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Geometry of warped product semi-slant submanifolds of Kenmotsu manifolds

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Abstract In this paper, we study semi-slant submanifolds and their warped products in Kenmotsu manifolds. The existence of such warped products in Kenmotsu manifolds is shown by an example and a characterization. A sharp relation is obtained as a lower bound of the squared norm of second fundamental form in terms of the warping function and the slant angle. The equality case is also considered in this paper. Finally, we provide some applications of our derived results.

Keywords Warped products · Slant submanifolds · Semi-slant submanifolds · Contact CR-warped products · Warped product semi-slant submanifolds · Kenmotsu manifolds

Mathematics Subject Classification 53C15 · 53C40 · 53C42 · 53B25

1 Introduction

Warped product manifolds were defined and studied by Bishop and ONeill [5] as a natural generalization of the Riemannian product manifolds. The geometrical aspects of these manifolds have been studied later by many mathematicians.

The study of warped product submanifolds from extrinsic view points was initiated by Chen [9,10]. He proved that there do not exist warped product CR-submanifolds of the form $M_\perp \times_f M_T,$ where M_T and $M_\perp$ are holomorphic (complex) and totally

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real submanifolds of a Kaehler manifold $\tilde{M}$, respectively. Then he studied warped products of the form $M_T \times_f M_\perp$, known as CR-warped products. Several fundamental results on CR-warped products in Kaehler manifolds were established in [9–11] by him. Motivated by Chens fundamental seminal work, many geometers studied warped product submanifolds in various Riemannian manifolds (see [1,12,14,15,19,20,27–30] and the references therein). For the most recent detailed survey on warped product manifolds and submanifolds, see [13].

The notion of slant submanifolds of almost Hermitian manifolds was introduced by Chen [8]. Later, Cabrerizo et al. studied in [7] slant immersions in K-contact and Sasakian manifolds. In particular, they provided interesting examples of slant submanifolds in both almost contact metric manifolds and Sasakian manifold. In [7], they also characterized slant submanifolds by means of the covariant derivative of the square of the tangent projection on the submanifold. In [6], they defined and studied semi-slant submanifolds of Sasakian manifolds.

The non-existence of warped product semi-slant submanifolds of Kaehler manifolds was proved in [25]. Later, Atceken [3] studied warped product semi-slant submanifolds and proved non-existence of such submanifolds in Kenmotsu manifolds. The warped product semi-slant submanifolds of Kenmotsu manifolds were also studied in [22,26]. For the survey on warped product submanifolds of Kenmotsu manifolds, we refer to [4,21–23].

In this paper, we continue the study of warped product semi-slant submanifolds in Kenmotsu manifolds. In the first part of this paper, we give some fundamental results for semi-slant submanifolds of Kenmotsu manifolds. Then we prove existence of such warped products by applying a characterization result. We also give an example of non-trivial proper warped product semi-slant submanifolds in Kenmotsu manifolds. In the second part, we derive a sharp inequality for the squared norm of the second fundamental form in terms of the warping function and the slant angle. We also investigate the equality case of this inequality. Several applications of our results are present in the last part.

2 Preliminaries

Let $\tilde{M}$ be an almost contact metric manifold with structure (φ, ξ, η, g), where φ is a $(1, 1)$ tensor field, ξ a vector field, η is a 1-form and g is a Riemannian metric on $\tilde{M}$ satisfying the following properties

$$\varphi^2 = -I + \eta \otimes \xi, \quad \varphi\xi = 0, \quad \eta \circ \varphi = 0, \quad \eta(\xi) = 1. \tag{1}$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y). \tag{2}$$

In addition, if the following relation [17]

$$(\tilde{\nabla}_X \varphi)Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X \tag{3}$$

holds for any X, Y on $\tilde{M}$, then $\tilde{M}$ is called a *Kenmotsu manifold*, where $\tilde{\nabla}$ is the Levi-Civita connection of g. It is easy to see from (3) that $\tilde{\nabla}_X\xi = X - \eta(X)\xi$. We shall use the symbol $\Gamma(T\tilde{M})$ for the Lie algebra of vector fields on the manifold $\tilde{M}$.

Let M be a submanifold of an almost contact metric manifold $\tilde{M}$ with induced metric g and if ∇ and $\nabla^{\perp}$ are the induced connections on the tangent bundle TM and the normal bundle $T^{\perp}M$ of M, respectively then Gauss–Weingarten formulas are respectively given by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{4}$$

$$\tilde{\nabla}_X N = -A_N X + \nabla^{\perp}_X N, \tag{5}$$

for each $X,\ Y \in \Gamma(TM)$ and $N \in \Gamma(T^{\perp}M)$, where h and A_N are the second fundamental form and the shape operator (corresponding to the normal vector field N) respectively for the immersion of M into $\tilde{M}$. They are related by the relation $g(A_N X, Y) = g(h(X, Y), N)$.

For any $X \in \Gamma(TM)$ and $N \in \Gamma(T^{\perp}M)$, we write

$$(a)\ \varphi X = PX + FX, \quad (b)\ \varphi N = tN + fN \tag{6}$$

where PX and tN are the tangential components of φX and φN, respectively and FX and fN are the normal components of φX and φN, respectively.

A submanifold M is said to be *invariant* if F is identically zero, that is, $\varphi X \in \Gamma(TM)$ for any $X \in \Gamma(TM)$. On the other hand, M is *anti-invariant* if P is identically zero, that is, $\varphi X \in \Gamma(T^{\perp}M)$, for any $X \in \Gamma(TM)$.

Let M be a submanifold tangent to the structure vector field ξ isometrically immersed into an almost contact metric manifold $\tilde{M}$. Then M is said to be a contact CR-submanifold if there exists a pair of orthogonal distributions $\mathcal{D} : p \to \mathcal{D}_p$ and $\mathcal{D}^{\perp} : p \to \mathcal{D}^{\perp}_p$, $\forall\ p \in M$ such that

(i) $TM = \mathcal{D} \oplus \mathcal{D}^{\perp} \oplus \langle\xi\rangle$, where $\langle\xi\rangle$ is the 1-dimensional distribution spanned by the structure vector field ξ,
(ii) $\mathcal{D}$ is invariant, i.e., $\varphi\mathcal{D} = \mathcal{D}$,
(iii) $\mathcal{D}^{\perp}$ is anti-invariant, i.e., $\varphi\mathcal{D}^{\perp} \subseteq T^{\perp}M$.

The invariant and anti-invariant submanifolds are the special cases of a contact CR-submanifold. If we denote the dimensions of $\mathcal{D}$ and $\mathcal{D}^{\perp}$ by d_1 and d_2, respectively then M is *invariant* (resp. *anti-invariant*) if $d_2 = 0$ (resp. $d_1 = 0$).

There is another class of submanifolds which are known as slant submanifolds which we define as follows:

For each non zero vector X tangent to M at p, such that X is not proportional to ξ, we denote by $\theta(X)$, the angle between φX and T_pM, for all $p \in M$. Then, M is said to be *slant* [7] if the angle $\theta(X)$ is constant for all $X \in TM - \{\xi\}$ and $p \in M$ i.e., $\theta(X)$ is independent of the choice of the vector field X and the point $p \in M$. The angle $\theta(X)$ is called the *slant angle*. Obviously, if $\theta = 0$ then, M is invariant and if $\theta = \pi/2$ then, M is an anti-invariant submanifold. If the slant angle of M is neither 0 nor $\pi/2$, then it is called *proper slant*.

A characterization of slant submanifolds was given in [7] as follows:

Theorem 1 [7] *Let M be a submanifold of an almost contact metric manifold $\tilde{M}$ such that $\xi \in \Gamma(TM)$. Then M is slant if and only if there exists a constant $\lambda \in [0, 1]$ such that*

$$P^2 = \lambda(-I + \eta \otimes \xi). \tag{7}$$

Furthermore, in such case, if θ is slant angle, then $\lambda = \cos^2 \theta$.

The following relations are straight forward consequences of (7)

$$g(PX, PY) = \cos^2 \theta[g(X, Y) - \eta(X)\eta(Y)] \tag{8}$$

$$g(FX, FY) = \sin^2 \theta[g(X, Y) - \eta(X)\eta(Y)] \tag{9}$$

for any X, Y tangent to M.

3 Semi-slant submanifolds

In [24], semi-slant submanifolds were defined and studied by Papaghiuc as a natural generalization of CR-submanifolds of almost Hermitian manifolds in terms of the slant distribution. Later on, Cabrerizo et al. [6] studied these submanifolds in contact geometry. They defined these submanifolds as follows:

Definition 1 [6] A Riemannian submanifold M of an almost contact manifold $\tilde{M}$ is said to be a *semi-slant submanifold* if there exist two orthogonal distributions $\mathcal{D}$ and $\mathcal{D}^\theta$ such that $TM = \mathcal{D} \oplus \mathcal{D}^\theta \oplus \langle\xi\rangle$, the distribution $\mathcal{D}$ is invariant i.e., $\varphi\mathcal{D} = \mathcal{D}$ and the distribution $\mathcal{D}^\theta$ is slant with slant angle $\theta \neq \frac{\pi}{2}$.

If we denote the dimensions of $\mathcal{D}$ and $\mathcal{D}^\theta$ by d_1 and d_2 respectively, then it is clear that contact CR-submanifolds and slant submanifolds are semi-slant submanifolds with $\theta = \frac{\pi}{2}$ and $d_1 = 0$, respectively. If neither $d_1 = 0$ nor $\theta = \frac{\pi}{2}$, then M is a *proper semi-slant* submanifold.

Moreover, if ν is the $\varphi-$invariant subspace of the normal bundle $T^\perp M$, then in case of a semi-slant submanifold, the normal bundle $T^\perp M$ can be decomposed as $T^\perp M = F\mathcal{D}^\theta \oplus \nu$.

First, we give the following non-trivial example of a semi-slant submanifold of an almost contact metric manifold.

Example 1 Consider a submanifold M of $\mathbb{R}^9$ with the cartesian coordinates $(x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4, z)$ and the contact structure

$$\varphi\left(\frac{\partial}{\partial x_i}\right) = -\frac{\partial}{\partial y_i}, \quad \varphi\left(\frac{\partial}{\partial y_j}\right) = \frac{\partial}{\partial x_j}, \quad \varphi\left(\frac{\partial}{\partial z}\right) = 0, \quad 1 \leq i, j \leq 4.$$

It is easy to show that (φ, ξ, η, g) is an almost contact metric structure on $\mathbb{R}^9$ with $\xi = \frac{\partial}{\partial z}$, $\eta = dz$ and g, the Euclidean metric of $\mathbb{R}^9$. Consider an immersion ψ on $\mathbb{R}^9$ defined by

$$\psi(\theta,\,\phi,\,v,\,w,\,z)$$
$$=\left(\cos(\theta+\phi),\,\theta-\phi,\,\frac{1}{2}\theta+\phi,\,v+w,\,\sin(\theta+\phi),\,\phi-\theta,\,\theta+\frac{1}{2}\phi,\,w-v,\,z\right).$$

If we put

$$X_1=-\sin(\theta+\phi)\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_2}+\frac{1}{2}\frac{\partial}{\partial x_3}+\cos(\theta+\phi)\frac{\partial}{\partial y_1}-\frac{\partial}{\partial y_2}+\frac{\partial}{\partial y_3},$$
$$X_2=-\sin(\theta+\phi)\frac{\partial}{\partial x_1}-\frac{\partial}{\partial x_2}+\frac{\partial}{\partial x_3}+\cos(\theta+\phi)\frac{\partial}{\partial y_1}+\frac{\partial}{\partial y_2}+\frac{1}{2}\frac{\partial}{\partial y_3},$$
$$X_3=\frac{\partial}{\partial x_4}-\frac{\partial}{\partial y_4},\;X_4=\frac{\partial}{\partial x_4}+\frac{\partial}{\partial y_4},\;X_5=\frac{\partial}{\partial z}$$

then the restriction of $\{X_1, X_2, X_3, X_4, X_5\}$ to M forms an orthogonal frame fields of the tangent bundle TM. Clearly, we have

$$\varphi X_1=\sin(\theta+\phi)\frac{\partial}{\partial y_1}-\frac{\partial}{\partial y_2}-\frac{1}{2}\frac{\partial}{\partial y_3}+\cos(\theta+\phi)\frac{\partial}{\partial x_1}-\frac{\partial}{\partial x_2}+\frac{\partial}{\partial x_3},$$
$$\varphi X_2=\sin(\theta+\phi)\frac{\partial}{\partial y_1}+\frac{\partial}{\partial y_2}-\frac{\partial}{\partial y_3}+\cos(\theta+\phi)\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_2}+\frac{1}{2}\frac{\partial}{\partial x_3},$$
$$\varphi X_3=-\frac{\partial}{\partial y_4}-\frac{\partial}{\partial x_4},\;\varphi X_4=-\frac{\partial}{\partial y_4}+\frac{\partial}{\partial x_4},\;\varphi X_5=0.$$

It is easy to verify that $\mathcal{D}=\mathrm{Span}\{\mathrm{X}_3,\,\mathrm{X}_4\}$ is an invariant distribution and $\mathcal{D}^\theta=\mathrm{Span}\{\mathrm{X}_1,\,\mathrm{X}_2\}$ is a slant distribution of M with slant angle $\theta=\cos^{-1}\left(\frac{3}{17}\right)$ such that $X_5=\xi=\frac{\partial}{\partial z}$ is tangent to M. Thus, M is a proper semi-slant submanifold of $\mathbb{R}^9$.

Now, we give some basic results for semi-slant submanifolds of Kenmotsu manifolds which are useful to the next section.

Lemma 1 *Let M be a proper semi-slant submanifold of a Kenmotsu manifold $\tilde{M}$ with invariant and proper slant distributions $\mathcal{D}\oplus\langle\xi\rangle$ and $\mathcal{D}^\theta$, respectively. Then we have:*

$$\sin^2\theta\, g(\nabla_Y X, Z)=g(A_{FZ}\varphi X-A_{FPZ}X, Y) \tag{10}$$

for any $X, Y\in\Gamma(\mathcal{D}\oplus\langle\xi\rangle)$ and $Z\in\Gamma(\mathcal{D}^\theta)$.

Proof For any $X, Y\in\Gamma(\mathcal{D}\oplus\langle\xi\rangle)$ and $Z\in\Gamma(\mathcal{D}^\theta)$, we have

$$g(\nabla_Y X, Z)=g(\tilde{\nabla}_Y X, Z)=g(\varphi\tilde{\nabla}_Y X, \varphi Z).$$

Using the covariant derivative property of φ and the relation (6), we get

$$g(\nabla_Y X, Z)=g(\tilde{\nabla}_Y\varphi X, PZ)+g(\tilde{\nabla}_Y\varphi X, FZ)-g((\tilde{\nabla}_Y\varphi)X, \varphi Z).$$

Then from (3), (4) and the fact that φX and PZ are orthogonal vector fields, we derive

$$\begin{aligned} g(\nabla_Y X, Z) &= -g(\varphi X, \tilde{\nabla}_Y PZ) + g(h(\varphi X, Y), FZ) \\ &= g(X, \tilde{\nabla}_Y \varphi PZ) - g(X, (\tilde{\nabla}_Y \varphi)PZ) + g(A_{FZ}\varphi X, Y) \\ &= g(X, \tilde{\nabla}_Y P^2 Z) + g(X, \tilde{\nabla}_Y FPZ) - g((\tilde{\nabla}_Y \varphi)PZ, X) + g(A_{FZ}\varphi X, Y). \end{aligned}$$

Again using (3), (5) and (7) and the fact that $\xi \in \Gamma(\mathcal{D})$, we find

$$g(\nabla_Y X, Z) = -\cos^2\theta g(X, \tilde{\nabla}_Y Z) - g(A_{FPZ}X, Y) + g(A_{FZ}\varphi X, Y).$$

Hence, the result follows from the last relation. □

The following corollary is an immediate consequence of the above lemma.

Corollary 1 *Let M be a proper semi-slant submanifold of a Kenmotsu manifold $\tilde{M}$. Then, the distribution $\mathcal{D} \oplus \langle \xi \rangle$ defines a totally geodesic foliation if and only if*

$$g(A_{FZ}\varphi X - A_{FPZ}X, Y) = 0$$

for any $X, Y \in \Gamma(\mathcal{D} \oplus \langle \xi \rangle)$ and $Z \in \Gamma(\mathcal{D}^\theta)$.

Also, we have the following results for the leaves of the slant distribution.

Lemma 2 *Let M be a proper semi-slant submanifold of a Kenmotsu manifold $\tilde{M}$. Then we have*

$$g(\nabla_Z W, X) = \csc^2\theta \left(g(A_{FPW}X - A_{FW}\varphi X, Z)\right) - \eta(X)g(Z, W) \tag{11}$$

for any $X \in \Gamma(\mathcal{D} \oplus \langle \xi \rangle)$ and $Z, W \in \Gamma(\mathcal{D}^\theta)$.

Proof For any $X \in \Gamma(\mathcal{D} \oplus \langle \xi \rangle)$ and $Z, W \in \Gamma(\mathcal{D}^\theta)$, we find

$$g(\nabla_Z W, X) = g(\tilde{\nabla}_Z \varphi W, \varphi X) + \eta(X)g(\tilde{\nabla}_Z W, \xi).$$

Using (3) and (6), we get

$$g(\nabla_Z W, X) = g(\tilde{\nabla}_Z PW, \varphi X) + g(\tilde{\nabla}_Z FW, \varphi X) - \eta(X)g(Z, W).$$

Then from (2) and (5), we arrive at

$$g(\nabla_Z W, X) = -g(\varphi\tilde{\nabla}_Z PW, X) - g(A_{FW}Z, \varphi X) - \eta(X)g(Z, W).$$

By using the covariant derivative property of φ and the symmetry of the shape operator A, we obtain

$$\begin{aligned} g(\nabla_Z W, X) = g((\tilde{\nabla}_Z \varphi)PW, X) - g(\tilde{\nabla}_Z \varphi PW, X) - g(A_{FW}\varphi X, Z) \\ - \eta(X)g(Z, W). \end{aligned}$$

Again using (3) and (6), we derive

$$g(\nabla_Z W, X) = -\, g(\tilde{\nabla}_Z P^2 W, X) - g(\tilde{\nabla}_Z FPW, X) + g(PZ, PW)\eta(X) \\ - g(A_{FW}\varphi X, Z) - \eta(X)g(Z, W).$$

From Theorem 1 and the relation (5), we find

$$g(\nabla_Z W, X) = \cos^2\theta g(\tilde{\nabla}_Z W, X) + g(A_{FPW}Z, X) + \cos^2\theta g(Z, W)\eta(X) \\ - g(A_{FW}\varphi X, Z) - \eta(X)g(Z, W).$$

Since A is self-adjoint, then by using trigonometric identities, the result follows from the last relation. □

From the above lemma, we have the following results.

Corollary 2 *Let M be a proper semi-slant submanifold of a Kenmotsu manifold $\tilde{M}$. Then the slant distribution $\mathcal{D}^\theta$ defines a totally geodesic foliation if and only if*

$$g(A_{FPZ}X - A_{FZ}\varphi X, W) = \eta(X)g(Z, W)$$

for any $X \in \Gamma(\mathcal{D} \oplus \langle\xi\rangle)$ and $Z, W \in \Gamma(\mathcal{D}^\theta)$.

Lemma 3 *Let M be a proper semi-slant submanifold of a Kenmotsu manifold $\tilde{M}$. Then we have:*

$$\sin^2\theta\, g([Z, W], X) = g(A_{FZ}\varphi X - A_{FPZ}X, W) - g(A_{FW}\varphi X - A_{FPW}X, Z)$$

for any $X \in \Gamma(\mathcal{D} \oplus \langle\xi\rangle)$ and $Z, W \in \Gamma(\mathcal{D}^\theta)$.

Proof From (11), we have

$$\sin^2\theta\, g(\tilde{\nabla}_Z W, X) = g(A_{FPW}X - A_{FW}\varphi X, Z) - \sin^2\theta\, \eta(X)g(Z, W) \tag{12}$$

for any $X \in \Gamma(\mathcal{D} \oplus \langle\xi\rangle)$ and $Z, W \in \Gamma(\mathcal{D}^\theta)$. Interchanging Z and W in (12), we find

$$\sin^2\theta\, g(\tilde{\nabla}_W Z, X) = g(A_{FPZ}X - A_{FZ}\varphi X, W) - \sin^2\theta\, \eta(X)g(Z, W). \tag{13}$$

Thus, the result follows from (12) and (13).

4 Warped product semi-slant submanifolds

In this section, we study warped product semi-slant submanifolds of Kenmotsu manifolds, by considering that one of the factor is a slant submanifold. In the following first, we give brief introduction of warped product manifolds.

Let M_1 and M_2 be two Riemannian manifolds with Riemannian metrics g_1 and g_2, respectively, and f be a positive differentiable function on M_1. Then, $M = M_1 \times_f M_2 = (M_1 \times M_2, g)$, is a warped product manifold of M_1 and M_2 such that

$$g(X, Y) = g_1(\pi_{1\star}X, \pi_{1\star}Y) + (f \circ \pi_1)^2 g_2(\pi_{2\star}X, \pi_{2\star}Y) \tag{14}$$

where the vector fields X and Y are tangent to $M = M_1 \times_f M_2$ at (p, q) and π_1 and π_2 are the canonical projections of $M = M_1 \times M_2$ onto M_1 and M_2, respectively and $\star$ is the symbol for the tangent map. A warped product manifold $M = M_1 \times_f M_2 = (M_1 \times M_2, g)$ is said to be *trivial* or simply a *Riemannian product* if the warping function f is constant. If X is a vector field on M_1 and V is an another vector field on M_2, then from Lemma 7.3 of [5], we have

$$\nabla_X V = \nabla_V X = X(\ln f)V \tag{15}$$

where ∇ denotes the Levi-Civita connection on M. It is well-known that M_1 is a totally geodesic submanifold and M_2 is a totally umbilical submanifold of M (cf. [5,11]).

The gradient ∇f of a function f on M is defined as $g(\nabla f, X) = X(f)$, for any vector field X on M. If $\{e_1, \dots e_n\}$ is an orthonormal frame field of the tangent space of M, then we have

$$\|\nabla f\|^2 = \sum_{i=1}^{n} (e_i(f))^2. \tag{16}$$

In this paper, we study warped product semi-slant submanifolds of the form $M_T \times_f M_\theta$ of a Kenmotsu manifold $\tilde{M}$, where M_T and M_θ are invariant and proper slant submanifolds of $\tilde{M}$, respectively. First, we consider M_1 and M_2 be two Riemannian submanifolds of a Kenmotsu manifold $\tilde{M}$. Then their warped product submanifold M is of the form $M_1 \times_f M_2$. Since, we consider the structure vector field ξ is tangent to M, therefore two possible cases arise:

Case 1 When the structure vector field ξ is tangent to M_2, then for any $X \in \Gamma(TM_1)$, we have $\tilde{\nabla}_X \xi = X$. From (4) and (15), we find that $X(\ln f)\xi = X$, by taking the inner product with ξ, we observe that f is constant, i.e., the warped product $M_1 \times_f M_2$ becomes a Riemannian product (trivial).

Case 2 When the structure vector field ξ is tangent to M_1, then for any $Z \in \Gamma(TM_2)$, we have

$$\tilde{\nabla}_Z \xi = \nabla_Z \xi + h(Z, \xi).$$

Then from (3) and (15), we find that

$$(i)\ \ \xi \ln f = 1, \qquad\qquad (ii)\ \ h(Z, \xi) = 0, \ \ \forall\, Z \in \Gamma(TM_2). \tag{17}$$

Now, in the following we consider the warped products of the form $M_T \times_f M_\theta$, called warped product semi-slant submanifolds of a Kenmotsu manifold $\tilde{M}$ such that

ξ is tangent to M_T, where M_T and M_θ are invariant and proper slant submanifolds of $\tilde{M}$, respectively. If neither dim M_T is zero nor the slant angle of M_θ is $\frac{\pi}{2}$, then the warped product semi-slant submanifold is called proper. It is clear that the contact CR-warped product submanifolds are the special cases of warped product semi-slant submanifolds.

First, we give the following non-trivial example of warped product semi-slant submanifolds in Kenmotsu manifolds.

Example 2 Consider the complex Euclidean space $\mathbb{C}^4$ with its usual Kaehler structure and the real global coordinates $(x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4)$. Let $\tilde{M} = \mathbb{R} \times_f \mathbb{C}^4$ be a warped product manifold between the product of real line $\mathbb{R}$ and the complex space $\mathbb{C}^4$. In fact, $\tilde{M}$ is a Kenmotsu manifold with the almost contact metric structure (φ, ξ, η, g) such that

$$\varphi\left(\frac{\partial}{\partial x_i}\right) = -\frac{\partial}{\partial y_i}, \quad \varphi\left(\frac{\partial}{\partial y_j}\right) = \frac{\partial}{\partial x_j}, \quad \varphi\left(\frac{\partial}{\partial z}\right) = 0, \quad 1 \leq i, j \leq 4,$$

and

$$\xi = e^z\left(\frac{\partial}{\partial z}\right), \quad \eta = e^z dz, \qquad g = e^{2z}\langle , \rangle$$

where $\langle , \rangle$ denotes the Euclidean metric tensor of $\mathbb{R}^9$. Consider a submanifold M defined by the immersion ϕ as follows

$$\phi(u_1, u_2, u_3, u_4, z) = (u_1, 0, u_3, 0, u_2, 0, u_4 \cos\theta, u_4 \sin\theta, z)$$

with $\theta \in (0, \pi/2)$. Then the tangent space TM of M at any point is spanned by the following vectors

$$X_1 = \frac{\partial}{\partial x_1} + y_1\left(\frac{\partial}{\partial z}\right), \quad X_2 = \frac{\partial}{\partial y_1}, \quad X_3 = \frac{\partial}{\partial x_3} + y_3\left(\frac{\partial}{\partial z}\right),$$
$$X_4 = \cos\theta\frac{\partial}{\partial y_3} + \sin\theta\frac{\partial}{\partial y_4}, \quad X_5 = \frac{\partial}{\partial z} = \frac{1}{e^z}\xi.$$

Then, we find

$$\varphi X_1 = -\frac{\partial}{\partial y_1}, \quad \varphi X_2 = \frac{\partial}{\partial x_1}, \quad \varphi X_3 = -\frac{\partial}{\partial y_3},$$
$$\varphi X_4 = \cos\theta\frac{\partial}{\partial x_3} + \sin\theta\frac{\partial}{\partial x_4}, \quad \varphi X_5 = 0.$$

Thus, M is a proper semi-slant submanifold tangent to the structure vector field ξ with invariant and proper slant distributions $\mathcal{D} = \mathrm{Span}\{\mathrm{X}_1, \mathrm{X}_2\}$ and $\mathcal{D}^\theta = \mathrm{Span}\{\mathrm{X}_3, \mathrm{X}_4\}$ respectively with slant angle θ. It is easy to see that the distributions $\mathcal{D}$ and $\mathcal{D}^\theta$ are integrable. Consider, the integral manifolds corresponding to the distributions $\mathcal{D}$ and

$\mathcal{D}^\theta$ by M_T and M_θ, respectively. Then it is easy to check that $M = M_T \times_f M_\theta$ is a proper warped product semi-slant submanifold isometrically immersed in $\tilde{M}$ with warping function $f = e^z$, $z \in \mathbb{R}$.

Now, we have the following useful lemma for later use.

Lemma 4 *Let $M = M_T \times {}_f M_\theta$ be a warped product semi-slant submanifold of a Kenmotsu manifold $\tilde{M}$ such that ξ is tangent to M_T, where M_T is an invariant submanifold and M_θ is a proper slant submanifold of $\tilde{M}$. Then, we have*

$$g(h(X, Z), FW) = \big(\eta(X) - X(\ln f)\big)g(Z, PW) - \varphi X(\ln f)\, g(Z, W) \tag{18}$$

for any $X \in \Gamma(TM_T)$ and $Z, W \in \Gamma(TM_\theta)$.

Proof For any $X \in \Gamma(TM_T)$ and $Z, W \in \Gamma(TM_\theta)$, we have

$$\begin{aligned} g(h(X, Z), FW) &= g(\tilde{\nabla}_Z X, \varphi W) - g(\tilde{\nabla}_Z X, PW) \\ &= -g(\varphi\tilde{\nabla}_Z X, W) - g(\nabla_Z X, PW). \end{aligned}$$

Using a covariant derivative property of φ and (15), we find

$$g(h(X, Z), FW) = g((\tilde{\nabla}_Z \varphi)X, W) - g(\tilde{\nabla}_Z \varphi X, W) - X(\ln f)\, g(Z, PW).$$

Again using (3), (4) and (15), we get the desired result. □

The following relations can be easily obtained by interchanging X by φX and Z by PZ and W by PW in (18), for any $X \in \Gamma(TM_T)$ and $Z, W \in \Gamma(TM_\theta)$

$$\begin{aligned} g(h(\varphi X, Z), FW) &= \big(X(\ln f) - \eta(X)\big)g(Z, W) - \varphi X(\ln f)\, g(Z, PW), \\ g(h(X, PZ), FW) &= \varphi X(\ln f)\, g(Z, PW) \\ &\quad - \cos^2\theta\big(X(\ln f) - \eta(X)\big)g(Z, W), \qquad (19) \\ g(h(X, Z), FPW) &= \cos^2\theta\big(X(\ln f) - \eta(X)\big)g(Z, W) \\ &\quad -\varphi X(\ln f)\, g(Z, PW), \qquad (20) \end{aligned}$$

and

$$\begin{aligned} g(h(X, PZ), FPW) = &-\cos^2\theta\varphi X(\ln f) g(Z, W) \\ &- \cos^2\theta\big(X(\ln f) - \eta(X)\big)\, g(Z, PW). \end{aligned} \tag{21}$$

From (19) and (20), we have

$$g(h(X, PZ), FW) = -g(h(X, Z), FPW).$$

On the other hand, for any $X, Y \in \Gamma(TM_T)$ and $Z \in \Gamma(TM_\theta)$, we have by Lemma 4.1 (i) of [22]

$$g(h(X,Y), FZ) = 0. \tag{22}$$

In order to give a characterization result for semi-slant submanifolds of a Kenmotsu manifold, we recall the following result of Hiepko [16]:

Hiepko's Theorem. Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be two orthogonal distribution on a Riemannian manifold M. Suppose that $\mathcal{D}_1$ and $\mathcal{D}_2$ both are involutive such that $\mathcal{D}_1$ is a totally geodesic foliation and $\mathcal{D}_2$ is a spherical foliation. Then M is locally isometric to a non-trivial warped product $M_1 \times_f M_2$, where M_1 and M_2 are integral manifolds of $\mathcal{D}_1$ and $\mathcal{D}_2$, respectively.

Now, we are able to prove the following main result of this section.

Theorem 2 *Let M be a proper semi-slant submanifold with invariant distribution $\mathcal{D} \oplus \langle\xi\rangle$ and proper slant distribution $\mathcal{D}^\theta$ of a Kenmotsu manifold $\tilde{M}$. Then M is locally a warped product submanifold of the form $M_T \times_f M_\theta$ if and only if*

$$A_{FZ}\varphi X - A_{FPZ}X = \sin^2\theta\big(X(\mu) - \eta(X)\big)Z \tag{23}$$

for any $X \in \Gamma(\mathcal{D} \oplus \langle\xi\rangle)$ and $Z \in \Gamma(\mathcal{D}^\theta)$ and for some function μ on M satisfying $W\mu = 0$, for any $W \in \Gamma(\mathcal{D}^\theta)$.

Proof If $M = M_T \times_f M_\theta$ is a warped product submanifold of a Kenmotsu manifold $\tilde{M}$ such that M_T is an invariant submanifold and M_θ is a proper slant submanifold of $\tilde{M}$, then from (22), we have $g(A_{FZ}\varphi X, Y) = 0$, for any $X, Y \in \Gamma(TM_T)$ and $Z \in \Gamma(TM_\theta)$, i.e., $A_{FZ}\varphi X$ has no components in TM_T. Also, if we interchange Z by PZ in (22), then we have $g(A_{FPZ}X, Y) = 0$, i.e., $A_{FPZ}X$ also has no components TM_T. Therefore, $A_{FZ}\varphi X - A_{FPZ}X$ lies in TM_θ only. On the other hand, for any $X, Y \in \Gamma(TM_T)$ and $Z, W \in \Gamma(TM_\theta)$, we have

$$g(A_{FZ}\varphi X, W) = g(h(\varphi X, W), FZ) = g(\tilde{\nabla}_W \varphi X, FZ).$$

From the covariant derivative proper of φ, we find

$$g(A_{FZ}\varphi X, W) = g((\tilde{\nabla}_W \varphi)X, FZ) + g(\varphi\tilde{\nabla}_W X, \varphi Z) - g(\varphi\tilde{\nabla}_W X, PZ).$$

Using (2), (3), (6), (9) and (15), we obtain

$$\begin{aligned} g(A_{FZ}\varphi X, W) = &-\sin^2\theta\,\eta(X)\,g(Z,W) + X(\ln f)g(Z,W) + g(\tilde{\nabla}_W X, P^2 Z) \\ &+ g(\tilde{\nabla}_W X, FPZ). \end{aligned}$$

Then from (4), Theorem 1 and (15), we find

$$\begin{aligned} g(A_{FZ}\varphi X, W) = &-\sin^2\theta\,\eta(X)\,g(Z,W) + X(\ln f)g(Z,W) \\ &-\cos^2\theta\,X(\ln f)\,g(Z,W) + g(A_{FPZ}X, W). \end{aligned}$$

Then (23) follows from the above relation with $\mu = \ln f$.

Conversely, if M is a proper semi-slant submanifold of a Kenmotsu manifold $\tilde{M}$ such that (23) holds, then from Lemma 1 and the given condition (23), we conclude that $\sin^2\theta\, g(\nabla_Y X, Z) = 0$, for any $X, Y \in \Gamma(\mathcal{D} \oplus \langle\xi\rangle)$ and $Z \in \Gamma(\mathcal{D}^\theta)$. Since M is a proper semi-slant submanifold, then $\sin^2\theta \neq 0$, therefore $g(\nabla_Y X, Z) = 0$, i.e., the leaves of the distribution $\mathcal{D} \oplus \langle\xi\rangle$ are totally geodesic in M. Also, from Lemma 3 and the given condition (23), we find that $\sin^2\theta\, g([Z, W], X) = 0$, for any $X \in \Gamma(\mathcal{D} \oplus \langle\xi\rangle)$ and $Z, W \in \Gamma(\mathcal{D}^\theta)$. Since, M is a proper semi-slant submanifold, thus we have $g([Z, W], X) = 0$, i.e., the slant distribution $\mathcal{D}^\theta$ is integrable. If we consider h^θ be the second fundamental form of a leaf M_θ of $\mathcal{D}^\theta$ in M, then for any $Z, W \in \Gamma(\mathcal{D}^\theta)$ and $X \in \Gamma(\mathcal{D} \oplus \langle\xi\rangle)$, we have

$$g\left(h^\theta(Z, W), X\right) = g(\nabla_Z W, X).$$

Using Lemma 2, we derive

$$g\left(h^\theta(Z, W), X\right) = -\csc^2\theta\, g(A_{FW}\varphi X - A_{FPW}X, Z) - \eta(X)g(Z, W).$$

From the given condition (23), we find

$$g\left(h^\theta(Z, W), X\right) = -X(\mu)g(Z, W).$$

Then, from the definition of gradient, we get

$$h^\theta(Z, W) = -\boldsymbol{\nabla}(\mu)\, g(Z, W),$$

which means that M_θ is totally umbilical in M with mean curvature vector $H^\theta = -\boldsymbol{\nabla}(\mu)$. Now we prove that H^θ is parallel corresponding to the normal connection D^N of M_θ in M. Consider any $Y \in \Gamma(\mathcal{D} \oplus \langle\xi\rangle)$ and $Z \in \Gamma(\mathcal{D}^\theta)$, then we find that $g(D^N_Z\boldsymbol{\nabla}\mu, Y) = g(\nabla_Z\boldsymbol{\nabla}\mu, Y) = Zg(\boldsymbol{\nabla}\mu, Y) - g(\boldsymbol{\nabla}\mu, \nabla_Z Y) = Z(Y(\mu)) - g(\boldsymbol{\nabla}\mu, [Z, Y]) - g(\boldsymbol{\nabla}\mu, \nabla_Y Z) = Y(Z\mu) + g(\nabla_Y\boldsymbol{\nabla}\mu, Z)\} = 0$, since $Z(\mu) = 0$, for all $Z \in \Gamma(\mathcal{D}^\theta)$ and thus $\nabla_Y\boldsymbol{\nabla}\mu \in \Gamma(\mathcal{D} \oplus \langle\xi\rangle)$. This means that the mean curvature of M_θ is parallel. Thus the leaves of the distribution $\mathcal{D}^\theta$ are totally umbilical in M with non-vanishing parallel mean curvature vector H^θ i.e., M_θ is an extrinsic sphere in M. Then from Hiepko's Theorem [16], M is a warped product submanifold, which proves the theorem completely.

Now, we construct the following orthonormal frame fields for a proper warped product semi-slant submanifold $M = M_T \times_f M_\theta$ of a $(2m+1)$-dimensional Kenmotsu manifold $\tilde{M}$ such that the structure vector field ξ is tangent to M_T. Consider $M = M_T \times_f M_\theta$ be an n-dimensional warped product semi-slant submanifold of a Kenmotsu manifold $\tilde{M}$. If $\dim M_T = 2t + 1$ and $\dim M_\theta = 2s$, then $n = 2t + 1 + 2s$. Let us consider the orthonormal frame fields of the corresponding tangent bundles $\mathcal{D}$ and $\mathcal{D}^\theta$ of M_T and M_θ, respectively as: $\{e_1, \ldots, e_t, e_{t+1} = \varphi e_1, \ldots, e_{2t} = \varphi e_t, e_{2t+1} = \xi\}$ is the orthonormal frame field of $\mathcal{D}$ and $\{e_{2t+2} = e_1^\star, \ldots, e_{2t+1+s} = e_s^\star, e_{2t+2+s} = e_{s+1}^\star = \sec\theta\, Pe_1^\star, \ldots, e_n = e_{2s}^\star = \sec\theta\, Pe_s^\star\}$ is the orthonormal frame field of $\mathcal{D}^\theta$.

Then the orthonormal frame fields in the normal bundle $T^{\perp}M$ of $F\mathcal{D}^{\theta}$ and invariant normal subbundle ν respectively are $\{e_{n+1} = \tilde{e}_1 = \csc\theta\, Fe_1^{\star} \ldots, e_{n+s} = \tilde{e}_s = \csc\theta\, Fe_s^{\star}, e_{n+s+1} = \tilde{e}_{s+1} = \csc\theta\sec\theta\, FPe_1^{\star}, \ldots, e_{n+2s} = \tilde{e}_{2s} = \csc\theta\sec\theta\, FPe_s^{\star}\}$ and $\{e_{n+2s+1} = \tilde{e}_{2s+1}, \ldots, e_{2m+1} = \tilde{e}_{2m+1-n-2s}\}$.

Next, we use the above constructed frame fields to find a relation (lower bound) for the squared norm of the second fundamental form $\|h\|^2$, in terms of the warping function and the slant angle of a proper warped product semi-slant submanifold of Kenmotsu manifolds.

Theorem 3 *Let $M = M_T \times_f M_\theta$ be a proper warped product semi-slant submanifold of a Kenmotsu manifold $\tilde{M}$ such that the structure vector field ξ is tangent to M_T, where M_T is an invariant submanifold and M_θ is a proper slant submanifold of $\tilde{M}$. Then*

(i) *The squared norm of the second fundamental form h of M satisfies*

$$\|h\|^2 \geq 4s\left(\csc^2\theta + \cot^2\theta\right)\left(\|\nabla^T(\ln f)\|^2 - 1\right) \tag{24}$$

where $\nabla^T(\ln f)$ is the gradient of the warping function $\ln f$ along M_T and $2s = \dim M_\theta$.

(ii) *If the equality sign in (i) holds, then M_T is a totally geodesic submanifold and M_θ is a totally umbilical submanifold of $\tilde{M}$. Furthermore, M is minimal in $\tilde{M}$.*

Proof From the definition of h, we have

$$\begin{aligned}
\|h\|^2 &= \sum_{i,j=1}^{n} g\left(h\left(e_i, e_j\right), h\left(e_i, e_j\right)\right) \\
&= \sum_{r=n+1}^{2m+1}\sum_{i,j=1}^{n} g\left(h\left(e_i, e_j\right), e_r\right)^2 \\
&= \sum_{r=1}^{2s}\sum_{i,j=1}^{n} g\left(h\left(e_i, e_j\right), \tilde{e}_r\right)^2 + \sum_{r=2s+1}^{2m+1-n-2s}\ \sum_{i,j=1}^{n} g\left(h\left(e_i, e_j\right), \tilde{e}_r\right)^2.
\end{aligned} \tag{25}$$

First term in the right hand side of (25) has $F\mathcal{D}^{\theta}$-components and the second term has ν-components. Let us compute $F\mathcal{D}^{\theta}$-components terms only by using the frame fields of $\mathcal{D}$ and $\mathcal{D}^{\theta}$. Then we have

$$\begin{aligned}
\|h\|^2 \geq \sum_{r=1}^{2s}\sum_{i,j=1}^{2t+1} g\left(h\left(e_i, e_j\right), \tilde{e}_r\right)^2 + 2\sum_{r=1}^{2s}\sum_{i=1}^{2t+1}\sum_{j=1}^{2s} g\left(h\left(e_i, e_j^{\star}\right), \tilde{e}_r\right)^2 \\
+ \sum_{r=1}^{2s}\sum_{i,j=1}^{2s} g\left(h\left(e_i^{\star}, e_j^{\star}\right), \tilde{e}_r\right)^2.
\end{aligned} \tag{26}$$

Then from (22), the first term in the right hand side of above inequality is identically zero. Let us compute just next term

$$\|h\|^2 \geq 2\sum_{r=1}^{2s}\sum_{i=1}^{2t+1}\sum_{j=1}^{2s} g\left(h\left(e_i, e_j^{\star}\right), \tilde{e}_r\right)^2.$$

$$= 2\sum_{r,j=1}^{2s}\sum_{i=1}^{2t} g\left(h\left(e_i, e_j^{\star}\right), \tilde{e}_r\right)^2 + 2\sum_{r,j=1}^{2s} g\left(h\left(\xi, e_j^{\star}\right), \tilde{e}_r\right)^2.$$

Since for a submanifold M of a Kenmotsu manifold $\tilde{M}$, we have $h(X, \xi) = 0$, for any $X \in \Gamma(TM)$. By using this fact the last term in above inequality is identically zero. Then from the assumed frame fields of $\mathcal{D}$, $\mathcal{D}^{\theta}$ and $F\mathcal{D}^{\theta}$, we derive

$$\begin{aligned}
\|h\|^2 \geq\ & 2\csc^2\theta \sum_{i=1}^{t}\sum_{r,j=1}^{s} g\left(h\left(e_i, e_j^{\star}\right), Fe_r^{\star}\right)^2 \\
& +2\csc^2\theta\sec^4\theta \sum_{i=1}^{t}\sum_{r,j=1}^{s} g\left(h\left(e_i, Pe_j^{\star}\right), FPe_r^{\star}\right)^2 \\
& +2\csc^2\theta \sum_{i=1}^{t}\sum_{r,j=1}^{s} g\left(h\left(\varphi e_i, e_j^{\star}\right), Fe_r^{\star}\right)^2 \\
& +2\csc^2\theta\sec^4\theta \sum_{i=1}^{t}\sum_{r,j=1}^{s} g\left(h\left(\varphi e_i, Pe_j^{\star}\right), FPe_r^{\star}\right)^2 \\
& +2\csc^2\theta\sec^2\theta \sum_{i=1}^{t}\sum_{r,j=1}^{s} g\left(h\left(e_i, Pe_j^{\star}\right), Fe_r^{\star}\right)^2 \\
& +2\csc^2\theta\sec^2\theta \sum_{i=1}^{t}\sum_{r,j=1}^{s} g\left(h\left(e_i, e_j^{\star}\right), FPe_r^{\star}\right)^2 \\
& +2\csc^2\theta\sec^2\theta \sum_{i=1}^{t}\sum_{r,j=1}^{s} g\left(h\left(\varphi e_i, Pe_j^{\star}\right), Fe_r^{\star}\right)^2 \\
& +2\csc^2\theta\sec^2\theta \sum_{i=1}^{t}\sum_{r,j=1}^{s} g\left(h\left(\varphi e_i, e_j^{\star}\right), FPe_r^{\star}\right)^2.
\end{aligned}$$

From the relations (18)–(21) and the fact that for an orthonormal frame field of $\mathcal{D}$, $\eta(e_i) = 0$, for $i = 1, \ldots, 2t$, we find

$$\begin{aligned}
\|h\|^2 \geq\ & 4\csc^2\theta \sum_{i=1}^{2t}\sum_{r,j=1}^{s} (\varphi e_i(\ln f))^2 g\left(e_j^{\star}, e_r^{\star}\right)^2 \\
& + 4\cot^2\theta \sum_{i=1}^{2t}\sum_{r,j=1}^{s} (e_i(\ln f))^2 g\left(e_j^{\star}, e_r^{\star}\right)^2.
\end{aligned}$$

Hence, to satisfy (16), we add and subtract the same term in the above relation and then we get

$$\begin{aligned}\|h\|^2 \geq 4(\csc^2\theta + \cot^2\theta) \sum_{r,j=1}^{s} \sum_{i=1}^{2t+1} (e_i(\ln f))^2 g\left(e_j^\star, e_r^\star\right)^2 \\ - 4(\csc^2\theta + \cot^2\theta) \sum_{r,j=1}^{s} (\xi(\ln f))^2 g\left(e_j^\star, e_r^\star\right)^2.\end{aligned}$$

Then from (16) and (17) (i), we obtain

$$\|h\|^2 \geq 4s\left(\csc^2\theta + \cot^2\theta\right)\left(\|\nabla^T(\ln f)\|^2 - 1\right)$$

which is the inequality (i). If the equality holds in (i), then from (25) and (26), we find

$$h(\mathcal{D}, \mathcal{D}) = 0, \quad h(\mathcal{D}^\theta, \mathcal{D}^\theta) = 0 \text{ and } h(\mathcal{D}, \mathcal{D}_\theta) \in F\mathcal{D}^\theta. \tag{27}$$

If h^θ is the second fundamental form of M_θ in M, then we have

$$g(h^\theta(Z, W), X) = g(\nabla_Z W, X) = -X(\ln f)\, g(Z, W) \tag{28}$$

for any $X \in \Gamma(\mathcal{D})$ and $Z, W \in \Gamma(\mathcal{D}^\theta)$. Since M_T is a totally geodesic submanifold in M [5,9], using this fact with the first condition of (27), we find that M_T is totally geodesic in $\tilde{M}$. Also, since M_θ is totally umbilical in M [5,9], using this fact with (28) and the second condition of (27), we observe that M_θ is totally umbilical in $\tilde{M}$. Moreover all conditions of (27) with the above fact show the minimality of M in $\tilde{M}$. This proves the theorem completely. □

Now, we have the following applications of our derived results.
1. If we assume $\theta = \frac{\pi}{2}$ in Theorem 2 and interchange X by φX in (23) for any $X \in \Gamma(\mathcal{D} \oplus \langle\xi\rangle)$, then the warped product semi-slant submanifold becomes a contact CR-warped product of the form $M = M_T \times_f M_\perp$ in a Kenmotsu manifold. Thus, Theorem 3.4 of [18] is a special case of Theorem 2 as follows:

Corollary 3 *(Theorem 3.4 of [18]) A proper contact CR-submanifold M of a Kenmotsu manifold $\tilde{M}$ is locally a contact CR-warped product if and only if*

$$A_{\varphi Z}X = -\varphi X(\mu)Z, \quad X \in \Gamma(\mathcal{D} \oplus \langle\xi\rangle), \quad Z \in \Gamma(\mathcal{D}^\perp)$$

for some smooth function μ on M satisfying $W(\mu) = 0$, for each $W \in \Gamma(\mathcal{D}^\perp)$.

2. Also, if we assume $\theta = \frac{\pi}{2}$ in Theorem 3, then warped product semi-slant submanifold is of the form $M = M_T \times_f M_\perp$ i.e., M becomes a contact CR-warped product. Thus, Theorem 3.1 of [2] is a special case of Theorem 3 as below:

Corollary 4 *(Theorem 3.1 of* [2]*) Let* $\tilde{M}$ *be a* $(2m+1)$*-dimensional Kenmotsu manifold and* $M = M_1 \times {}_f M_2$ *be an* n*-dimensional contact CR-warped product submanifold, such that* M_1 *is a* $(2t+1)$*-dimensional invariant submanifold tangent to* ξ *and* M_2 *is a* s*-dimensional anti-invariant submanifold of* $\tilde{M}$*. Then*

(i) *The squared norm of the second fundamental form of* M *satisfies*

$$\|h\|^2 \geq 2s\left(\|\nabla^T(\ln f)\|^2 - 1\right) \tag{29}$$

where $\nabla^T(\ln f)$ *is the gradient of* $\ln f$*.*

(ii) *If the equality sign in* (29) *holds identically, then* M_1 *is a totally geodesic submanifold and* M_2 *is a totally umbilical submanifold of* $\tilde{M}$*. Moreover,* M *is a minimal submanifold of* $\tilde{M}$*.*

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