Bulletin of Mathematical Sciences (2018) 8:603–617
https://doi.org/10.1007/s13373-018-0131-3

Minimizer of an isoperimetric ratio on a metric on $\mathbb{R}^2$ with finite total area

Shu-Yu Hsu[1]

Received: 23 August 2018 / Revised: 11 September 2018 / Accepted: 13 September 2018 /
Published online: 25 September 2018

Abstract
Let $g = (g_{ij})$ be a complete Riemmanian metric on $\mathbb{R}^2$ with finite total area and let I_g be the infimum of the quotient of the length of any closed simple curve γ in $\mathbb{R}^2$ and the sum of the reciprocal of the areas of the regions inside and outside γ respectively with respect to the metric g. Under some mild growth conditions on g we prove the existence of a minimizer for I_g. As a corollary we obtain a proof for the existence of a minimizer for $I_{g(t)}$ for any $0 < t < T$ when the metric $g(t) = g_{ij}(\cdot, t) = u\delta_{ij}$ is the maximal solution of the Ricci flow equation $\partial g_{ij}/\partial t = -2R_{ij}$ on $\mathbb{R}^2 \times (0, T)$ (Daskalopoulos and Hamilton in Commun Anal Geom 12(1):143–164, 2004) where $T > 0$ is the extinction time of the solution. This existence of minimizer result is assumed and used without proof by Daskalopoulos and Hamilton (2004).

Keywords Existence of minimizer · Isoperimetric ratio · Complete Riemannian metric on $\mathbb{R}^2$ · Finite total area

Mathematics Subject Classification Primary 58E99 · 49Q99; Secondary 58C99

1 Introduction

Isoperimetric inequalities arises in many problems in analysis and geometry such as the study of partial differential equations and Sobolev inequality [1,17]. In [8,11], Gage and Hamilton studied isoperimetric inequalities arising from the curve shortening flow. In [2] Daskalopoulos and Hamilton assumed the existence of a minimizer for an isoperimetric inequality corresponding to the maximal solution of the finite mass

Communicated by Neil Trudinger.

✉ Shu-Yu Hsu
shuyu.sy@gmail.com

1 Department of Mathematics, National Chung Cheng University, 168 University Road, Min-Hsiung, Chia-Yi 621, Taiwan, ROC

2-dimensional Ricci flow on $\mathbb{R}^2$ and studied various properties of this isoperimetric inequality. However there is no proof of the existence of this minimizer in [2] and there is also no proof of this important existence result in other papers. In [3,4,6], Daskalopoulos, Hamilton, Del Pino and Sesum used these properties to study the behaviour of ancient solution of Ricci flow and the extinction behavior of finite mass maximal solution of Ricci flow, which is an important tool in the classification of manifolds [14–16,18].

Since the existence of such minimizer is crucial to the proof of various theorems in [2,3,5,6], in this paper I will give a rigorous proof of this important existence result. In fact my existence result holds for any metrics that satisfies some structural conditions which include the maximal finite mass solution of the 2-dimensional Ricci flow as a special case.

Let $g = (g_{ij})$ be a complete Riemannian metric on $\mathbb{R}^2$ with finite total area $A = \int_{\mathbb{R}^2} dV_g$ satisfying

$$\lambda_1(|x|)\delta_{ij} \le g_{ij}(x) \le \lambda_2(|x|)\delta_{ij} \quad \forall |x| \ge r_0 \tag{1.1}$$

for some constant $r_0 > 1$ and positive monotone decreasing functions $\lambda_1(r)$, $\lambda_2(r)$, on $[r_0, \infty)$ that satisfy

$$\int_r^{c_0 r} \sqrt{\lambda_1(\rho)}\, d\rho \ge \pi r \sqrt{\lambda_2(r)} \quad \forall r \ge r_0, \tag{1.2}$$

$$r\sqrt{\lambda_1(c_0 r)} \ge b_1 \int_r^{\infty} \rho \lambda_2(\rho)\, d\rho \quad \forall r \ge r_0, \tag{1.3}$$

$$\int_r^{r^2} \sqrt{\lambda_1(\rho)}\, d\rho \ge b_2 \quad \forall r \ge r_0, \tag{1.4}$$

and

$$\lambda_1(c_0 r) \ge \delta \lambda_2(r) \quad \forall r \ge r_0 \tag{1.5}$$

for some constants $c_0 > 1$, $b_1 > 0$, $b_2 > 0$, $\delta > 0$, where $|x|$ is the distance of x from the origin with respect to the Euclidean metric. For any closed simple curve γ in $\mathbb{R}^2$, let (cf. [2])

$$I(\gamma) = L(\gamma)\left(\frac{1}{A_{in}(\gamma)} + \frac{1}{A_{out}(\gamma)}\right), \tag{1.6}$$

where $L(\gamma)$ is the length of the curve γ, $A_{in}(\gamma)$ and $A_{out}(\gamma)$ are the areas of the regions inside and outside γ respectively, with respect to the metric g. Let

$$I = I_g = \inf_{\gamma} I(\gamma) \tag{1.7}$$

where the infimum is over all closed simple curves γ in $\mathbb{R}^2$. In this paper we will prove that there exists a constant $b_0 > 0$ such that if $I_g < b_0$, then there exists a closed simple curve γ satisfying $I_g = I(\gamma)$. As a corollary we obtain a proof for the existence of a minimizer for the isoperimetric ratio $I_{g(t)}$ for any $0 < t < T$ when the metric $g(t) = g_{ij}(\cdot, t) = u\delta_{ij}$ is the maximal solution of the Ricci flow [2]

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij} \quad \text{on } \mathbb{R}^2 \times (0, T),$$

where $T > 0$ is the extinction time of the solution and u is a solution of

$$u_t = \Delta \log u \quad \text{on } \mathbb{R}^2 \times (0, T). \tag{1.8}$$

We will adapt and modify the techniques in [11,12] to prove the result. Since the domain under consideration in [11,12], is either the sphere S^2 ([12]) or a bounded domain, the minimizing sequences for the infimum of the isoperimetric ratios considered in those cases stay in a compact set. On the other hand since the isoperimetric ratio (1.6) is for any curve γ in $\mathbb{R}^2$, the minimizing sequence of curves for the infimum of the isoperimetric ratio (1.7) may not stay in a compact subset of $\mathbb{R}^2$ and may not have a limit at all. The technique for compact manifold [12] is not sufficient to prove this result. New technique and ideas are used in this paper to prove the result. We will show that there exists a constant such that this is impossible when I_g is less than this constant. After this we will use the curve shortening flow technique of [12] to modify the minimizing sequence of curves and show that they will converge to a minimizer of (1.7).

For any $x_0 \in \mathbb{R}^2$ and $r > 0$ let $B_r(x_0) = \{x \in \mathbb{R}^2 : |x - x_0| < r\}$ and $B_r = B_r(0)$. The main results of the paper are as follows.

Theorem 1.1 *Suppose g satisfies (1.1) for some constant $r_0 > 1$ where $\lambda_1(r)$, $\lambda_2(r)$, are positive monotone decreasing functions on $[r_0, \infty)$ that satisfy (1.2), (1.3), (1.4) and (1.5) for some constants $c_0 > 1$, $b_1 > 0$, $b_2 > 0$ and $\delta > 0$. Then there exists a constant $b_0 > 0$ depending on b_1, b_2 and A such that the following holds:*

If

$$I_g < b_0, \tag{1.9}$$

then there exists a closed simple curve γ in $\mathbb{R}^2$ such that $I_g = I(\gamma)$. Hence $I_g > 0$.

Proposition 1.2 *Suppose $g = (g_{ij})$ satisfies*

$$\frac{C_1}{r^2(\log r)^2}\delta_{ij} \le g_{ij} \le \frac{C_2}{r^2(\log r)^2}\delta_{ij} \quad \forall r \ge r_1$$

for some constants $C_2 \ge C_1 > 0$, $r_1 > 1$. Then there exist constants $c_0 > 1$, $\delta > 0$, $b_1 > 0$, $b_2 > 0$, and $r_0 \ge r_1$ such that (1.2), (1.3), (1.4) and (1.5) hold.

Corollary 1.3 *Let $g_{ij}(x, t) = u(x, t)\delta_{ij}$ where u is the maximal solution of (1.8) with initial value $0 \le u_0 \in L^p(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$, $u_0 \not\equiv 0$, for some $p > 1$ satisfying*

$$u_0(x) \le \frac{C}{|x|^2(\log|x|)^2} \quad \forall |x| > 1 \tag{1.10}$$

given by [4,13] *where* $T = (1/4\pi)\int_{\mathbb{R}^2} u_0\,dx$. *Then for any* $0 < t_1 < T$ *there exists a constant* $b_0 > 0$ *such that the following holds:*

For any $t_1 \le t < T$, *if* $I_{g(t)} < b_0$, *then there exists a closed simple curve* γ *that satisfies* $I_{g(t)} = I(\gamma)$.

By an argument similar to the proof of Proposition 1.2 we also have the following result.

Remark 1.4 Suppose $g = (g_{ij})$ satisfies

$$\frac{C_1}{r^2(\log r)^2(1+\log r)^2}\delta_{ij} \le g_{ij} \le \frac{C_2}{r^2(\log r)^2(1+\log r)^2}\delta_{ij} \quad \forall r \ge r_1$$

for some constants $C_2 \ge C_1 > 0$, $r_1 > 1$. Then there exist constants $c_0 > 1$, $\delta > 0$, $b_1 > 0$, $b_2 > 0$, and $r_0 \ge r_1$ such that (1.2), (1.3), (1.4) and (1.5) hold. The growth condition for g here is different from that of Proposition 1.2. Hence Theorem 1.1 is more general than the minimizer result used by Daskalopoulos and Hamilton [2].

2 The proof of the main results

Proof of Proposition 1.2 Let $\lambda_i(r) = C_i(r\log r)^{-2}$, $i = 1, 2$,

$$c_0 = 2e^{\pi\sqrt{C_2/C_1}}, \tag{2.1}$$

and $\delta = C_1/(2c_0^2C_2)$. We choose $r_2 \ge r_1$ such that

$$\frac{\log r}{\log(c_0 r)} \ge \frac{1}{\sqrt{2}} \quad \forall r \ge r_2. \tag{2.2}$$

Then by (2.1) and (2.2),

$$\frac{\lambda_1(c_0 r)}{\lambda_2(r)} = \frac{C_1}{c_0^2C_2}\left(\frac{\log r}{\log(c_0 r)}\right)^2 \ge \frac{C_1}{2c_0^2C_2} = \delta \quad \forall r \ge r_2. \tag{2.3}$$

We next note that

$$\lim_{r\to\infty}\left((\log r)\log\left(\frac{\log(c_0 r)}{\log r}\right)\right) = \lim_{z\to 0}\frac{\log((\log c_0)z + 1)}{z} = \log c_0. \tag{2.4}$$

By (2.1) and (2.4) there exists $r_0 \ge r_2$ such that

$$(\log r)\log\left(\frac{\log(c_0 r)}{\log r}\right) > \pi\sqrt{C_2/C_1} \quad \forall r \ge r_0. \tag{2.5}$$

By (2.3) and (2.5), we get (1.2) and (1.5). By (2.2) and a direct computation (1.3) and (1.4) holds with $b_1 = \sqrt{C_1}/(\sqrt{2}c_0C_2)$, $b_2 = \sqrt{C_1}\log 2$, and the proposition follows. □

Proof of Corollary 1.3 By (1.10) and the results of [7] there exists a constant $C_2 > 0$ such that

$$u(x,t) \le \frac{C_2}{|x|^2(\log|x|)^2} \quad \forall |x| > 1, 0 < t < T \tag{2.6}$$

and for any $t_0 \in (0, T)$ there exists a constant $r_1 > 1$ such that

$$u(x,t) \ge \frac{(3/2)t}{|x|^2(\log|x|)^2} \quad \forall |x| \ge r_1, 0 < t \le t_0. \tag{2.7}$$

By (2.6), (2.7), Theorem 1.1 and Proposition 1.2, the corollary follows. □

Henceforth we will assume that g is a metric on $\mathbb{R}^2$ with finite total area that satisfies (1.1), (1.2), (1.3), (1.4) and (1.5) for some constants $r_0 > 1$, $c_0 > 1$, $b_1 > 0$, $b_2 > 0$, $\delta > 0$ where $\lambda_1(r)$, $\lambda_2(r)$, are positive monotone decreasing functions on $[r_0, \infty)$. Let $b_0 = \min(b_1, 4b_2/A)$. Suppose (1.9) holds. Let $\{\gamma_k\}_{k=1}^{\infty}$ be a sequence of closed simple curves on $\mathbb{R}^2$ such that

$$I(\gamma_k) \to I \quad \text{as } k \to \infty \quad \text{and} \quad I(\gamma_k) < b_0 \quad \forall k \in \mathbb{Z}^+. \tag{2.8}$$

We will show that the sequence $\{\gamma_k\}_{k=1}^{\infty}$ is contained in some compact set of $\mathbb{R}^2$. Let Ω_k be the region inside γ_k and $r_k = \min_{x\in\gamma_k}|x|$. Let $L_e(\gamma_k)$ be the length of γ_k and $|\Omega_k|$ be the area of Ω_k with respect to the Euclidean metric. We choose $r_0' > r_0$ such that

$$\mathrm{Vol}_g\left(\mathbb{R}^2\setminus B_{r_0'}\right) \le \frac{A}{4} \quad \forall k \in \mathbb{Z}^+. \tag{2.9}$$

Lemma 2.1 *The sequence r_k is uniformly bounded.*

Proof Suppose the lemma is not true. Then there exists a subsequence of r_k which we may assume without loss of generality to be the sequence itself such that

$$r_k > r_0' \quad \forall k \in \mathbb{Z}^+ \tag{2.10}$$

and $r_k \to \infty$ as $k \to \infty$. Let $\widetilde{\gamma}_k = \partial B_{r_k}$. We choose a point $x_k \in \gamma_k \cap \partial B_{r_k}$ and let $\gamma_k : [0, 2\pi] \to \mathbb{R}^2$ be a parametrization of the curve γ_k such that $x_k = \gamma_k(0) = \gamma_k(2\pi)$. Since for any $k \in \mathbb{Z}^+$ either $0 \in \Omega_k$ or $0 \in \mathbb{R}^2\setminus\Omega_k$ holds, thus either

$$0 \in \Omega_k \quad \text{for infinitely many k} \tag{2.11}$$

or

$$0 \in \mathbb{R}^2\setminus\Omega_k \quad \text{for infinitely many k} \tag{2.12}$$

holds. We need the following result for the proof of the lemma.

Claim 1 There exists only finitely many k such that $\gamma_k \cap (\mathbb{R}^2 \backslash \overline{B}_{c_0 r_k}) \neq \emptyset$.

Proof of Claim 1 Suppose claim 1 is false. Then there exists infinitely many k such that $\gamma_k \cap (\mathbb{R}^2 \backslash \overline{B}_{c_0 r_k}) \neq \emptyset$. Without loss of generality we may assume that

$$\gamma_k \cap \left(\mathbb{R}^2 \backslash \overline{B}_{c_0 r_k}\right) \neq \emptyset \quad \forall k \in \mathbb{Z}^+. \tag{2.13}$$

By (2.13) there exists $\phi_0 \in (0, 2\pi)$ such that

$$|\gamma_k(\phi_0)| > c_0 r_k.$$

Hence there exists $0 < \phi_1 < \phi_0 < \phi_2 < 2\pi$ such that

$$\gamma_k(\phi_1) = \gamma_k(\phi_2) = c_0 r_k$$

and

$$r_k \le |\gamma_k(\phi)| \le c_0 r_k \quad \forall \phi \in (0, \phi_1) \cup (\phi_2, 2\pi).$$

Then by (1.1),

$$\begin{aligned} L(\gamma_k) &= \int_0^{2\pi} \left(g_{ij} \dot{\gamma}_k^i \dot{\gamma}_k^j\right)^{\frac{1}{2}} \, d\phi \\ &\ge \left(\int_0^{\phi_1} + \int_{\phi_2}^{2\pi}\right) \left(g_{ij} \dot{\gamma}_k^i \dot{\gamma}_k^j\right)^{\frac{1}{2}} \, d\phi \\ &\ge \left(\int_0^{\phi_1} + \int_{\phi_2}^{2\pi}\right) \sqrt{\lambda_1(r)} \sqrt{\left(\frac{dr}{d\phi}\right)^2 + r^2 \left(\frac{d\theta}{d\phi}\right)^2} \, d\phi \\ &\ge 2 \int_{r_k}^{c_0 r_k} \sqrt{\lambda_1(r)} \, dr \end{aligned} \tag{2.14}$$

and

$$2\pi r_k \sqrt{\lambda_1(r_k)} \le L(\widetilde{\gamma}_k) = \int_0^{2\pi} \left(g_{ij} \dot{\widetilde{\gamma}}_k^i \dot{\widetilde{\gamma}}_k^j\right)^{\frac{1}{2}} \, d\phi \le 2\pi r_k \sqrt{\lambda_2(r_k)}. \tag{2.15}$$

By (1.2), (2.14) and (2.15),

$$L(\widetilde{\gamma}_k) \le L(\gamma_k). \tag{2.16}$$

Suppose (2.11) holds. Without loss of generality we may assume that $0 \in \Omega_k$ for all $k \in \mathbb{Z}^+$. Then $B_{r_k} \subset \Omega_k$ for all $k \in \mathbb{Z}^+$. Hence by (2.9), (2.10),

$$A_{out}(\gamma_k) \le \mathrm{Vol}_g \left(\mathbb{R}^2 \backslash B_{r_k}\right) \le \frac{A}{4} \quad \forall k \in \mathbb{Z}^+ \tag{2.17}$$

and

$$\frac{3A}{4} \le \mathrm{Vol}_g\left(B_{r_k}\right) \le A_{in}(\gamma_k) \le A \quad \forall k \in \mathbb{Z}^+. \tag{2.18}$$

We will now show that the circle $\widetilde{\gamma}_k = \partial B_{r_k}$ satisfies

$$I(\widetilde{\gamma}_k) \le I(\gamma_k). \tag{2.19}$$

Let $\varepsilon = A_{out}(\widetilde{\gamma}_k) - A_{out}(\gamma_k)$. Then $\varepsilon = A_{in}(\gamma_k) - A_{in}(\widetilde{\gamma}_k)$. Since $\widetilde{\gamma}_k \subset \overline{\Omega}_k$ and the region between γ_k and $\widetilde{\gamma}_k$ is contained in $\mathbb{R}^2\setminus B_{r_k}$, by (2.17),

$$0 \le \varepsilon \le \frac{A}{4}. \tag{2.20}$$

Hence by (2.17) and (2.20),

$$\begin{aligned}\frac{1}{A_{in}(\widetilde{\gamma}_k)} + \frac{1}{A_{out}(\widetilde{\gamma}_k)} &= \frac{A}{A_{in}(\widetilde{\gamma}_k)A_{out}(\widetilde{\gamma}_k)} = \frac{A}{(A_{in}(\gamma_k)-\varepsilon)(A_{out}(\gamma_k)+\varepsilon)}\\ &\le \frac{A}{A_{in}(\gamma_k)A_{out}(\gamma_k)} = \frac{1}{A_{in}(\gamma_k)} + \frac{1}{A_{out}(\gamma_k)}.\end{aligned} \tag{2.21}$$

By (2.16) and (2.21) we get (2.19). Now by (1.1),

$$A_{out}(\widetilde{\gamma}_k) = \int_{\mathbb{R}^2\setminus B_{r_k}} \sqrt{\det g_{ij}}\, dx \le 2\pi \int_{r_k}^{\infty} \rho\lambda_2(\rho)\, d\rho. \tag{2.22}$$

By (1.3), (2.15), (2.19) and (2.22),

$$I(\gamma_k) \ge \frac{L(\widetilde{\gamma}_k)}{A_{out}(\widetilde{\gamma}_k)} + \frac{L(\widetilde{\gamma}_k)}{A_{in}(\widetilde{\gamma}_k)} \ge b_1. \tag{2.23}$$

Letting $k \to \infty$ in (2.23),

$$I \ge b_1. \tag{2.24}$$

This contradicts (1.9) and the definition of b_0. Hence (2.11) does not hold.

Suppose (2.12) holds. Without loss of generality we may assume that $0 \in \mathbb{R}^2\setminus\Omega_k$ for all $k \in \mathbb{Z}^+$. Then by (2.10) $0 \in \mathbb{R}^2\setminus\overline{\Omega}_k$ and $B_{r_k} \subset \mathbb{R}^2\setminus\overline{\Omega}_k$ for any $k \in \mathbb{Z}^+$. By an argument similar to the proof of (2.17) and (2.18) but with the role of $A_{in}(\gamma_k)$ and $A_{out}(\gamma_k)$ being interchanged in the proof we get

$$\begin{cases} A_{in}(\gamma_k) \le \mathrm{Vol}_g\left(\mathbb{R}^2\setminus B_{r_k}\right) \le \dfrac{A}{4} & \forall k \in \mathbb{Z}^+ \\ \dfrac{3A}{4} \le A_{out}(\gamma_k) \le A & \forall k \in \mathbb{Z}^+. \end{cases} \tag{2.25}$$

Similarly by interchanging the role of $A_{in}(\gamma_k)$ and $A_{out}(\gamma_k)$ and replacing ε by $\varepsilon' = A_{out}(\widetilde{\gamma}_k) - A_{in}(\gamma_k) = A_{out}(\gamma_k) - A_{in}(\widetilde{\gamma}_k)$ in the proof of (2.19)–(2.23) above, we get that $0 \le \varepsilon' \le A/4$ and (2.19), (2.23), still holds. Letting $k \to \infty$ in (2.23), we get (2.24). This again contradicts (1.9) and the definition of b_0. Thus (2.12) does not hold and Claim 1 follows. □

We will now continue with the proof of the lemma. By Claim 1 there exists $k_0 \in \mathbb{Z}^+$ such that

$$\begin{aligned} &\gamma_k \cap \left(\mathbb{R}^2 \setminus \overline{B}_{c_0 r_k}\right) = \emptyset \quad \forall k \ge k_0 \\ &\Rightarrow \gamma_k \subset \overline{B}_{c_0 r_k} \setminus B_{r_k} \quad \forall k \ge k_0. \end{aligned} \tag{2.26}$$

Note that either (2.11) or (2.12) holds. Suppose (2.11) holds. Without loss of generality we may assume that $0 \in \Omega_k$ for all $k \ge k_0$. Then $B_{r_k} \subset \Omega_k$ for all $k \ge k_0$. Hence by (1.1) and (2.26),

$$\begin{aligned} L(\gamma_k) &= \int_0^{2\pi} \left(g_{ij}\dot{\gamma}_k^i \dot{\gamma}_k^j\right)^{\frac{1}{2}} d\phi \\ &\ge \sqrt{\lambda_1(c_0 r_k)} \int_0^{2\pi} \left(\left(\frac{dr}{d\phi}\right)^2 + r^2\left(\frac{d\theta}{d\phi}\right)^2\right)^{\frac{1}{2}} d\phi \\ &\ge 2\pi r_k \sqrt{\lambda_1(c_0 r_k)} \quad \forall k \ge k_0 \end{aligned} \tag{2.27}$$

and

$$A_{out}(\gamma_k) \le \int_{\mathbb{R}^2 \setminus B_{r_k}} \sqrt{\det g_{ij}}\, dx \le 2\pi \int_{r_k}^{\infty} \rho \lambda_2(\rho)\, d\rho \quad \forall k \ge k_0. \tag{2.28}$$

By (1.3), (2.27) and (2.28),

$$I(\gamma_k) \ge \frac{L(\gamma_k)}{A_{out}(\gamma_k)} \ge \frac{r_k \sqrt{\lambda_1(c_0 r_k)}}{\int_{r_k}^{\infty} \rho \lambda_2(\rho)\, d\rho} \ge b_1 \quad \forall k \ge k_0. \tag{2.29}$$

Letting $k \to \infty$ in (2.29), we get (2.24). Since (2.24) contradicts (1.9) and the definition of b_0, (2.11) does not hold. Hence (2.12) holds. By (2.10) and (2.12) we may assume without loss of generality that $0 \in \mathbb{R}^2 \setminus \overline{\Omega}_k$ for all $k \ge k_0$. Then $B_{r_k} \subset \mathbb{R}^2 \setminus \overline{\Omega}_k$ for all $k \ge k_0$. Hence Ω_k is contractible to a point in $\overline{B}_{c_0 r_k} \setminus B_{r_k}$ for all $k \ge k_0$. By (1.1),

$$L(\gamma_k) = \int_0^{2\pi} \left(g_{ij}\dot{\gamma}_k^i \dot{\gamma}_k^j\right)^{\frac{1}{2}} d\phi \ge \sqrt{\lambda_1(c_0 r_k)} L_e(\gamma_k) \quad \forall k \ge k_0. \tag{2.30}$$

By the isoperimetric inequality,

$$4\pi |\Omega_k| \le L_e(\gamma_k)^2. \tag{2.31}$$

Then by (2.30) and (2.31),

$$L(\gamma_k) \geq 2(\pi\lambda_1(c_0 r_k)|\Omega_k|)^{\frac{1}{2}} \quad \forall k \geq k_0. \tag{2.32}$$

Now

$$A_{in}(\gamma_k) = \int_{\Omega_k} \sqrt{\det g_{ij}}\, dx \leq \lambda_2(r_k)|\Omega_k| \quad \forall k \geq k_0. \tag{2.33}$$

By (1.5), (2.32) and (2.33),

$$L(\gamma_k) \geq 2\pi^{\frac{1}{2}} \left(\frac{\lambda_1(c_0 r_k)}{\lambda_2(r_k)}\right)^{\frac{1}{2}} A_{in}(\gamma_k)^{\frac{1}{2}} \geq 2(\pi\delta)^{\frac{1}{2}} A_{in}(\gamma_k)^{\frac{1}{2}} \quad \forall k \geq k_0$$

$$\Rightarrow \quad I(\gamma_k) \geq \frac{L(\gamma_k)}{A_{in}(\gamma_k)} \geq 2(\pi\delta)^{\frac{1}{2}} A_{in}(\gamma_k)^{-\frac{1}{2}} \quad \forall k \geq k_0. \tag{2.34}$$

Since $\Omega_k \subset \mathbb{R}^2 \setminus B_{r_k}$,

$$A_{in}(\gamma_k) \to 0 \quad \text{as } k \to \infty. \tag{2.35}$$

Letting $k \to \infty$ in (2.34) by (2.35) we get $I = \infty$. This contradicts (1.9). Hence (2.12) does not hold and the lemma follows. □

By Lemma 2.1 there exists a constant $a_1 > r_0$ such that

$$r_k \leq a_1 \quad \forall k \in \mathbb{Z}^+. \tag{2.36}$$

Lemma 2.2 $\gamma_k \in \overline{B}_{a_1^2} \quad \forall k \in \mathbb{Z}^+$.

Proof Let $\rho_k = \max_{\gamma_k} |x|$. Suppose the lemma does not hold. Then there exists a subsequence of ρ_k which we may assume without loss of generality to be the sequence itself such that

$$\rho_k > a_1^2 \quad \forall k \in \mathbb{Z}^+. \tag{2.37}$$

By (1.1), (1.4), (2.36), (2.37) and an argument similar to the proof of (2.14),

$$L(\gamma_k) \geq \int_{a_1}^{a_1^2} \sqrt{\lambda_1(\rho)}\, d\rho \geq b_2 \quad \forall k \in \mathbb{Z}^+. \tag{2.38}$$

Hence by (2.38),

$$I(\gamma_k) = \frac{AL(\gamma_k)}{A_{in}(\gamma_k)A_{out}(\gamma_k)} \geq \frac{Ab_2}{(A/2)^2} = \frac{4b_2}{A} \quad \forall k \in \mathbb{Z}^+$$

$$\Rightarrow I \geq \frac{4b_2}{A} \quad \text{as } k \to \infty.$$

This contradicts (1.9) and the definition of b_0. Hence the lemma follows. □

Let $L_k = L(\gamma_k)$. Since $\overline{B}_{a_1^2}$ is compact, there exists constants $c_2 > c_1 > 0$ such that

$$c_1\delta_{ij} \le g_{ij} \le c_2\delta_{ij} \quad \text{on } \overline{B}_{a_1^2}. \tag{2.39}$$

Lemma 2.3 *There exists a constant* $\delta_1 > 0$ *such that* $L_k \ge \delta_1 \quad \forall k \in \mathbb{Z}^+$.

Proof By (2.39),

$$\begin{cases} c_1^{\frac{1}{2}} L_e(\gamma_k) \le L_k \le c_2^{\frac{1}{2}} L_e(\gamma_k) & \forall k \in \mathbb{Z}^+ \\ c_1|\Omega_k| \le A_{in}(\gamma_k) \le c_2|\Omega_k| & \forall k \in \mathbb{Z}^+. \end{cases} \tag{2.40}$$

By (2.8), (2.31) and (2.40),

$$b_0 > \frac{L_k}{A_{in}(\gamma_k)} \ge \frac{c_1^{\frac{1}{2}} L_e(\gamma_k)}{c_2|\Omega_k|} \ge \frac{c_1^{\frac{1}{2}}}{c_2} \cdot \frac{L_e(\gamma_k)}{(L_e(\gamma_k))^2/4\pi} \ge \frac{4\pi c_1^{\frac{1}{2}}}{c_2 L_e(\gamma_k)} \quad \forall k \in \mathbb{Z}^+$$
$$\Rightarrow \quad L_k \ge c_1^{\frac{1}{2}} L_e(\gamma_k) \ge \frac{4\pi c_1}{c_2 b_0} \quad \forall k \in \mathbb{Z}^+$$

and the lemma follows. □

By the proof of Lemma 2.3 we have the following corollary.

Corollary 2.4 *For any constant* $C_1 > 0$ *there exists a constant* $\delta_1 > 0$ *such that*

$$L(\gamma) > \delta_1$$

for any simple closed curve $\gamma \subset \overline{B}_{a_1^2}$ *satisfying*

$$I(\gamma) < C_1. \tag{2.41}$$

By (1.6) and Corollary 2.4 we have the following corollary.

Corollary 2.5 *For any constant* $C_1 > 0$ *there exists a constant* $\delta_2 > 0$ *such that*

$$A_{in}(\gamma) > \delta_2 \quad \text{and} \quad A_{out}(\gamma) > \delta_2$$

for any simple closed curve $\gamma \subset \overline{B}_{a_1^2}$ *satisfying* (2.41).

Lemma 2.6 *There exists a constant* $C_2 > 0$ *such that the following holds. Suppose* $\beta \subset \overline{B}_{a_1^2}$ *is a closed simple curve. Then under the curve shrinking flow*

$$\frac{\partial \beta}{\partial \tau}(s,\tau) = k\vec{N} \tag{2.42}$$

with $\beta(s,0)=\beta(s)$ where for each $\tau\ge 0$, $k(\cdot,\tau)$ is the curvature, $\vec{N}$ is the unit inner normal, and s is the arc length of the curve $\beta(\cdot,\tau)$ with respect to the metric g, there exists $\tau_0\ge 0$ such that the curve $\beta^{\tau_0}=\beta(\cdot,\tau_0)\subset\overline{B}_{a_1^2}$ satisfies $I(\beta^{\tau_0})\le I(\beta)$ and

$$\int k(s,\tau_0)^2\,ds\le C_2.$$

Proof Since the proof is similar to the proof of [2] and the Lemma on P.197 of [12], we will only sketch the proof here. Let $\beta^\tau=\beta(\cdot,\tau)$ and write

$$L(\tau)=L_g(\beta(\cdot,\tau)),\ \ I(\tau)=I(\beta^\tau)=I_g(\beta(\cdot,\tau)),$$

and the areas

$$A_{in}(\tau)=A_{in}(\beta(\cdot,\tau)),\ \ A_{out}(\tau)=A_{out}(\beta(\cdot,\tau)),$$

with respect to the metric g. Let $T_1>0$ be the maximal existence time of the solution of (2.42). Then

$$\beta^\tau\subset\overline{B}_{a_1^2}\quad\forall 0\le\tau<T_1. \tag{2.43}$$

Similar to the result on P.196 of [12] we have

$$\frac{\partial A_{in}}{\partial\tau}=-\int k\,ds,\quad\frac{\partial A_{out}}{\partial\tau}=\int k\,ds,\quad\frac{\partial L}{\partial\tau}=-\int k^2\,ds \tag{2.44}$$

and

$$\int k\,ds+\int_{\Omega(\tau)}K\,dV_g=2\pi \tag{2.45}$$

by the Gauss-Bonnet theorem where K is the Gauss curvature with respect to g and $\Omega(\tau)\subset\overline{B}_{a_1^2}$ is the region enclosed by the curve $\beta(s,\tau)$. Let $C_1=2I(\beta)$. By continuity there exists a constant $0<\delta_0<T_1$ such that

$$I(\tau)<C_1\quad\forall 0\le\tau\le\delta_0. \tag{2.46}$$

By (2.46), Corollary 2.4, and Corollary 2.5 there exist constants $\delta_1>0$, $\delta_2>0$, such that

$$L(\tau)>\delta_1,\quad A_{in}(\tau)>\delta_2,\quad A_{out}(\tau)>\delta_2\quad\forall 0\le\tau\le\delta_0. \tag{2.47}$$

Now

$$\frac{\partial}{\partial\tau}(\log I(\tau))=\frac{1}{L}\frac{\partial L}{\partial\tau}-\frac{1}{A_{in}}\frac{\partial A_{in}}{\partial\tau}-\frac{1}{A_{out}}\frac{\partial A_{out}}{\partial\tau}+\frac{1}{A}\frac{\partial A}{\partial\tau}. \tag{2.48}$$

By (2.43) and (2.45) $\int k\,ds$ is uniformly bounded for all $0 \le \tau < T_1$. Then by (2.44), (2.45), (2.47), and (2.48), there exists a constant $C_2 > 0$ independent of δ_0 such that

$$\frac{\partial}{\partial \tau}(\log I(\tau)) < 0$$

for any $\tau \in (0, \delta_0]$ satisfying

$$\int k(s,\tau)^2\,ds > C_2.$$

If

$$\int k(s,0)^2\,ds \le C_2,$$

we set $\tau_0 = 0$ and we are done. If

$$\int k(s,0)^2\,ds > C_2,$$

then either there exists $\tau_0 \in (0, \delta_0]$ such that

$$\int k(s,\tau_0)^2\,ds = C_2 \quad \text{and} \quad \int k(s,\tau)^2\,ds > C_2 \quad \forall 0 \le \tau < \tau_0 \tag{2.49}$$

or

$$\int k(s,\tau)^2\,ds > C_2 \quad \forall 0 \le \tau \le \delta_0. \tag{2.50}$$

If (2.49) holds, since $I(\tau_0) \le I(0)$ we are done. If (2.50) holds, since $I(\delta_0) \le I(0)$ we can repeat the above the argument a finite number of times. Then either
(a) there exists $\tau_0 \in (0, T_1)$ such that (2.49) holds
or
(b)

$$\int k(s,\tau)^2\,ds > C_2 \quad \forall 0 \le \tau < T_1 \tag{2.51}$$

holds.

If (b) holds, then similar to the proof of the Lemma on P.197 of [12] by (2.47) we get a contradiction to the Grayson theorem ([9,10,12]) for curve shortening flow. Hence (a) holds. Since $I(\tau_0) \le I(0)$, the lemma follows. □

To complete the proof of Theorem 1.1 we also need the following technical lemma (see [12]).

Lemma 2.7 *For any positive numbers $\alpha_1, \alpha_2, A_1, A_2, A_3$ we have*

$$(\alpha_1+\alpha_2)\left(\frac{1}{A_2}+\frac{1}{A_1+A_3}\right) \ge \min\left\{\alpha_1\left(\frac{1}{A_1}+\frac{1}{A_2+A_3}\right), \alpha_2\left(\frac{1}{A_3}+\frac{1}{A_1+A_2}\right)\right\}. \tag{2.52}$$

Proof Suppose (2.52) does not hold. Then

$$\begin{aligned}(\alpha_1+\alpha_2)\left(\frac{1}{A_2}+\frac{1}{A_1+A_3}\right) &\le \alpha_1\left(\frac{1}{A_1}+\frac{1}{A_2+A_3}\right)\\ \Rightarrow \quad \frac{A_1(A_2+A_3)}{A_2(A_1+A_3)} &\le \frac{\alpha_1}{\alpha_1+\alpha_2}\end{aligned} \tag{2.53}$$

and

$$\begin{aligned}(\alpha_1+\alpha_2)\left(\frac{1}{A_2}+\frac{1}{A_1+A_3}\right) &\le \alpha_2\left(\frac{1}{A_3}+\frac{1}{A_1+A_2}\right)\\ \Rightarrow \quad \frac{A_3(A_1+A_2)}{A_2(A_1+A_3)} &\le \frac{\alpha_2}{\alpha_1+\alpha_2}.\end{aligned} \tag{2.54}$$

Summing (2.53) and (2.54),

$$\frac{2A_1A_3}{A_2(A_1+A_3)} \le 0 \quad \Rightarrow \quad A_1 = 0 \text{ or } A_3 = 0.$$

Contradiction arises. Hence (2.52) holds and the lemma follows. □

We are now ready for the proof of Theorem 1.1.

Proof of Theorem 1.1 Since the proof is similar to the proof of [11,12] we will only sketch the argument here. Let $C_2 > 0$ be given by Lemma 2.6 and $\delta_1 > 0$ be given by Corollary 2.4 with $C_1 = b_0$. By Lemma 2.2, Lemma 2.3, Corollary 2.4, Lemma 2.6 and an argument similar to the proof of [12] for each $j \in \mathbb{Z}^+$ there exists a closed simple curve $\overline{\gamma}_j \subset \overline{B_{a_1^2}}$ satisfying

$$I(\overline{\gamma}_j) \le I(\gamma_j) \quad \text{and} \quad L(\overline{\gamma}_j) \ge \delta_1 \quad \forall j \in \mathbb{Z}^+$$

and

$$\int_{\overline{\gamma}_j} k^2\, ds \le C_2, \tag{2.55}$$

where k is the curvature of $\overline{\gamma}_j$. By (2.55) and the same argument as that on P. 197-199 of [12] $\overline{\gamma}_j$ are locally uniformly bounded in L_2^1 and $C^{1+\frac{1}{2}}$. Hence $\overline{\gamma}_j$ has a sequence which we may assume without loss of generality to be the sequence itself that converges uniformly in L^1_p for any $1 < p < 2$ and in $C^{1+\alpha}$ for any $0 < \alpha < 1/2$ as $j \to \infty$ to some closed immersed curve $\gamma \subset \overline{B}_{a_1^2}$. Moreover γ satisfies

$$I = I(\gamma) \quad \text{and} \quad L(\gamma) \geq \delta_1.$$

Since γ is the limit of embedded curves, γ cannot cross itself and at worst it will be self tangent. Suppose γ is self tangent. Without loss of generality we may assume that γ is only self tangent at one point. Then $\gamma = \beta_1 \cup \beta_2$ with $\beta_1 \cap \beta_2$ being a single point where β_1, β_2, are simple closed curves. Then $A_{in}(\gamma) = A_{in}(\beta_1) + A_{in}(\beta_2)$, $A_{out}(\beta_1) = A_{out}(\gamma) + A_{in}(\beta_2)$, $A_{out}(\beta_2) = A_{out}(\gamma) + A_{in}(\beta_1)$, and $L(\gamma) = L(\beta_1) + L(\beta_2)$. Let $L_1 = L(\beta_1)$ and $L_2 = L(\beta_2)$. By Lemma 2.7,

$$\begin{aligned}
&(L_1 + L_2)\left(\frac{1}{A_{out}(\gamma)} + \frac{1}{A_{in}(\beta_1) + A_{in}(\beta_2)}\right) \\
&\quad \geq \min\left\{L_1\left(\frac{1}{A_{in}(\beta_1)} + \frac{1}{A_{out}(\gamma) + A_{in}(\beta_2)}\right), L_2\left(\frac{1}{A_{in}(\beta_2)} + \frac{1}{A_{out}(\gamma) + A_{in}(\beta_1)}\right)\right\}.
\end{aligned}$$

Hence

$$\begin{aligned}
&L(\gamma)\left(\frac{1}{A_{out}(\gamma)} + \frac{1}{A_{in}(\gamma)}\right) \\
&\geq \min\left\{L_1\left(\frac{1}{A_{in}(\beta_1)} + \frac{1}{A_{out}(\beta_1)}\right), L_2\left(\frac{1}{A_{in}(\beta_2)} + \frac{1}{A_{out}(\beta_2)}\right)\right\} \\
&\Rightarrow \quad I(\gamma) \geq \min(I(\beta_1), I(\beta_2)) \\
&\Rightarrow \quad I(\gamma) = \min(I(\beta_1), I(\beta_2)).
\end{aligned}$$

Without loss of generality we may assume that $I(\gamma) = I(\beta_1)$. Then by Corollary 2.4 β_1 is a simple closed curve which attains the minimum. Similar to the proof of [12], by a variation argument β_1 has constant curvature

$$k = L\left(\frac{1}{A_{in}} - \frac{1}{A_{out}}\right).$$

Hence β_1 is smooth and the theorem follows. □

References

1. Bandle, C.: Isoperimetric Inequalities and Applications, Monographs and Studies in Mathematics, vol. 7. Pitman, Boston (1980)

2. Daskalopoulos, P., Hamilton, R.S.: Geometric estimates for the logarithmic fast diffusion equation. Commun. Anal. Geom. **12**(1), 143–164 (2004)
3. Daskalopoulos, P., Hamilton, R.S., Sesum, N.: Classification of compact ancient solutions to the Ricci flow on surfaces. J. Differ. Geom. **91**(2), 171–214 (2012)
4. Daskalopoulos, P., del Pino, M.A.: On a singular diffusion equation. Commun. Anal. Geom. **3**(3–4), 523–542 (1995)
5. Daskalopoulos, P., del Pino, M.A.: Type II collapsing of maximal solutions to the Ricci flow in $\mathbb{R}^2$. Ann. I. H. Poincaré **24**, 851–874 (2007)
6. Daskalopoulos, P., Sesum, N.: Type II extinction profile of maximal solutions to the Ricci flow in $\mathbb{R}^2$. J. Geom. Anal. **20**, 565–591 (2010)
7. Esteban, J.R., Rodriguez, A., Vazquez, J.L.: The maximal solution of the logarithmic fast diffusion equation in two space dimensions. Adv Differ Equ **2**(2), 867–894 (1997)
8. Gage, M.: An isoperimetric inequality with applications to curve shortening. Duke Math. J. **50**, 1225–1229 (1983)
9. Grayson, M.: The heat equation shrinks embedded plane curves to points. J. Differ. Geom. **26**, 285–314 (1987)
10. Grayson, M.: Shortening embedded curves. Ann. Math. **129**, 71–111 (1989)
11. Hamilton, R.S.: Isoperimetric estimates for the curve shrinking flow in the plane. In: Modern Methods in Complex Analysis. Princeton, pp. 201–222 (1992); Annals of Mathematics Studies, vol. 137. Princeton University Press, Princeton (1995)
12. Hamilton, R.S.: An isoperimetric estimate for the Ricci flow on the two-sphere. In: Modern Methods in Complex Analysis. Princeton, pp. 191–200 (1992); Annals of Mathematics Studies, vol. 137. Princeton University Press, Princeton (1995)
13. Hui, K.M.: Existence of solutions of the equation $u_t = \Delta \log u$. Nonlinear Anal. TMA **37**(7), 875–914 (1999)
14. Morgan, J., Tang, G.: Ricci Flow and the Poincaré Conjecture, Clay Mathematics Monographs, vol. 3. American Mathematical Society, Providence (2007)
15. Perelman, G.: The entropy formula for the Ricci flow and its geometric applications. arXiv:math/0211159
16. Perelman, G.: Ricci flow with surgery on three-manifolds. arXiv:math/0303109
17. Schoen, R., Yau, S.T: Lectures on differential geometry. In: Conference Proceedings and Lecture Notes in Geometry and Topology, vol. 1. International Press, Boston, Cambridge (1994)
18. Zhang, Q.S.: Some gradient estimates for the heat equation on domains and for an equation by Perelman. Int. Math. Res. Not. Art. ID 92314, p. 39 (2006)