\frac{4b^2}{\eta^2} \left(Y(t) + \frac{\eta^2}{2b^2} + \rho \frac{\sigma\eta}{ab} \right). \end{aligned}$$

We start with the case $a < b$. We choose c_B as

$$c_B > \max \left\{ 1; -\frac{4b^2}{\eta^2} \left(\frac{a}{a-b} (r(t) - \theta) + \frac{\eta^2}{2b^2} + \rho \frac{\sigma\eta}{ab} \right) \right\}$$

and c_A for $\rho \leq 0$ as

$$c_A \leq \min \left\{ -14; -\frac{a^2\eta^2}{b^2\sigma^2} c_B \right\}$$

and for $\rho > 0$ as

$$c_A \leq \min \left\{ -14; -\frac{4a^2}{\sigma^2} \left(\frac{\eta^2}{4b^2} c_B + \rho \frac{\sigma\eta}{ab} \right) \right\}.$$

The first derivative of $y(t, x)$ with respect to x at $x = 0$ is given by (see the proof of Theorem 3)

$$\lim_{x \downarrow 0} \frac{\partial y(t, x)}{\partial x} = -\frac{1}{2} (a(r(t) - \theta) - (a - b)Y(t)).$$

Solving the defining equation of c_B to $Y(t)$ yields

$$Y(t) = -\frac{\eta^2}{4b^2} c_B - \frac{\eta^2}{2b^2} - \rho \frac{\sigma\eta}{ab}.$$

With this we have

$$\lim_{x \downarrow 0} \frac{\partial y(t, x)}{\partial x} = -\frac{1}{2} \left(a(r(t) - \theta) + (a - b) \left(\frac{\eta^2}{4b^2} c_B + \frac{\eta^2}{2b^2} + \rho \frac{\sigma \eta}{ab} \right) \right).$$

Since we choose $c_B > -\frac{4b^2}{\eta^2} \left(\frac{a}{a-b} (r(t) - \theta) + \frac{\eta^2}{2b^2} + \rho \frac{\sigma \eta}{ab} \right)$, it holds

$$\lim_{x \downarrow 0} \frac{\partial y(t, x)}{\partial x} > 0.$$

Next, we consider the difference between $y(t, x)$ and y_{asympt} . For this, note

$$\frac{B_a(x)}{x} > \frac{B_b(x)}{x} > \frac{B_{a+b}(x)}{x} > 0, \quad x > 0. \quad (1)$$

With these inequalities, $c_B > 1$, and $1 - e^{-zx} > 0$ for $x, z > 0$, we can estimate the difference as follows:

$$\begin{aligned} y(t, x) - y_{asympt} &= -\frac{\sigma^2}{4a^2} \frac{B_a(x)}{x} \left(c_A - (1 - e^{-ax}) \right) \\ &\quad - \frac{\eta^2}{4b^2} \frac{B_b(x)}{x} \left(c_B - (1 - e^{-bx}) \right) - \rho \frac{\sigma \eta}{ab} \frac{B_{a+b}(x)}{x} \\ &> -\frac{B_a(x)}{x} \left(\frac{\sigma^2}{4a^2} c_A + \frac{\eta^2}{4b^2} c_B \right) - \rho \frac{\sigma \eta}{ab} \frac{B_{a+b}(x)}{x}. \end{aligned}$$

For $\rho \leq 0$ we thus have

$$y(t, x) - y_{asympt} \geq -\frac{B_a(x)}{x} \left(\frac{\sigma^2}{4a^2} c_A + \frac{\eta^2}{4b^2} c_B \right).$$

With the choice of $c_A \leq -\frac{a^2 \eta^2}{b^2 \sigma^2} c_B$, we obtain $y(t, x) - y_{asympt} > 0$ for all $x > 0$.

For $\rho > 0$ we have

$$\begin{aligned} &-\frac{B_a(x)}{x} \left(\frac{\sigma^2}{4a^2} c_A + \frac{\eta^2}{4b^2} c_B \right) - \rho \frac{\sigma \eta}{ab} \frac{B_{a+b}(x)}{x} \\ &> -\frac{B_a(x)}{x} \left(\frac{\sigma^2}{4a^2} c_A + \frac{\eta^2}{4b^2} c_B + \rho \frac{\sigma \eta}{ab} \right). \end{aligned}$$

Due to the choice of $c_A \leq -\frac{4a^2}{\sigma^2} \left(\frac{\eta^2}{4b^2} c_B + \rho \frac{\sigma \eta}{ab} \right)$, we obtain $y(t, x) - y_{asympt} > 0$ for all $x > 0$.

Consequently, $y(t, x)$ is greater than y_{asympt} for the choice of c_A and c_B .

Since $r(t)$ is greater than or equal to y_{asympt} , $\lim_{x \downarrow 0} \frac{\partial y(t, x)}{\partial x}$ is positive, $y(t, x)$ converges towards y_{asympt} on the long end, and $y(t, x)$ lies above y_{asympt} , the yield curve has an odd number of local extreme values.

We write the first order condition for a local extreme value of $y(t, x)$ in case of $\rho \leq 0$ as

$$\frac{\partial K_{c_A}(x)}{\partial x} + \frac{\partial R_1(x)}{\partial x} = -\frac{\partial K_{c_B}(x)}{\partial x}$$

and in case of $\rho > 0$ as

$$\frac{\partial K_{c_A}(x)}{\partial x} = -\left(\frac{\partial K_{c_B}(x)}{\partial x} + \frac{\partial R_1(x)}{\partial x}\right).$$

$\frac{\partial K_{c_A}(x)}{\partial x}$ is strictly concave and $\frac{\partial K_{c_B}(x)}{\partial x}$ is strictly convex due to the choice of $c_A \leq -14$ and $c_B > 1$ (see Lemma 2). $\frac{\partial R_1(x)}{\partial x}$ is strictly concave for $\rho < 0$, it equals zero for $\rho = 0$, and it is strictly convex for $\rho > 0$. Consequently, $\frac{\partial K_{c_A}(x)}{\partial x} + \frac{\partial R_1(x)}{\partial x}$ and $-\frac{\partial K_{c_B}(x)}{\partial x}$ have at most two intersections for $\rho \leq 0$ according to Lemma 1. The same holds for $\frac{\partial K_{c_A}(x)}{\partial x}$ and $-\left(\frac{\partial K_{c_B}(x)}{\partial x} + \frac{\partial R_1(x)}{\partial x}\right)$ with $\rho > 0$. Therefore, the derivative of $y(t, x)$ has at most two zeros and thus $y(t, x)$ at most two local extreme values.

Combined with the odd number of local extreme values that the yield curve must have, it follows that the yield curve is humped for the above choice of $r(t)$, c_A , and c_B .

For $a > b$ we choose c_A as

$$c_A \geq 1$$

and c_B for $\rho \leq 0$ as

$$c_B < \min\left\{-14; -\frac{4b^2}{\eta^2}\left(\frac{a}{a-b}(r(t) - \theta) + \frac{\eta^2}{2b^2} + \rho\frac{\sigma\eta}{ab}\right); -\frac{b^2\sigma^2}{a^2\eta^2}c_A\right\}$$

and for $\rho > 0$ as

$$c_B < \min\left\{-14; -\frac{4b^2}{\eta^2}\left(\frac{a}{a-b}(r(t) - \theta) + \frac{\eta^2}{2b^2} + \rho\frac{\sigma\eta}{ab}\right); -\frac{4b^2}{\eta^2}\left(\frac{\sigma^2}{4a^2}c_A + \rho\frac{\sigma\eta}{ab}\right)\right\}.$$

Analogously to $a < b$, it can be shown that $\lim_{x \downarrow 0} \frac{\partial y(t, x)}{\partial x} > 0$ and $y(t, x) > y_{asymp}$ for the choice of c_A and c_B . Due to the strict curvature of the derivative of $K_{c_A}(x)$, $K_{c_B}(x)$, and $R_1(x)$, $y(t, x)$ has a maximum of two local extreme values according to Lemma 1. Together with $r(t) \geq y_{asymp}$, $\lim_{x \downarrow 0} \frac{\partial y(t, x)}{\partial x} > 0$, $\lim_{x \uparrow \infty} y(t, x) = y_{asymp}$ as well as $y(t, x) > y_{asymp}$, it follows that $y(t, x)$ is humped for the above choice of $r(t)$, c_A , and c_B .

(d) The case for dipped yield curves follows with obvious modifications. We now consider a short rate $r(t) \leq y_{asymp}$. For $a < b$ we choose c_B as

$$c_B < \min\left\{-14; -\frac{4b^2}{\eta^2}\left(\frac{a}{a-b}(r(t) - \theta) + \frac{\eta^2}{2b^2} + \rho\frac{\sigma\eta}{ab}\right)\right\}$$

and c_A for $\rho \leq 0$ as

$$c_A \geq \max \left\{ 1; -\frac{4a^2}{\sigma^2} \left(\frac{\eta^2}{4b^2} (c_B - 1) + \rho \frac{\sigma\eta}{ab} \right) + 1 \right\}$$

and for $\rho > 0$ as

$$c_A \geq \max \left\{ 1; -\frac{a^2\eta^2}{b^2\sigma^2} (c_B - 1) + 1 \right\}.$$

For $a > b$ we choose c_A as

$$c_A \leq -14$$

and c_B for $\rho \leq 0$ as

$$c_B > \max \left\{ 1; -\frac{4b^2}{\eta^2} \left(\frac{a}{a-b} (r(t) - \theta) + \frac{\eta^2}{2b^2} + \rho \frac{\sigma\eta}{ab} \right); -\frac{4b^2}{\eta^2} \left(\frac{\sigma^2}{4a^2} (c_A - 1) + \rho \frac{\sigma\eta}{ab} \right) + 1 \right\}$$

and for $\rho > 0$

$$c_B > \max \left\{ 1; -\frac{4b^2}{\eta^2} \left(\frac{a}{a-b} (r(t) - \theta) + \frac{\eta^2}{2b^2} + \rho \frac{\sigma\eta}{ab} \right); -\frac{b^2\sigma^2}{a^2\eta^2} (c_A - 1) + 1 \right\}.$$

It can be shown that we have

$$\lim_{x \downarrow 0} \frac{\partial y(t, x)}{\partial x} < 0 \text{ and } y(t, x) < y_{\text{asympt}}.$$

In particular, $y(t, x)$ has an odd number of local extreme values. Due to the strict curvature of the derivative of $K_{c_A}(x)$, $K_{c_B}(x)$, and $R_1(x)$, $y(t, x)$ has at most two local extreme values according to Lemma 1. Together with

$$r(t) \leq y_{\text{asympt}}, \lim_{x \downarrow 0} \frac{\partial y(t, x)}{\partial x} < 0, \lim_{x \uparrow \infty} y(t, x) = y_{\text{asympt}}, \text{ and } y(t, x) < y_{\text{asympt}},$$

it follows that $y(t, x)$ is dipped for the above choice of $r(t)$, c_A , and c_B . $\square$

Acknowledgements We would like to thank Ralf Werner for alerting us to the mistake in the published version of [1].

References

1. Diez F, Korn R (2020) Yield curve shapes of Vasicek interest rate models, measure transformations and an application for the simulation of pension products. *Eur. Actuar. J.* 10:91–120. <https://doi.org/10.1007/s13385-019-00214-0>

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