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Decomposition of ordinary differential equations

Fritz Schwarz[1]

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Abstract Decompositions of *linear* ordinary differential equations (ode's) into components of lower order have successfully been employed for determining their solutions. Here this approach is generalized to *nonlinear* ode's. It is not based on the existence of Lie symmetries, in that it is a genuine extension of the usual solution algorithms. If an equation allows a Lie symmetry, the proposed decompositions are usually more efficient and often lead to simpler expressions for the solution. For the vast majority of equations without a Lie symmetry decomposition is the only available systematic solution procedure. Criteria for the existence of diverse decomposition types and algorithms for applying them are discussed in detail and many examples are given. The collection of Kamke of solved equations, and a tremendeous compilation of random equations are applied as a benchmark test for comparison of various solution procedures. Extensions of these proceedings for more general types of ode's and also partial differential equations are suggested.

Keywords Ordinary differential equations · Decomposition · Exact solutions

Mathematics Subject Classification Primary 34A05 · 34A34; Secondary 34C14 · 34B60

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✉ Fritz Schwarz
fritz.schwarz@scai-extern.fraunhofer.de

1 Fraunhofer Institute for Algorithms and Scientific Computing SCAI, 53754 Sankt Augustin, Germany

1 Introduction: description of the problem

Ever since its introduction more than 300 years ago the concept of a differential equation, and connected to it finding its solutions, has been a fundamental problem in mathematics and its applications in natural sciences. Despite intensive efforts over the centuries there are numerous open problems. In this article ordinary differential equations (ode's) will be discussed, i.e. equations containing a single unknown function depending on a single variable.

There are various notions of a solution of a differential equation. On the one hand, there are numerical and graphical solutions; their main advantage is that almost always they may be determined; they are not considered here. On the other hand, there are so-called exact or closed form solutions, a detailed discussion of this concept may be found in [1]. Thereby the goal is to obtain finite expressions in terms of known functions like elementary, Liouvillian or special functions that annihilate the given differential equation upon substitution; they exist only in exceptional cases. The *general solution* of an equation of order n contains n undetermined constants; special solutions contain less than n constants, or no constants at all. Furthermore, there may be *first integrals*; for an equation of order n, a first integral is a relation containing derivatives of order lower than n. In addition there may be singular integrals that may not be obtained by specialization of the constants; they are not considered in this article. As a prerequisite, general introductions into the theory of ordinary differential equations may be found e.g. in the books by Ince [9], Kamke [12] and Forsyth [6,7], or more recent books by Coddington [2] or Wirkus and Swift [21].

A complete solution scheme must allow to prove the existence or non-existence of a particular type of solution. In the former case, it should be possible to design an algorithm that returns this solution explicitly. If none is returned this should be equivalent to the proof that such a solution does not exist. Finally, if such an algorithm cannot be found, a complete answer should provide a proof that the existence of certain types of solutions is undecidable. For general differential equations this goal is out of reach at present. Only for linear equations a fairly detailed solution scheme along these lines is available [20]. It is based on the decomposition of the differential operator that is associated with the differential equation.

For nonlinear equations the most important solution procedure right now is based on Lie's symmetry analysis; in Appendix E of [18] it has been shown that almost all solutions of non-linear second order equations listed in Kamke's collection [12] are based on the existence of nontrivial Lie symmetries. Yet the situation is not completely satisfactory. Drach [3] has described this situation as follows: *"Lie's théorie des groupes à l'intégration des équations n'est pas la véritable généralization de la méthode employée par Galois pour les équations algébriques"*.[1] This judgement is based on the following observation. There are equations with a large group of Lie symmetries that may not be utilized for solving it as may be seen in several examples given below. On the other hand, there are equations without any Lie symmetry that have a fairly simple closed-form solution. Consequently, a theory for solving ode's based

[1] Translation by the author: "...Lie's group theory for the integration of equations is not the true generalization of the method applied by Galois for algebraic equations".

on Lie's symmetry analysis will not be able to design a complete solution scheme as described above.

This situation suggests trying a different approach based on factorization and decomposition as it has been applied successfully to linear equations. In this article this project will be performed in full detail for quasilinear equations of second order. On the one hand, equations of this type occur in numerous applications, and they are special enough such that a fairly complete treatement of the subject may be given. On the other hand, the scope is sufficiently general such that the applied methods may serve as a guideline for equations not linear in the highest derivative or of higher order.

In Sect. 2 second-order equations that are of first degree in the second derivative, polynomial of any degree in the first derivative, and rational in the dependent and the independent variable are discussed in detail. It is shown there how to determine first-order components of various types and how they faciliate the solution procedure. In Sect. 3 it is shown that the existence of a first-order component of a particular type is not limited to an individual equation but applies to the full equivalence class of the structure invariance group of the respective component. Section 4 deals with the computational problems that have to be solved in order to determine particular decompositions; furthermore the relation between solution procedures based on Lie symmetries versus decompositions are discussed. In the final Sect. 5 the extension of the methods described in this article to more general classes of ordinary and partial differential equations is discussed and several examples are given.

2 Decomposing quasilinear equations of second order

The general solution of a second-order ode depends on two independent constants C_1 and C_2, it generates a two-parameter family of plane curves $\omega(x, y, C_1, C_2) = 0$. For linear equations this may be written as $y = C_1\varphi_1(x) + C_2\varphi_2(x)$ where φ_1, φ_2 are a fundamental system; the function field containing these basis elements is determined by the Galois group corresponding to the differential equation, details may be found in the book by Magid [15].

For nonlinear equations Lie's symmetry analysis is based on the following observation; for details see Kapitel 16 and 17 in [13]. It may occur that a transformation of x and y just interchanges the members of the family, i.e. the above expression is transformed into $\omega(x, y, \bar{C_1}, \bar{C_2}) = 0$ where the new constants $\bar{C_1}$ and $\bar{C_2}$ are functions of C_1 and C_2. Because ω is not changed by this operation, the same is true for the corresponding differential equation; Lie calls a transformation with this property a *symmetry*. A simple example is the family of parabolas $y = C_1^2x^2 + C_1C_2x + C_2^2$. The two-parameter group of transformations $x = a\bar{x}$, $y = b\bar{y}$ changes it into $\bar{y} = C_1^2\frac{a^2}{b}\bar{x}^2 + C_1C_2\frac{a}{b}\bar{x} + \frac{1}{b}C_2^2$. Defining $\bar{C_1} \equiv \frac{a}{\sqrt{b}}C_1$ and $\bar{C_2} \equiv \frac{1}{\sqrt{b}}C_2$, the family of parabolas has the form $\bar{y} = \bar{C_1}^2\bar{x}^2 + \bar{C_1}\bar{C_2}\bar{x} + \bar{C_2}^2$. The relation between $\bar{x}$ and $\bar{y}$ is the same as between x and y with new constants $\bar{C_1}$ and $\bar{C_2}$, i.e. the members of the two-parameter family of parabolas are swapped by the transformations of the group. Consequently both obey the same second-order ode $y''^2 - \frac{2}{x}y'y'' - \frac{2}{x^2}yy'' + \frac{4}{3x^2}y'^2 = 0$, or the same equation with the barred variables. Obviously this approach is suggested by Galois' method for solving algebraic equations.

In the above mentioned reference Lie describes in detail how a symmetry may be utilized for solving differential equations of any order. The main limitation of this proceeding is the fact that symmetries are extremely rare, and even the existence of a symmetry is no guarantee that the equation may be solved; details will be given below in Sect. 4.2.

The following reasoning appears to be more natural, it applies to *any* differential equation. If the parameters in $\omega(x, y, C_1, C_2) = 0$ are constrained by a relation $\varphi(C_1, C_2) = 0$, the resulting expression contains effectively a single constant C; it may satisfy a first order ode which is a component of the original second-order equation. Obviously there are infinitely many of them corresponding to the choice of φ. If $F(x, y, y', y'') = 0$ is a second-order equation, a decomposition may have the form

$$F(x, y, y', y'') = \begin{cases} f(x, z, z')\big(z \equiv g(x, y, y')\big) \\ f(x, z, z', C)\big(z \equiv g(x, y, y', C)\big) \end{cases} \tag{1}$$

This notation is applied througout this article. It means that substituting the expression $z \equiv g$ into f yields the given second-order ode F; g is called a *right component* of F. Consequently, any solution of $g = 0$ is a solution of $F = 0$. If g contains a constant C and may be integrated, the general solution of $F = 0$ may be obtained in this way. If g does not contain a constant and f does not contain y explicitly it may be possible to proceed with the solution procedure by solving $g = z_1$ where z_1 is a solution of $f = 0$; this resembles the case of solving a *linear* ode by decomposition, details are given in Chapters 4 and 5 of Schwarz [20]. If f does contain the dependent variable y this leads to an integro-differential equation; in this case only the right component may be applied for the solution procedure; it may allow determining special solutions. The existence of first-order components is based on the following lemma.

Lemma 1 *Let the given second-order equation be* $y'' + R(x, y, y') = 0$ *where* $R \in \mathbb{Q}(x, y)[y']$. *A first-order component* $y' + r(x, y) = 0$ *exists if* r *is a solution of*

$$r_x - rr_y - R(x, y, -r) = 0; \tag{2}$$

its general solution contains an undetermined function of x *and* y.

Proof Reduction of the given second-order equation w.r.t. $y' + r = 0$ and $y'' + r_x - rr_y = 0$ yields the above constraint. According to Kamke [11], Sects. 5.1 and 5.4, (2) may be transformed into a linear partial differential equation $w_x - rw_y + R(x, y, -r)w_r = 0$ for a function $w(x, y, r)$. If $\varphi_1(x, y, r)$ and $\varphi_2(x, y, r)$ is an integral basis, its general solution is $\Phi(\varphi_1, \varphi_2)$ where Φ is an undetermined function of its two arguments. □

Equation (2) is called the *determining equation* for the component $y' + r(x, y) = 0$. The undetermined function Φ in its general solution corresponds to the function φ mentioned at the beginning of this section.

In order to obtain an algorithmic procedure for determining a first-order component the given second-order equation has to be suitably specified; a polynomial in y' with rational coefficients in x and y, i.e. $R \in \mathbb{Q}(x, y)[y']$ will be chosen.

The subsequent proposition is the basis for generating quasilinear first-order components; throughout this article $y' \equiv \frac{dy}{dx}$ and $D \equiv \frac{d}{dx}$.

Proposition 1 *Let a quasilinear second-order differential equation*

$$y'' + \sum_{k=0}^{K} c_k(x, y)y'^k = 0 \quad \text{with} \quad c_k \in \mathbb{Q}(x, y),\, K \in \mathbb{N} \tag{3}$$

be given. A first-order component $y' + r(x, y) = 0$ *exists if* $r(x, y)$ *satisfies*

$$r_x - rr_y - \sum_{k=0}^{K}(-1)^k c_k r^k = 0. \tag{4}$$

Then the original second-order equation may be written as

$$\left(z' - r_y z + \sum_{k=1}^{K} c_k \left((z - r)^k + (-1)^{k+1} r^k\right)\right)\left(z \equiv y' + r\right) = 0. \tag{5}$$

Proof The constraint (4) follows immediately from the preceding lemma. The term $z' - r_y z$ in (5) originates from the second derivative in (3). The sums in (4) and in the first bracket of (5) originate from the algebraic quotient of the sum in (3) by $y' + r$, its existence is assured if condition (4) is satisfied. □

When the coefficients c_k are given, (4) is a first-order partial differential equation for $r(x, y)$. In general its solution cannot be obtained in closed form without further specification. Because the main objective is finding explicit solutions, components that may lead to solvable first-order equations are of particular interest. A rather lcomplete discussion of solvable first-order equations is given in Kamke [10], §1, Sect. 4. They are the basis for the decomposition procedures described in this section.

Depending on the decomposition type the computational problem for determining $r(x, y)$ is different. In any case, there will be a system of equations for the coefficients that occur in the component, it will called the *determining system*. It may be an algebro-differential system, an algebraic system or a system of first-order ordinary differential equations.

2.1 Linear components

As a first case, linear first-order right components of the form $z \equiv y' + a(x)y + b(x)$ are searched for; with the above notation $r(x, y) = a(x)y + b(x)$. The following proposition describes how they may be obtained.

Proposition 2 *Let a second-order quasilinear equation* (3) *be given. In order that it has a linear first-order component* $z \equiv y' + a(x)y + b(x)$ *the coefficients* $a(x)$ *and* $b(x)$ *have to be a solution of*

$$(a' - a^2)y + b' - ab - \sum_{k=0}^{K} (-1)^k c_k (ay + b)^k = 0. \tag{6}$$

Then (3) *may be written as follows.*

$$\left(z' - az + \sum_{k=1}^{K} c_k \left((z - ay - b)^k - (-ay - b)^k\right)\right)\left(z \equiv y' + ay + b\right) = 0. \tag{7}$$

The coefficients a and b may be obtained from a first-order algebro-differential system; its degree in a and b is bounded by K except for $K = 1$ *where the degree in a is 2. For low values of K Eqs.* (6) *and* (7) *are explicitly given as follows.*

$$\begin{aligned}
K = 1 &: \begin{cases} (a' - a^2)y + b' - ab + c_1(ay + b) - c_0 = 0, \\ (z' - az + c_1 z)(z \equiv y' + ay + b) = 0; \end{cases} \\
K = 2 &: \begin{cases} (a' - a^2)y + b' - ab - c_2(ay + b)^2 + c_1(ay + b) - c_0 = 0, \\ \left(z' - az + c_2 z^2 - 2c_2(ay + b)z + c_1 z\right)(z \equiv y' + ay + b) = 0; \end{cases} \\
K = 3 &: \begin{cases} (a' - a^2)y + b' - ab + c_3(ay + b)^3 - c_2(ay + b)^2 + c_1(ay + b) \\ -c_0 = 0, \\ \left(z' - az + c_3 z^3 - 3c_3(ay + b)z^2 + c_2 z^2 + 3c_3(ay + b)^2 z \right. \\ \left. -2c_2(ay + b)z + c_1 z\right) \\ (z \equiv y' + ay + b) = 0. \end{cases}
\end{aligned} \tag{8}$$

Proof Substituting $r = a(x)y + b(x)$ into (4) yields (6). Decomposing it w.r.t. powers of the undetermined variable y yields sufficient conditions for the vanishing of (6). They form an algebro-differential system for a and b; the first order in a and b is obvious from (6). The minimal second degree of a arises from the coefficient of y in (6); the powers of a and b originate from the sum with highest power K. Substitution of $y' = z - ay - b$ into (5) yields (7). As a result, the sums at the left-hand side of (7) are a polynomial in z the coefficients of which may depend explicitly on y. □

The solutions for a and b determine the decomposition and finally the solutions of the originally given second-order equation. Subsequently these results will be illustrated by several examples. They are compared with the possible outcome of Lie's symmetry analysis whenever there is any symmetry.

Example 1 The equation

$$(y - x)y'' + yy' + xy - x = 0 \tag{9}$$

with $K = 1$ and coefficients $c_1 = \frac{y}{y - x}$, $c_0 = \frac{x(y - 1)}{y - x}$ has the determining system

$$a' - a^2 + a = 0,\ (a' - a^2 + 1)x - b' + ab - b = 0, \quad \text{and} \quad b' - ab - 1 = 0.$$

Its general solution is

$$a = \frac{C}{e^x + C}, \quad b = \frac{xe^x}{e^x + C} - 1;$$

C is an undetermined constant. It leads to the decomposition

$$\left(z' - \frac{ye^x + Cx}{(x-y)(e^x + C)} z\right)\left(z \equiv y' + \frac{C}{e^x + C} y + \frac{xe^x}{e^x + C} - 1\right) = 0.$$

Integrating the first-order component the general solution

$$y = \left(1 + C_1 e^{-x}\right)\left(C_1^2 \int \frac{dx}{(C_1 + e^x)^2} - \log\left(C_1 + e^x\right) - \frac{1}{2}x^2 + C_2\right) + 2x$$

of (9) follows; this equation does not have any Lie symmetry. □

Example 2 For the equation

$$y'' - \left(1 + \frac{2}{y}\right) y'^2 + \frac{1}{x}(3y + 4)y' - \frac{2}{x^2} y(y+1) = 0 \tag{10}$$

$K = 2$, its coefficients are $c_2 = -\left(1 + \frac{2}{y}\right)$, $c_1 = \frac{1}{x}(3y+4)$ and $c_0 = -\frac{2}{x^2} y(y+1)$, the determining system is

$$a' + a^2 + \frac{4}{x}a + \frac{2}{x^2} = 0, \quad a^2 + \frac{3}{x}a + \frac{2}{x^2} = 0 \quad \text{and} \quad b = 0.$$

The two solutions $a = -\frac{1}{x}$ and $a = -\frac{2}{x}$ yield the decompositions

$$\left(D - \left(1 + \tfrac{2}{y}\right)y' + \tfrac{1}{x}(2y+3)\right)\left(y' - \tfrac{1}{x}y\right),$$

$$\left(D - \left(1 + \tfrac{2}{y}\right)y' + \tfrac{1}{x}(y+2)\right)\left(y' - \tfrac{2}{x}y\right).$$

They lead to the special solutions $y_1 = C_1 x$ and $y_2 = C_2 x^2$ of (10), its extension to the general solution is not provided by them. The single Lie symmetry generator $x\partial_x$ leads to a complicated transformation to canonical form from which the general solution cannot be obtained. □

Example 3 The equation

$$y'' - \frac{1}{x^2} y'^2 - 2xy' + \frac{4}{x} yy' - \frac{1}{x} y' - 4y^2 = 0 \tag{11}$$

with $K = 2$ has the coefficients $c_2 = \frac{1}{x^2}$, $c_1 = -2x + \frac{4}{x}y - \frac{1}{x}$ and $c_0 = -4y^2$. They lead to the system

$$a' - a^2 - \left(2x + \frac{1}{x}\right) a = 0, \quad (a + 2x)^2 = 0 \quad \text{and}$$
$$b' + \frac{1}{x^2} b^2 - \left(2x + \frac{1}{x}\right) b - ab = 0$$

for a and b. It yields the inhomogeneous component $y' - 2xy - \dfrac{x}{log(x + C)} = 0$ from which the general solution

$$y = \exp{(x^2)} \left(\int \frac{x \exp(-x^2) dx}{\log{(x)} + C_1} + C_2 \right)$$

of (11) follows. This equation has a one-parameter group of Lie symmetries with generator $\exp{(x^2)} \partial_y$ that yields the same answer. However, the solution by decomposition is much more efficient. □

Example 4 The equation

$$yy'' - y'^2 + \frac{2}{3} y' - \frac{1}{x} y^2 = 0 \tag{12}$$

with $K = 2$ has the coefficients $c_2 = -\frac{1}{y}$, $c_1 = \frac{2}{3y}$ and $c_0 = -\frac{y}{x}$; they generate the system

$$a' + \frac{1}{x} = 0, \quad b' + ab + \frac{2}{3} a = 0, \quad b^2 + \frac{2}{3} b = 0$$

with solution $a = -\log{(x)} + C$, $b = -\frac{2}{3}$ from which the decomposition

$$\left(z' - \frac{1}{y} z^2 - \frac{1}{y} \left(\log{(x)} y - Cy + \frac{2}{2} \right) z \right) \left(z \equiv y' - (\log{(x)} - C) y - \frac{2}{3} \right) = 0$$

is obtained. Integration of the first-order component leads to the general solution

$$y = \exp\left(\frac{1}{3} x^3 + C_1 x \right) \left(C_2 + 2 \int \exp\left(-C_1 x - \frac{1}{3} x^3 \right) dx \right)$$

of (12). This equation does not have any Lie symmetry. □

2.2 Riccati components

A component containing a quadratic nonlinearity is considered next. Like the linear component considered in the preceding section any such component guarantees the existence of a special solution. If the Riccati component contains a constant, the general solution of the corresponding second-order equation may be obtained from it.

Proposition 3 *Let a second-order quasilinear equation (3) be given. In order that it has a first-order Riccati component* $z \equiv y' + a(x)y^2 + b(x)y$ *the coefficients* $a(x)$ *and* $b(x)$ *have to be solutions of*

$$(a' - 2a^2 y)y^2 - 3aby^2 + (b' - b^2)y - \sum_{k=0}^{K}(-1)^k c_k (ay^2 + by)^k = 0. \tag{13}$$

Then (3) may be written as follows:

$$\left(z' - (2ay + b)z + \sum_{k=1}^{K} c_k\big((z - ay^2 - by)^k + (-1)^{k+1}(ay^2 + by)^k\big)\right) \big(z \equiv y' + ay^2 + by\big) = 0. \tag{14}$$

The coefficients a and b satisfy a first-order algebro-differential system; its degree in a and b is bounded by K except for K = 1 where the degree in a is 2. For low values of K Eqs. (13) and (14) are explicitly given as follows.

$$
\begin{aligned}
&K = 1 : \begin{cases} (a' - 2a^2 y)y^2 - 3aby^2 + (b' - b^2)y + c_1(ay^2 + by) - c_0 = 0, \\ \big(z' - (2ay + b)z + c_1 z\big)(z \equiv y' + ay^2 + by) = 0; \end{cases} \\
&K = 2 : \begin{cases} (a' - 2a^2 y)y^2 - 3aby^2 + (b' - b^2)y - c_2(ay^2 + by)^2 + c_1(ay^2 + by) \\ -c_0 = 0, \\ \big(z' - (2ay + b)z + c_2 z^2 - 2c_2(ay^2 + by)z + c_1 z\big)(z \equiv y' + ay^2 + by) \\ = 0; \end{cases} \\
&K = 3 : \begin{cases} (a' - 2a^2 y)y^2 - 3aby^2 + (b' - b^2)y + c_3(ay^2 + by)^3 - c_2(ay^2 + by)^2 \\ +c_1(ay^2 + by) - c_0 = 0, \\ \big(z' - (2ay + b)z + c_3 z^3 - 3c_3(ay^2 + by)z^2 + c_2 z^2 + 3c_3(ay^2 + by)^2 z \\ -2c_2(ay^2 + by)z + c_1 z\big) \\ (z \equiv y' + ay^2 + by) = 0. \end{cases}
\end{aligned}
\tag{15}
$$

The proof is similar as for Proposition 2 and is therefore omitted. The subsequent examples apply the preceding proposition.

Example 5 For the equation

$$y'' + 2yy' + \frac{2}{x}y' + \frac{2}{x}y^2 = 0 \tag{16}$$

$K = 1$, $c_1 = 2y + \frac{2}{x}$ and $c_0 = \frac{2}{x}y^2$. According to Proposition 2 there exists the decomposition $\left(z' + \frac{1}{x}z + 2yz\right)\left(z = y' + \frac{1}{x}y\right) = 0$ from which only the special solution $y = \frac{C}{x}$ follows. Proposition 3 yields the system

$$a^2 - a = 0, \quad a' + \frac{2}{x}a - \frac{2}{x} - (3a - 2)b = 0, \quad b' - b^2 + \frac{2}{x}b = 0.$$

The alternative $a = 0, b = \frac{1}{x}$ reproduces the linear component given above; the other alternative $a = 1, b = 0$ leads to the decomposition $(z' + \frac{2}{x}z)(z \equiv y' + y^2) = 0$; the equation $z' + \frac{2}{x}z = 0$ has the general solution $z = \frac{C_1}{x^2}$. It leads to the equation $y' + y^2 = \frac{C_1}{x^2}$ with the solution

$$y = \frac{1}{x}\left(\frac{1}{2} - C_1 \tan(C_1 \log x + C_2)\right)$$

where the constant C_1 has been redefined; this is the general solution of (16). It has a two-parameter group of Lie symmetries with generators

$$U_1 = x\partial_x - y\partial_y, \quad U_2 = x \log(x)\partial_x - \left(y \log(x) + y - \frac{1}{2x}\right)\partial_y$$

which is difficult to apply for solving it. □

Example 6 Consider the equation

$$yy'' - y'^2 - (x^2 + 1)y^2 y' - 2xy^3 = 0 \tag{17}$$

with $K = 2$. Its coefficients are $c_2 = -\frac{1}{y}$, $c_1 = -(x^2 + 1)y$ and $c_0 = -2xy^2$. By Proposition 2 a linear first-order component does not exist. According to Proposition 3 the system

$$a(a + x^2 + 1) = 0, \quad a' - (a + x^2 + 1)b + 2x = 0 \quad \text{and} \quad b' = 0$$

for a and b is obtained. Its general solution is $a = -(x^2 + 1)$ and $b = C$ where C is a constant. It yields the decomposition

$$\left(z' - \frac{1}{y}z^2 - (y + x^2 y - C)z\right)\left(y' - (x^2 + 1)y^2 + Cy = 0\right) = 0.$$

Integrating the first-order component the general solution

$$y = \frac{C_1^3}{(C_1 x + 1)^2 + C_1^2 + 1 + C_2 \exp(C_1 x)}$$

of Eq. (17) follows. It may be shown that it does not have any Lie symmetry. To this end, a Janet basis of the determining system for its symmetries has to be computed that is rather complex. □

2.3 Bernoulli components

Similar as a Riccati component in the preceding subsection, a Bernoulli component containing a nonlinearity y^n, and in addition a linear term proportional to y guarantees the existence of a special solution; if the Bernoulli component contains a constant the general solution follows. The main result is given next.

Proposition 4 *Let a second-order quasilinear equation (3) be given. In order that it has a first-order Bernoulli component $z \equiv y' + a(x)y^n + b(x)y$, $n > 2$, the coefficients a and b have to satisfy*

$$(a' - na^2y^{n-1})y^n - (n+1)aby^n + (b' - b^2)y - \sum_{k=0}^{K}(-1)^k c_k (ay^n + by)^k = 0. \quad (18)$$

Then (3) may be written as follows.

$$\left(z' - (nay^{n-1} + b)z + \sum_{k=1}^{K}\Big((z - ay^n - by)^k + (-1)^{k+1}(ay^n + by)^k\Big)\right)$$
$$\big(z \equiv y' + ay^n + by\big) = 0. \quad (19)$$

The coefficients a and b may be obtained from a first-order algebro-differential system; its degree in a and b is bounded by K except for $K = 1$ where the degree in a is 2. For low values of K (18) and (19) are explicitly given as follows.

$$K = 1 : \begin{cases} (a' - na^2y^{n-1})y^n - (n+1)aby^n + (b' - b^2)y + c_1(ay^n + by) - c_0 = 0, \\ \big(z' - (nay^{n-1} + b)z + c_1 z\big)(z \equiv y' + ay^n + by) = 0; \end{cases}$$
$$K = 2 : \begin{cases} (a' - na^2y^{n-1})y^n - (n+1)aby^n + (b' - b^2)y - c_2(ay^n + by)^2 \\ +c_1(ay^n + by) - c_0 = 0, \\ \big(z' - (nay^{n-1} + b)z + c_2z^2 - 2c_2(ay^n + by)z + c_1 z\big) \\ (z \equiv y' + ay^n + by) = 0; \end{cases}$$
$$K = 3 : \begin{cases} (a' - na^2y^{n-1})y^n - (n+1)aby^n + (b' - b^2)y + c_3(ay^n + by)^3 \\ -c_2(ay^n + by)^2 + c_1(ay^n + by) - c_0 = 0, \\ \big(z' - (nay^{n-1} + b)z + c_3z^3 - 3c_3(ay^n + by)z^2 + c_2z^2 \\ +3c_3(ay^n + by)^2 z \\ -2c_2(ay^n + by)z + c_1 z\big)(z \equiv y' + ay^n + by) = 0. \end{cases} \quad (20)$$

The proof is similar as for Proposition 2 and is therefore omitted. The equation of the subsequent example does not have any Lie symmetry, i.e. decomposition is the only means for solving it.

Example 7 The equation

$$yy'' - y'^2 + 2y^3y' + xy^2 = 0 \quad (21)$$

with $K = 2$ has coefficients $c_2 = -\frac{1}{y}$, $c_1 = 2y^2$ and $c_0 = xy$. A linear or Riccati component does not exist. The lowest Bernoulli component of interest corresponds to $n = 3$. For these values Proposition 4 yields the system

$$a^2 - a = 0, \quad a' + 2ab - 2b = 0, \quad b' - x = 0.$$

Its solution $a = 1$, $b = \frac{1}{2}x^2 + C$ leads to the decomposition

$$\left(z' - \frac{1}{y}z^2 + \left(y^2 + \frac{1}{2}x^2 + C\right)z\right)\left(z \equiv y' + y^3 + \left(\frac{1}{2}x^2 + C\right)y\right) = 0.$$

Integrating the right component the general solution

$$y = \frac{1}{\sqrt{(2)}\exp\left(C_1 x + \frac{1}{6}x^3\right)\left(\int \exp\left(-2C_1 x - \frac{1}{3}x^3\right)dx + \frac{1}{2}C_2\right)^{1/2}}$$

of equation (21) is obtained. It does not have any Lie symmetry. □

2.4 Abel components

Abel's equations of the first and second kind were amongst the first systematically investigated ode's. Therefore they are included in this chapter although they are only in exceptional cases integrable in closed form; a rather detailed discussion may be found in [18], pp. 179–185. At first Abel's equations of the first kind are considered.

Proposition 5 *Let a second-order quasilinear equation* (3) *be given. In order that it has an Abel component of first kind* $z \equiv y' + a(x)y^3 + b(x)y^2 + c(x)y + d(x)$ *the coefficients* a, b, c *and* d *have to be solutions of*

$$\begin{aligned}&(a' - 3a^2y^2)y^3 + (b' - 2b^2y)y^2 + (c' - c^2)y + d'\\&\quad -(5by^2 + 4cy + 3d)ay^2 - (3by^2 + d)c\\&\quad -\sum_{k=0}^{K}(-1)^k c_k(ay^3 + by^2 + cy + d)^k = 0.\end{aligned} \tag{22}$$

Then (3) *may be written as follows:*

$$\begin{aligned}&\Big(z' - (3ay^2 + 2by + c)z + \sum_{k=1}^{K} c_k\big((z - ay^3 - by^2 - cy - d)^k\\&\quad +(-1)^{k+1}(ay^3 + by^2 + cy + d)^k\big)\Big)\\&\quad \big(z \equiv y' + ay^3 + by^2 + cy + d\big) = 0.\end{aligned} \tag{23}$$

The coefficients a, b, c *and* d *may be obtained from a first-order algebro-differential system; its degree in these coefficients is bounded by* K *except for* $K = 1$ *where the degree in* a, b *and* c *is bounded by 2. For low values of* K *Eqs.* (22) *and* (23) *are explicitly given as follows.*

$$K=1:\begin{cases}(a'-3a^2y^2)y^3+(b'-2b^2y)y^2+(c'-c^2)y+d'-(5by^2+4cy+3d)ay^2\\ -(3by^2+d)c+c_1(ay^3+by^2+cy+d)-c_0=0,\\ \Big(z'-(3ay^2+2by+c)z+c_1z\Big)\big(z\equiv y'+ay^3+by^2+cy+d\big)=0;\end{cases}$$

$$K=2:\begin{cases}(a'-3a^2y^2)y^3+(b'-2b^2y)y^2+(c'-c^2)y+d'-(5by^2+4cy+3d)ay^2\\ -(3by^2+d)c-c_2\big(a^2y^6+2aby^5+(2ac+b^2)y^4+2(ad+bc)y^3\\ +(2bd+c^2)y^2+2xdy+d^2\big)c_1(ay^3+by^2+cy+d)-c_0=0,\\ \Big(z'-(3ay^2+2by+c)z^2+c_2z^2-2c_2(ay^2+by^2+cy+d)z\Big)\\ \big(z\equiv y'+ay^3+by^2+cy+d\big)=0;\end{cases} \qquad (24)$$

The proof is similar as for Proposition 2 and is therefore again omitted. Due to the larger number of unknown coefficients in the Abel component the system for its determination usually comprises more equations as the following example shows.

Example 8 The equation

$$y''+\frac{3}{x^5}y^2y'-\frac{1}{x}y'-\frac{6}{x^6}y^3=0 \qquad (25)$$

with $K=1$ has coefficients $c_1=\frac{3}{x^5}y^2-\frac{1}{x}$ and $c_0=-\frac{6}{x^6}y^3$. The preceding proposition leads to the following system for a, b, c and d.

$$a\big(a-\frac{1}{x^5}\big)=0,\quad b\big(5a-\frac{3}{x^5}\big)=0,$$
$$a'+\frac{3}{x^5}c-\frac{1}{x}a+\frac{6}{x^6}-2b^2-4ac=0,\quad b'+3ad+3bc-\frac{3}{x^5}d+\frac{1}{x}b=0,$$
$$c'-c^2-\frac{1}{x}c=0,\quad d'-cd-\frac{1}{x}d=0.$$

Its solution is $a=\frac{1}{x^5}$, $b=c=0$ and $d=Cx$ where C is an undetermined constant. It leads to the decomposition

$$\left(z'-\frac{1}{x}z\right)\left(z\equiv y'+\frac{1}{x^5}y^3+Cx\right)=0$$

of Eq. (25). Integration of the right component finally yields its general solution in the form $y=\frac{1}{\sqrt{C_1}}x^2u$ where u is determined by

$$\int\frac{du}{u^3-2C_1u+1}=\log x+C_2.$$

The one-parameter group of Lie symmetries with generator $x\partial_x + 2y\partial_y$ leads to a similar answer, albeit much less efficiently. □

Abel equations of the second kind are considered next. It is assumed that there is a quadratic term in y. Although it may always be removed by a variable change, this is in general only possible by an extension of the base field that is inconvenient for the further proceeding.

Proposition 6 *Let a second-order quasilinear equation* (3) *be given. In order that it has an Abel component of second kind* $z \equiv yy' + a(x)y^2 + b(x)y + c(x)$ *the coefficients* $a(x)$, $b(x)$ *and* $c(x)$ *have to be solutions of*

$$(a' - a^2)y^4 + (b' - ab)y^3 + c'y^2 + bcy + c^2 - \sum_{k=0}^{K}(-1)^k c_k y^{3-k}(ay^2 + by + c)^k = 0. \tag{26}$$

Then (3) *may be written as follows:*

$$\begin{aligned}\Bigg(z' - \left(a - \frac{c}{y^2}\right)z + \sum_{k=1}^{K} c_k \frac{1}{y^k}\Big(&(yz - ay^2 - by - c)^k \\ &+ (-1)^{k+1}(ay^2 + by + c)^k\Big)\Bigg)\Big(z \equiv y' + ay + b + \frac{c}{y}\Big).\end{aligned} \tag{27}$$

The coefficients a, b and c may be obtained from a first-order algebro-differential system; its degree is bounded by K except for $K = 1$ *where the degree in a and c is 2. For low values of K Eqs.* (26) *and* (27) *are explicitly given as follows.*

$$K = 1: \begin{cases} (a' - a^2)y^4 + (b' - ab)y^3 + c'y^2 + bcy + c^2 + c_1y^2(ay^2 + by + c) - c_0y^3 = 0, \\ \Big(z' - \big(a - \frac{c}{y^2}\big)z + c_1z\Big)\big(z \equiv y' + ay + b + \frac{c}{y}\big) = 0; \end{cases}$$

$$K = 2: \begin{cases} (a' - a^2)y^4 + (b' - ab)y^3 + c'y^2 + bcy + c^2 \\ -c_2y(ay^2 + by + c)^2 + c_1y^2(ay^2 + by + c) - c_0y^3 = 0, \\ \Big(z' - \big(a - \frac{c}{y^2}\big)z + c_1z + c_2z^2 - 2c_2\big(ay + b + \frac{c}{y}\big)z\Big)\big(z \equiv y' + ay + b + \frac{c}{y}\big) = 0; \end{cases} \tag{28}$$

Proof The computations are performed with $y' + ay + b + \frac{c}{y}$. At first $r = ay + by + \frac{c}{y}$ is substituted into (4) and (5), then the substitution $y' = \frac{z}{y} - ay - b - \frac{c}{y}$ is performed. As a result the sums at the left-hand side of (26) are a polynomial in z, its coefficients may depend explicitly on y. □

Example 9 The equation

$$yy'' + y'^2 + \frac{2}{x}yy' + 1 = 0 \tag{29}$$

with $K = 2$ has coefficients $c_2 = 1$, $c_1 = \dfrac{2y}{x}$ and $c_0 = 1$. The preceding proposition leads to the following system for a, b and c.

$$a' - 2a^2 + \tfrac{2}{x}a = 0, \quad b' - 3ab + \tfrac{2}{x}b = 0,$$
$$c' + \tfrac{2}{x}c - 2ac - b^2 - 1 = 0, \quad bc = 0.$$

The alternative $c = 0$ from the last equation leads to $b = \pm i$ from the third equation, then the second equation yields $a = \frac{2}{3x}$ which does not satisfy the first equation; therefore this alternative is excluded. The alternative $b = 0$ leaves the two equations

$$a' - 2a^2 + \frac{2}{x}a = 0, \quad c' + \frac{2}{x}c - 2ac - 1 = 0.$$

Two special solutions are $a = \frac{1}{2x}$, $c = \frac{1}{2}x + \frac{C}{x}$ and $a = 0$, $c = \frac{1}{3}x + \frac{C}{x^2}$; they lead to the decompositions

$$\left(z' + \frac{1}{x}z\right)\left(z \equiv yy' + \frac{1}{2x}y^2 + \frac{1}{2}x + \frac{C}{x}\right) = 0 \quad \text{and}$$
$$\left(z' + \frac{2}{x}z\right)\left(z \equiv yy' + \frac{1}{3}x + \frac{C}{x^2}\right) = 0$$

respectively. Integrating either one of them the general solution

$$y = \left(\frac{C_1 + C_2 x - \frac{1}{3}x^3}{x}\right)^{1/2}$$

follows. The eight-parameter group of Lie symmetries leads to the same answer, albeit much less efficiently. □

2.5 Separable components

This is the first decomposition type where an unspecified dependence on y occurs in the coefficients of the first-order component. As a consequence, the determining system comprises ode's w.r.t. to both x and y. Details are given in the following proposition.

Proposition 7 *Let a second-order quasilinear equation* (3) *be given. A first-order component $y' + s(x)r(y)$ exists if $s(x)$ and $r(y)$ satisfiy*

$$s^2 r\frac{dr}{dy} - \frac{ds}{dx}r + \sum_{k=0}^{k=K}(-1)^k c_k s^k r^k = 0. \tag{30}$$

Then (3) *may be written as follows.*

$$\left(z' - s\frac{dr}{dy}z + \sum_{k=0}^{K} c_k\left((z - sr)^k + (-1)^{k+1}(sr)^k\right)\right)(z \equiv y' + sr) = 0. \tag{31}$$

For low values of K this means:

$$K=1:\left\{s^2r\frac{dr}{dy}-s'r-c_1sr+c_0=0,\quad \big(z'-s\frac{dr}{dy}z+c_1z\big)(z\equiv y'+sr)=0;\right.$$

$$K=2:\begin{cases} s^2r\dfrac{dr}{dy}-s'r+c_2s^2r^2-c_1sr+c_0=0,\\ \Big(z'-s\dfrac{dr}{dy}z+c_2z^2-(2c_2sr-c_1)z\Big)(z\equiv y'+sr)=0;\end{cases}$$

$$K=3:\begin{cases} s^2r\dfrac{dr}{dy}-s'r-c_3s^3r^3+c_2s^2r^2-c_1sr+c_0=0,\\ \Big(z'-s\dfrac{dr}{dy}z+c_3z^3-(3c_3sr-c_2)z^2+(3c_3s^2r^2-2c_2sr+c_1)z\Big)\\ (z\equiv y'+sr)=0.\end{cases}\tag{32}$$

Proof Reduction of y'' yields the first two terms of (30), and the term $z'-sr\frac{dr}{dy}z$ to the left component in (31). The remaining terms follow from algebraic reduction of y'^k w.r.t. $y'+sr$. □

Due to the explicit dependence of the coefficient $r(y)$ on y the constraint (30) may not be separated w.r.t. powers of y, i.e. there is a single equation involving $s(x)$ and $r(y)$. There does not seem to exist a solution procedure for general equations of this kind. For the special case $K=2$, $c_2=f(y)$, $c_1=g(x)$ and $c_0=0$, i.e. $y''+f(y)y'^2+g(x)y'=0$ however the constraint for r and s may be rewritten in the form $\frac{dr}{dy}+f(y)r=\frac{1}{s^2}\left(\frac{ds}{dx}+g(x)s\right)$, reducing the problem to solving two linear first-order equations. The subsequent examples apply this scheme.

Example 10 The equation

$$y''-\frac{y}{y^2-1}y'^2+\frac{x}{x^2-1}y'=0\tag{33}$$

with $K=2$ has coefficients $c_2=-\frac{y}{y^2-1}$, $c_1=\frac{x}{x^2-1}$ and $c_0=0$. According to Proposition 7 the equation

$$s^2r\frac{dr}{dy}-\frac{ds}{dx}r-\frac{y}{y^2-1}s^2r^2-\frac{x}{x^2-1}sr=0$$

follows. It may be written as

$$\frac{dr}{dy}-\frac{y}{y^2-1}r=\frac{1}{s^2}\left(\frac{ds}{dx}+\frac{x}{x^2-1}s\right).$$

Its solution $r = C\sqrt{y^2-1}$, $s = \dfrac{1}{\sqrt{x^2-1}}$ leads to the decomposition

$$\left(z'\right)\left(z \equiv y' + C\frac{\sqrt{y^2-1}}{\sqrt{x^2-1}}\right) = 0.$$

The right component yields the general solution

$$y = \frac{C_1^2\left(\sqrt{x^2-1}+x\right)^{2C_2}+1}{2C_1\left(\sqrt{x^2-1}+x)\right)^{C_2}}$$

of Eq. (33). Although it has an eight-parameter group of Lie symmetries, it does not allow its integration due the rather complicated generators of the symmetry algebra. □

Example 11 The equation

$$y'' + yy'^2 + xy' = 0 \tag{34}$$

with $K = 2$ has coefficients $c_2 = y$, $c_1 = x$ and $c_0 = 0$. According to Proposition 7 the equation

$$s^2 r\frac{dr}{dy} - \frac{ds}{dx} + ys^2 r - xsr = 0$$

follows. It may be written as

$$\frac{dr}{dy} + yr = \frac{1}{s^2}\left(\frac{ds}{dx} + xs\right) = 0.$$

Its solution $r = C\exp\left(\frac{1}{2}y^2\right)$, $s = \exp\left(\frac{1}{2}x^2\right)$ leads to the decomposition

$$\left(z'\right)\left(z \equiv y' + C\exp\frac{1}{2}(x^2+y^2)\right).$$

The right component yields the general solution

$$\mathrm{erf}\left(\frac{y}{\sqrt{2}}\right) + iC_1\mathrm{erf}\left(\frac{ix}{\sqrt{2}}\right) = C_2$$

of Eq. (34). The same remarks apply to the use of its eight-parameter group of Lie symmetries as in the preceding example. □

If an equation does not contain x explicitly, its order may be reduced by exchange of x and y. This ad hoc method may be replaced by a decomposition of the type as described in the preceding proposition with $s(x) = 1$. The resulting constraints for $r(y)$ are much more manageable as the following corollary shows.

Corollary 1 *Let a second-order quasilinear equation* (3) *be given. A first-order component* $y' + r(y)$ *exists if* $r(y)$ *satisfies*

$$r\frac{dr}{dy} + \sum_{k=0}^{K} (-1)^k c_k r^k = 0. \tag{35}$$

Then (3) *may be written as follows.*

$$\left(z' - \frac{dr}{dy} z + \sum_{k=0}^{K} c_k \left((z-r)^k + (-1)^{k+1} r^k \right) \right) (z \equiv y' + r) = 0. \tag{36}$$

For low values of K this means:

$$\begin{aligned}
K = 0 &: \left\{ r\frac{dr}{dy} + c_0 = 0, \quad \left(z' - \frac{dr}{dy}z\right)(z \equiv y' + r) = 0; \right. \\
K = 1 &: \left\{ r\frac{dr}{dy} - c_1 r + c_0 = 0, \quad \left(z' - \frac{dr}{dy}z + c_1 z\right)(z \equiv y' + r) = 0; \right. \\
K = 2 &: \left\{ \begin{array}{l} r\frac{dr}{dy} + c_2 r^2 - c_1 r + c_0 = 0, \\ \left(z' - \frac{dr}{dy}z + c_2 z^2 - (2c_2 r - c_1)z\right)(z \equiv y' + r) = 0; \end{array} \right. \\
K = 3 &: \left\{ \begin{array}{l} r\frac{dr}{dy} - c_3 r^3 + c_2 r^2 - c_1 r + c_0 = 0, \\ \left(z' - \frac{dr}{dy}z + c_3 z^3 - (3c_3 r - c_2)z^2 + (3c_3 r^2 - 2c_2 r + c_1)z\right)(z \equiv y' + r) = 0. \end{array} \right.
\end{aligned} \tag{37}$$

Proof Reduction of y'' yields the contribution $r\frac{dr}{dy}$ to the constraint for r, and the term $z' - \frac{dr}{dy}z$ to the left component. The remaining terms follow from algebraic reduction of y'^k w.r.t. $y' + r$. □

Example 12 The equation

$$yy'' - \frac{5}{4}y'^2 + \frac{2}{3}y^3 = 0 \tag{38}$$

with $K = 2$ has coefficients $c_2 = -\frac{5}{4y}$, $c_1 = 0$ and $c_0 = \frac{2}{3}y^2$. According to the preceding corollary the equation $r\frac{dr}{dy} - \frac{5}{4y}r^2 + \frac{2}{3}y^2 = 0$ for r follows with general solution $r = y\sqrt{C\sqrt{y} - \frac{8}{3}y}$; it leads to the decomposition

$$\left(z' - \frac{5}{4y}z^2 - \frac{\frac{5}{4}C\sqrt{y} - \frac{8}{3}y}{\sqrt{C\sqrt{y} - \frac{8}{3}y}} z \right) \left(z \equiv y' + y\sqrt{C\sqrt{y} - \frac{8}{3}y} \right) = 0.$$

Integration of the right component yields the general solution of (38) in the form

$$y = \frac{36C_1^2}{(6C_1^2C_2^2 + 12C_1^2C_2x + 6C_1^2x^2 + 1)^2}.$$

The three-parameter Lie symmetry group with generators ∂_x, $x^2\partial_x - 4xy\partial_y$ and $x\partial_x - 2y\partial_y$ yields the same answer. □

2.6 Homogeneous components

In Sect. 4.6 of Kamke [10] various types of first-order equations are considered that may be transformed into an homogeneous equation; here only case 4.6(c) is considered.

Proposition 8 *Let a second-order quasilinear equation* (3) *be given. In order for a first-order component*

$$y' + \frac{ax + by + c}{\alpha x + \beta y + \gamma} = 0 \tag{39}$$

to exist the constant coefficients a, b, c, α, β *and* γ *have to satisfy an algebraic system. For* $K = 2$ *or* $K = 3$ *it comprises at least 10 equations; for* $K > 3$ *the number of equations is greater or equal than* $\frac{1}{2}(K+1)(K+2)$.

Proof Reduction of (3) w.r.t. (39) and the second-order expression for y'' following from it leads to a polynomial condition in x, y and the variables a, b, c, α, β and γ. The vanishing of this polynomial requires that the coefficients of the monomials in x and y vanish, this yields the algebraic system. The number of different monomials is minimal if the coefficients c_k in (3) are constant. For $K \geq 4$ they are determined by the highest power K that occurs in a trinomial expansion. For $K = 2$ and $K = 3$ the third power in the denominator of the derivative of (39) yields the highest power three. □

This result shows that the number of equations is considerably larger than the number of unknowns which is 6. As a consequence, the existence of components of this type is very rare. Because linear components have been considered before, the constraint $\beta \neq 0$ is imposed; this leads in many cases to a fast exclusion of a component (39). Furthermore, the equations originating from high monomials $y^m x^n$ are often reducible; this property makes a Gröbner basis algorithm that applies factorization very efficient. The subsequent example is typical for this behavior.

Example 13 The equation

$$yy'' + y'^2 - \frac{1}{x}yy' = 0 \tag{40}$$

with $K=2$ has coefficients $c_2=\frac{1}{y}, c_1=-\frac{1}{x}$ and $c_0=0$. According to the preceding proposition, the following system for the undetermined coefficients is obtained.

$$
\begin{gathered}
b\beta=0, \quad (2\alpha+b)b\beta=0, \quad (2b\gamma+\beta c)\beta=0,\\
a\alpha\beta+ab\beta+2\alpha^2b+2\alpha b^2=0, \quad 2\alpha b\gamma+3\alpha\beta c+2b^2\gamma+b\beta c=0,\\
(b\gamma+2\beta c)\gamma=0, \quad (\alpha+3b)\alpha b=0, \quad a\alpha\gamma+3ab\gamma+2\alpha^2c+3\alpha bc=0,\\
(\alpha\gamma+b)c=0, \quad c\gamma=0, \quad a\alpha=0, \quad a\gamma+2\alpha c=0, \quad 2a\gamma+\alpha c=0, c\gamma=0.
\end{gathered}
$$

Simplification due to the constraint $\beta\neq 0$ and the requirement that the trivial component $y'=0$ is excluded leads in a few steps to $b=c=\alpha=\gamma=0$ leaving the quotient $\frac{ax}{\beta y}$. Introducing the new constant $C\equiv\frac{a}{\beta}$ yields finally

$$
\left(z'+\frac{1}{y}z^2-\left(C\frac{x}{y}+\frac{1}{x}\right)z\right)\left(y'+C\frac{x}{y}\right)=0.
$$

Integrating the right component, the general solution $y=C_1(x^2+C_2)^{1/2}$ of (40) is obtained. In addition (40) allows also a decomposition with a linear first-order component.

$$
\left(z'+\frac{1}{y}z^2-\frac{C-2x^2}{Cx(x^2+1)}z\right)\left(z\equiv y'-\frac{x}{x^2+C}y\right)=0.
$$

The solution of (40) may also be obtained from its 8-parameter group of Lie symmetries. □

3 Decomposition of equivalence classes

The symmetry type of an ode is invariant under point transformations. As a consequence, the full equivalence class of an equation has the same symmetry type. By contrast, the decomposition type of an ode is not invariant under general point transformations because in general it changes the type of the right component. For example, if Eq. (17) of Example 6 is transformed by $x\to xy$ and $y\to\frac{1}{y}$ it has the form

$$
x^2yy''-x^2y'^2-x^3(x^2+2x+2)y^2y'-x^2(3x^2+4x+1)y^3-y^2=0
$$

that does no longer have a Riccati component. However, the equation transformed by $x\to x+1$ and $y\to xy$ does have a Riccati component as shown in Example 15 below. The reason is that this latter transformetion belongs to the structure invariance group of the Riccati component. This means, the solution procedure that applies to the originally given equation applies to the full equivalence class of the respective structure invariance group.

In the subsequent proposition the structure invariant groups of the right components considered in the preceding section are determined.

Proposition 9 *The first-order right components given in the preceding section have the following structure invariance groups. F, G and H are undetermined functions of its argument; u and $v(u)$ are the transformed independent and dependent variables respectively.*

- *The linear component $y' + a(x)y + b(x) = 0$ and the general Riccati component $y' + a(x)y^2 + b(x)y + c(x) = 0$ are invariant w.r.t. $x = F(u)$, $y = G(y)v + H(u)$.*
- *The Bernoulli component $y' + a(x)y^n + b(x)y = 0$ and for $n = 2$ the special Riccati component are invariant w.r.t. $x = F(u)$, $y = G(u)v$.*
- *The separable component $y' + r(y) = 0$ is invariant w.r.t. $x = Cu + F(v)$, $y = G(v)$; C is an undetermined constant.*
- *The separable component $y' + s(x)r(y) = 0$ is invariant w.r.t. $x = F(u)$ and $y = G(v)$.*
- *The homogeneous component $y' + \dfrac{ax + by + c}{\alpha x + \beta y + \gamma} = 0$ is invariant w.r.t. $x = \bar{a}u + \bar{b}v + \bar{c}$ and $y = \bar{\alpha}u + \bar{\beta}v + \bar{\gamma}$; the barred variables are undetermined constants.*
- *The homogeneous component $y' + \dfrac{a\left(\frac{y}{x}\right)^2 + b\frac{y}{x} + c}{\alpha\left(\frac{y}{x}\right)^2 + \beta\frac{y}{x} + \gamma} = 0$ is invariant w.r.t. $x = \bar{a}u$ and $y = \bar{\alpha}u$; $\bar{a}$ and $\bar{\alpha}$ are undetermined constants.*

Proof The proof follows closely [18], Chapters 4.1 and 4.2. Let u, $v(u)$ be the new independent and dependent variables respectively, and

$$x = \varphi(u, v), \quad y = \psi(u, v) \quad \text{and} \quad y' = \frac{\psi_u + \psi_v v'}{\varphi_u + \varphi_v v'}. \tag{41}$$

Substitution into the linear component $y' + a(x)y + b(x)$ yields

$$v' + \frac{\psi_u + \big(a(\varphi)\psi + b(\varphi)\big)\varphi_u}{\psi_v + \big(a(\varphi)\psi + b(\varphi)\big)\varphi_v} = 0.$$

In order to avoid the occurence of an unspecified dependence on v via the coefficients a and b, $\varphi_v = 0$ is required, i.e. $\varphi \equiv F(u)$. The denominator must be independent of v, this requires $\psi_v = G(u)$, consequently $\psi = G(u)v + H(u)$. Under these constraints the above expression simplifies to the linear component

$$v' + \left(\frac{G'}{G} + a(F)F'\right)v + \frac{H' + (a(F)H + b(F))\,F'}{G} = 0,$$

i.e. the structure invariance group is $x = F(u)$, $y = G(u)v + H(u)$. The proof for the Riccati component is similar and is therefore omitted.

Performing the same steps for the Bernoulli component $y' + a(x)y^n + b(x)y = 0$ for $n \geq 2$ leads to an expression containing the terms $a(Gv + H)^n$ and $b(Gv + H)$; in order to avoid powers of v different from n and 1, $H = 0$ is required. The resulting component in the new variables is

$$v' + aG^{n-1}v^n + \left(\frac{G'}{G} + bF'\right)v = 0.$$

The separable component $y' + r(y)y = 0$ is transformed by (41) into $v' + \dfrac{\psi_u + r(\psi)\varphi_u}{\psi_v + r(\psi)\varphi_v} = 0$. In order to avoid a dependence on u of the fraction via r, ψ must be independent of u, i.e. $\psi \equiv F(v)$. For the same reason $\varphi_{uu} = \varphi_{uv} = 0$ is required, it leads to $\varphi = Cu + G(v)$, C a constant and the transformed equation becomes

$$v' + \frac{Cr(G)}{G' + F'r(G)} = 0.$$

The general separable component $y' + s(x)r(y) = 0$ is transformed into $v' + \dfrac{\psi_u + s(\varphi)r(\psi)\varphi_u}{\psi_v + s(\varphi)r(\psi)\varphi_v} = 0$. Similar reasoning as in the preceding case leads to the constraints $\varphi_v = 0$ and $\psi_u = 0$, i.e. $\varphi = F(y)$ and $\psi = G(v)$ and the transformed equation becomes

$$v' + s\big(F(u)\big)r\big(G(v)\big)\frac{F'(u)}{G'(v)} = 0.$$

Transforming the homogeneous component $y' + \dfrac{ax + by + c}{\alpha x + \beta y + \gamma} = 0$ by (41) yields

$$v' + \frac{(\alpha\varphi + \beta\psi + \gamma)\psi_u + (a\varphi + b\psi + c)\varphi_u}{(\alpha\varphi + \beta\psi + \gamma)\psi_v + (a\varphi + b\psi + c)\varphi_v} = 0.$$

In order that the fraction is a linear homogeneous expression in u and v, the transformation functions must be linear functions in these variables as given above.

Finally, substituting (41) into $y' + \dfrac{a\left(\frac{y}{x}\right)^2 + b\frac{y}{x} + c}{\alpha\left(\frac{y}{x}\right)^2 + \beta\frac{y}{x} + \gamma} = 0$ yields a rather complicated first-order component for $v(u)$; it is easy to see that it has the required structure if $\varphi = \bar{a}u$, $\psi = \bar{b}v$ where $\bar{a}$ and $\bar{b}$ are undetermined constants. □

This result will be applied to several examples that have been considered previously. At first an equation with a linear component is shown.

Example 14 The equation of Example 3 has a linear first-order component. According to the preceding proposition it is transformed by $x \to x + 1$, $y \to xy$ with the result

$$\begin{aligned} &y'' - \frac{x}{(x+1)^2}y'^2 + \frac{4x^2 + 4x + 2}{(x+1)^2}yy' - \frac{(2x^3 + 4x^2 + x - 2)^2}{x(x+1)}y' \\ &-\frac{(2x^2 + 2x - 1)^2}{x(x+1)^2}y^2 - \frac{2x^2 + 4x + 3}{x(x+1)}y = 0. \end{aligned} \tag{42}$$

The first order component is transformed into

$$y' - \frac{2x^2 + 2x - 1}{x} y + \frac{x + 1}{\log(x + 1) + C} = 0.$$

Integration yields the general solution of (42).

$$y = \frac{1}{x} \exp(x^2 + 2x) \left(C_1 - \int \frac{(x + 1) \exp(-x^2 - 2x) dx}{\log(x + 1) + C_2} \right).$$

□

Example 15 In Example 6 an equation has been considered that does not have any Lie symmetry. Its special Riccati component allows the same transformation as in the preceding example with the result

$$yy'' - y'^2 - x(x^2 + 2x + 2) y^2 y' - (3x^2 + 4x + 2) y^3 - \frac{1}{x} y^2 = 0. \tag{43}$$

Its component $y' - x(x^2 + 2x + 2) y^2 + \left(\frac{1}{x} + C\right) y = 0$ yields the general solution

$$y = \frac{C_1^3}{x\left((x^2 + 2x + 2) C_1^2 + 2(x + 1) C_1 + 2\right) + C_2 x \exp(C_1 x)}.$$

□

These examples show that the equivalence classes of the structure invariance groups of decomposable equations may contain equations of enormous complexity. Their decomposition and therefore its general solution may easily be determined; most of the time there is no other solution procedure available.

4 Algorithmic and computational issues

In order to faciliate a decomposition by pencil-and-paper, the determining systems for the coefficients of a component have been given explicitly in Sect. 2 for low values of K because they occur frequently in applications. Furthermore, the decomposition procedures described in this article are implemented in the computer algebra system ALLTYPES that is available in the internet [19]. Various aspects that are relevant in this connection are discussed subsequently. Thereafter the relationship between symmetry analysis and decomposition is discussed in some detail.

4.1 Solving the determining system

The determining system for a first-order component depends on the type of decomposition. For linear, Riccati, Bernoulli and Abel components its coefficients are functions of the independent variable x. They are determined by an algebro-differential system with the following properties according to Sect. 2.

- For quasilinear second-order equations all derivatives are of first order and first degree; for equations of order n its order is $n-1$.
- The degree of the coefficients is bounded by the order K of the first derivative in the given second-order equation if $K>1$, and by 2 if $K=1$.

The details of this system depend on the given ode. In any case, by standard elimination procedures it is always possible to decide its consistency. If a system is inconsistent the corresponding component does not exist even if a universal field is allowed for the coefficients. Otherwise the structure of the system determines the function field where a closed-form solution may exist. For $K=2$ the differential equation is usually a Riccati equation, the existence of a component with rational coefficients may be decided and effectively determined. For $K=3$ it is an Abel equation.

For homogeneous components considered in Proposition 8 the undetermined elements are constants, consequently the determining system is a system of algebraic equations. Initially it is usually quite voluminous. The following two features make the problem manageable.

- Because the vanishing of β leads to a linear component that has been considered before, the constraint $\beta \neq 0$ is included in the solution procedure.
- A Gröbner basis algorithm with factorization and consequently branching into smaller problems is applied whenever possible.

Extensive tests have shown that proceeding in this way inconsistent systems are quickly discovered and may be discarded. The remaining interesting cases comprise only a few simple equations.

4.2 Smmetry analysis versus decomposition

In order to compare the power of various solution schemes it is necessary to apply them to representative collections of equations. To begin with, this is the collection of solved equations by Kamke [10]. Chapter 6 contains about 250 equations of second order, 185 of them are quasilinear, and about 130 are polynomial in y' and rational in x and y, i.e. they have the structure $y''+R(x,y,y')$ where $R \in \mathbb{Q}(x,y)[y']$. The result of applying the decomposition schemes of this article to these equations are listed in detail in the "Appendix". A summary is given next.

The general solution may be obtained most of the time by a decomposition for the following equations.

6.1, 6.2, 6.4, 6.7, 6.10. 6.12, 6.37, 6.39, 6.40, 6.41, 6.42, 6.44, 6.45, 6.50, 6.52, 6.56, 6.57, 6.63, 6.71, 6.78, 6.80, 6.81, 6.86, 6.89, 6.93, 6.99, 6.104, 6.107, 6.109, 6.110, 6.111, 6.117, 6.122, 6.123, 6.124, 6.125, 6.126, 6.127, 6.128, 6.129, 6.131, 6.133, 6.134, 6.135, 6.136, 6.137, 6.138, 6.139, 6.140, 6.141, 6.143, 6.146, 6.150, 6.151, 6.153, 6.154, 6.155, 6.157, 6.158, 6.159, 6.160, 6.162, 6.163, 6.164, 6.165, 6.167, 6.168, 6.169, 6.170, 6.173, 6.174, 6.175, 6.176, 6.178, 6.179, 6.180, 6.181, 6.182, 6.183, 6.184, 6.185, 6.187, 6.188, 6.191, 6.192, 6.193, 6.194, 6.195, 6.196, 6.197, 6.200, 6.201, 6.202, 6.204, 6.206, 6.209, 6.212, 6.214, 6.220, 6.224, 6.226, 6.227, 6.231, 6.232, 6.235, 6.236, 6.238, 6.239, 6.240.

One or more special solutions involving a single parameter may be obtained for the following equations.

6.24, 6,30, 6.31, 6.32, 6.33, 6.35, 6.51, 6.58, 6.60, 6.79, 6.87, 6.98, 6.116, 6.189, 6.190, 6.208.

Finally, no decomposition is obtained for the remaining 37 equations so that the decomposition procedures described in Sect. 2 are of no avail for their solution.

6.3, 6.5, 6.6, 6.8, 6.9, 6.21, 6.23, 6.26, 6.27, 6.34, 6.36, 6.38, 6.46, 6.47, 6.72, 6.73, 6.74, 6.91, 6.95, 6.100, 6.105, 6.106, 6.108, 6.114, 6.115, 6.142, 6.147, 6.148, 6.149, 6.152, 6.156, 6.172, 6.186, 6.205, 6.210, 6.211, 6.219.

In [18], Appendix E it is shown that almost all equations in Kamke's collection have a nontrivial group of Lie symmetries. Many of them may be applied for solving the equation although this is not mentioned by Kamke. Often it remains unclear how the given solution is obtained. For the 37 equations for which no decomposition has been found a closed-form solution is not known. Often solutions in terms of elliptic integrals, Painlevé transcendents or series expansion are given; it is not clear how they have been obtained.

Summing up, it is obvious that by decomposition almost all known closed form solutions may be obtained, in general much more efficiently than applying Lie's symmetry analysis. What is missing for a complete solution scheme as mentioned in the introduction is a proof that a closed form solution possibly does not exist in those cases where none is found.

These remarks apply to equations with nontrivial Lie symmetries. In order to broaden the scope of decomposition methods a systematic study of equations with no symmetries has been performed. In this context the following question arises: how special are equations with Lie symmetries? To this end a random generator for second-order equations of the type considered in this article is applied. Its parameter space is such that the equations in Chapter 6 of Kamke [10] are essentially covered; for the maximal order K of y' the value $K = 2$ is chosen, with rational coefficients of maximal degree 3 or 2 in y and x respectively.

The first result may be described as follows. In a test run with 10^4 random ode's in this parameter space there occur about 100 with a symmetry, i.e. about 1%; about 80 of them are one-parameter symmetries of some kind, there is a single one with a 2- or 3-parameter group, and about 20 with an 8-parameter projective symmetry. This result shows that equations with a symmetry are extremely special. The reason why most equations in Kamke's collection have a symmetry is probably that for equations without a symmetry solutions are rarely known.

When it comes to determining closed form solutions for equations *without* a Lie symmetry, decomposition is the only systematic solution procedure at hand. In order to get an estimate of their power, 10^4 equations in the above parameter space without any Lie symmetry have been randomly generated and the decomposition scheme of Proposition 2 has been applied to them with the following result. On average about 3–5 linear first-order components comprising an undetermined constant have been found, they yield the general solution of the corresponding second-order equation. In about 100 cases one or more linear components without a parameter are obtained, they yield special solutions containing a single parameter for the second-order equation. In general, nonlinear first-order components are less frequent, it is estimated that all decomposition types considered in this article will allow finding the general solution in about one out of one thousend cases; special solutions containing a single parameter will occur about ten times as often.

5 Summary and conclusions

The results of this article show that decomposition is a powerful tool for solving ordinary differential equations. For equations allowing a Lie symmetry it is usually more efficient to search for a decomposition than to go through the laborious steps involved in symmetry analysis; often a solution found from a decomposition is in simpler form than that obtained by symmetry analysis. If an equation does not have a Lie symmetry, decomposition seems to be the only systematic method for determining closed form solutions. This is all the more important due to the rareness of equations with a symmetry as shown above. Furthermore, it would be of interest to determine the relation between decomposition and Lie's symmetry analysis in general.

Due to the efficiency of decomposition procedures it is advantageous to start a solution procedure always by applying them first. If decomposition fails, symmetry analysis may be another possibility to proceed. This strategy is implemented in the computer algebra system ALLTYPES [19].

There are numerous possible extensions of the results described in this article. As explained in the introduction, there are infinitely many possible first order components; here the simplest and most obvious ones have been applied. If they do not exist, or do not lead to a solution, more sophisticated components may lead to additional solvable cases. A systematic investigation of the existence of such first-order components and its use for solving an equation would be highly desirable.

The equations considered here are quasilinear of second order, polynomial in the first derivative and rational in the dependent and the independent variable. It is obvious that the described decomposition methods may be applied to equations of any order and any degree in the variables, a detailed discussion of these more general cases will be given elsewhere; an algorithm for decomposing equations of any order into rational components has been given by Gao and Zhang [8].

An example of decomposing an equation that is of second degree in the highest derivative y'' and of fourth degree in y' is shown next.

Example 16 The equation

$$y^2y''^2 - 2yy'^2y'' + \frac{2}{x^2}y^3y'' + y'^4 - 4yy'^3 + \frac{12x-2}{x^2}y^2y'^2 - \frac{12}{x^2}y^2y' + \frac{4x+1}{x^4}y^4 = 0 \tag{44}$$

is quadratic in the second derivative and of degree four in the first one. It has a right component that yields its general solution

$$z \equiv y' - \left(\frac{1}{x} + \frac{1}{(x+C)^2}\right)y = 0 \Longrightarrow y = C_1 x \exp\left(\frac{x}{C_2(x+C_2)}\right).$$

Although Eq. (44) has a three-parameter symmetry group, the correspondig transformation to canonical form is very complicated and cannot be applied for solving it. □

A slightly generalized third-order equation taken from [4], Sect. 3.4.1, a a constant, is shown in the next example; it describes a special flow in fluid dynamics.

Example 17 The velocity components of a stagnation-point flow are determined by a function $f(\eta)$ that obeys a third-order equation $f''' + 2aff'' - af'^2 + a = 0$. It may be decomposed as

$$\left(w'' + \left(2af + \frac{2}{\eta + C}\right)w' - aw^2 \pm 2aw\right)\left(w \equiv f' - \frac{2}{\eta + C}f \pm 1\right) = 0 \quad (45)$$

from wich the solution $f = C_1\big(\eta + C_2\big)^2 \pm \frac{1}{C_2}\eta\big(\eta + C_2\big)$ follows. This is a partial solution containing two undetermined constants C_1 and C_2 whereas the general solution would contain three constants. It is interesting that the above solution does not depend on a. □

Additional decompositions of equations that are not quasilinear or of third-order are included in the "Appendix".

Of particular interest will be the extension of decomposition methods to partial differential equations. An example containing two independent variables due to Forsyth and Liouville [5,14] is shown next.

Example 18 The equation $z_{x,y} - zz_x = 0$ for $z(x, y)$ may be written as $w_x\big(w \equiv z_y - \frac{1}{2}z^2\big) = 0$. Writing the general solution of $w_x = 0$ as Schwarzian derivative of an undetermined function $G(y)$ leads to

$$z_y - \frac{1}{2}z^2 = \left(\frac{G''(y)}{G'(y)}\right)' - \frac{1}{2}\left(\frac{G''(y)}{G'(y)}\right)^2;$$

this Riccati equation for the y-dependence of z has the obvious special solution $z_0 = \frac{G''(y)}{G'(y)}$. Applying the standard procedure for solving Riccati equations yields the general solution

$$z(x, y) = \frac{G''(y)}{G'(y)} - \frac{2G'(y)}{F(x) + G(y)};$$

$F(x)$ and $G(y)$ are undetermined functions of its argument. The proceeding shown in this example should be compared with the ad hoc methods described in [5]. □

Appendix

The equations in Kamke's collection [10] are an important benchmark test for solution procedures. For each entry the specification of parameters that have applied here are given; they are chosen such that special cases like e.g. the vanishing of terms are avoided. For each entry its symmetry class as defined in [18] is given; $\mathcal{S}_0^2$ or $\mathcal{S}_0^3$ means that an equation does not have any nontrivial Lie symmetry. The result of solving an equation by applying its Lie symmetries may be found in Appendix E of [18], it may be used for comparison with the result from a decomposition given here. If a decomposition exists, its right components are given. The corresponding left components are omitted because they are often rather voluminous and are usuallynot necessary for proceeding with the solution procedure. They may be obtained by applying the procedures provided in the ALLTYPES system. There exist more collections of solved differential equations like e.g. [16,17]. Applying decomposition to randomly chosen equations of them leads to a similar conclusion as shown here.

Second-order equations of chapter 6

6.1 Symmetry class $\mathcal{S}_{2,2}^2$; $y'' - y^2 = 0$; $z \equiv y' \pm \sqrt{2C + \frac{2}{3}y^3)} = 0$.
6.2 Symmetry class $\mathcal{S}_{2,2}^2$; $y'' - 6y^2 + 4y = 0$; $z \equiv y' \pm 2\sqrt{y^3 - y^2 + C)} = 0$.
6.3 Symmetry class $\mathcal{S}_0^2$; $y'' - 6y^2 - x = 0$; no decomposition found.
6.4 Symmetry class $\mathcal{S}_1^2$; $y'' - 6y^2 = 0$; $z \equiv y' \pm 2\sqrt{y^3 + C)} = 0$.
6.5 $a = b = c = 1$, Symmetry class $\mathcal{S}_0^2$; $y'' + y^2 + x + 1 = 0$; no decomposition found.
6.6 $a = 1$, Symmetry class $\mathcal{S}_0^2$; $y'' - 2y^3 - xy + 1 = 0$; no decomposition found.
6.7 Symmetry class $\mathcal{S}_{2,2}^2$; $y'' - y^3 = 0$; $z \equiv y' \pm \frac{1}{2}\sqrt{2}\sqrt{y^4 + C)} = 0$.
6.8 $a = b = 1$, Symmetry class $\mathcal{S}_0^2$; $y'' - 2y^3 + 2xy - 1 = 0$; no decomposition found.
6.9 $a = b = c = d = 1$, Symmetry class $\mathcal{S}_0^2$; $y'' + y^3 + (x + 1)y + 1 = 0$; no decomposition found.
6.10 Symmetry class $\mathcal{S}_1^2$; $y'' + y^3 + y^2 + y + 1 = 0$; $z \equiv y' \pm \sqrt{2C + 4y^4 - 4y^3} = 0$.
6.11 $a = 1$, $y'' + x^2y^3 = 0$; no decomposition found.
6.12 Symmetry class $\mathcal{S}_{2,2}^2$; $y'' + 2y^3 = 0$; $z \equiv y' \pm \sqrt{2C - y^4} = 0$.
6.21 Symmetry class $\mathcal{S}_1^2$; $y'' - 3y' - y^2 - 2y = 0$; no decomposition found.
6.23 Symmetry class $\mathcal{S}_1^2$; $y'' + 5y' - 6y^2 + 6y = 0$; no decomposition found.
6.24 $a = 1$, Symmetry class $\mathcal{S}_{2,2}^2$; $y'' + 3y' - 2y^3 + 2y = 0$; $z \equiv y' \pm y^2 + y = 0$.
6.26 $a = 2, b = 1, n = 2$ Symmetry class $\mathcal{S}_1^2$; $y'' + y' + y^2 + \frac{3}{4}y = 0$; no decomposition found.
6.27 $a = b = 1$, $\nu = 2$, $n = 2$, Symmetry class $\mathcal{S}_0^2$; $y'' + y' + x^2y^2 = 0$; no decomposition found.
6.30 Symmetry class $\mathcal{S}_{2,2}^2$; $y'' + yy' - y^3 = 0$; $z \equiv y' - \frac{1}{2}y^2 = 0$; $z \equiv y' + y^2 = 0$.
6.31 $a = 1$, Symmetry class $\mathcal{S}_1^2$; $y'' + yy' - y^3 - y = 0$; $z \equiv y' - \frac{1}{2}y^2 - \frac{1}{2} = 0$, $z \equiv y' + y^2 + 1 = 0$.

6.32 $a = 1$, Symmetry class $\mathcal{S}^2_{2,2}$; $y'' + yy' - y^3 + 3y' - y^3 + y^2 + 2y = 0$; $z \equiv y' - \frac{1}{2}y^2 + y = 0$, $z \equiv y' + y^2 + y = 0$

6.33 $f(x) = x$, Symmetry class $\mathcal{S}^2_{2,2}$; $y''+yy'-y^3+3xy'-y^3+xy^2+(2x^2+1)y = 0$;
$z \equiv y' - \frac{1}{2}y^2 + xy = 0$, $z \equiv y' + y^2 + xy = 0$.

6.34 $f(x) = x$, Symmetry class $\mathcal{S}^2_0$; $y'' + 2yy' + xy' + y = 0$; no decomposition found.

6.35 $f(x) = x$, Symmetry class $\mathcal{S}^2_1$; $y''+yy'-\frac{3}{2x}y'-y^3-\frac{1}{2x}y^2-\frac{x^3-1}{x^2}y = 0$;
$z \equiv y' - \frac{1}{2}y^2y - \frac{1}{2}x = 0$, $z \equiv y' + y^2 - \frac{1}{2x}y + x = 0$.

6.36 $a = b = 1$, $f(x) = x$, Symmetry class $\mathcal{S}^2_0$; $y'' + yy' + xy' + \frac{3x^2+3}{x}y' - y^3 - \frac{x^2+1}{x}y^2 + \frac{x^4+3x^2+3}{x^2} + x^2 = 0$; no decomposition found.

6.37 $f(x) = x$, $g(x) = 0$, Symmetry class $\mathcal{S}^2_0$; $y'' + 2yy' + xy' + xy^2 = 0$; $z \equiv y' + y^2 + C\exp\left(-\frac{1}{2}x^2\right) = 0$.

6.38 $f(x) = g(x) = x$, Symmetry class $\mathcal{S}^2_8$; $y'' + 3yy' + y^3 + xy + x = 0$; no decomposition found.

6.39 $f(x) = x$, Symmetry class $\mathcal{S}^2_8$; $y'' + 3yy'^2 + xy' + y^3 + xy^2 = 0$;

$$z \equiv y' + y^2 - \frac{2\exp\left(-\frac{1}{2}x^2\right)y}{\sqrt{2\pi}\,\mathrm{erf}\left(\frac{x}{\sqrt{2}}\right) - 2C} = 0.$$

6.40 $a = b = 1$, Symmetry class $\mathcal{S}^2_{2,2}$; $y'' - 3yy' - 3y^2 - 4y - 1 = 0$; $z \equiv y' - \frac{3}{2}y^2 - 2y + C\exp(-2x) - \frac{1}{2} = 0$.

6.41 $f(x) = x$, Symmetry class $\mathcal{S}^2_8$; $y'' + 3yy'^2 + xy' + y^3 + xy^2 = 0$;

$$z \equiv y' - y^2 - \frac{2i\exp\left(-\frac{1}{2}x^2\right)y}{\sqrt{2\pi}\,\mathrm{erf}\left(\frac{ix}{\sqrt{2}}\right) + 2C} = 0.$$

6.42 $a = 1$, Symmetry class $\mathcal{S}^2_1$; $y'' - 2yy' - 1 = 0$; $z \equiv y' - y^2 - x + C = 0$.

6.43 $a = 2$, $b = 1$, Symmetry class $\mathcal{S}^2_{2,2}$; $y'' + 2yy' + y^3$; $z \equiv y' + r(y) = 0$, r implicit function of x and y.

6.44 $f(y) = y$, $f(x) = x$, Symmetry class $\mathcal{S}^2_0$; $y'' + yy' + x$; $z \equiv y' + \frac{1}{2}y^2 + C + \frac{1}{2}x^2 = 0$.

6.45 $a = b = 1$, Symmetry class $\mathcal{S}^2_1$; $y'' + y'^2 + y = 0$; $z \equiv y' \pm \exp(-y)\sqrt{\left(y - \frac{1}{2}\right)\exp(2y) - Ci} = 0$.

6.46 $a = b = c = 1$, Symmetry class $\mathcal{S}^2_1$; $y''+yy'^2+y'+y = 0$; no decomposition found.

6.47 Symmetry class $\mathcal{S}^2_1$; $y'' + y'^2 + y' + y = 0$; no decomposition found.

6.50 Symmetry class $\mathcal{S}^2_1$; $y'' + yy'^2 + y = 0$;
$z \equiv y' + \pm\exp\left(-\frac{1}{2}y^2\right)\sqrt{\exp(y^2) - Ci} = 0$.

6.51 $f(y) = y, g(x) = x$, Symmetry class $\mathcal{S}_8^2$; $y'' + yy'^2 + xy' = 0$;
$z \equiv y' + \exp\left(\frac{1}{2}x^2 + \frac{1}{2}y^2\right) = 0.$

6.52 Symmetry class $\mathcal{S}_8^2$; $yy'' - y'^2 + xyy' + xy^2 = 0$;
$z \equiv y' + \left(1 + C\exp\left(-\frac{1}{2}x^2\right)\right)y = 0.$

6.56 $a = 1$, Symmetry class $\mathcal{S}_1^2$; $y'' + yy'^4 + 2yy'^2 + y = 0$;
$$z \equiv y' \pm \frac{\sqrt{y^2 - 2C}}{\sqrt{2C - y^2 - 1}} = 0.$$

6.57 $a = 1, \nu = 3$, Symmetry class $\mathcal{S}_{3,2}^2$; $y'' - x^3y'^3 + 3x^2yy'^2 - 3xy^2y' + y^3 = 0$;
$z \equiv y' \pm \dfrac{1}{x\sqrt{C - x^2}} = 0.$

6.58 $a = b = c = k = 1$, Symmetry class $\mathcal{S}_1^2$; $y'' - xyy' = 0$; $z \equiv y' = 0$.

6.60 $a = 1$, Symmetry class $\mathcal{S}_{2,1}^2$; $y''^2 - y'^2 - 1 = 0$;
$z \equiv y' - \dfrac{1}{2}\dfrac{\exp(2C) - \exp(2x)}{\exp(x + C)} = 0,$
$z \equiv y' \pm \sqrt{(y - C)^2 - 1} = 0.$

6.63 $a = 1$, Symmetry class $\mathcal{S}_{3,1}^2$; $y''^2 - y'^6 - 3y'^4 - 3y'^2 - 1 = 0$;
$z \equiv y' - \dfrac{C - x - 1}{\sqrt{2(C - 1)x - (C - x)^2}}.$

6.71 $a = 1$, Symmetry class $\mathcal{S}_{2,2}^2$; $y'' + \frac{9}{8}y'^4 = 0$; $z \equiv y' \pm \dfrac{2/3}{\sqrt{y - C}} = 0.$

6.72 $R(y') = y'^2$, Symmetry class $\mathcal{S}_0^2$; $y'' + y'^2 + xy = 0$; no decomposition found.

6.73 $n = 2, 3, 4$, $xy'' + 2y' - xy^n = 0$, Symmetry class $\mathcal{S}_1^2$; no decomposition found.

6.74 $a = 1, \nu = 1, n = 2$, $xy'' + 2y' + xy^2 = 0$, Symmetry class $\mathcal{S}_1^2$; no decomposition found.

6.78 Symmetry class $\mathcal{S}_{2,2}^2$; $y'' + \frac{1}{x}yy' - \frac{1}{x}y' = 0$; $z \equiv y' - \frac{1}{2x}y^2 - \frac{2}{x}y + \frac{C}{x} = 0.$

6.79 Symmetry class $\mathcal{S}_1^2$; $y'' - xy'^2 + \frac{2}{x}y' + \frac{1}{x}y^2 = 0$; $z \equiv y' - \frac{1}{x}y - \dfrac{1}{x^2} = 0,$
$z \equiv y' + \frac{1}{x}y = 0,$

$$z \equiv yy' - \frac{1}{2}\frac{\log(x) + 1}{x\log(x)} + \frac{1}{x}y^2 - \frac{1}{2}\frac{\log(x) + 1}{x^2\log(x)}y - \frac{1}{2x^3\log(x)} = 0.$$

6.80 Symmetry class $\mathcal{S}_{3,2}^2$; $y'' + xy'^2 - 2yy' + \frac{1}{x}y^2 = 0$;
$z \equiv y' - \frac{1}{x}y - \dfrac{1}{x(x + C)} = 0.$

6.81 Symmetry class $\mathcal{S}_{3,2}^2$; $y'' + \frac{1}{2x}y'^3 + \frac{1}{2x}y' = 0$; $z \equiv y' \pm \dfrac{1}{\sqrt{Cx - 1}} = 0.$

6.86 $a = 1, b = 0$, Symmetry class $\mathcal{S}_{2,2}^2$; $y'' + y'^2 + -\frac{2}{x}yy' + \frac{1}{x^2}y^2 = 0$;
$z \equiv y' - \frac{1}{x}y - \dfrac{1}{x(\log(x) - C)} = 0.$

6.87 $a = 1, b = 0$, Symmetry class $\mathcal{S}_1^2$; $y'' + \frac{1}{x^2}yy' = 0$; $z \equiv y' = 0$.

6.89 Symmetry class $\mathcal{S}_1^2$; $y'' + \frac{1}{x^2+1}y'^2 + \frac{1}{x^2+1} = 0$; $z \equiv y' + \frac{Cx-1}{C+x} = 0$.

6.90 Symmetry class $\mathcal{S}_1^2$; $y'' - \frac{1}{4}x^2y'^2 + \frac{1}{x^2}y = 0$; no decomposition found.

6.91 $a = 1$, Symmetry class $\mathcal{S}_1^2$; $y'' + +\frac{1}{9x^2}y^3 + \frac{2}{9x^2}y = 0$; no decomposition found.

6.93 $a = 1$, Symmetry class $\mathcal{S}_8^2$; $y'' - \frac{1}{x}y'^2 + \frac{2}{x^2}yy' - \frac{1}{x^2}y^2 = 0$;
$z \equiv y' - \frac{1}{x}y - \frac{C}{x+C} = 0$.

6.95 Symmetry class $\mathcal{S}_0^2$; $y'' + yy' + \frac{\frac{9}{2}x^2}{x^3 - \frac{1}{4}} - \frac{x^3 - \frac{3}{2}x^2 - \frac{1}{4}}{x^3 - \frac{1}{4}} + \frac{\frac{1}{8}x}{x^3 - \frac{1}{4}} + \frac{\frac{1}{8}}{x^3 - \frac{1}{4}} = 0$; no decomposition found.

6.97 Symmetry class $\mathcal{S}_{2,2}^2$; $y'' - \frac{2}{x^3}yy' - \frac{1}{x}y' + \frac{4}{x^2}y^2 = 0$; $z \equiv y' - \frac{1}{x^3}y^2 + Cx = 0$, $z \equiv y' - \frac{2}{x}y = 0$.

6.98 Symmetry class $\mathcal{S}_1^2$; $y'' - \frac{1}{x^2}y'^2 - \frac{1}{x}y' + \frac{4}{x^4}y^2$; $z \equiv y' + \frac{2}{x}y + 2x = 0$.

6.99 Symmetry class $\mathcal{S}_8^2$; $y'' + \frac{1}{x}y'^3 - \frac{3}{x^2}yy'^2 + \frac{3}{x^2}y^2y' - \frac{1}{x^4}y^3 = 0$;
$z \equiv y' - \frac{1}{x}y \pm \frac{1}{\sqrt{Cx^2-1}} = 0$.

6.100 Symmetry class $\mathcal{S}_1^2$; $\sqrt{x}y'' - y^{2/3} = 0$; no decomposition found.

6.104 Symmetry class $\mathcal{S}_{2,2}^2$; $yy'' + 1 = 0$; $z \equiv y' \pm \sqrt{C - 2\log(y)} = 0$.

6.105 $a = 1$, Symmetry class $\mathcal{S}_1^2$; $yy'' - x = 0$; no decomposition found.

6.106 $a = 1$, Symmetry class $\mathcal{S}_1^2$; $yy'' - x^2 = 0$; no decomposition found.

6.107 $a = 1$, Symmetry class $\mathcal{S}_8^2$; $yy'' + y'^2 - 1 = 0$; $z \equiv yy' - \frac{1}{2}C - x = 0$, $z \equiv yy' - \frac{1}{2x}y^2 + \frac{C-x^2}{2x} = 0$.

6.108 $a = b = 1$, Symmetry class $\mathcal{S}_0^2$; $yy'' + y^2 - x + 1 = 0$; no decomposition found.

6.109 Symmetry class $\mathcal{S}_{2,2}^2$; $yy'' + y'^2 - y' = 0$; $z \equiv yy' - y + C = 0$.

6.110 Symmetry class $\mathcal{S}_{2,2}^2$; $yy'' - y'^2 + 1 = 0$; $z \equiv y' \pm \sqrt{Cy^2+1} = 0$.

6.111 Symmetry class $\mathcal{S}_{2,2}^2$; $yy'' - y'^2 - 1 = 0$; $z \equiv y' \pm \sqrt{Cy^2-1} = 0$.

6.114 $f(x) = x$, Symmetry class unknown; $yy'' - y'^2 - y' + xy^3 - \frac{1}{x^2}y^2 = 0$; no decomposition found.

6.115 $f(x) = x$, Symmetry class unknown; $yy'' - y'^2 + xy' - y^3 - y = 0$; no decomposition found.

6.116 $f(x) = x$, Symmetry class $\mathcal{S}_0^2$; $yy'' - y'^2 + y' - y^4 + xy^3 = 0$;
$z \equiv y' \pm y^2 \mp xy - 1 = 0$.

6.117 $a = b = 1$, Symmetry class $\mathcal{S}_8^2$; $yy'' - y'^2 + yy' + y^2 = 0$;
$z \equiv y' + (1 + C\exp(-x))y = 0$.

6.122 $f(x) = g(x) = x$, Symmetry class $\mathcal{S}^2_8$; $yy'' - y'^2 + xyy' + xy^2 = 0$; $z \equiv y' + \big(1 + C\exp(-\frac{1}{2}x^2)\big)y = 0$.

6.123 $f(x) = g(x) = x$, Symmetry class unknown; $yy'' - y'^2 + xy^2y' + xy' + y^3 - y = 0$; $z \equiv y' + xy^2 + Cy - x = 0$.

6.124 Symmetry class $\mathcal{S}^2_8$; $yy'' - 3y'^2 + 3yy' - y^2 = 0$;

$$z \equiv y' - \frac{1}{2}\frac{C\exp(x) + 2}{C\exp(x) + 1}y = 0, z \equiv y' + Cy^3 - \sqrt{C}\sqrt{Cy^2 - 1}y^2 - y = 0.$$

6.125 $a = 1$, Symmetry class $\mathcal{S}^2_8$; $yy'' - y'^2 = 0$; $z \equiv y' + Cy = 0$;

6.126 Symmetry class $\mathcal{S}^2_8$; $yy'' + y'^2 + 1 = 0$; $z \equiv yy' + x + C = 0, z \equiv yy' - \frac{1}{2x}y^2 + \frac{1}{2}x = 0$.

6.127 $a = b = 1$, Symmetry class $\mathcal{S}^2_{2,2}$; $yy'' + y'^2 + y^3 = 0$; $z \equiv yy' + \sqrt{C - \frac{2}{5}y^5} = 0$.

6.128 $a = -1, b = c = d = 1$, Symmetry class $\mathcal{S}^2_8$; $yy'' - y'^2 + yy' + 2y^2 = 0$; $z \equiv y' + (2 + C\exp(-x))y = 0$.

6.129 $a = 1, f(x) = x, g(x) = 1$, Symmetry class $\mathcal{S}^2_8$; $yy'' + y'^2 + xyy' + y^2 = 0$;

$$z \equiv y' - \frac{1}{2}\frac{\exp\left(\frac{1}{2}x^2\right)y}{x^2\left(\int \exp\left(\frac{1}{2}x^2\right)\frac{dx}{x^2} + C\right)} - \frac{1}{2}x(x^2 - 1)y = 0.$$

6.130 $a = b = c = 1$, Symmetry class $\mathcal{S}^2_{2,2}$; $yy'' + y'^2 + y^2y' + y^4$; $z \equiv y' + r(y) = 0$, r implicit function of x and y.

6.131 $a = -1, f(x) = 1$, Symmetry class $\mathcal{S}^2_1$;

$$y'' - \frac{2}{y}y'^2 - xyy' - x^2y^3 + y^2 = 0; z \equiv y' + xy^2 + \frac{1}{x + C}y = 0.$$

6.133 Symmetry class $\mathcal{S}^2_{3,2}$; $(y + x)y'' + y'^2 - y' = 0$; $z \equiv y' - \frac{1}{2}\frac{x + y}{x + C} = 0$, $z \equiv yy' + xy' - 2y + C = 0$.

6.134 Symmetry class $\mathcal{S}^2_8$; $(y - x)y'' - 2y'^2 - 2y' = 0$; $z \equiv y' + \frac{y + C}{x + C} = 0$, $z \equiv y' + \frac{y(y + C)}{x(x + C)} = 0$.

6.135 Symmetry class $\mathcal{S}^2_8$; $(y - x)y'' + y'^3 + y'^2 + y' + 1 = 0$;

$$z \equiv y' + \frac{x + C}{y + C} = 0, z \equiv y' + 1 = 0, z \equiv y' + \theta = 0, \theta^2 + 1 = 0.$$

6.136 Symmetry class $\mathcal{S}^2_{2,2}$; $(y - x)y'' + y' = 0$; $z \equiv y' + \frac{x - y}{x + C} = 0$.

6.137 Symmetry class $\mathcal{S}^2_{2,2}$; $yy'' + \frac{1}{2}y'^2 + \frac{1}{2} = 0$; $z \equiv y' \pm \frac{\sqrt{C - y}}{\sqrt{y}} = 0$.

6.138 $a = 1$, Symmetry class $\mathcal{S}^2_8$; $yy'' - \frac{1}{2}y'^2 + \frac{1}{2} = 0$; $z \equiv y' - \frac{2}{x+C}y \pm 1 = 0$.

6.139 $a = 0$, $f(x) = \frac{1}{x^2}$, Symmetry class $\mathcal{S}^2_8$; $yy'' - \frac{1}{2}y'^2 + \frac{1}{2x^2}y^2 = 0$;
$z \equiv y' - \frac{\log(x)C + 1}{x(\log(x) + C)} = 0$.

6.140 $a = b = 1$, Symmetry class $\mathcal{S}^2_{2,2}$; $yy'' - \frac{1}{2}y'^2 + \frac{1}{2}y^3 + \frac{1}{2}y^2 = 0$;
$z \equiv y' \pm \sqrt{Cy - \frac{1}{2}y^3 - y^2} = 0$.

6.141 Symmetry class $\mathcal{S}^2_1$; $yy'' - \frac{1}{2}y'^2 - 4y^3 - 2y^2 = 0$; $z \equiv y' \pm \sqrt{Cy + 4y^3 + 4y^2} = 0$.

6.142 Symmetry class $\mathcal{S}^2_0$; $yy'' - \frac{1}{2}y'^2 - 4y^3 - 2xy^2 = 0$; no decomposition found.

6.143 Symmetry class $\mathcal{S}^2_1$; $yy'' - \frac{1}{2}y'^2 - \frac{3}{2}y^4 = 0$; $z \equiv y' \pm \sqrt{y(y^3 + C)} = 0$.

6.146 Symmetry class $\mathcal{S}^2_{2,2}$; $yy'' - \frac{1}{2}y'^2 - \frac{3}{2}y^4 = 0$; $z \equiv y' \pm \sqrt{y(y^3 + C)} = 0$.

6.147 $a = b = 1$, Symmetry class unknown; $yy'' - \frac{1}{2}y'^2 - \frac{3}{2}y^4 - 4xy^3 - (2x^2 + 2))y^2 + \frac{1}{2} = 0$; no decomposition found.

6.148 Symmetry class unknown; $yy'' - \frac{1}{2}y'^2 + 2y^2y' + \frac{1}{2}xy^2 + \frac{1}{2} = 0$; no decomposition found.

6.149 Symmetry class $\mathcal{S}^2_0$; $yy'' - \frac{1}{2}y'^2 + \frac{3}{2}xyy' - 4y^3 + (x^2 + 1)y^2 = 0$; no decomposition found.

6.150 $f(x) = x$, Symmetry class $\mathcal{S}^2_8$; $yy'' - \frac{3}{2}y'^2 = 0$; $z \equiv y' + \frac{2}{x+C}y = 0$,
$z \equiv y' + Cy^{\frac{3}{2}} = 0$.

6.151 $f(x) = x$, Symmetry class $\mathcal{S}^2_8$; $yy'' - \frac{3}{2}y'^2 - 2y^2 = 0$;
$z \equiv y' - 2\tan(x + C)y = 0$.

6.152 $f(x) = x$, Symmetry class $\mathcal{S}^2_8$; $yy'' - \frac{3}{2}y'^2 + \frac{1}{2}xy^2 = 0$; no decomposition found.

6.153 $a = 1$, Symmetry class $\mathcal{S}^2_1$; $yy'' - 3y'^2 + \frac{1}{2}y^5 + \frac{1}{2}y^2 = 0$;
$z \equiv y' \pm \frac{\sqrt{y}}{\sqrt{C - y}} = 0$.

6.154 Symmetry class $\mathcal{S}^2_{2,2}$; $yy'' - \frac{1}{2}y'^4 - \frac{1}{2}y'^2 = 0$; $z \equiv y' \pm \sqrt{Cy^4 + y^3 + \frac{1}{4}y} = 0$.

6.155 Symmetry class $\mathcal{S}^2_8$; $(y - 1)y'' + \frac{1}{2}y'^2 + \frac{1}{2} = 0$; $z \equiv y' \pm \frac{\sqrt{C - y}}{\sqrt{y - 1}} = 0$.

6.156 $a = b = c = 1$, Symmetry class $\mathcal{S}^2_1$; $yy'' - \frac{2}{3}df(y, x)^2 - \frac{1}{3}(x^2 + x + 1) = 0$; no decomposition found.

6.157 Symmetry class $\mathcal{S}^2_8$; $yy'' - \frac{5}{3}y'^2 = 0$; $z \equiv y' + \frac{3/2}{x+C}y = 0$, $z \equiv y' + Cy^{\frac{5}{3}} = 0$.

6.158 Symmetry class $\mathcal{S}^2_{3,2}$; $yy'' - \frac{3}{4}y'^2 + y = 0$; $z \equiv y' \pm \sqrt{y}\sqrt{C\sqrt{y} + 4} = 0$.

6.159 Symmetry class $\mathcal{S}^2_{2,2}$; $yy'' - \frac{3}{4}y'^2 - 3y^3 = 0$; $z \equiv y' \pm \sqrt{y}\sqrt{C\sqrt{y} + 4y^2} = 0$.

6.160 Symmetry class $\mathcal{S}^2_1$; $yy'' - \frac{3}{4}y'^2 + \frac{1}{4}y^3 + \frac{1}{4}y^2 + \frac{1}{4}y = 0$;
$z \equiv y' \pm \sqrt{Cy\sqrt{y} - \frac{1}{3}y^3 - y^2 + y} = 0$.

6.162 Symmetry class $\mathcal{S}^2_{3,2}$; $yy'' - \frac{5}{4}y'^2 + \frac{1}{4}y^3 = 0$; $z \equiv y' + \pm y\sqrt{C\sqrt{y} - y} = 0$.

6.163 Symmetry class $\mathcal{S}^2_{3,2}$; $yy'' - \frac{5}{4}y'^2 + \frac{2}{3}y^3 = 0$; $z \equiv y' \pm y\sqrt{C\sqrt{y} - \frac{8}{3}y} = 0$.

6.164 Symmetry class $\mathcal{S}^2_8$;

$$yy'' - \frac{n-1}{n}y'^2 = 0;\ z \equiv y' - \frac{n}{x+C}y = 0,\ z \equiv y' + C\exp\Big(-\frac{1}{n}\log(y)\Big)y = 0.$$

6.165 Symmetry class $\mathcal{S}^2_8$; $yy'' + y'^2 + y^4 + y^3 + y^2 + y + 1 = 0$;
$z \equiv yy' \pm \sqrt{C - \frac{1}{3}y^6 - \frac{2}{5}y^5 - \frac{1}{2}y^4 - \frac{2}{3}y^3 - y^2} = 0$.

6.167 $a = -1$, $f(x) = x$, Symmetry class $\mathcal{S}^2_1$; $yy'' - 2y'^2 - xy^2y' - x^2y^4 + y^3 = 0$;
$z \equiv y' + xy^2 + \frac{1}{x+C} = 0$.

6.168 $a = b = c = 1$, Symmetry class $\mathcal{S}^2_8$;

$$(y+1)y'' + y'^2 = 0;\ z \equiv y' - \frac{1/2}{x+C}(y+1) = 0,\ z \equiv yy' + y' + C = 0.$$

6.169 Symmetry class $\mathcal{S}^2_8$; $yy'' + y'^2 - \frac{1}{x}yy' = 0$; $z \equiv y' - \frac{x}{C+x^2}y$,
$z \equiv yy' + Cx = 0$.

6.170 Symmetry class $\mathcal{S}^2_8$; $yy'' + y'^2 + \frac{2}{x}yy' + 1 = 0$;
$z \equiv yy' + \frac{1}{x} = 0$, $z \equiv yy' + \frac{1}{2x} + \frac{1}{2}x + \frac{C}{x}$, $z \equiv yy' + \frac{1}{3}x + \frac{C}{x^2}$.

6.172 Symmetry class $\mathcal{S}^2_{2,2}$; $yy'' - y'^2 + \frac{1}{x}yy' + y^3 = 0$; no decomposition found.

6.173 $a = 1$, Symmetry class $\mathcal{S}^2_8$; $yy'' + 2y'^2 + \frac{1}{x}yy' = 0$; $z \equiv y' - \frac{1/3}{x(\log(x) - C)}y$.

6.174 Symmetry class $\mathcal{S}^2_{2,2}$; $yy'' - 2y'^2 + \frac{1}{x}yy' + \frac{1}{x}y' = 0$; $z \equiv y' + \frac{C}{x}y^2 - \frac{1}{2x} = 0$.

6.175 Symmetry class $\mathcal{S}^2_8$; $yy'' - 2y'^2 + \frac{1}{x}yy' = 0$; $z \equiv y' + \frac{1}{x(\log(x) + C)}y = 0$.

6.176 Symmetry class $\mathcal{S}^2_8$; $yy'' + y'^2 - 4y'^2 + \frac{4}{x}yy' = 0$; $z \equiv y' - \frac{1}{x(1 - Cx^3)}y = 0$.

6.178 Symmetry class $\mathcal{S}^2_8$; $(y+x)y'' + y'^2 - \frac{1}{x}yy' + y' - \frac{1}{x}y = 0$;
$z \equiv yy' + xy' + y + Cx = 0$.

6.179 Symmetry class $\mathcal{S}^2_8$; $yy'' - \frac{1}{2}y'^2 + \frac{1}{2x}yy' = 0$; $z \equiv y' - \frac{1}{x - C\sqrt{x}}y = 0$.

6.180 Symmetry class $\mathcal{S}^2_8$;

$$(y-1)y'' - 2y'^2 - \frac{2}{x}yy' + \frac{2}{x}y' - \frac{2}{x^2}y^3 - \frac{4}{x^2}y^2 - \frac{2}{x^2}y = 0;$$
$$z \equiv y' - \frac{2x+C}{x(x+C)}y(y-1) = 0.$$

6.181 Symmetry class $\mathcal{S}^2_8$; $(y+x)y'' - y'^2 - \frac{2}{x}yy' - \frac{1}{x^2}y^2 = 0$;
$z \equiv y' - \frac{x-C}{x^2}y + \frac{C}{x} = 0$.

6.182 $a=1$, Symmetry class $\mathcal{S}^2_8$; $(y-x)y''-y'^2+\frac{2}{x}yy'-\frac{1}{x^2}y^2=0$;
$z\equiv y'-\frac{x+C}{x^2}y-\frac{C}{x}=0$.

6.182 $a=-1$, Symmetry class $\mathcal{S}^2_8$; $(y-x)y''+y'^2-\frac{2}{x}yy'+\frac{1}{x^2}y^2=0$;
$z\equiv y'-\frac{\frac{1}{2}C+x}{x(x+C)}y-\frac{\frac{1}{2}C}{C+x}=0$.

6.183 Symmetry class $\mathcal{S}^2_{3,2}$; $yy''-\frac{1}{2}y'^2+\frac{1}{2x^2}y^2-\frac{1}{2}=0$;
$z\equiv y'-\frac{\log(x)+C+2}{x(\log(x)+C)}y+\theta=0,\ \theta^2+1=0$.

6.184 Symmetry class $\mathcal{S}^2_8$; $yy''+y'^2+\frac{1}{x}yy'+\frac{1}{x^2}y^2=0$;
$z\equiv y'-\frac{\theta\exp(4\theta)+2C}{x(\exp(4\theta)+4C\theta)}=0,\ \theta^2+\frac{1}{2}=0$.

6.185 $a=1$, Symmetry class $\mathcal{S}^2_8$; $yy''-y'^2+\frac{2}{x}yy'+\frac{x+2}{x(x+1)^2}y^2=0$;
$z\equiv y'+\frac{x^2+Cx+C}{x^2(x+1)}=0$.

6.186 Symmetry class $\mathcal{S}^2_8$; $yy''-\frac{1}{2}y'^2+\frac{3/2x^2}{x^2-1}yy'-\frac{3/8x}{x^3-1}y^2=0$; no decomposition found.

6.187 $f(x)=g(x)=h(x)=k(x)=1$, Symmetry class $\mathcal{S}^2_8$;

$$yy''+y'^2+yy'+\frac{1}{x}y^2=0;\ I_1\equiv\int\frac{e^x dx}{x(x-1)^2},\ I_2\equiv\int\frac{e^x dx}{(x-1)^2},$$

$$z\equiv y'-\frac{\frac{1}{2}(x-2)e^x+(x-2)(x^2-4x+2)(C+2I_1-\frac{1}{2}I_2)}{(x-2)e^x+x(x-2)^2(2C+4I_1-I_2)}=0.$$

6.188 $a=1$, Symmetry class $\mathcal{S}^2_{2,2}$; $y^2y''-1=0$; $z\equiv y'+\sqrt{\frac{2(Cy-1)}{y}}=0$.

6.189 $a=1$, Symmetry class $\mathcal{S}^2_1$; $y^2y''+yy'^2+x=0$; $z\equiv yy'-y+x=0$.

6.190 $a=1$, Symmetry class $\mathcal{S}^2_1$; $y^2y''+yy'^2-x-1=0$; $z\equiv yy'+y+x+1=0$.

6.191 Symmetry class $\mathcal{S}^2_8$; $(y^2+1)y''-2yy'^2+y'^2=0$;
$z\equiv y'-\frac{y^2+1}{x+C}=0,\ z\equiv y'+C(y^2+1)\exp(-\arctan(y))=0$.

6.192 Symmetry class $\mathcal{S}^2_8$; $(y^2+1)y''-3yy'^2=0$;
$z\equiv y'-\frac{y^3+y}{x+C}=0,\ z\equiv y'+C(y^2+1)^{\frac{3}{2}}=0$.

6.193 Symmetry class $\mathcal{S}^2_8$; $(y^2+x)y''+2y^2y'^3-2xy'^3+4yy'^2+y'=0$;
$z\equiv y'+(y+2x^2)C+2x$.

6.194 Symmetry class $\mathcal{S}^2_8$; $(y^2+x^2)y''-xy'^3+yy'^2-xy'+y=0$;
$z\equiv y'-\frac{Cy-x}{Cx+y}=0$.

6.195 Symmetry class $\mathcal{S}^2_8$; $(y^2+x^2)y''-2xy'^3+2yy'^2-2xy'+2y=0$;
$z\equiv y'+\frac{C(y^2-x^2)+2xy}{y^2-x^2-2Cxy}$.

6.196 $f(x) = x$, Symmetry class $\mathcal{S}^2_8$; $y'' - \frac{1}{2}\frac{2y-1}{y(y-1)}y'^2 + \frac{1}{2}xy' = 0$;
$z \equiv y' + C\exp\left(-\frac{1}{4}x^2\right)\sqrt{y(y-1)} = 0$.

6.197 $f(y) = 0$, Symmetry class $\mathcal{S}^2_8$; $y^2y'' - yy'' - \frac{3}{2}yy'^2 + \frac{2}{2}y'^2 = 0$;
$z \equiv y' + \sqrt{y}(y-1)C = 0$.

6.198 $f(y) = 0$, Symmetry class unknown; $z \equiv y' + (y-1)^{\frac{2}{3}}y^{\frac{2}{3}}C = 0$.

6.200 $f(y) = 0$, Symmetry class $\mathcal{S}^2_8$; $y^2y'' - yy'' - \frac{4}{3}yy'^2 + \frac{2}{2}y'^2 = 0$;
$z \equiv y' + (y-1)^{\frac{2}{3}}y^{\frac{2}{3}}C = 0$.

6.201 Symmetry class $\mathcal{S}^2_8$; $(y-1)y'' - 6yy'^2 + 3y'^2 = 0$;
$z \equiv y' + C(y-1)^3\exp(6y) = 0$.

6.202 $a = b = c = 1$, $f(y) = 0$, Symmetry class $\mathcal{S}^2_8$; $(y^2 - y)y'' + yy'^2 + y'^2 = 0$;
$z \equiv y^2y' - 2yy' + y' + Cy = 0$.

6.204 $a = 1, b = -1$, $f(x) = x$, Symmetry class $\mathcal{S}^2_8$;

$$y^2y'' + \frac{1}{2}xy^2 = 1;\ z \equiv y' - \frac{2\exp(1/2x^2)}{\sqrt{2\pi}i\operatorname{erf}\frac{ix}{\sqrt{2}} + 2C}(y-1) = 0.$$

6.205 $a = 1$, Symmetry class $\mathcal{S}^2_8$; $y^2y'' - \frac{1}{x} = 0$; no decomposition found.

6.206 $a = 1$, Symmetry class $\mathcal{S}^2_8$;

$$(y^2-1)y'' - yy'^2 + \frac{x}{x^2-1}y^2y' - \frac{x}{x^2-1}y' = 0;\ z \equiv y' + C\frac{\sqrt{y^2-1}}{\sqrt{x^2-1}} = 0.$$

6.208 Symmetry class $\mathcal{S}^2_8$;

$$y^2y'' + (y+x)y'^3 - \frac{3}{x}y^2y'^2 - 3yy'^2 + \frac{3}{x^2}y^3y' + \frac{3}{x}y^2y' - \frac{1}{x^3}y^4 - \frac{1}{x^2}y^3 = 0;$$
$$z \equiv y' - \frac{1}{x}y = 0.$$

6.209 Symmetry class $\mathcal{S}^2_{3,2}$; $y^3y'' - 1 = 0$; $z \equiv yy' \pm \sqrt{Cy^2 - 1} = 0$.

6.210 $a = b = c = 1$, Symmetry class $\mathcal{S}^2_8$. $y^3y'' + \frac{1}{2}y^2y'^2 - \frac{1}{2}x^2 - \frac{1}{2}x - \frac{1}{2} = 0$;
no decomposition found.

6.211 Symmetry class $\mathcal{S}^2_0$; $y^3y'' + \frac{1}{2}y^3 - \frac{1}{2}xy = 0$; no decomposition found.

6.212 Symmetry class $\mathcal{S}^2_0$; $y^3y'' + yy'' - 3y^2y'^2 + y'^2 = 0$; $z \equiv y' - \frac{1}{2}\frac{y^3+y}{x+C}$;
$z \equiv yy' + (y^2+1)^2C = 0$.

6.214 $g_2 = g_3 = 1$, Symmetry class $\mathcal{S}^2_8$; $y^3y'' - \frac{1}{4}yy'' - \frac{1}{4}y'' - \frac{3}{2}y^2y'^2 + \frac{1}{8}y'^2 = 0$;
$z \equiv y' + 2C\sqrt{y^3 - 1/4y - 1/4} = 0$.

6.219 $a = 1, b = c = 0, d = 1$, Symmetry class $\mathcal{S}^2_1$; $y^4y'' + 2x^2y^2y'' + x^4y'' + y = 0$;
no decomposition found.

6.220 $a=1$, Symmetry class $\mathcal{S}^2_{2,2}$; $yy''^2-1=0$; $z\equiv y'+2\sqrt{\sqrt{y}+\frac{1}{2}C}$.

6.224 $f(y)=g(y)=y$, Symmetry class $\mathcal{S}^2_{2,2}$; $yy''+y'^2+y=0$;
$z\equiv yy'\pm\sqrt{C-\frac{2}{3}y^3}=0$.

6.226 Symmetry class $\mathcal{S}^2_{2,1}$; $y'y''-x^2yy'-xy^2=0$; $z\equiv y'^2-x^2y^2+C=0$.

6.227 Symmetry class $\mathcal{S}^2_{2,1}$; $(xy'-y)y''+4y'^2=0$;
$$z\equiv y+x\frac{\log(y')+C-4}{\log(y')+C}-\frac{\sqrt{y}'}{\log(y')+C}.$$

6.231 Symmetry class $\mathcal{S}^2_1$; $y^2y'y''+\frac{1}{2}x^2y''+yy'^3+\frac{3}{2}xy'+\frac{1}{2}y=0$;
$$z\equiv y'^2+\frac{x^2}{y^2}y'+\frac{x}{y}+C=0.$$

6.232 Symmetry class $\mathcal{S}^2_{2,1}$; $y'^2y''+y^2y''+y^3=0$;
$$z\equiv y'-\frac{\frac{1}{2}\sqrt{3}-\sqrt{\tan\left(\sqrt{3}x+C\right)^2+\frac{3}{4}}}{\tan\left(\sqrt{3}x+C\right)}y=0.$$

6.235 $f(y')=y'$, $g(y)=y$, $h(x)=x$, Symmetry class $\mathcal{S}^2_0$; $y'y''+yy'+x=0$;
$z\equiv y'^2+y^2+x^2+C=0$.

6.236 $a=1, b=1$, Symmetry class $\mathcal{S}^2_{2,2}$; y''^2-y-1; $z\equiv y'\pm\frac{2}{\sqrt{3}}\sqrt{(y+1)^{3/2}+C}$.

6.238 Symmetry class $\mathcal{S}^2_0$; $y''^2-\frac{2x}{x^2+1}y'y''-\frac{\frac{1}{2}x^2}{x^2+1}y''+\frac{1}{x^2+1}(y'^2+xy'-y)=0$;
$$z\equiv y'^2-(2C-\tfrac{1}{2}x^3)y'-x^2y+C^2x^2+C^2+\tfrac{1}{2}Cx^3,$$
$$z\equiv y'^2-(4Cx-x)y'-y+4C^2x^2+4C^2-Cx^2.$$

6.239 Symmetry class $\mathcal{S}^2_{2,1}$; $y''^2-\frac{2}{x}y'y''-\frac{2}{3x^2}yy''+\frac{4}{3x^2}y'^2=0$;
$$z\equiv y'-\frac{C+2x}{C^2+Cx+x^2}y=0.$$

6.240 Symmetry class $\mathcal{S}^2_1$; $y''^2-\frac{4x-\frac{2}{3}}{x^2-\frac{2}{9}}y'y''-\frac{\frac{2}{3}}{x^3-\frac{2}{9}x^2}yy''+\frac{4}{x^2-\frac{2}{9}x}y'^2=0$;
$$z\equiv y'^2-\frac{6}{x}yy'+\frac{2C^2(3x-1)}{x^2}y'+\frac{9}{x^2}y^2-\frac{2C^2(9x-1)}{x^3}y+\frac{C^4(9x-2)}{x^3}=0,$$
$$z\equiv y'^2-C^2x(6x-1)y'+C^2y+C^4x^3(9x-2)=0.$$

6.243 $a=b=1$, Symmetry class $\mathcal{S}^2_1$; $(y^2-1)y''^2-2yy'^2y''+(y^2-1)y'^2=0$;
$z\equiv y'+Cy+\sqrt{C^2-1}=0$.

6.244 Symmetry class $\mathcal{S}^2_{3,1}$; $(x^2yy''-x^2y'^2+y^2)^2-4xy(xy'-y)^3=0$;
$$z\equiv y'-\left(\frac{1}{x}+\frac{1}{(x+C)^2}\right)y=0.$$

Third-order equations of chapter 7

7.4 $a = 1$, Symmetry class $\mathcal{S}^3_{2,2}$; $y''' + yy'' = 0$; no decomposition found.

7.8 Symmetry class $\mathcal{S}^3_{3,7}$; $y^2y''' - \frac{9}{2}yy'y'' + \frac{15}{4}y'^3 = 0$;
$$z \equiv y' + \frac{C_1 + x}{\frac{1}{4}x^2 + \frac{1}{2}C_1x + C_2} = 0.$$

7.9 Symmetry class $\mathcal{S}^3_{3,7}$; $y^2y''' - 5yy'y'' + \frac{40}{9}y'^3 = 0$;
$$z \equiv y' + \frac{C_1 + x}{\frac{1}{3}x^2 + \frac{2}{3}C_1x + C_2} = 0.$$

7.10 Symmetry class $\mathcal{S}^3_{3,6}$; $y'y''' - \frac{3}{2}y''^2 = 0$; $z \equiv y' + \dfrac{C_1x + C_2 + \frac{1}{2}}{C_1x + C_2 - \frac{1}{2}}y$.

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