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Inverse problems for general second order hyperbolic equations with time-dependent coefficients

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Abstract We study the inverse problems for the second order hyperbolic equations of general form with time-dependent coefficients assuming that the boundary data are given on a part of the boundary. The main result of this paper is the determination of the time-dependent Lorentzian metric by the boundary measurements. This is achieved by the adaptation of a variant of the boundary control method developed by Eskin (Inverse Probl 22(3):815–833, 2006; Inverse Probl 23:2343–2356, 2007).

Keywords Inverse problems · Hyperbolic equations · Geometric optics

Mathematics Subject Classification 35R30 · 35L10

1 Introduction

Consider a second order hyperbolic equation in $\mathbb{R}^{n+1}$ of the form

$$\sum_{j,k=0}^{n} \frac{1}{\sqrt{(-1)^n g(x)}} \Big(-i\frac{\partial}{\partial x_j} - A_j(x)\Big)\sqrt{(-1)^n g(x)}\, g^{jk}(x)\Big(-i\frac{\partial}{\partial x_k} - A_k(x)\Big)u(x) = 0, \tag{1.1}$$

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where $x = (x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1}$, x_0 is the time variable. In (1.1) $g(x) = \det[g_{jk}(x)]_{j,k=0}^n$, where $[g_{jk}(x)]_{j,k=0}^n = \left([g^{jk}]_{j,k=0}^n\right)^{-1}$ is the metric tensor, $A(x) = (A_0(x), A_1(x), \dots, A_n(x))$ is the vector potential. We assume that all coefficients in (1.1) belong to $C^\infty(\mathbb{R}^{n+1})$ and that

$$g^{00}(x) \geq c_0 > 0, \quad \forall x \in \mathbb{R}^{n+1}. \tag{1.2}$$

Let $(\xi_0, \xi_1, \dots, \xi_n)$ be dual variables to $(x_0, x_1, \dots, x_n)$. The strict hyperbolicity of (1.1) with respect to ξ_0 means that the quadratic equation

$$\sum_{j,k=0}^n g^{jk}(x)\xi_j\xi_k = 0 \tag{1.3}$$

has two real distinct roots $\xi_0^-(\xi_1, \dots, \xi_n) < \xi_0^+(\xi_1, \dots, \xi_n)$ for all $(\xi_1, \dots, \xi_n) \neq (0, \dots, 0)$ and all $x \in \mathbb{R}^{n+1}$. We have

$$\begin{aligned} &\xi_0^\pm(\xi_1, \dots, \xi_n) \\ &\quad = \frac{-\sum_{j=1}^n g^{j0}(x)\xi_j \pm \sqrt{\left(\sum_{j=1}^n g^{j0}(x)\xi_j\right)^2 - g^{00}(x)\sum_{j,k=1}^n g^{jk}(x)\xi_j\xi_k}}{g^{00}}. \end{aligned} \tag{1.4}$$

The strict hyperbolicity implies that

$$\left(\sum_{j=1}^n g^{j0}(x)\xi_j\right)^2 - g^{00}(x)\sum_{j,k=1}^n g^{jk}(x)\xi_j\xi_k > 0 \tag{1.5}$$

for all $(\xi_1, \dots, \xi_n) \neq 0,\ x \in \mathbb{R}^n$.

In this paper we assume a more restrictive condition that

$$\sum_{j,k=1}^n g^{jk}(x)\xi_j\xi_k \leq -c_1 \sum_{j=1}^n \xi_j^2, \tag{1.6}$$

i.e. we assume that the spatial part of the equation (1.1) is elliptic for any $x \in \mathbb{R}^{n+1}$.

Note that the quadratic form (1.3) has the signature $(+1, -1, \dots, -1)$. Therefore $(-1)^n g(x) > 0$. We assume also that $A_j(x), 0 \leq j \leq n$, are real-valued. Thus the operator in (1.1) is formally self-adjoint.

We consider the following class of domains $D \subset \mathbb{R}^{n+1}$. Let $D_t = D \cap \{x_0 = t\}$ be the intersection of D with the plane $\{t = x_0\}$, $t \in \mathbb{R}$. We assume that D_t is a smooth closed bounded domain in $\mathbb{R}^n$ smoothly dependent and uniformly bounded in t and such that D_t is diffeomorphic to D_0 for all $t \in \mathbb{R}$. More precisely we assume that there exists a diffeomorphism

$$y_0 = x_0, \quad y_j = \hat{y}_j(x_0, x_1, \dots, x_n), \quad 1 \leq j \leq n, \tag{1.7}$$

that maps D_{x_0} onto D_0 and smoothly depends on x_0. We shall call such domains D admissible.

Let $S(x_0, x_1, \dots, x_n) = 0$ be the equation of $\partial D = \bigcup_{t\in\mathbb{R}} \partial D_t$. Sometimes we denote ∂D by S. We assume that S is a time-like smooth surface in $\mathbb{R}^{n+1}$, i.e.

$$\sum_{j,k=0}^{n} g^{jk}(x)\nu_j\nu_k < 0, \tag{1.8}$$

where $x \in S$ and $(\nu_0, \nu_1, \dots, \nu_n)$ is a normal vector to S. The vector $(\nu_0, \dots, \nu_n)$ satisfying (1.8) is called a space-like vector. Also, the surface Σ in $\mathbb{R}^{n+1}$ is called space-like if $\sum_{j,k=0}^{n} g^{jk}(x)\nu_j(x)\nu_k(x) > 0$, where $x \in \Sigma$ and $(\nu_0(x), \dots, \nu_n(x))$ is the normal vector to Σ.

Consider the initial–boundary value problem

$$Lu = 0 \quad \text{in} \quad D, \tag{1.9}$$

$$u = 0 \quad \text{for} \quad x_0 \ll 0 \quad \text{in} \quad D, \tag{1.10}$$

$$u\big|_S = f, \tag{1.11}$$

where f is a smooth function on $S = 0$ with compact support, $Lu = 0$ is the same as in (1.1).

It is well known that the initial-value problem (1.9), (1.10), (1.11) is well-posed (cf. [15]), assuming that (1.2), (1.5) and (1.8) are satisfied.

Let $S_0 \subset S$ be a part of S such that (see Fig. 1) $S_{0t} = \partial D_t \cap S_0$ has a nonempty interior for all $t \in \mathbb{R}$. We assume also that for any $x^{(0)} \in \partial S_0$ a vector $\tau^{(1)}$, tangent to S and normal to S_0, is not parallel to $(1, 0, \dots, 0)$.

We include in the definition of the admissibility of D (see (1.7)) that the map $y = \hat{y}(x_0, x)$ is such that

$$\hat{y}(x_0, x) = x \quad \text{on} \quad S_{0x_0}, \quad \forall x_0 \in \mathbb{R}. \tag{1.12}$$

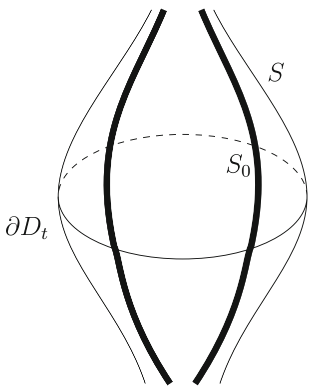


Fig. 1 S_0 is part of the boundary S, $S_0 \cap \partial D_t \neq \emptyset$ for $\forall t$

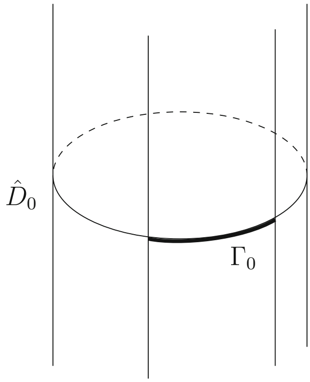


Fig. 2 $\hat{D} = \hat{D}_0 \times \mathbb{R}$, $\hat{S}_0 = \Gamma_0 \times \mathbb{R}$ are cylindrical domains

Note that (1.7) maps the admissible domain D onto $D = D_0 \times \mathbb{R}$, S_0 onto $\hat{S}_0 = \Gamma_0 \times \mathbb{R}$ (cf. (1.12)), where $\Gamma_0 = S_0 \cap \partial D_0$, i.e. $\hat{D}$, $\hat{S}_0$ are cylindrical domains (see Fig. 2).

The Dirichlet-to-Neumann operator Λ that maps the Dirichlet data to the Neumann data on ∂D is defined as

$$\Lambda f = \sum_{j,k=0}^{n} g^{jk}(x)\left(\frac{\partial u}{\partial x_j} - iA_j(x)u\right)\nu_k(x)\left(\sum_{p,r=0}^{n} g^{pr}(x)\nu_r\nu_p\right)^{-\frac{1}{2}}\Bigg|_S, \tag{1.13}$$

where $u(x)$ is the solution of (1.9), (1.10), (1.11) and $(\nu_0(x), \dots, \nu_n(x))$ is the unit outward normal to S.

Denote by

$$y = y(x) \tag{1.14}$$

any proper diffeomorphism of $\overline{D}$ onto some domain $\overline{\hat{D}}$ such that

$$y = x \quad \text{on} \quad S_0. \tag{1.15}$$

We call a diffeomorphism of $\overline{D}$ onto $\overline{\hat{D}}$ proper if for any $[t_1, t_2] \subset \mathbb{R}$ the image of $\overline{D} \cap \{t_1 \le x_0 \le t_2\}$ is a domain $\overline{\hat{D}} \cap \{S_-^{(t_1)}(y_1, \dots, y_n) \le y_0 \le S_+^{(t_2)}(y_1, \dots, y_n)\}$, where $y_0 = S_+^{(t_1)}$, $y_0 = S_-^{(t_0)}$ are space-like surfaces.

Let $\hat{L}\hat{u} = 0$ be the Eq. (1.1) in y-coordinates, $y \in \hat{D}$. We have

$$\hat{L}\hat{u} \equiv \sum_{j,k=0}^{n} \frac{1}{\sqrt{(-1)^n \hat{g}(y)}}\Big(-i\frac{\partial}{\partial y_j} - \hat{A}_j(y)\Big)\sqrt{(-1)^n\hat{g}(y)}\hat{g}^{jk}(y)$$
$$\cdot\Big(-i\frac{\partial}{\partial y_k} - \hat{A}_k(y)\Big)\hat{u} = 0, \tag{1.16}$$

where

$$\hat{g}^{jk}(y) = \sum_{p,r=0}^{n} g^{pr}(x)\frac{\partial y_j}{\partial x_p}\frac{\partial y_k}{\partial x_r}, \tag{1.17}$$

$$A_j(x) = \sum_{k=0}^{n} \hat{A}_k(y)\frac{\partial y_k}{\partial x_j}, \quad 0 \le j \le n, \tag{1.18}$$

Here $\hat{u}(y) = u(x)$, $y = y(x)$, $\hat{g}(y) = \det[\hat{g}_{jk}(y)]_{j,k=0}^{n}$, $[\hat{g}_{jk}(y)]_{j,k=0}^{n} = \left([\hat{g}^{jk}(y)]_{j,k=0}^{n}\right)^{-1}$.

Note that (1.17), (1.18) are equivalent to the equalities

$$\sum_{k=0}^{n} A_k(x)dx_k = \sum_{k=0}^{n} \hat{A}_k(y)dy_k, \tag{1.19}$$

$$\sum_{j,k=0}^{n} \hat{g}_{jk}(y)dy_j dy_k = \sum_{j,k=0}^{n} g_{jk}(x)dx_j dx_k, \tag{1.20}$$

where y and x are related by (1.14). Metric tensors $[g_{jk}(x)]_{j,k=0}^{n}$ and $[\hat{g}_{jk}(x)]_{j,k=0}^{n}$, related by (1.20), are called isometric.

We assume that conditions (1.2), (1.6) hold also in y-coordinates, i.e.

$$\hat{g}^{00}(y) \ge C_0, \quad \sum_{j,k=1}^{n} \hat{g}^{jk}(y)\xi_j\xi_k \le -C_1\sum_{j=1}^{n}\xi_j^2. \tag{1.21}$$

Let $c(x) \in C^\infty(\overline{D})$ be such that

$$|c(x)| = 1, \quad x \in \overline{D}, \quad c(x) = 1 \text{ on } S_0. \tag{1.22}$$

The group $G_0(\overline{D})$ of such $c(x)$ is called the gauge group.

If $Lu = 0$ then $u' = c^{-1}(x)u(x)$ satisfies the equation of the form (1.1) with $A_j(x)$ replaced by

$$\begin{aligned} A'_j(x) &= A_j(x) - ic^{-1}(x)\frac{\partial c}{\partial x_j}, \quad 1 \le j \le n, \\ A'_0(x) &= A_0(x) + ic^{-1}(x)\frac{\partial c}{\partial x_0}. \end{aligned} \tag{1.23}$$

We shall call potentials $(A'_0, \ldots, A'_n(x))$ and $(A_0(x), \ldots, A_n(x)))$ related by (1.23) gauge equivalent. Note that when D is simply connected then $c(x) = \exp i\varphi$ where $\varphi(x) \in C^\infty(\overline{D})$, $\varphi(x)$ is real-valued and $\varphi(x) = 0$ on S_0.

Let $y = y(x)$ be the change of variables, such that $y(x) = x$, $x \in S_0$, transforming the equation $Lu = 0$ in D to the equation of the form (1.16) in $\hat{D}$. We consider the initial–boundary value problem

$$\hat{L}\hat{u} = 0 \quad \text{in} \quad \hat{D}, \tag{1.24}$$

$$\hat{u} = 0 \quad \text{for} \quad y_0 \ll 0, \quad y \in \hat{D}, \tag{1.25}$$

$$\hat{u}\big|_{\hat{S}} = f. \tag{1.26}$$

Note that $\hat{S}_0 = S_0$, since $\hat{y}(x) = x$ on S_0.

Since (1.21) holds, the initial–boundary value problem (1.24), (1.25), (1.26) is also well-posed. Let $\hat{c}(y) \in G_0(\overline{\hat{D}})$. Make the gauge transformation $u'(y) = \hat{c}^{-1}(y)\hat{u}(y)$ and let L' be such that $L'u' = 0$. We have

$$\hat{L}'u' = 0 \quad \text{in} \quad \hat{D}, \tag{1.27}$$

$$u' = 0 \quad \text{for} \quad y_0 \ll 0, \quad y \in \hat{D}, \tag{1.28}$$

$$u'\big|_{\hat{S}} = f. \tag{1.29}$$

Note that $u' = \hat{u}$ on $\hat{S}_0$ since $\hat{c}(y) = 1$ on $\hat{S}_0$ and $L'u'$ has the form

$$\begin{aligned} L'u' = \sum_{j,k=0}^{n} \frac{1}{\sqrt{(-1)^n \hat{g}(y)}} \Big(-i\frac{\partial}{\partial y_j} - A'_j(y)\Big) \\ \sqrt{(-1)^n \hat{g}(y)}\hat{g}^{jk}(y)\Big(-i\frac{\partial}{\partial y_k} - A'_k(y)\Big)u'(y) = 0, \end{aligned} \tag{1.30}$$

$A'_j(y),\ 0 \le j \le n$, are potentials gauge equivalent to $\hat{A}_j(y),\ 0 \le j \le n$.

Let Λ' be the DN operator for (1.27), (1.28), (1.29)

$$\Lambda' f = \sum_{j,k=0}^{n} \hat{g}^{jk}(y)\Big(\frac{\partial u'}{\partial y_j} - iA'_j(y)u'\Big)\nu_k(y)\Big(\sum_{p,r=0}^{n} \hat{g}^{pr}(y)\nu_r(y)\nu_p(y)\Big)^{-\frac{1}{2}}\Big|_{\hat{S}}, \tag{1.31}$$

where f is the same as in (1.10) and (1.29).

It can be shown that

$$\Lambda f\big|_{S_0} = \Lambda' f\big|_{S_0}, \quad \forall f \in C_0^\infty(S_0), \tag{1.32}$$

if the operator L' is obtained from L by the change of variables (1.14), (1.15) and the gauge transformation $c(y)$ such that (1.22) holds.

Therefore the inverse problem of the determination of the coefficient of (1.1) can be solved only up to the changes of variables (1.14), (1.15) and the gauge transformations (1.22).

We shall formulate now some conditions which will be required to solve the inverse problem.

(1) Real analyticity in the time variable

One of the crucial steps in solving the inverse problem will be the use of the following unique continuation theorem of Tataru and Robbiano and Zuily (cf. [25,29]) that requires the analyticity in x_0:

Theorem 1.1 *Let the coefficients of (1.1) be analytic in x_0. Consider the equation $Lu = 0$ in a neighborhood U_0 of a point P_0. Let $\Sigma = 0$ be a noncharacteristic surface with respect to L passing through P_0. If $u = 0$ in $U_0 \cap \{\Sigma < 0\}$ then $u = 0$ in $U_0 \cap \{\Sigma > 0\}$ near $\Sigma = 0$.*

We assume also that the gauge $c(x)$ and the map (1.7) are analytic in x_0.

Let $y = \varphi(x)$ be a diffeomorphism of neighborhood U_0 onto the neighborhood $\overline{V}_0 = \varphi(\overline{U}_0)$. Here $\varphi(x)$ is smooth but not analytic in any variable. It is clear that if the unique continuation property for the operator L holds in U_0 then it holds in V_0 for the operator $\tilde{L} = \varphi \circ L$, though the coefficients of $\tilde{L}$ are not analytic. Here $\varphi \circ L$ is the operator L in y-coordinates (cf. (1.16)). Therefore the following more general class of operators L with non-analytic coefficients has the unique continuation property: For each point $x^{(0)}$ on D there is a neighborhood U_0 and the diffeomorphism $\psi(x)$ of U_0 onto $V_0 = \psi(U_0)$ such that the coefficients of the operators $\psi \circ L$ in V_0 are analytic in x_0. Thus, the unique continuation property holds for L in U_0.

(2) The Bardos–Lebeau–Rauch condition

Consider the initial–boundary value problem

$$Lu = 0, \quad u = 0 \quad \text{for} \quad x_0 \ll 0, \quad u\big|_{\partial D_0 \times \mathbb{R}} = f$$

in the cylindrical domain $D_0 \times \mathbb{R}$, f has a compact support in $\Gamma_0 \times \mathbb{R}$, $\Gamma_0 \subset \partial D_0$. We say that BLR condition holds on $[t_0, T_{t_0}]$ if the bounded map from $f \in H_1(\Gamma_0 \times (t_0, T_{t_0}))$ to $\left(u\big|_{x_0 = T_{t_0}}, \frac{\partial u}{\partial x_0}\big|_{x_0 = T_{t_0}}\right) \in H_1(D_0) \times L_2(D_0)$, is onto in $H_1(D_0) \times L_2(D_0)$, where $u = 0$ for $x_0 < t_0$, $f = 0$ for $x_0 < t_0$.
Note that BLR condition obviously holds on $[t_0, T]$ for any $T > T_{t_0}$ if it holds on $[t_0, T_{t_0}]$.
Let $\{x = x(s), \xi = \xi(s)\} \in T_0^*(\overline{D}_0 \times [t_0, T_{t_0}])$, where

$$\begin{aligned} \frac{dx_j}{ds} &= \frac{\partial L_0(x(s), \xi(s))}{\partial \xi_j}, \quad x_j(0) = y_j, \quad 0 \le j \le n, \\ \frac{d\xi_j}{ds} &= -\frac{\partial L_0(x(s), \xi(s))}{\partial x_j}, \quad \xi_j(0) = \eta_j, \quad 0 \le j \le n, \end{aligned} \tag{1.33}$$

be the equations of null-bicharacteristics. Here $L_0(x, \xi) = \sum_{j,k=0}^n g^{jk}(x)\xi_j\xi_k$, $L_0(y, \eta) = 0$.

We assume that for any t_0 there exists T_{t_0} depending continuously on t_0 such that the BLR condition holds on $[t_0, T_{t_0}]$. It follows from [1] that BLR condition holds if any null bicharacteristic in $T_0^*(\overline{D}_0 \times [t_0, T])$ intersects $T_0^*(\overline{\Gamma}_0 \times [t_0, T])$ when $T \ge T_{t_0}$.

(3) Domains of dependence

Let $G(x,\xi) = \sum_{j,k=0}^{n} g_{jk}(x)\xi_j\xi_k$, $[g_{jk}]_{j,k=0}^{n} = \left([g^{jk}]_{j,k=0}^{n}\right)^{-1}$. We say that $x = x(\tau)$ is a forward time-like ray in $D_0 \times \mathbb{R}$ if $x = x(\tau)$ is piece-wise smooth, $G(x(\tau), \frac{dx(\tau)}{d\tau}) > 0$ and $\frac{dx_0}{d\tau} > 0,\ 0 \leq \tau$. If $G(x(\tau), \frac{dx(\tau)}{d\tau}) > 0$ and $\frac{dx_0}{d\tau} < 0$ the ray $x = x(\tau)$ is called the backward time-like ray.

One can show (cf [7]) that the forward domain of influence $D_+(F)$ of a closed set $F \subset D_0 \times \mathbb{R}$ is the closure of the union of all piece-wise smooth forward time-like rays in $D_0 \times \mathbb{R}$ starting on F.

Analogously, the backward domain of influence $D_-(F)$ of the closed set $F \subset D_0 \times \mathbb{R}$ is the closure of the union of all backward time-like piece-wise smooth rays in $D_0 \times \mathbb{R}$ starting at F. The domain of dependence of F is the intersection $D_+(F) \cap D_-(F)$.

Let $\Gamma \subset \partial D_0$ and let $Lu = 0$ in $D_0 \times \mathbb{R}$. A consequence of the unique continuation property is that $u\big|_{\Gamma\times(t_1,t_2)} = \frac{\partial u}{\partial \nu}\Big|_{\Gamma\times(t_1,t_2)} = 0$ implies $u = 0$ in the domain of dependence of $\Gamma \times [t_1, t_2]$. Here $\frac{\partial}{\partial \nu}$ is the normal derivative to Γ. This fact follows from [19] in the case of time-independent coefficients. The proof in the time-dependent case is similar.

The following fact follows from the BLR condition:

Consider $\Gamma \times [t_1, t_2]$, $\Gamma \subset \partial D_0$. Suppose $[t_1, t_2]$ is arbitrary large. Then the domain of dependence of $\overline{\Gamma} \times [t_1, t_2]$ contains $\overline{D}_0 \times [t_1 + \delta, t_2 - \delta]$ for some $\delta > 0$ dependent of the metric and the domain.

In this paper we will not attempt to estimate $\delta > 0$ since $[t_0 + \delta, t_2 - \delta]$ is also arbitrary large if $[t_1, t_2]$ is arbitrary large. □

Now we shall state the main result of this paper.

Consider an admissible domain D in $\mathbb{R}^{n+1}$ and an initial–boundary value problem in D.

Using the map of the form (1.7) defining the admissibility of the domain D we get a cylindrical domain $D_0 \times \mathbb{R}$ with $S_0 = \Gamma_0 \times \mathbb{R}$ (cf. Fig. 2) and the initial–boundary value problem

$$Lu = 0 \quad \text{in} \quad D_0 \times \mathbb{R}, \tag{1.34}$$

$$u = 0 \quad \text{when} \quad x_0 \ll 0, \tag{1.35}$$

$$u\big|_{\partial D_0 \times \mathbb{R}} = f, \tag{1.36}$$

where L has the form (1.1) and f has a compact support in $\overline{\Gamma}_0 \times \mathbb{R}$. Consider another admissible domain $\hat{D}$. Making again the change of variables (1.7) we get a cylindrical domain $\hat{D}_0 \times \mathbb{R}$ and another initial–boundary value problem

$$L'u' = 0 \quad \text{in} \quad \hat{D}_0 \times \mathbb{R}, \tag{1.37}$$

$$u' = 0 \quad \text{when} \quad y_0 \ll 0, \tag{1.38}$$

$$u'\big|_{\partial \hat{D}_0 \times \mathbb{R}} = f', \tag{1.39}$$

where $L'u'$ has the form (1.30), f' has a compact support in $\overline{\Gamma}_0 \times \mathbb{R}$. Therefore the inverse problems for the admissible domains are reduced to the inverse problems in cylindrical domains.

We shall prove the following theorem:

Theorem 1.2 *Consider two initial–boundary value problems (1.34), (1.35), (1.36) and (1.37), (1.38), (1.39) in domains $D_0 \times \mathbb{R}$ and $\hat{D}_0 \times \mathbb{R}$, respectively. Suppose $A_j(x), A'_j(y), 0 \leq j \leq n$, are real-valued. Assume that $\Gamma_0 \subset \partial D_0 \cap \partial \hat{D}_0$ is nonempty and open. Let Λ and Λ' be the corresponding DN operators for L and L'. Assume that $\Lambda f\big|_{\Gamma_0 \times \mathbb{R}} = \Lambda' f\big|_{\Gamma_0 \times \mathbb{R}}$ for all smooth f with compact support in $\overline{\Gamma}_0 \times \mathbb{R}$. Suppose the conditions (1.2), (1.6) hold for L and L'. Assume that the coefficients of L and L' are analytic in x_0 and y_0, respectively. Suppose also that BLR condition holds for (1.34), (1.35), (1.36) on $[t_0, T_{t_0}]$ for each $t_0 \in \mathbb{R}$. Then there exists a proper map $y = y(x)$ of $\overline{D}_0 \times \mathbb{R}$ onto $\overline{\hat{D}_0} \times \mathbb{R}$, $y = x$ on $\Gamma_0 \times \mathbb{R}$, and there exists a gauge transformation with the gauge $c'(y) \in G_0(\overline{\hat{D}_0} \times \mathbb{R})$, $c'(y) = 1$ on $\overline{\Gamma}_0 \times \mathbb{R}$ such that $L' = c' \circ y^* L$. Here $y^* \circ L$ is the operator with $[\hat{g}^{jk}(y)]_{j,k=0}^n$ and $\hat{A}_k(y)$, $0 \leq k \leq n$ as in (1.17), (1.18), $c' \circ y^* \circ L$ is the operator with potentials $A'_j(y)$, $0 \leq j \leq n$, gauge equivalent to $\hat{A}_k(y)$, $0 \leq k \leq n$.*

We end the introduction with the outline of the previous work and a short description of the content of the paper.

The first result on inverse hyperbolic problems with the data on the part of the boundary was obtained by Isakov [17]. The powerful boundary control (BC) method was discovered by Belishev [2] and was further developed by Belishev [3–5], Belishev and Kurylev [6], Kurylev and Lassas [21,22] and others (see [19,20]). In [8,9] the author proposed a new approach to hyperbolic inverse problems that uses substantially the idea of BC method. This approach was extended in [10] to some class of time-independent metrics with time-dependent vector potentials and in [11] to the case of hyperbolic equations of general form with time-independent coefficients without vector potentials. The generalization to the case of Yang–Mills potentials was considered in [14]. The inverse problems for the D'Alambert equation with the time-dependent scalar potentials were considered earlier by Stefanov [S] and Ramm and Sjostrand [24] (see also Isakov [18]). The case of the D'Alambert equation with time-dependent vector potentials was studied by Salazar [26,27]:

As it was mentioned in [1] the study of hyperbolic equations with time-dependent coefficients is very important because the linearization of basic nonlinear hyperbolic equations of mathematical physics leads to time-dependent linear hyperbolic equations.

The main result of the present paper is the determination of the time-dependent Lorentzian metric by the boundary measurements given on the part of the boundary. We consider the second order hyperbolic equations of general form (1.1) with time-dependent coefficients and vector potentials. The method is the extension of the approach in [8,9,11] to the case of time-dependent metrics. We adapt some lemmas of [8–11] to the time-dependent situation and simultaneously give sharper and simpler proofs.

The main step in the proof is the local step of solving the inverse problem in a small neighborhood near the boundary. This is done in Sects. 2–6.

In Sect. 2 we make a change of variables in a neighborhood $U_0 \subset \mathbb{R}^{n+1}$ of a point $x^{(0)} \in S_0$. We called the new coordinates the Goursat coordinates since they are similar to coordinates arising in a solution of the Goursat problem in the case of hyperbolic equations with one space variable. The Goursat coordinates allow to simplify the equation (1.1). Another advantage of the Goursat coordinates is that the characteristic surface is a plane in these coordinates. Also it is proven in Sect. 2 that the original DN operator determines the new DN operator corresponding to the equation in Goursat coordinates.

In Sect. 3 we derive the Green formula in Goursat coordinates and prove the crucial density lemma (Lemma 3.1).

In Sect. 4 we establish the main formula that states that some integrals of solutions of the initial–boundary value problems are determined by the DN operator [see Theorem 4.5, formula (4.29)]. To establish this formula one needs to compare Sobolev norms on the characteristic surfaces corresponding to different operators having the same DN data. This is done by using the BLR condition. Note that in the case of time-independent coefficients there is an additional energy identity that allows to avoid the use of the BLR condition (see Remark 4.1). Also note that proofs in Sect. 4 require that the hyperbolic operators are formally self-adjoint. Thus the vector potentials $A = (A_0, A_1, \dots, A_n)$ are required to be real-valued.

In Sect. 5 we construct geometric optics type solutions depending on a large parameter k. Substituting the geometric optics solutions in the main formula, we prove in Sects. 5 and 6 the local version (Theorem 6.2) of the main theorem (Theorem 1.2). In the last Sect. 7 we consider the global case. At first we study the case of a finite time interval (Theorem 7.7) and then finally prove Theorem 1.2.

2 The Goursat coordinates

We shall prove first the Theorem 1.2 in the small neighborhood of the boundary ∂D.

Let $x^{(0)} \in S_0$ and let $U_0 \subset \mathbb{R}^{n+1}$ be a small neighborhood of $x^{(0)}$.

Suppose that we already did the change of variables (1.7) to make ∂D and S_0 cylindrical, i.e. $\partial D = \partial D_0 \times \mathbb{R}$ and $S_0 = \Gamma_0 \times \mathbb{R}$. We assume that we have chosen the coordinates (x_0, x', x_n), $x' = (x_1, \dots, x_{n-1})$ in U_0 such that $x_n = 0$ is the equation of $U_0 \cap \partial D$ and $U_0 \cap D$ is contained in the half-space $x_n > 0$. Let $(x_0^{(0)}, x_1^{(0)}, \dots, x_{n-1}^{(0)}, 0)$ be the coordinates of the point $x^{(0)}$. Let $T_1 < x_0^{(0)} < T_2$, $T_2 - T_1$ is small.

Consider the initial–boundary value problem in $U_0 \cap D$:

$$Lu = 0, \quad x_n > 0, \quad T_1 < x_0 < T_2, \tag{2.1}$$

$$u\Big|_{x_0 = T_1} = 0, \quad \frac{\partial u}{\partial x_0}\Big|_{x_0 = T_1} = 0, \tag{2.2}$$

$$u\Big|_{x_n = 0} = g(x_0, x'). \tag{2.3}$$

We assume that L has the form (1.1). For the simplicity, we shall not change the notations when choosing the local coordinates such that the equation of $U_0 \cap S_0$ is $x_n = 0$. Assume that $\operatorname{supp} g \subset U_0 \cap (\Gamma_0 \times [T_1, T_2])$, $g = 0$ for $x_0 < T_1$. Note that $\operatorname{supp} u(x_0, x', x_n) \cap [T_1, T_2] \subset U_0 \cap [T_1, T_2]$ for $x_n > 0$ if $T_2 - T_1$ is small.

We introduce new coordinates to simplify the operator L (cf. [11, pp. 327–329]) that we called the Goursat coordinates.

Denote by $\psi^{\pm}(x)$, $x = (x_0, x', x_n)$ the solutions of the eikonal equations

$$\sum_{j,k=0}^{n} g^{jk}(x_0, x', x_n)\psi^{\pm}_{x_j}(x)\psi^{\pm}_{x_k}(x) = 0, \quad x_n > 0, \tag{2.4}$$

with initial conditions

$$\psi^{+}\big|_{x_n=0} = x_0 - T_1, \quad \psi^{-}\big|_{x_n=0} = T_2 - x_0. \tag{2.5}$$

It is well known that the solution $\psi^{\pm}(x)$ of (2.4), (2.5) exists for $0 \le x_n \le \varepsilon$, where $\varepsilon > 0$ is small (see, for example, [12], §64).

Since (2.4) is a quadratic equation in $\psi^{\pm}_{x_n}$ one has to specify the sign of the square root. We have

$$g^{nn}(\psi^{\pm}_{x_n})^2 + 2\sum_{j=0}^{n-1} g^{nj}\psi^{\pm}_{x_j}\psi^{\pm}_{x_n} + \sum_{j,k=0}^{n-1} g^{jk}\psi^{\pm}_{x_j}\psi^{\pm}_{x_k} = 0.$$

We will need below that $\psi^{+}_{x_n} + \psi^{-}_{x_n} < 0$ for $x_n > 0$. So we choose the plus sign of the square root:

$$\psi^{\pm}_{x_n} = \frac{-\sum_{j=0}^{n-1} g^{nj}\psi^{\pm}_{x_j} + \sqrt{\left(\sum_{j=0}^{n-1} g^{nj}\psi^{\pm}_{x_j}\right)^2 - g^{nn}\left(\sum_{j,k=0}^{n-1} g^{jk}\psi^{\pm}_{x_j}\psi^{\pm}_{x_k}\right)}}{g^{nn}(x)} \tag{2.6}$$

Note that $g^{nn}(x) < 0$, $\psi^{\pm}_{x_0}\big|_{x_n=0} = \pm 1$. Therefore $\psi^{\pm}_{x_n}\big|_{x_n=0} = \frac{\mp g^{n0} + \sqrt{(g^{n0})^2 - g^{nn}g^{00}}}{g^{nn}}$. The solutions $\psi^{\pm}(x)$ exists for $0 < x_n < \delta$, δ is small. For given T_1, T_2 we assume that δ is such that surfaces $\psi^{+} = 0$ and $\psi^{-} = 0$ intersect when $x_n < \delta$ and are inside U_0 when $x_n < \delta$.

Let $\varphi_j(x_0, x', x_n)$, $1 \le j \le n-1$, be solutions of the linear equation

$$\sum_{p,k=0}^{n} g^{pk}(x_0, x', x_n)\psi^{-}_{x_p}\varphi_{jx_k} = 0, \quad x_n > 0, \tag{2.7}$$

with initial condition

$$\varphi_j(x_0, x', 0) = x_j, \quad 1 \le j \le n-1. \tag{2.8}$$

Make the following change of variables in $U_0 \cap [T_1, T_2]$:

$$
\begin{aligned}
s &= \psi^+(x_0, x', x_n),\\
\tau &= \psi^-(x_0, x', x_n),\\
y_j &= \varphi_j(x_0, x', x_n), \quad 1 \le j \le n-1.
\end{aligned}
\tag{2.9}
$$

Equation (1.1) has the following form in (s, τ, y') coordinates where $y' = (y_1, \dots, y_{n-1})$

$$
\begin{aligned}
\hat{L}\hat{u} \stackrel{def}{=} &\frac{2}{\sqrt{|\hat{g}|}}\Big(\frac{\partial}{\partial s} + i\hat{A}_+(s,\tau,y')\Big)\sqrt{|\hat{g}|}\,\hat{g}^{+,-}(s,\tau,y')\Big(\frac{\partial}{\partial \tau} + i\hat{A}_-\Big)\hat{u}\\
&+ \frac{2}{\sqrt{|\hat{g}|}}\Big(\frac{\partial}{\partial \tau} + i\hat{A}_-(s,\tau,y')\Big)\sqrt{|\hat{g}|}\,\hat{g}^{+,-}(s,\tau,y')\Big(\frac{\partial}{\partial s} + i\hat{A}_+\Big)\hat{u}\\
&- \sum_{j=1}^{n-1}\frac{2}{\sqrt{|\hat{g}|}}\Big(\frac{\partial}{\partial y_j} - i\hat{A}_j(s,\tau,y')\Big)\sqrt{|\hat{g}|}\,\hat{g}^{+,j}(s,\tau,y')\Big(\frac{\partial}{\partial s} + i\hat{A}_+\Big)\hat{u}\\
&- \sum_{j=1}^{n-1}\frac{2}{\sqrt{|\hat{g}|}}\Big(\frac{\partial}{\partial s} + i\hat{A}_+(s,\tau,y')\Big)\sqrt{|\hat{g}|}\,\hat{g}^{+,j}(s,\tau,y')\Big(\frac{\partial}{\partial y_j} - i\hat{A}_j\Big)\hat{u}\\
&- \sum_{j,k=1}^{n-1}\frac{1}{\sqrt{|\hat{g}|}}\Big(\frac{\partial}{\partial y_j} - i\hat{A}_j(s,\tau,y')\Big)\sqrt{|\hat{g}|}\,\hat{g}^{jk}(s,\tau,y')\Big(\frac{\partial}{\partial y_k} - i\hat{A}_k\Big)\hat{u},
\end{aligned}
\tag{2.10}
$$

where

$$
\hat{g} = -(2\hat{g}^{+,-})^{-2}\big(\det[\hat{g}^{jk}]_{j,k=1}^{n-1}\big)^{-1}. \tag{2.11}
$$

Note that terms containing $\frac{\partial^2}{\partial s^2}, \frac{\partial^2}{\partial \tau^2}, \frac{\partial^2}{\partial y_j \partial \tau}$ vanished because of (2.4), (2.7), and

$$
\begin{aligned}
2\hat{g}^{+,-} &= -\sum_{j,k=0}^{n} g^{jk}\psi^+_{x_j}\psi^-_{x_k},\\
2\hat{g}^{+,j} &= \sum_{p,r=0}^{n} g^{pr}\psi^+_{x_p}\varphi_{jx_r}, \quad 1 \le j \le n-1,\\
\hat{g}^{jk} &= \sum_{p,r=0}^{n} g^{pr}\varphi_{jx_p}\varphi_{kx_r}, \quad 1 \le j,k \le n-1,
\end{aligned}
\tag{2.12}
$$

It follows from (2.6) for $x_n = 0$ that $g^{+,-} > 0$.

In (2.10) $\hat{u}(s, \tau, y') = u(x_0, x', x_n)$,

$$A_k(x) = \sum_{j=1}^{n-1} \hat{A}_j(s, \tau, y')\varphi_{jx_k} - \hat{A}_+\psi^+_{x_k} - \hat{A}_-\psi^-_{x_k}, \quad 0 \le k \le n. \tag{2.13}$$

Now we shall introduce a new system of coordinates (cf. [11])

$$\begin{aligned} y_0 &= \frac{s - \tau + T_2 + T_1}{2} = \frac{\psi^+ - \psi^- + T_2 + T_1}{2}, \\ y_j &= \varphi_j(x), \quad 1 \le j \le n-1, \\ y_n &= \frac{T_2 - T_1 - s - \tau}{2} = \frac{T_2 - T_1 - \psi^+(x) - \psi^-(x)}{2}, \end{aligned} \tag{2.14}$$

where $\psi^+, \psi^-, \varphi_j$, $1 \le j \le n-1$, are the same as in (2.4), (2.7).

Note that

$$\begin{aligned} y_0\big|_{x_n=0} &= \frac{x_0 - T_1 - (T_2 - x_0) + T_2 + T_1}{2} = x_0, \\ y_j\big|_{x_n=0} &= x_j, \quad 1 \le j \le n-1, \\ y_n\big|_{x_n=0} &= \frac{T_2 - T_1 - s - \tau}{2} = \frac{T_2 - T_1 - \psi^+(x) - \psi^-(x)}{2} = 0, \end{aligned} \tag{2.15}$$

Therefore $y = \varphi(x) = (\varphi_0(x_1), \varphi_1(x), \ldots, \varphi_n(x))$ is the identity on $x_n = 0$:

$$\varphi(x) = I \quad \text{when} \quad x_n = 0. \tag{2.16}$$

Here

$$\varphi_0 = \frac{\psi^+(x) - \psi^-(x) + T_2 + T_1}{2}, \quad \varphi_n = \frac{T_2 - T_1 - \psi^+ - \psi^-}{2}.$$

Note that $y_n = \varphi_n(x) > 0$ when $x_n > 0$ since $\psi^+_{x_n} + \psi^-_{x_n} < 0$ (cf. (2.6)),

$$u_s = \frac{1}{2}(u_{y_0} - u_{y_n}), \quad u_\tau = -\frac{1}{2}(u_{y_0} + u_{y_n}). \tag{2.17}$$

Thus one can easily rewrite (2.10) in (y_0, y', y_n) coordinates.

We shall further simplify (2.10) by making a gauge transformation

$$u' = e^{-id(s,\tau,y')}\hat{u}. \tag{2.18}$$

Then u' satisfies the equation

$$L'u' = 0, \tag{2.19}$$

where L' is the same as $\hat{L}$ with $\hat{A}_j$, $\hat{A}_+$, $\hat{A}_-$ replaced by A'_j, A'_+, A'_-, $1 \leq j \leq n-1$, where

$$A'_j = \hat{A}_j - \frac{\partial d}{\partial y_j}, \quad 1 \leq j \leq n-1,$$
$$A'_+ = \hat{A}_+ - \frac{\partial d}{\partial s}, \quad A'_- = \hat{A}_- - \frac{\partial d}{\partial \tau}. \tag{2.20}$$

We choose $d(s, \tau, y')$ such that

$$A'_+ = -\frac{\partial d}{\partial s} + \hat{A}_+ = 0 \quad \text{for} \quad y_n > 0,$$
$$d\big|_{y_n=0} = 0. \tag{2.21}$$

Let

$$g_1 = \left| \det[\hat{g}^{jk}]_{j,k=1}^{n-1} \right|^{-1}, \quad A = \ln(g_1)^{\frac{1}{4}}. \tag{2.22}$$

Note that

$$\frac{\partial A}{\partial y_j} = \frac{g_{1y_j}}{4g_1} = \frac{1}{2}\frac{1}{\sqrt{g_1}}\frac{\partial}{\partial y_1}\sqrt{g_1}, \quad 1 \leq j \leq n-1,$$
$$\frac{\partial A}{\partial s} = \frac{g_{1s}}{4g_1} = \frac{1}{2}\frac{1}{\sqrt{g_1}}\frac{\partial}{\partial s}\sqrt{g_1},$$
$$\frac{\partial A}{\partial \tau} = \frac{g_{1\tau}}{4g_1} = \frac{1}{2}\frac{1}{\sqrt{g_1}}\frac{\partial}{\partial \tau}\sqrt{g_1}. \tag{2.23}$$

Since $\sqrt{|\hat{g}|} = \frac{\sqrt{g_1}}{2\hat{g}^{+,-}}$ (cf (2.11)) we can rewrite $L'u' = 0$ in the form (cf. [8]):

$$\begin{aligned}
L'u' = {} & 2\hat{g}^{+,-}\Big(\frac{\partial}{\partial s} + \frac{\partial A}{\partial s}\Big)\Big(\frac{\partial}{\partial \tau} + iA'_- + \frac{\partial A}{\partial \tau}\Big) \\
& + 2\hat{g}^{+,-}\Big(\frac{\partial}{\partial \tau} + iA'_- + \frac{\partial A}{\partial \tau}\Big)\Big(\frac{\partial}{\partial s} + \frac{\partial A}{\partial s}\Big)u' \\
& - 2\hat{g}^{+,-}\sum_{k=1}^{n-1}\Big(\frac{\partial}{\partial s} + \frac{\partial A}{\partial s}\Big)\frac{\hat{g}^{+,k}}{\hat{g}^{+,-}}\Big(\frac{\partial}{\partial y_k} - iA'_k + \frac{\partial A}{\partial y_k}\Big)u' \\
& - 2\hat{g}^{+,-}\sum_{k=1}^{n-1}\Big(\frac{\partial}{\partial y_k} - iA'_k + \frac{\partial A}{\partial y_k}\Big)\frac{\hat{g}^{+,k}}{\hat{g}^{+,-}}\Big(\frac{\partial}{\partial s} + \frac{\partial A}{\partial s}\Big)u' \\
& - \sum_{j,k=1}^{n-1}\hat{g}^{+,-}\Big(\frac{\partial}{\partial y_j} - iA'_j + \frac{\partial A}{\partial y_j}\Big)\frac{\hat{g}^{jk}}{\hat{g}^{+,-}}\Big(\frac{\partial}{\partial y_k} - iA'_k + \frac{\partial A}{\partial y_k}\Big)u' \\
& + \hat{g}^{+,-}V_1u' = 0,
\end{aligned} \tag{2.24}$$

where

$$V_1 = -\sum_{j,k=1}^{n-1}\Big(\frac{\hat g^{jk}}{\hat g^{+,-}}\frac{\partial A}{\partial y_j}\frac{\partial A}{\partial y_k} + \frac{\partial}{\partial y_k}\Big(\frac{\hat g^{jk}}{\hat g^{+,-}}\frac{\partial A}{\partial y_j}\Big)\Big)$$
$$+4\frac{\partial^2 A}{\partial s\partial\tau} + 4\frac{\partial A}{\partial s}\frac{\partial A}{\partial \tau} - 4\sum_{j=1}^{n-1}\frac{\hat g^{+,j}}{\hat g^{+,-}}\frac{\partial A}{\partial s}\frac{\partial A}{\partial y_j}$$
$$-2\sum_{j=1}^{n-1}\Big(\frac{\partial}{\partial s}\Big(\frac{\hat g^{+,j}}{\hat g^{+,-}}\frac{\partial A}{\partial y_j}\Big) + \frac{\partial}{\partial y_j}\Big(\frac{\hat g^{+,j}}{\hat g^{+,-}}\frac{\partial A}{\partial s}\Big)\Big). \tag{2.25}$$

Make the change of unknown function

$$u_1 = g_1^{\frac{1}{4}}u', \tag{2.26}$$

where $g_1 = |\det[\hat g^{jk}]_{j,k=1}^{n-1}|^{-1}$ (cf. (2.11)). Then dividing $L'u' = 0$ by $\hat g^{+,-}$ we get (cf. [11])

$$L_1u_1 = 0,$$

where $L_1u_1 = 0$ has the form (cf. (2.24))

$$L_1u_1 = 2\frac{\partial}{\partial s}\Big(\frac{\partial}{\partial\tau} + iA'_-\Big)u_1 + 2\Big(\frac{\partial}{\partial\tau} + iA'_-\Big)\frac{\partial}{\partial s}u_1$$
$$-2\sum_{j=1}^{n-1}\frac{\partial}{\partial s}\Big(g_0^{+,j}\Big(\frac{\partial}{\partial y_j} - iA'_j\Big)u_1\Big) - \sum_{j=1}^{n-1}2\Big(\frac{\partial}{\partial y_j} - iA'_j\Big)g_0^{+,j}\frac{\partial u_1}{\partial s}$$
$$-\sum_{j,k=1}^{n-1}\Big(\frac{\partial}{\partial y_j} - iA'_j\Big)g_0^{jk}\Big(\frac{\partial}{\partial y_k} - iA'_k\Big)u_1 + V_1u_1 = 0, \tag{2.27}$$

where

$$g_0^{jk} = \frac{\hat g^{jk}}{\hat g^{+,-}}, \quad g_0^{+,j} = \frac{\hat g^{+,j}}{\hat g^{+,-}},$$

and V_1 is the same as in (2.25). Using that $\frac{\partial}{\partial\tau} + iA'_- = \frac{1}{2}\big(-\frac{\partial}{\partial y_0} - \frac{\partial}{\partial y_n}\big) + iA'_- = -\frac{1}{2}\big[\big(\frac{\partial}{\partial y_0} - iA'_-\big) + \big(\frac{\partial}{\partial y_n} - iA'_-\big)\big]$ and $\frac{\partial}{\partial s} = \frac{1}{2}\big(\frac{\partial}{\partial y_0} - \frac{\partial}{\partial y_n}\big) = \frac{1}{2}\big[\big(\frac{\partial}{\partial y_0} - iA'_-\big) - \big(\frac{\partial}{\partial y_n} - iA'_-\big)\big]$ we can rewrite L_1u_1 in (y_0, y', y_n) coordinates:

$$L_1u_1 = -\Big(\frac{\partial}{\partial y_0} - iA'_-\Big)^2u_1 + \Big(\frac{\partial}{\partial y_n} - iA'_-\Big)^2u_1$$
$$-\sum_{j=1}^{n-1}\Big(\frac{\partial}{\partial y_0} - iA'_-\Big)g_0^{+,j}\Big(\frac{\partial}{\partial y_j} - iA'_j\Big)u_1$$

$$
\begin{aligned}
&-\sum_{j=1}^{n-1}\Big(\frac{\partial}{\partial y_j}-iA'_j\Big)g_0^{+,j}\Big(\frac{\partial}{\partial y_0}-iA'_-\Big)u_1\\
&+\sum_{j=1}^{n-1}\Big(\frac{\partial}{\partial y_n}-iA'_-\Big)g_0^{+,j}\Big(\frac{\partial}{\partial y_j}-iA'_j\Big)u_1\\
&+\sum_{j=1}^{n-1}\Big(\frac{\partial}{\partial y_j}-iA'_j\Big)g_0^{+,j}\Big(\frac{\partial}{\partial y_n}-iA'_-\Big)u_1\\
&-\sum_{j,k=1}^{n}\Big(\frac{\partial}{\partial y_j}-iA'_j\Big)g_0^{jk}\Big(\frac{\partial}{\partial y_k}-iA'_k\Big)u_1+V_1u_1=0. \qquad (2.28)
\end{aligned}
$$

Note that we transformed the equation $Lu = 0$ to the equation $L_1u_1 = 0$ in two steps. First, we transformed $Lu = 0$ to $L'u' = 0$ by making the change of variables $y = \varphi(x)$ of the form (2.15) and gauge transformation with the gauge $e^{-id(s,\tau,y')}$ belonging to the group G_0 (cf. (1.22). Then we transform $L'u' = 0$ to $L_1u_1 = 0$ by using the change of variables (2.26), i.e. by using a gauge e^A, where $A = \ln g_1^{\frac{1}{4}}$ and then dividing $L'u' = 0$ by $\hat{g}^{+,-}$. □

The DN operator for L has the form

$$
\Lambda g = -\sum_{j=0}^{n} g^{jn}(x)\Big(\frac{\partial u}{\partial x_j}-iA_j(x)u\Big)(-g^{nn}(x))^{-\frac{1}{2}}\Big|_{x_n=0}
$$

since the outward normal to $x_n = 0$ is $(0, 0, \dots, -1)$.

Rewrite $L'u' = 0$ in (y_0, y', y_n) coordinates using (2.17).

Denote $-\hat{g}^{00} = \hat{g}^{nn} = \hat{g}^{+,-}$, $\hat{g}^{nj} = \hat{g}^{jn} = \hat{g}^{+,j}$, $1 \le j \le n-1$. Note that $\hat{g}^{+,-} > 0$.

The DN operator for $L'u' = 0$ has the following form in (y_0, y', y_n) coordinates:

$$
\Lambda' g = (\hat{g}^{+,-})^{\frac{1}{2}}\Big[\Big(\frac{\partial u'}{\partial y_n}-iA'_-u'\Big)+\sum_{j=1}^{n-1}\frac{\hat{g}^{nj}}{\hat{g}^{+,-}}\Big(\frac{\partial u'}{\partial y_j}-iA'_ju'\Big)\Big]\Big|_{y_n=0}, \qquad (2.29)
$$

where (cf. (2.18))

$$
u'(y) = e^{-id(y)}u(\varphi^{-1}(y)),
$$

$y = y(x)$ is the same as in (2.14).

Since L' is obtained from (1.1) by the change of variables (2.14)and the gauge transformation (2.18) and since (2.15), (2.21) hold, we have $\Lambda g = \Lambda' g$ on $\{y_n = 0\} \cap U_0$ for all g with supp g in $(\Gamma_0 \times [T_1, T_2]) \cap U_0$. Using the expression of $L_1u_1 = 0$ in (y_0, y', y_n) coordinates (see (2.28)) we get that DN operator $\Lambda_1 g$ has the form

$$\Lambda_1 g = \left(\frac{\partial u_1}{\partial y_n} - iA'_{-}u_1 + \sum_{j=0}^{n-1} g_0^{+,j} \Big(\frac{\partial u_1}{\partial y_j} - iA'_j u_1 \Big) \right) \Big|_{y_n=0}, \tag{2.30}$$

where $g_0^{+,j} = \frac{\hat{g}^{nj}}{\hat{g}^{+,-}}$.

We shall show that the DN operators Λ' determines the DN operator Λ_1 in $U_0 \cap \Gamma_0$.

The following lemma is well known, especially in the elliptic case (cf. [12,23, § 57]). For the hyperbolic case see [8, Remark 2.2].

Lemma 2.1 *The DN operator Λ' determines*

$$\hat{g}^{+,-}\big|_{y_n=0}, \quad \frac{\hat{g}^{nj}}{\hat{g}^{+,-}}\Big|_{y_n=0}, \quad \frac{\hat{g}^{jk}}{\hat{g}^{+,-}}\Big|_{y_n=0}, \quad 1 \le j \le n-1, \ 1 \le k \le n-1, \tag{2.31}$$

and the derivatives of (2.31) in y_n at $y_n = 0$.

Proof The principal symbol of operator L' has the form $\hat{g}^{+,-}p(y,\eta)$, where (cf. (2.10) in y-coordinates)

$$p(y,\eta) = \eta_0^2 - \eta_n^2 + 2\sum_{j=1}^{n-1} g_0^{+,j}(\eta_0 - \eta_n)\eta_j + \sum_{j,k=1}^{n-1} g_0^{jk}\eta_j\eta_k, \tag{2.32}$$

where

$$g_0^{+.j} = \frac{\hat{g}^{+,j}}{\hat{g}^{+,-}}, \quad g_0^{jk} = \frac{\hat{g}^{jk}}{\hat{g}^{+,-}}. \tag{2.33}$$

Since (1.6) holds the quadratic form $\sum_{j,k=1}^{n-1} g_0^{jk}\eta_j\eta_k$ is negative definite. Therefore for $\varepsilon > 0$ in the region $\Sigma = \{\eta_0^2 + \big(\sum_{j=1}^{n-1} g_0^{+,j}\eta_j\big)^2 - \varepsilon\sum_{j=1}^{n-1}\eta_j^2 < 0\}$ of the cotangent space $T^* = U_0 \times (\mathbb{R}^{n+1} \setminus \{0\})$ the operator $p(y,\eta)$ is elliptic. We shall call Σ the elliptic region.

There is a parametrix of the Dirichlet problem in the elliptic region and DN operator microlocally in Σ is a pseudodifferential operator on $y_n = 0$. We shall find the principal symbol of this operator in Σ. Let $\lambda_\pm$ be the roots in η_n of $p(y, \eta_0, \eta', \eta_n) = 0$:

$$\begin{aligned}\lambda_\pm &= -\sum_{j=1}^{n-1} g_0^{+,j}\eta_j \\ &\quad \pm \sqrt{\Big(\sum_{j=1}^{n-1} g_0^{+,j}\eta_j\Big)^2 + \Big(\eta_0^2 + 2\sum_{j=1}^{n-1} g_0^{+,j}\eta_j\eta_0 + \sum_{j,k=1}^{n-1} g_0^{jk}\eta_j\eta_k\Big)} \\ &\overset{def}{=} -\sum_{j=1}^{n-1} g_0^{+,j}\eta_j \pm \sqrt{Q},\end{aligned} \tag{2.34}$$

where $\Im\lambda_+ > 0$ in Σ. Therefore the symbol of DN in Σ is (cf. [12, § 57]):

$$\big(\hat{g}^{+,-}\big)^{\frac{1}{2}}\sqrt{Q}. \tag{2.35}$$

Knowing Λ' we know the symbol (2.35) for all η_0, η'. In particular, we can find (2.31). Computing the next term of the parametrix (cf. [12, § 57]) we can find the normal derivatives of (2.31). $\square$

We have

$$\Lambda_1 f' = \left(\frac{\partial u_1}{\partial y_n} - iA'\_u_1 + \sum_{j=1}^{n-1} g_0^{+,j}\Big(\frac{\partial u_1}{\partial y_j} - iA'_j u_1\Big)\right)\Big|_{y_n=0}, \tag{2.36}$$

where $u_1\big|_{y_n=0} = f',\ u_1 = g_1^{\frac{1}{4}} u',\ f' = g_1^{\frac{1}{4}}\big|_{y_n=0} f$. Note that $\frac{\partial}{\partial y_k} u_1 = g_1^{\frac{1}{4}} \frac{\partial}{\partial y_k} u' + \big(\frac{\partial}{\partial y_k} g_1^{\frac{1}{4}}\big) u'$. Therefore

$$\Lambda_1 f' = g_1^{\frac{1}{4}} (g^{+,-})^{-\frac{1}{2}} \Lambda' f + \left(\frac{\partial g_1^{\frac{1}{4}}}{\partial y_n} + \sum_{j=1}^{n-1} g_0^{+,j} \frac{\partial g_1^{\frac{1}{4}}}{\partial y_j}\right) g_1^{-\frac{1}{4}} f\Big|_{y_n=0}. \tag{2.37}$$

It follows from the Lemma 2.1 that $g_1, \frac{\partial g_1}{\partial y_n}, \hat{g}^{+,-}, g_0^{+,j}$ are known on $y_n = 0$ if Λ' is known. Therefore knowing $\Lambda' f$ we can determine $\Lambda_1 f'$. Note that

$$u_1 = g_1^{\frac{1}{4}} e^{-id(y)} u(\varphi^{-1}(y)), \tag{2.38}$$

where $y = \varphi(x)$ is given by (2.14), (2.15).

3 The Green's formula

First, we introduce some notations.

Let $\Gamma_1 \subset U_0 \cap \Gamma_0$. Denote by D_{1T_1} the forward domain of influence of $\overline{\Gamma}_1 \times [T_1, T_2]$ in the half-space $y_n \geq 0$. We shall define Γ_2 as the intersection $D_{1T_1} \cap \{y_n = 0\} \cap \{y_0 = T_2\}$. Analogously, let $\Gamma_3 = D_{2T_1} \cap \{y_n = 0\} \cap \{y_0 = T_2\}$, where D_{2T_1} is the forward domain of influence of $\overline{\Gamma}_2 \times [T_1, T_2]$. We assume that $\Gamma_1 \subset \Gamma_2 \subset \Gamma_3 \subset (\Gamma_0 \cap U_0)$.

Let D_{js_0} be the forward domain of influence of $\overline{\Gamma}_j \times [s_0, T_2]$, $1 \leq j \leq 3$, where $T_1 \leq s_0 \leq T_2$. Denote by Y_{js_0} the intersection of D_{js_0} with the plane $\tau = T_2 - y_n - y_0 = 0$ (cf. Fig. 3). Let X_{js_0} be the part of D_{js_0} below Y_{js_0} and let $Z_{js_0} = \partial X_{js_0} \setminus (Y_{js_0} \cup \{y_n = 0\})$.

We assume also that $D_{3,T_1} \cap \{y_n = 0\} \subset \Gamma_0 \cap U_0$ and that D_{3T_1} does not intersect $\Gamma_0 \times [T_1, T_2]$ outside of $y_n = 0$.

Consider the following initial–boundary value problem:

$$\begin{aligned} &L_1 u^f = 0 \\ &u^f = u^f_{y_0} = 0 \quad \text{for} \quad y_0 = T_1, \quad y_n > 0, \\ &u^f\big|_{y_n=0} = f, \end{aligned} \tag{3.1}$$

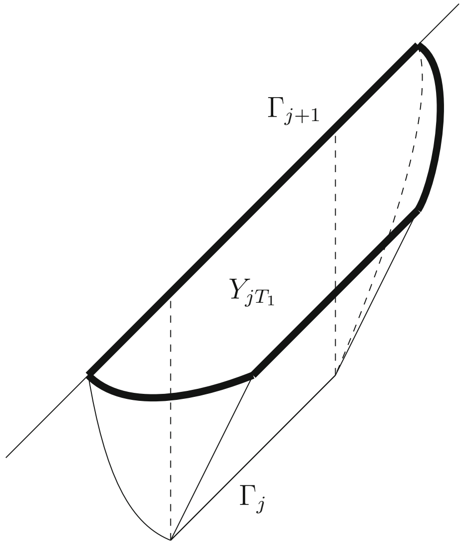


Fig. 3 Y_{jT_1} is the intersection of the plane $\tau = 0$ with D_{jT_1}, Γ_{j+1} is the intersection of Y_{jT_1} with the plane $y_n = 0$

where supp $f \subset \overline{\Gamma}_3 \times [T_1, T_2]$. Also let v^g be such that

$$
\begin{aligned}
&L_1 v^g = 0 \quad \text{for} \quad y_n > 0, \\
&v^g = v^g_{y_0} = 0 \quad \text{for} \quad y_0 = T_1, \quad y_n > 0, \\
&v^g\big|_{y_n=0} = g,
\end{aligned}
\tag{3.2}
$$

where supp $g \subset \overline{\Gamma}_3 \times [T_1, T_2]$.

Note that $L_1^* = L_1$, i.e. L_1 is formally self-adjoint.

Let (u, v) be the L_2 inner product $\int_{X_{3T_1}} u(y)\overline{v}(y)dy$. We have

$$
(L_1 u^f, v^g) - (u^f, L_1 v^g) = 0, \tag{3.3}
$$

since $L_1 u^f = 0$, $L_1 v^g = 0$. The Jacobian $\frac{\partial(y_n, y_0)}{\partial(s,\tau)}$ is equal to $\frac{1}{2}$. Thus $dy_0 dy_n = \frac{1}{2} ds d\tau$. Integrating by parts in s we get

$$
\begin{aligned}
&-\int\limits_{X_{3T_1}} \frac{\partial}{\partial s}\Big(\frac{\partial}{\partial \tau} + iA'_-\Big)u^f \overline{v^g} ds d\tau \\
&\quad = \int\limits_{X_{3T_1}} \Big(\frac{\partial}{\partial \tau} + iA'_-\Big)u^f \overline{\frac{\partial v^g}{\partial s}} ds d\tau - \int\limits_{y_n=0} \Big(\frac{\partial}{\partial \tau} + iA'_-\Big)u^f \overline{v^g} d\tau.
\end{aligned}
\tag{3.4}
$$

Integrating by parts in τ we get

$$
\begin{aligned}
&-\int\limits_{X_{3T_1}} \frac{\partial^2}{\partial\tau\partial s} u^f \overline{v^g} ds d\tau \\
&\quad = \int\limits_{X_{3T_1}} \frac{\partial u^f}{\partial s} \frac{\overline{\partial v^g}}{\partial\tau} ds d\tau - \int\limits_{y_n=0} \frac{\partial u^f}{\partial s} \overline{v^g} ds + \int\limits_{\tau=0} \frac{\partial u^f}{\partial s} \overline{v^g} ds. \qquad (3.5)
\end{aligned}
$$

We used in (3.4), (3.5) that u^f, v^g are equal to zero on Z_{3T_1}. Note that $s = y_0 - T_1$, $\tau = T_2 - y_0$ on $y_n = 0$, and $\frac{\partial}{\partial s} = \frac{1}{2}\big(\frac{\partial}{\partial y_0} - \frac{\partial}{\partial y_n}\big)$, $\frac{\partial}{\partial \tau} = -\frac{1}{2}\big(\frac{\partial}{\partial y_0} + \frac{\partial}{\partial y_n}\big)$. Therefore, making changes of variable $\tau = T_2 - y_0$ in the first integral and $s = y_0 - T_1$ in the second, we get

$$
-\int\limits_{y_n=0} \Big(\frac{\partial}{\partial\tau} + iA'_-\Big) u^f \overline{v^g} d\tau - \int\limits_{y_n=0} \frac{\partial u^f}{\partial s} \overline{v^g} ds = \int_{y_n=0} \Big(\frac{\partial}{\partial y_n} - iA'_-\Big) u^f \overline{v^g} dy_0 \qquad (3.6)
$$

Analogously, integrating by parts other terms of $\int_{X_{3T_1}} (L_1 u^f)\overline{v^g} ds d\tau$ we get (cf. [10], p. 316)

$$
\begin{aligned}
0 &= (L_1 u^f, v^g) - (u^f, L_1 v^g) \\
&= \int\limits_{Y_{3T_1}} \Big(\frac{\partial u^f}{\partial s}\overline{v^g} - u^f \frac{\overline{\partial v^g}}{\partial s}\Big) dy' ds + \int\limits_{\Gamma_3 \times [T_1, T_2]} (\Lambda_1 f \overline{g} - f \overline{\Lambda_1 g}) dy' dy_0, \qquad (3.7)
\end{aligned}
$$

where Λ_1 has the form (2.30)

$$
\Lambda_1 f = \Big(\frac{\partial u^f}{\partial y_n} - iA'_- u^f\Big) - \sum_{j=1}^{n-1} g_0^{+,j} \Big(\frac{\partial u^f}{\partial y_j} - iA'_j u^f\Big)\Big|_{y_n=0}. \qquad (3.8)
$$

It follows from (3.7) that

$$
\int\limits_{Y_{3T_1}} (u_s^f \overline{v^g} - u^f \overline{v_s^g}) ds dy' \qquad (3.9)
$$

is determined by the boundary data, i.e. by the DN operator on $\Gamma^{(3)} \times (T_1, T_2)$.

We shall denote the L_2 inner product in Y_{3T_1} by $(u, v)_{Y_{3T_1}}$, or simply (u, v) when it is clear what is the domain of integration.

Let D_j^- be the backward domain of influence of $\overline{\Gamma}_j \times [T_1, T_2]$. Thus $D_{jT1} \cap D_j^-$ is the domain of dependence of $\overline{\Gamma}_j \times [T_1, T_2]$. Denote by Q_j the intersection of D_j^- with $\tau = 0$. Let $R_{js_0} = Y_{js_0} \cap Q_j$ be the rectangle $\{s_0 - T_1 \le s \le T_2 - T_1, \tau = 0, y' \in \overline{\Gamma}_j\}$.

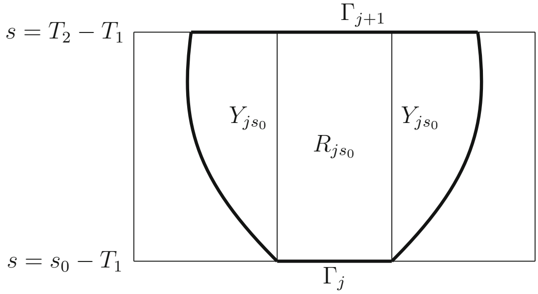


Fig. 4 The rectangle $R_{js_0} = \{s_0 - T_1 \le s \le T_2 - T_1, \tau = 0, y' \in \Gamma_j\}$, Y_{js_0} is the intersection of the domain of influence of $[s_0, T_2] \times \overline{\Gamma}_j$ with the plane $\tau = 0$. Note that $R_{js_0} \subset Y_{js_0} \subset R_{j+1.s_0}$

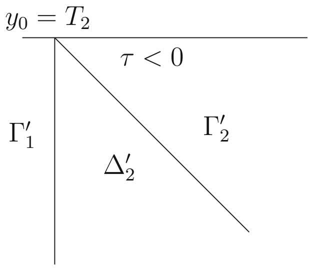


Fig. 5 The domain Δ_2' is bounded by Γ_1' and Γ_2'

Note that R_{js_0} belongs to the domain of dependence of $\overline{\Gamma}_j \times [s_0, T_2]$. Let $H_0^1(R_{js_0})$ be the subspace of the Sobolev space $H^1(R_{js_0})$ consisting of $w \in H^1(R_{js_0})$ such that $w = 0$ on $\partial R_{j0} \setminus \{y_n = 0\}$. Analogously, let $H_0^1(Y_{js_0})$ be the subspace of $H^1(Y_{js_0})$ consisting of $v \in H^1(Y_{js_0})$ such that $v = 0$ on $\partial Y_{js_0} \setminus \{y_n = 0\}$. Note that $R_{js_0} \subset Y_{js_0} \subset R_{j+1,s_0}$ (cf. Fig. 4).

Note that $H_0^1(R_{js_0})$ is a subspace of $H_0^1(Y_{js_0})$.

Lemma 3.1 (Density lemma) *For any $w \in H_0^1(R_{js_0})$ there exists a sequence $\{u^{f_n}\}$ where u^{f_n} are solutions of the initial–boundary value problem (3.1), $f_n(y_0, y') \in H_0^1(\Gamma_j \times [s_0, T_2])$, such that $\|w - u^{f_n}\|_{1,Y_{js_0}} \to 0$ when $n \to \infty$, $j = 1, 2, 3$.*

Here $\|w\|_{1,Y_{js_0}}$ is the norm in $H_0^1(Y_{js_0})$ and $f \in H_0^1(\Gamma_j \times [s_0, T_2])$, i.e. $f = 0$ on $\partial(\Gamma_j \times [s_0, T_2]) \setminus (\Gamma_j \times \{y_0 = T_2\})$.

Proof The proof of Lemma 3.1 is a simplification of the proof of Lemmas 2.2 and 3.2 in [10]. We shall prove Lemma 3.1 for the case $s_0 = T_1$. The proof for the case $T_1 < s_0 < T_2$ is identical.

Denote by Δ_2' the domain bounded by the half-plaines $\Gamma_1' = \{y_n = 0, y_0 < T_2, y' \in \mathbb{R}^{n-1}\}$ and $\Gamma_2' = \{\tau = T_2 - y_n - y_0 = 0, s < T_2, y' \in \mathbb{R}^{n-1}\}$ (cf. Fig. 5). Let Γ_∞' be the plane $\tau = 0$. Denote by $H_0^{-1}(\Gamma_2')$ the Sobolev space of $h \in H^{-1}(\Gamma_\infty')$ such that $\operatorname{supp} h \subset \overline{\Gamma}_2'$, i.e. $h(s, y') = 0$ when $s > T_2$. Note that $H_0^{-1}(\Gamma_2')$ is dual to $H^1(\Gamma_2')$ with respect to the extension of the L_2 inner product on Γ_∞' (cf. [12]). ◻

Lemma 3.2 *For any $h(s, y') \in H_0^{-1}(\Gamma_2')$ there exists a distribution $u(s, \tau, y')$ such that*

$$L_1 u = 0 \quad in \quad \Delta_2', \tag{3.10}$$

$$\left.\frac{\partial u}{\partial s}\right|_{\Gamma_2'} = h, \tag{3.11}$$

$$u\big|_{y_n=0} = 0. \tag{3.12}$$

Proof Since $h(s, y') = 0$ for $s > T_2$, there exists $v(s, y') = 0$ for $s > T_2$, $v(s, y')$ belongs to L_2 in s and to H^{-1} in y' and such that $\frac{\partial v}{\partial s} = h$ in Γ_∞'. We can define $v(s, y')$ by the formula

$$v(s, y') = \lim_{\varepsilon \to 0} e^{\varepsilon(s-T_2)} F^{-1} \frac{\tilde{h}(z_0, \xi_1, \dots, \xi_n)}{z_0 + i\varepsilon},$$

where $\tilde{h}(z_0, \xi_1, \dots, \xi_{n-1})$ is the Fourier transform of $h(s, y')$ and F^{-1} is the inverse Fourier transform, z_0 is the dual variable to s.

The distribution $\theta(-\tau)u$ satisfies the equation

$$L_1(\theta(-\tau)u) = 4h\delta(-\tau) \tag{3.13}$$

in the half-space $y_n > 0$ with the boundary condition

$$\theta(-\tau)u\big|_{y_n=0} = 0, \tag{3.14}$$

where $\theta(s) = 1$ for $s > 0$ and $\theta(s) = 0$ for $s < 0$.

We look for $\theta(-\tau)u$ in the form

$$\theta(-\tau)u = \theta(-\tau)v + w, \tag{3.15}$$

where w satisfies

$$L_1 w = \varphi, \tag{3.16}$$

$$w\big|_{y_n=0} = -\theta(-\tau)v\big|_{y_n=0}, \tag{3.17}$$

$$\varphi = L_{11}(\theta(-\tau)v), \tag{3.18}$$

where $L_{11} = L_1 + \frac{4\partial^2}{\partial s \partial \tau}$. Note that L_{11} is a differential operator in $\frac{\partial}{\partial s}, \frac{\partial}{\partial y_k}, 1 \le k \le n-1$.

We impose the zero initial conditions on w requiring that

$$w = 0 \quad \text{for} \quad y_0 > T_2. \tag{3.19}$$

Therefore w is the solution of the hyperbolic equation $L_1 w = \varphi$ in the half-space $y_n > 0$ with the boundary condition (3.17) and the zero initial conditions (3.19). It follows from ([15] and [13]) that initial–boundary value problem has a unique solution in appropriate Sobolev space of negative order.

Since φ belongs to L_2 in τ and to Sobolev spaces of negative order in s and y', we get that w belongs to H^1 in τ. Therefore $w\big|_{\tau=\tau_0}$ is continuous function of τ_0 with the values in Sobolev's spaces of negative order in (s, y'). Since $\varphi = 0$ for $\tau < 0$ we have that $w = 0$ for $\tau < 0$ by the domain of influence argument. Therefore by the continuity $w\big|_{\tau=0} = 0$ and $\frac{\partial w}{\partial s}\big|_{\tau=0} = 0$.

Therefore $u = v(s) + w(s, \tau, y')$ is the distribution solution of (3.10), (3.11), (3.12) in Δ_2'.

Note that the restrictions of any distribution solution of $L_1 u = 0$ to $y_n = 0$ exists since $y_n = 0$ is not a characteristic surface for L_1. This property is called the partial hypoellipticity (cf., for example, [12]). $\square$

Now using Lemma 3.2 we can prove Lemma 3.1. If $\{u^f, f \in H_0^1(\Gamma_{jT_1}), \Gamma_{jT_1} = \Gamma_j \times [T_1, T_2]\}$ is not dense in $H_0^1(R_{jT_1})$ then there exists nonzero $h \in H_0^{-1}(R_{jT_1})$ such that $(u^f, h) = 0, \forall u^f, f \in H_0^1(\Gamma_{jT_1})$. Let v be such that $\frac{\partial v}{\partial s} = h$. Then $\left(u^f, \frac{\partial v}{\partial s}\right)_{Y_{jT_1}} = 0, \forall f \in H_0^1(\Gamma_{jT_1})$. Let u be the same as in Lemma 3.2, i.e. $L_1 u = 0$ in Δ_2', $u\big|_{Y_{T_1}} = v$, $u\big|_{y_n=0} = 0$. Then

$$
-2\left(u^f, \frac{\partial v}{\partial s}\right)_{Y_{jT_1}} = \int\limits_{\Gamma_{jT_1}} f \overline{\frac{\partial u}{\partial y_n}} dy' dy_0, \quad \forall f \in H_0^1(\Gamma_{jT_1}). \tag{3.20}
$$

Note that $(u^f, h)_{Y_{jT_1}}$ is understood as the extension of L_2 inner product in (u_1, h) in Γ_∞', where u_1 is an arbitrary extension of u^f for $s > T_2$. Analogously for the right hand side of (3.20). (Note that $u = 0$ for $y_0 > T_2$).

To justify (3.20) we take a sequence $h_j \in C_0^\infty(\Gamma_2')$, $h_j \to h$ in $H_0^{-1}(\Gamma_2')$. By Lemma 3.2 there exists smooth v_j such that $L_1 v_j = 0$ in Δ_2', $v_j\big|_{y_n=0} = 0$, $\frac{\partial v_j}{\partial s}\big|_{\tau=0} = h_j$. Applying the Green's formula (3.7) to u^f and v_j we get

$$
\int\limits_{Y_{3T_1}} \left(\frac{\partial u^f}{\partial s}\overline{v}_j - u^f \overline{\frac{\partial v_j}{\partial s}}\right) dy' ds = \int\limits_{\Gamma_{3T_1}} f \overline{\frac{\partial v_j}{\partial y_n}} dy' dy_0,
$$

since $v_j\big|_{y_n=0} = 0$. Integrating by parts we get

$$
\int\limits_{Y_{3T_1}} \frac{\partial u^f}{\partial s}\overline{v}_j ds dy' = -\int\limits_{Y_{3T_1}} u^f \overline{\frac{\partial v_j}{\partial s}} ds dy' + u^f \overline{v}_j\big|_{y_n=0, y_0=T_2}.
$$

Since $v_j = 0$ for $s > T_2$ we have that $v_j\big|_{y_n=0, y_0=T_2} = 0$. Therefore, taking the limit when $j \to \infty$ we get (3.20).

Since f is arbitrary and $\left(u^f, \frac{\partial v}{\partial s}\right) = 0$ we get that $\frac{\partial u}{\partial y_n} = 0$ on Γ_{jT_1}. Therefore $L_1 u = 0$ in Δ_2' and u has zero Cauchy data on Γ_{2T_1}. Then $u = 0$ in the domain of dependence $D_j^- \cap D_{jT_1}$, in particular, $u = 0$ on R_{jT_1}. Thus $v = 0$ on R_{jT_1} and this contradicts the assumption that $h \neq 0$. $\square$

We shall prove two more theorems in this section that will be used in Sect. 4.

We shall need some known results on the initial–boundary hyperbolic problem. The following theorem holds:

Lemma 3.3 *Let* $L_1u = F$ *in* $\mathbb{R}^n_+ \times (-\infty, T_2)$ *where* $F \in H^s_+(\mathbb{R}^n \times (-\infty, T_2), \mathbb{R}^n_+ = \{y_n > 0, y' \in \mathbb{R}^{n-1}\}$. *Let* $u\big|_{y_n=0} = f$, *where* $f \in H^{s+1}_+(\mathbb{R}^{n-1} \times (-\infty, T_2))$. *Then for any* $s \geq 0$ *and any* $f \in H^{s+1}_+(\mathbb{R}^{n-1} \times (-\infty, T_2))$ *and* $F \in H^s_+(\mathbb{R}^n_+ \times (-\infty, T_2))$ *there exists a unique* $u \in H^{s+1}_+(\mathbb{R}^n_+ \times (-\infty, T_2))$ *such that*

$$\|u\|_{s+1} \leq C([f]_{s+1} + \|F\|_s). \tag{3.21}$$

Moreover,

$$\Big[\frac{\partial u(y_0, y', 0)}{\partial y_n}\Big]_s \leq C(\|F\|_s + [f]_{s+1}). \tag{3.22}$$

Here $H^s_+(\mathbb{R}^n_+ \times (-\infty, T_2))$ *is the Sobolev's space* $H^s(\mathbb{R}^n_+ \times (-\infty, T_2))$ *with norm* $\|u\|_s$ *consisting of* $u(y)$ *with the support in* $y_0 \geq T_1,\ [f]_s$ *is the norm in* $H^s_+(\mathbb{R}^{n-1} \times (-\infty, T_2))$.

We assume that $F(y)$ *and* $f(y_0, y')$ *have compact supports in* y'. *Note that* $f = 0,\ F = 0,\ u = 0$ *for* $y_0 < T_1$.

Then $u(y_0, y', y_n)$ *has also a compact support in* y'.

The proof of Lemma 3.3 in the case of time-dependent coefficients is given in [15] and [13].

Note that Lemma 3.3 holds also in the case when $\mathbb{R}^n_+$ is replaced by an arbitrary smooth domain $\Omega \subset \mathbb{R}^n$.

The following lemma follows from Lemma 3.3.

Lemma 3.4 *Let, for the simplicity,* $F = 0$. *For any* $f \in H^1_+(\mathbb{R}^{n-1} \times (-\infty, T_2])$, $f = 0$ *for* $y_0 \leq T_1$, *there exists* $u \in C(H^1(\mathbb{R}^n_+), [T_1, T_2]) \cap C^1(L_2(\mathbb{R}^n_+), [T_1, T_2])$ *such that* $L_1u = 0$ *in* $\mathbb{R}^n_+ \times (-\infty, T_2]$, $u = 0$ *for* $y_0 < T_1$,

$$\max_{T_1 \leq y_0 \leq T_2} \|u(y_0, y', y_n)\|_1^2 + \max_{T_1 \leq y_0 \leq T_2} \|\frac{\partial u}{\partial y_0}(y_0, y', y_n)\|_0^2 \leq C[f]_1^2. \tag{3.23}$$

Here $C(H^1(\mathbb{R}^n_+), [T_1, T_2]) \cap C^1(L_2(\mathbb{R}^n_+), (T_1, T_2))$ *means that* $u(y)$, $\frac{\partial u(y)}{\partial y_0}$ *are continuous functions of* y_0 *with values in* $H^1(\mathbb{R}^n_+), L_2(\mathbb{R}^n_+)$, *respectively.*

Proof Take $s > \frac{3}{2}$. Consider the equation $L_1u = 0$ in $\mathbb{R}^n_+ \times (-\infty, T_2)$, $u = 0$ for $y_0 < T_1$, using (y_0, y', y_n) coordinates (cf. (2.28)). We have

$$g_0^{00} = -g_0^{nn} = 1, \quad g_0^{0j} = g_0^{nj}, \quad A_0' = A_n' = A_-'. \tag{3.24}$$

Let $(u, v)_{T'}$ be the L_2-inner product in $\mathbb{R}^{n+1}_+ \times (T_1, T')$, $T' \leq T_2$. Integrating by parts the identity

$$0 = (L_1u, u_{y_0})_{T'} + (u_{y_0} L_1u)_{T'},$$

we get (cf. [11])

$$E_{T'}(u,u)+\Lambda_0(f,f)+I_1=0, \tag{3.25}$$

where

$$\begin{aligned}E_{T'}(u,u)=&\int\limits_{\mathbb{R}^n_+}\Big(|u_{y_0}-iA_0'u|^2-\sum_{j,k=1}^n g_0^{jk}(u_{y_j}-iA_j'u)\\&\times\overline{(u_{y_k}-iA_ku')}+V_1|u|^2\Big)dy'dy_n\Big|_{y_0=T'},\end{aligned} \tag{3.26}$$

$$\Lambda_0(f,f)=\int\limits_{T_1}^{T_2}\int\limits_{\mathbb{R}^{n-1}}\big[(\Lambda_1 f)\overline{f_{y_0}}+f_{y_0}\overline{\Lambda_1 f}\big]dy'dy_0, \tag{3.27}$$

$\Lambda_1 f$ is the same as in (2.36),

$$|I_1|\le C\int\limits_{\mathbb{R}^n_+\times[T_1,T']}\sum_{k=0}^n|u_{y_k}|^2dy_0dy'dy_n. \tag{3.28}$$

Note that $I_1=0$ when the coefficients of L_1 do not depend on y_0.

Let $\|u\|_{s,T'}$ be the norm in $H^s(\mathbb{R}^n_+)$ when $y_0=T'$. We have

$$|I_1|\le C\int\limits_{T_1}^{T'}(\|u\|_{1,t}^2+\|u_{y_0}\|_{0,t}^2)dt\le C\int\limits_{T_1}^{T'}|[u]|_t^2dt \tag{3.29}$$

where

$$|[u]|_t^2=\|u\|_{1,t}^2+\|u_{y_0}\|_{0,t}^2. \tag{3.30}$$

We have

$$E_{T'}(u,u)\ge C|[u]|_{1,T'}^2 \tag{3.31}$$

if T_2-T_1 is small.

Since T_2-T_1 is small, (3.25) implies

$$\max_{T_1\le T'\le T_2}|[u]|_{1,T'}^2\le C(T_2-T_1)\left(\max_{T_1\le T'\le T_2}|[u]|_{1,T'}^2+|\Lambda_0(f,f)|\right) \tag{3.32}$$

Note that

$$|\Lambda_0(f,f)|\le C\Big([f]_1^2+\Big[\frac{\partial u(y_0,y',0)}{\partial y_n}\Big]_0\Big). \tag{3.33}$$

Therefore (3.22), (3.25), (3.31), (3.32), (3.33) imply (3.23).

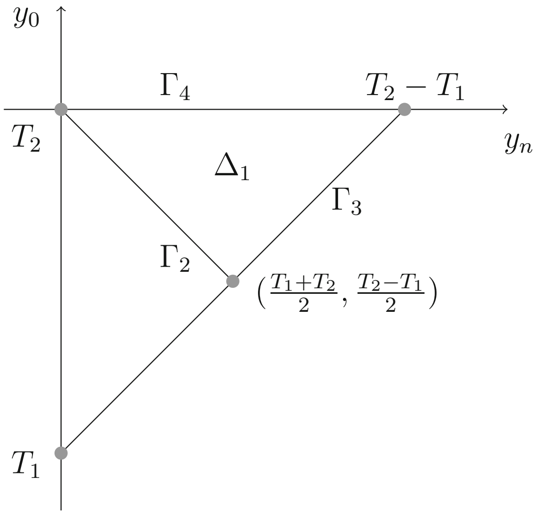


Fig. 6 Domain Δ_1 is bounded by the planes $\Gamma_2, \Gamma_3, \Gamma_4$

Since H_+^s is dense in H_+^1 when $s > 1$ we can approximate $f \in H_+^1(\mathbb{R}^{n-1} \times (-\infty, T_2))$ by functions from $H_+^s(\mathbb{R}^{n-1} \times (-\infty, T_2))$, $s > \frac{3}{2}$ and therefore the inequality (3.23) holds for $f \in H_+^1$. $\square$

We shall study the Goursat problem (see Fig. 6).

We use the same notations that we used in the proof of the Lemma 3.1 in [10]: Let Γ_2, Γ_3 and Γ_4 be the following planes:

$$\Gamma_2 = \left\{\tau = T_2 - y_0 - y_n = 0, 0 \le y_n \le \frac{T_2 - T_1}{2}, y' \in \mathbb{R}^{n-1}\right\},$$
$$\Gamma_3 = \left\{s = y_0 - y_n - T_1 = 0, \frac{T_2 + T_1}{2} \le y_0 \le T_2, y' \in \mathbb{R}^{n-1}\right\},$$
$$\Gamma_4 = \left\{y_0 = T_2, 0 \le y_n \le T_2 - T_1, y' \in \mathbb{R}^{n-1}\right\}.$$

Let Δ_1 be the domain bounded by $\Gamma_2, \Gamma_3, \Gamma_4$. The following lemma is similar to Lemma 3.1 in [10].

Lemma 3.5 *For any $v_0 \in H^1(\Gamma_4)$, $v_1 \in L_2(\Gamma_4)$ there exists $u \in H^1(\Delta_1)$ such that $L_1 u = 0$ in Δ_1, $u\big|_{\Gamma_4} = v_0, u_{y_0}\big|_{\Gamma_4} = v_1$. Moreover, the traces $\varphi = u\big|_{\Gamma_2}, \psi = u\big|_{\Gamma_3}$ exists and belongs to $H^1(\Gamma_2)$, $H^1(\Gamma_3)$, respectively. The following estimate holds:*

$$\left\|u\big|_{\Gamma_2}\right\|_{1,\Gamma_2}^2 + \left\|u\big|_{\Gamma_3}\right\|_{1,\Gamma_3}^2 \le C\Big(\left\|u\big|_{\Gamma_4}\right\|_{1,\Gamma_4}^2 + \left\|u_{y_0}\big|_{\Gamma_4}\right\|_{0,\Gamma_4}^2\Big). \tag{3.34}$$

Vice versa, for any $\varphi \in H^1(\Gamma_2)$, $\psi \in H^1(\Gamma_3)$, $\varphi = \psi$ at $y_0 = \frac{T_2+T_1}{2}$ there exists $u \in H^1(\Delta_1)$, $L_1 u = 0$ in Δ_1 such that $u\big|_{\Gamma_2} = \varphi$, $u\big|_{\Gamma_3} = \psi$ and the following estimate holds:

$$\|u|_{\Gamma_4}\|^2_{1,\Gamma_4} + \|u_{y_0}|_{\Gamma_4}\|^2_{0,\Gamma_4} \leq C\Big(\|u|_{\Gamma_2}\|^2_{1,\Gamma_2} + \|u|_{\Gamma_3}\|^2_{1,\Gamma_3}\Big). \tag{3.35}$$

Proof Let $\Delta_{1,T'}$ be the domain bounded by Γ_2, Γ_3 and $\Gamma_{4,T'}$, where $\Gamma_{4,T'}$ is the plane $y_0 = T'$, $\frac{T_1+T_2}{2} \leq T' \leq T_2$. Denote by $(u, v)_{\Delta_{1,T'}}$ the L_2-inner product in $\Delta_{1,T'}$. Integrating by parts the identity

$$(L_1 u, u_{x_0})_{\Delta_{1T'}} + (u_{x_0}, L_1 u)_{\Delta_{1T'}} = 0$$

we get, as in [11]:

$$E_{T'}(u,u) + Q_{T'}(u,u) + Q^{(1)}_{T'}(u,u) = I_2, \tag{3.36}$$

where $E_{T'}(u,u)$ is the same as in (3.26),

$$\begin{aligned} Q_{T'}(u,u) = \frac{1}{2}\int\limits_{\Gamma_{2T'}} \Bigg[& 4|u_s|^2 - \sum_{j,k=1}^{n-1} g_0^{jk}\Big(\frac{\partial u}{\partial y_j} - iA'_j u\Big)\overline{\Big(\frac{\partial u}{\partial y_k} - iA'_k u\Big)} \\ & - 2\sum_{j=1}^{n-1}\Bigg(g_0^{0j}\Big(\frac{\partial u}{\partial s} + iA'_{\_}u\Big)\overline{\Big(\frac{\partial u}{\partial y_j} - iA'_j u\Big)} \\ & + g_0^{0j}\Big(\frac{\partial u}{\partial y_j} - iA'_j u\Big)\overline{\Big(\frac{\partial u}{\partial s} + iA'_{\_}u\Big)}\Bigg) + V_1|u|^2\Bigg]dy'ds, \end{aligned} \tag{3.37}$$

(cf. (3.22) in [11]),

$$\begin{aligned} & Q^{(1)}_{T'}(u,u) \\ & = \frac{1}{2}\int\limits_{\Gamma_{3T'}} \Bigg(|u_\tau + iA'_{\_}u|^2 - \sum_{j,k=1}^{n-1} g_0^{jk}(u_{y_j} - iA'_j u)\overline{(u_{y_k} - iA'_k u)} + V_1|u|^2\Bigg)dy'd\tau, \end{aligned} \tag{3.38}$$

$$|I_2| \leq C\int\limits_{\Delta_{1T'}} \sum_{j=0}^{n} \Big|\frac{\partial u}{\partial y_j}\Big|^2 dy_0 dy' dy_n. \tag{3.39}$$

Here $\Gamma_{2T'}$, $\Gamma_{3T'}$ are parts of Γ_2, Γ_3 for $\frac{T_1+T_2}{2} \leq T'$. When $T_2 - T_1$ is small, $Q_{T'}(u,u)$ is positive definite (cf. [11, 3.23]). Therefore

$$C_1\|u\|^2_{1,\Gamma_{2T'}} \leq Q_{T'}(u,u) \leq C_2\|u\|^2_{1,\Gamma_{2T'}}. \tag{3.40}$$

Analogously,

$$C'_1\|u\|^2_{1,\Gamma_{3T'}} \leq Q^{(1)}_{T'}(u,u) \leq C'_2\|u\|^2_{1,\Gamma_{3T'}}. \tag{3.41}$$

Having (3.31), (3.39), (3.40), (3.41) we can complete the proof of Lemma 3.5 exactly as the proof of Lemma 3.1 in [10]. □

Combining Lemmas 3.4 and 3.5 we can prove the following lemma:

Lemma 3.6 *The map $f \to u^f$ is a bounded operator from $H_0^1(\Gamma_j \times [s_0, T_2])$ to $H_0^1(Y_{js_0})$:*

$$\|u^f\|_{1,Y_{js_0}} \le C[f]_1. \tag{3.42}$$

Proof It follows from Lemma 3.4 that

$$\|u^f\|_{1,\Gamma_4}^2 + \|u_{y_0}^f\|_{0,\Gamma_4}^2 \le C[f]_1^2. \tag{3.43}$$

Then (3.34) gives

$$\|u^f\|_{1,\Gamma_2}^2 \le C\Big(\|u^f\|_{1,\Gamma_4} + \|u_{y_0}^f\|_{0,\Gamma_4}^2\Big). \tag{3.44}$$

Combining (3.43) and (3.44) and taking into account that $\operatorname{supp} u^f\big|_{\Gamma_2} = Y_{js_0}$, we get (3.42). □

4 The main formula

Let $L_1^{(i)}$, $i = 1, 2$, be two operators of the form (2.27) such that the corresponding DN operators $\Lambda_1^{(1)}$ and $\Lambda_1^{(2)}$ are equal on $U_0 \cap \{y_n = 0\}$. We choose $\Gamma_1^{(1)} = \Gamma_1^{(2)} = \Gamma_1$ in a neighborhood of $x^{(0)}$ in $U_0 \cap \{y_n = 0\}$. Let $\Gamma_j^{(i)}$, $j = 2, 3$, $i = 1, 2$, be defined as before (see Fig. 3) for $i = 1, 2$, respectively.

Lemma 4.1 *We have* $\Gamma_j^{(1)} = \Gamma_j^{(2)}$, $j = 2, 3$ (cf. [8]).

Proof Let $\Delta_{2T_1}^{(i)}$ be the intersection of the domain of influence $D_{2T_1}^{(i)}$, $i = 1, 2$, with the plane $y_n = 0$. Note that $\Delta_{2T_1}^{(i)}$ is the intersection of $y_n = 0$ with the closure of the union $\bigcup \operatorname{supp} u_i^f$ where the union is taken over all $f \in H_0^1(\Gamma_1 \times [T_1, T_2])$, $L_1^{(i)} u_i^f = 0$.

Let $\tilde\Delta_{2T_1}^{(i)}$ be the closure of the union $\bigcup \operatorname{supp} \Lambda_1^{(i)} f$, where the union is taken also over all $f \in H_0^1(\Gamma_1 \times [T_1, T_2])$. We shall show that $\tilde\Delta_{2T_1}^{(i)} = \Delta_{2T_1}^{(i)}$.

If $x^{(0)} \notin \Delta_{2T_1}^{(i)}$ then $u_i^f = 0$, $\forall f$, in some neighborhood of $x^{(0)}$ in U_0. Then $\Lambda_1^{(i)} f = 0$ in a neighborhood of $x^{(0)}$, $\forall f$. Thus $x^{(0)} \notin \tilde\Delta_{2T_1}^{(i)}$, i.e. $\tilde\Delta_{2T_1}^{(i)} \subset \Delta_{2T_1}^{(i)}$. Let now $x_0' \notin \tilde\Delta_{2T_1}^{(i)}$. Then $\Lambda_1^{(i)} f = 0$ in a neighborhood of x_0' for any $f \in H_0^1(\Gamma_1 \times [T_1, T_2])$ and also $f = 0$ in a neighborhood of x_0'. Then by the uniqueness of the Cauchy problem (see [25,29]) we have that all $u^f = 0$ in a neighborhood of x_0' in $\mathbb{R}^{n+1}$. Therefore $x_0' \notin \Delta_{2T_1}^{(i)}$. Thus $\Delta_{2T_1}^{(1)} = \tilde\Delta_{2T_1}^{(1)}$. Since $\Lambda_1^{(1)} = \Lambda_1^{(2)}$, we have $\tilde\Delta_{2T_1}^{(1)} = \tilde\Delta_{2T_1}^{(2)}$. Therefore $\Delta_{2T_1}^{(1)} = \Delta_{2T_1}^{(2)}$, i.e. $\Gamma_2^{(1)} = \Gamma_2^{(2)}$. Analogously one shows that $\Gamma_3^{(1)} = \Gamma_3^{(2)}$. □

Since $\Gamma_j^{(1)} = \Gamma_j^{(2)}$ we shall write Γ_j, $1 \le j \le 3$, instead of $\Gamma_j^{(i)}$. It follows from (3.7) that (3.9) is determined by the boundary data. Integrating by part we have

$$\int\limits_{Y_{3T_1}} (u_s^f \overline{v^g} - u^f \overline{v_s^g}) ds dy' = 2 \int\limits_{Y_{3T_1}} u_s^f \overline{v^g} ds dy' - \int\limits_{\partial Y_{3T_1} \cap \{y_n = 0\}} u^f (T_2 - T_1, 0, y') \overline{v^g} (T_2 - T_1, 0, y') dy'. \tag{4.1}$$

Since $u^f (T_2 - T_1, 0, y') = f(T_2, y')$, $v^g(T_2 - T_1, 0, y') = g(T_2, y')$, we have that

$$(u_s^f, v^g) = \int\limits_{Y_{3T_1}} u_s^f \overline{v^g} ds dy' \tag{4.2}$$

is also determined by the boundary data.

Lemma 4.2 *Let $f \in H_0^1(\Gamma_1 \times [T_1, T_2])$. For any $s_0 \in [T_1, T_2)$ there exists $u_0 \in H_0^1(R_{2s_0})$ such that*

$$(u_s^f, v') = (u_{0s}, v') \tag{4.3}$$

for any $v' \in H_0^1(Y_{3s_0})$. Note that $R_{2s_0} = \{\tau = 0, s_0 - T_1 \le s \le T_2 - T_1, y' \in \Gamma_2\}$ (cf. Fig. 7).

Proof Note that $Y_{1T_1} \cap \{s_0 - T_1 \le s \le T_2 - T_1\} \subset R_{2s_0}$. Let w_1 be such that $w_{1s} = 0$ in R_{2s_0}, $w_1 = u^f$ when $s = s_0 - T_1$, $y' \in \Gamma_2$. Then $u_0 = u^f - w_1$ for $s \ge s_0 - T_1$, $u_0 = 0$ for $s \le s_0 - T_1$, belongs to $H_0^1(R_{2s_0})$ and solves (4.3). □

If $v' = v^{g'}$ where $g' \in H_0^1(\Gamma_2 \times [s_0, T_2])$ then $(u_0, v^{g'}) = (u^f, v^{g'})$ is determined by the DN operator. Let $g \in H_0^1(\Gamma_1 \times [T_1, T_2])$. We shall show that still (u_0, v^g) is determined by the DN operator. The following theorem holds.

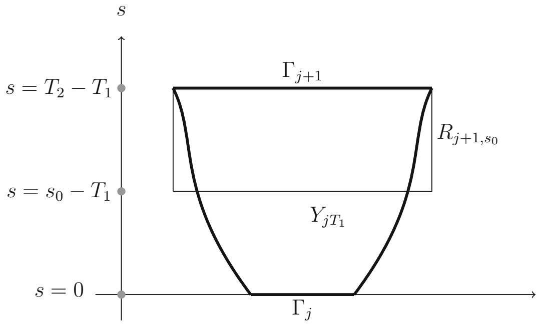


Fig. 7 R_{j+1,s_0} is the rectangle $\{s_0 - T_1 \le s \le T_2 - T_1, \tau = 0, y' \in \Gamma_{j+1}\}$, $Y_{jT_1} \cap \{s_0 - T_1 \le s \le T_2 - T_1\} \subset R_{j+1,s_0}$

Theorem 4.3 *Let $L_1^{(i)}, i = 1, 2$, be two operators of the form (2.27). Let f be in $H_0^1(\Gamma_1 \times [T_1, T_2])$ and let $u_0^{(i)}$ be the same as in (4.3) for $i = 1, 2$. Then*

$$(u_{0s}^{(1)}, v_1^g)\big|_{Y_{2s_0}^{(1)}} = (u_{0s}^{(2)}, v_2^g)\big|_{Y_{2s_0}^{(2)}} \tag{4.4}$$

for all $g \in H_0^1(\Gamma_2 \times [T_1, T_2])$.

Here u_i^f, v_i^g are the same as in (3.1), (3.2) for $i = 1, 2$, respectively. Operators $L^{(i)}$ and, consequently, $L_1^{(i)}$ are formally self-adjoint, $(u_0^{(i)}, v_i^g)_{Y_{2s_0}^{(i)}}$ is the L_2-inner product over $Y_{2s_0}^{(i)}, i = 1, 2$.

To prove Theorem 4.3 we will need the Density Lemma 3.1 and the following lemma that uses the BLR condition:

Lemma 4.4 *Let $L^{(1)}$ and $L^{(2)}$ be two operators in $D \cap [t_0, T_2]$ having the same DN operator on $\Gamma_0 \times [t_0, T_2]$. Suppose $L^{(1)}$ satisfies the BLR condition on $[t_0, T_2]$.*

Let $L_1^{(i)}, u_i^f, X_{2s_0}$ be the same as in (3.1), $i = 1, 2$, $f \in H_0^1(\Gamma_{2s_0})$, where $\Gamma_{2s_0} = \Gamma_2 \times [s_0, T_2]$. Then

$$\|u_2^f\|_{1, Y_{2s_0}^{(2)}} \le C_2 \|u_1^f\|_{1, Y_{2s_0}^{(1)}}. \tag{4.5}$$

Proof of Lemma 4.4 (cf. Lemma 2.3 in [10]): Suppose that BLR condition (see [1]) is satisfied for $L^{(1)}$ on $[t_0, T_{t_0}]$ and $t_0 < T_1$, $T_2 \ge T_{t_0}$. The BLR condition implies that the map $f \to \left(u_1^f(x)\big|_{D_{T_2}}, \frac{\partial u_1^f(x)}{\partial x_0}\big|_{D_{T_2}}\right)$ of $H_+^1(\Gamma_0 \times (t_0, T_2))$ to $H^1(D_{T_2}) \times L_2(D_{T_2})$ is onto, where $D_{T_2} = D_0^{(1)} \times \{x_0 = T_2\}$. It follows from [15] (cf. also Lemma 3.6) that

$$\|u_1^f(x)\|_{1, D_{T_2}}^2 + \left\|\frac{\partial u_1^f}{\partial x_0}\right\|_{0, D_{T_2}} \le C_0[f]_{1, \Gamma_0 \times (t_0, T_2)}. \tag{4.6}$$

By the closed graph theorem we have

$$\inf_{\mathcal{F}} [f']_{1, \Gamma_0 \times [t_0, T_2]} \le C_1 \Big(\|u_1^f\|_{1, D_{T_2}}^2 + \left\|\frac{\partial u_1^f}{\partial x_0}\right\|_{0, D_{T_2}}^2\Big), \tag{4.7}$$

where $\mathcal{F} \subset H_+^1(\Gamma_0 \times (t_0, T_2))$ is the set of f' such that

$$u_1^{f'}(x)\big|_{D_{T_2}} = u_1^f(x)\big|_{D_{T_2}}, \quad \frac{\partial u_1^{f'}(x)}{\partial x_0}\Big|_{D_{T_2}} = \frac{\partial u_1^f(x)}{\partial x_0}\Big|_{D_{T_2}}. \tag{4.8}$$

It follows from (4.7) that there exists $f_0 \in \mathcal{F}$, $f_0 = 0$ for $x_0 < t_0$ such that

$$[f_0]_{1, \Gamma_0 \times [t_0, T_2]} \le C \Big(\|u_1^f\|_{1, D_{T_2}}^2 + \left\|\frac{\partial u_1^f}{\partial x_0}\right\|_{0, D_{T_2}}^2\Big), \tag{4.9}$$

Note that

$$u_1^{f_0}(x)\big|_{D_{T_2}} = u_1^f(x)\big|_{D_{T_2}}, \quad \frac{\partial u_1^{f_0}(x)}{\partial x_0}\Big|_{D_{T_2}} = \frac{\partial u_1^f(x)}{\partial x_0}\Big|_{D_{T_2}}. \tag{4.10}$$

Let $\Lambda^{(i)}$ be the DN operator corresponding to $L^{(i)}$, $i = 1, 2$, and let $u_{i0}^{f_0}$ be the solution of $L^{(i)}u_{i0}^{f_0} = 0$ in $D \times [t_0, T_2)$, $u_{i0}^{f_0} = f_0$ on $\Gamma_0 \times [t_0, T]$, $i = 1, 2$. Let $L_1^{(i)}u_i^{f_0} = 0$ be the solutions in a neighborhood U_0 obtained from $L^{(i)}u_{i0}^{f_0} = 0$ as in Sect. 2 (cf. (2.38)) for $i = 1, 2$.

Consider the identity

$$(L_1^{(i)}u_i^{f_0}, v_i^g)\big|_{X_{js_0}^{(i)}} - (u_i^{f_0}, L_1^{(i)}v_i^g)\big|_{X_{js_0}^{(i)}} = 0, \tag{4.11}$$

where v^g is the same as in (3.2). Since $\operatorname{supp} v^g \subset D(\Gamma_{js_0})$, where $D(\Gamma_{js_0})$ is the domain of influence of Γ_{js_0} for $y_0 \le T_2$, we have that $v_i^g = 0$ on Z_{js_0}. Therefore integrating by parts in (4.11) we get as in (3.7):

$$(u_{1s}^{f_0}, v_1^g)\big|_{Y_{js_0}^{(1)}} - (u_1^{f_0}, v_{1s}^g)\big|_{Y_{js_0}^{(1)}} = -(\Lambda_1^{(1)}f_0, g)\big|_{\Gamma_{js_0}} + (f_0, \Lambda_1^{(1)}g)\big|_{\Gamma_{js_0}}.$$

Analogously, we have for $L^{(2)}u_2^{f_0} = 0$, $L^{(2)}v_2^g = 0$:

$$(u_{2s}^{f_0}, v_2^g)\big|_{Y_{js_0}^{(2)}} - (u_2^{f_0}, v_{2s}^g)\big|_{Y_{js_0}^{(2)}} = -(\Lambda_1^{(2)}f_0, g)\big|_{\Gamma_{js_0}} + (f_0, \Lambda_1^{(2)}g)\big|_{\Gamma_{js_0}}.$$

We have that $\Lambda^{(1)}f_0 = \Lambda^{(2)}f_0$ on $\Gamma_0 \times [t_0, T_2]$. Therefore (2.38) implies that $\Lambda_1^{(1)}f_0 = \Lambda_1^{(2)}f_0$ in Γ_{js_0}. Also $\Lambda_1^{(1)}g = \Lambda_1^{(2)}g$ in Γ_{js_0}. Integrating by parts we get

$$-(u_i^{f_0}, v_{is}^g) = (u_{is}^{f_0}, v_i^g) - \int\limits_{R^{n-1}} \left(u_i^{f_0}\overline{v_i^g}\big|_{s=T_2-T_1} - u_i^{f_0}v_i^g\big|_{s=0}\right)dy'.$$

Note that $v_i^g\big|_{s=0} = 0$ and $u_i^{f_0}\overline{v_1^g}\big|_{s=T_2-T_1} = f_0(T_2, y')\overline{g(T_2, y')}$. Therefore

$$(u_{1s}^{f_0}, v_1^g) = (u_{2s}^{f_0}, v_2^g) \tag{4.12}$$

for all $g \in H_0^1(\Gamma_{3s_0})$.

Let $\Gamma_2, \Gamma_3, \Gamma_4$ be the same as in Lemma 3.5. It was proven there that

$$\|u_1^f\|_{1,\Gamma_4}^2 + \|u_{1y_0}^f\|_{0,\Gamma_4}^2 \le C\Big(\|u_1^f\|_{1,\Gamma_2}^2 + \|u_1^f\|_{1,\Gamma_3}^2\Big), \tag{4.13}$$

$$\|u_1^f\|_{1,\Gamma_2}^2 + \|u_1^f\|_{1,\Gamma_3}^2 \le C\Big(\|u_1^f\|_{1,\Gamma_4}^2 + \|u_{1,y_0}^f\|_{0,\Gamma_4}^2\Big). \tag{4.14}$$

It follows from $u_1^f|_{\Gamma_4} = u_1^{f_0}|_{\Gamma_4}$, $u_{1y_0}^f|_{\Gamma_4} = u_{1y_0}^{f_0}|_{\Gamma_4}$ that

$$u_1^f|_{\Gamma_2} = u_1^{f_0}|_{\Gamma_2} \tag{4.15}$$

by the domain of dependence argument. Comparing (4.12) with $(u_{1s}^f, v_1^g) = (u_{2s}^f, v_2^g)$ and taking into account (4.15) we get

$$(u_{2s}^{f_0}, v_2^g)|_{Y_{2s_0}^{(2)}} = (u_{2s}^f, v_2^g)|_{Y_{2s_0}^{(2)}}, \quad \forall g \in H_0^1(\Gamma_3 \times (s_0, T_2)). \tag{4.16}$$

By Lemma 3.1 $\{v_2^g\}$ are dense in $H_0^1(R_{3s_0}^{(2)})$. Since $Y_{2s_0}^{(2)} \subset R_{3s_0}^{(2)}$ we get that $\{v_2^g\}$ are dense in $H_0^1(Y_{2s_0}^{(2)})$ and therefore $u_{2s}^{f_0} = u_{2s}^f$ in $Y_{2s_0}^{(2)}$. Since $u_2^f|_{s=T_2-T_1} = f(T_2, y') = u_1^f(T_2, y', 0)$, $u_2^{f_0}|_{s=T_2-T_1} = f_0(T_2, y') = u_1^{f_0}(T_2, y', 0)$ and since $u_1^{f_0}(T_2, y', 0) = u_1^f(T_2, y', 0)$ we get that $u_2^f|_{s=T_2-T_1} = u_2^{f_0}|_{s=T_2-T_1}$. Thus

$$u_2^{f_0} = u_2^f \quad \text{on} \quad Y_{2s_0}^{(2)}. \tag{4.17}$$

It follows from (4.13) that

$$\|u_1^f\|_{1,\Gamma_4}^2 + \|u_{1y_0}^f\|_{0,\Gamma_4}^2 \le C\|u_1^f\|_{1,\Gamma_2}^2, \tag{4.18}$$

since that $u_1^f = 0$ on Γ_3 by the domain of dependence argument.

Since $Y_{2s_0}^{(2)}$ belongs to the domain of dependence of D_{T_2} we get, similarly to (4.14), that

$$\|u_2^{f_0}\|_{1,Y_{2s_0}^{(2)}}^2 \le C_1\left(\|u_2^{f_0}\|_{1,D_{T_2}^{(2)}}^2 + \|u_{2y_0}^{f_0}\|_{0,D_{T_2}^{(2)}}^2\right), \tag{4.19}$$

where $D_{T_2}^{(2)} = D^{(2)} \cap \{y_0 = T_2\}$.

We also have (cf. Lemma 3.6)

$$\|u_2^{f_0}\|_{1,D_{T_2}^{(2)}}^2 + \left\|\frac{\partial u_2^{f_0}}{\partial x_0}\right\|_{0,D_{T_2}^{(2)}}^2 \le C\big[f_0\big]_{1,\Gamma_0\times[t_0,T_2]}^2. \tag{4.20}$$

Combining (4.18), (4.9) with (4.19), (4.20) and taking into account (4.17), we get

$$\|u_2^f\|_{1,Y_{2s_0}^{(2)}} \le C\|u_1^f\|_{1,Y_{2s_0}^{(1)}}. \tag{4.21}$$

Now we shall prove Theorem 4.3. □

Proof of Theorem 4.3 Since $u_0^{(1)} \in H_0^1(R_{2s_0}^{(1)})$ we get, using the Density Lemma 3.1, that there exists $u_1^{f_n}$, $f_n \in H_0^1(\Gamma_2 \times [s_0, T_2])$ such that $\|u_0^{(1)} - u_1^{f_n}\|_{1,Y_{2s_0}^{(1)}} \to 0$.

By Lemma 4.4 $\{u_2^{f_n}\}$ also converges in $H_0^1(Y_{2s_0}^{(2)})$ to some function $w \in H_0^1(Y_{2s_0}^{(2)})$. Passing to the limit in

$$\left(u_{1s}^{f_n}, v_1^g\right) = \left(u_{2s}^{f_n}, v_2^g\right), \tag{4.22}$$

we get

$$\left(u_{0s}^{(1)}, v_1^g\right) = \left(w_s, v_2^g\right) \quad \text{for any} \quad g \in H_0^1(\Gamma_{3T_1}), \tag{4.23}$$

where $\Gamma_{3T_1} = \Gamma_3 \times [T_1, T_2]$ Note that (4.22) and therefore (4.23) hold also for any $g' \in H_0^1(\Gamma_{3s_0})$, i.e.

$$\left(u_{0s}^{(1)}, v_1^{g'}\right) = \left(w_s, v_2^{g'}\right). \tag{4.24}$$

For such g' the equality (4.3) holds, i.e.

$$\left(u_{0s}^{(1)}, v_1^{g'}\right) = \left(u_{0s}^{(2)}, v_2^{g'}\right). \tag{4.25}$$

Comparing (4.24) and (4.25) we get

$$\left(u_{0s}^{(2)}, v_2^{g'}\right) = \left(w_s, v_2^{g'}\right), \tag{4.26}$$

Since $v_2^{g'} \in H_0^1(Y_{3s_0}^{(2)})$ are dense in $H_0^1(R_{3s_0}^{(2)})$ and $w \in H_0^1(Y_{2s_0}^{(2)}) \subset H_0^1(R_{3s_0}^{(2)})$, we have that $u_{0s}^{(2)} = w_s$. Since $u_0^{(2)}$ and w are zero on $\partial Y_{3s_0}^{(2)} \setminus \{y_n = 0\}$ we get that

$$u_0^{(2)} = w \ \text{ in } \ Y_{2s_0}^{(2)}. \tag{4.27}$$

Therefore (4.23) and (4.27) gives

$$\left(u_{0s}^{(1)}, v_1^g\right) = (u_{0s}^{(2)}, v_2^g\right) \tag{4.28}$$

for all $g \in H_0^1(\Gamma_{3T_1})$, i.e. (4.4) holds. □

The following formula will be the main tool in solving the inverse problem.

Theorem 4.5 *For any $T_1 \le s_0 \le T_2$ the integral*

$$\int\limits_{Y_{jT_1} \cap \{0 \le s \le s_0 - T_1\}} \frac{\partial u^f}{\partial s} \overline{v^g} ds dy', \quad \forall f \in H_0^1(\Gamma_{jT_1}), \quad \forall g \in H_0^1(\Gamma_{jT_1}), \quad j = 1, 2, \tag{4.29}$$

is determined by the DN operator on $\Gamma_{jT_1} = \Gamma_j \times [T_1, T_2]$.

Proof Since $u_0^{(i)}=\frac{\partial u^f}{\partial s}$ for $s\geq s_0-T_1,\ u_{0s}=0$ for $s\leq s_0-T_1$, formula (4.28) gives that $\int_{Y_{jT_1}\cap\{s>s_0-T_1\}}\frac{\partial u^f}{\partial s}\overline{v^g}dsdy'$ is determined by the DN operator on Γ_{jT_1}. The integral (4.29) is the difference $(u_s^f,v^g)-(u_{0s},v^g)$ thus (4.29) is determined by DN operator. □

Remark 4.1 When the coefficients of $L_1^{(i)}, i=1,2$, do not depend on y_0, we can obtain the estimate (4.5) without assuming the BLR condition. In this case we can derive, in addition to (3.3), another Green's formula (cf. (3.18) in [10]):

Consider the identity

$$0=(L_1u,v_{y_0})+(u_{y_0},L_1v). \tag{4.30}$$

Integrating by parts as in [10] and using that $\frac{\partial}{\partial y_0}$ and L_1 are commute, we get (cf. (3.20) in [10])

$$\tilde{Q}(u,v)=-\tilde{\Lambda}_0(f,g), \tag{4.31}$$

where

$$\begin{aligned}\tilde{Q}(u,v)=\frac{1}{2}\int\limits_{Y_{20}}\Big[&2(u_s+iA'\_u)\overline{v_s}+2u_s\overline{(v_s+iA'\_v)}\\&-2\sum_{j=1}^{n-1}\Big(g_0^{0j}\Big(\frac{\partial u}{\partial s}+iA'\_u\Big)\overline{\Big(\frac{\partial v}{\partial y_j}-iA'_jv\Big)}\\&+g_0^{0j}\Big(\frac{\partial u}{\partial y_j}-iA'_ju\Big)\overline{\Big(\frac{\partial v}{\partial s}+iA'\_v\Big)}\Big)\\&-\sum_{j,k=1}^{n-1}g_0^{jk}\Big(\frac{\partial}{\partial y_j}-iA'_j\Big)u\overline{\Big(\frac{\partial}{\partial y_k}-iA'_k\Big)v}+V_1u\overline{v}\Big]dsdy'\end{aligned} \tag{4.32}$$

and

$$\Lambda_0(f,g)=\int\limits_{\Gamma^{(2)}\times[0,T]}\big(\Lambda_1f\,\overline{g_{y_0}}+f_{y_0}\overline{\Lambda_1g}\big)dy'dy_0. \tag{4.33}$$

As in [10] (cf. (3.23) in [10]) we have that $\tilde{Q}(u,u)$ is a positive definite form when T_2-T_1 is small and

$$C_2\|u\|_{1,Y_{2s_0}}^2\leq\tilde{Q}(u,u)\leq C_1\|u\|_{1,Y_{2s_0}}^2. \tag{4.34}$$

Let $u_i^f, i=1,2$, be such that $L_1^{(i)}u_i^f=0$ in $X_{2s_0}^{(i)},\ u_i^f\big|_{y_n=0}=f,\ u_i^f=0$ for $y_0<T_1$, supp f is contained in $\Gamma_2\times(T_1,T_2]$. We assume that $\Lambda_1^{(1)}=\Lambda_1^{(2)}$ on $\Gamma_{2T_1}=\Gamma_2\times(T_1,T_2)$. It follows from (4.13), (4.31), (4.33) that

$$\tilde{Q}_1(u_1^f, u_1^f) = \tilde{Q}_2(u_2^f, u_2^f), \tag{4.35}$$

where $\tilde{Q}_i$ corresponds to $L_1^{(i)}$, $i = 1, 2$. Thus, (4.34) implies that

$$C_1 \| u_1^f \|_{1, Y_{2s0}^{(1)}} \le \| u_2^f \|_{1, Y_{2s0}^{(2)}} \le C_2 \| u_1^f \|_{1, Y_{2s0}^{(1)}}, \tag{4.36}$$

i.e. the estimate (4.5) is proven.

5 The geometric optics construction

It follows from Theorem 4.5 that the DN operator allows to determine $\int_{Y_{2s_0} \cap \{s \le s_0 - T_1\}} u_s^f \overline{v^g} ds dy'$ for all $f \in H_o^1(\Gamma_{jT_1})$, $g \in H_0^1(\Gamma_{jT_1})$, $j = 1, 2$, i.e. if u_i^f, v_i^g satisfy (3.1), (3.2), $i = 1, 2$, then

$$\int\limits_{Y_{2s_0}^{(1)} \cap \{s \le s_0 - T_1\}} u_{1s}^f \overline{v_1^g} ds dy' = \int\limits_{Y_{2s_0}^{(2)} \cap \{s \le s_0 - T_1\}} u_{2s}^f \overline{v_2^g} ds dy'$$

Let u_i be the solution of $L^{(i)} u_i = 0$ such that

$$u_i = u_N^{(i)} + u_i^{(N+1)}, \quad u_N^{(i)} = \sum_{p=0}^{N} \frac{a_p^{(i)}(s, \tau, y')}{(ik)^p} e^{ik(s - s_0')}, \quad s_0' = s_0 - T_1, \tag{5.1}$$

$u_i^{(N+1)}$ will be chosen below, k is a large parameter. We have the following equations for $a_p^{(i)}$, $0 \le p \le N$ (see [11] and [12], §64, for more details on the construction of geometric optics type solutions):

$$\begin{aligned}
&-4i\Big(\frac{\partial}{\partial \tau} + iA_-^{(i)}\Big) a_0^{(i)} + 2i \sum_{j=1}^{n-1} g_{i0}^{oj} \Big(\frac{\partial}{\partial y_j} - iA_j^{(i)}\Big) a_0^{(i)} \\
&\quad + 2i \sum_{j=1}^{n-1} \Big(\frac{\partial}{\partial y_j} - iA_j^{(i)}\Big)\big(g_{i0}^{0j} a_0^{(i)}\big) = 0, && (5.2) \\
&\quad - 4i\Big(\frac{\partial}{\partial \tau} + iA_-^{(i)}\Big) a_p^{(i)} + 2i \sum_{j=1}^{n-1} g_{i0}^{oj} \Big(\frac{\partial}{\partial y_j} - iA_j^{(i)}\Big) a_p^{(i)} \\
&\quad + 2i \sum_{j=1}^{n-1} \Big(\frac{\partial}{\partial y_j} - iA_j^{(i)}\Big)\big(g_{i0}^{0j} a_0^{(i)}\big) = -L_1^{(i)} a_{p-1}^{(i)}, \quad p \ge 1, && (5.3)
\end{aligned}$$

with the initial conditions

$$a_0^{(i)}(s, \tau, y')\big|_{\tau = \tau_0} = \chi_1(s)\chi_2(y'), \quad \tau_0 = T_2 - T_1 - s, \tag{5.4}$$

$$a_p^{(i)}(s,\tau,y')\big|_{\tau=\tau_0}=0,\quad p\geq 1, \tag{5.5}$$

where $\chi_1(s)=0$ for $|s-s_0'|>2\delta$, $\chi_1(s)=1$ for $|s-s_0'|\leq\delta$, $\chi_2(y')\in C_0^\infty(\Gamma_2)$, $\chi_2(y')\neq 0$ when $|y'-y_0'|<\delta$, $y_0'\in\Gamma_2$ is arbitrary, g_{i0}^{j0} corresponds to $L_1^{(i)}$, $i=1,2$. Note that $y_n=\frac{T_2-T_1-s-\tau}{2}=0$ when $\tau=\tau_0$.

Let $u_i^{(N+1)}$ be such that

$$L_1^{(i)}u_i^{(N+1)}=-\frac{1}{(ik)^N}\big(L_1^{(i)}a_N^{(i)}\big)e^{ik(s-s_0')},\quad y_n>0,\quad y_0<T_2, \tag{5.6}$$

$u_i^{(N+1)}=u_{iy_0}^{(N+1)}=0$ when $y_0=T_1$, $y_n>0$, $i=1,2$, $u_i^{(N+1)}\big|_{y_n=0}=0$, $y_n\leq T_2$. Such $u_i^{(N+1)}$ exists (cf. [15]) and $L_1^{(i)}(u_N^{(i)}+u_i^{(N+1)})=0$.

Since $\operatorname{supp}u_N^{(i)}$ is contained in a small neighborhood of the line $\{s=s_0-T_1,\ y'=y_0'\}$, we have that $\operatorname{supp}(u_N^{(i)}+u_i^{N+1})\subset D_+(\Gamma_2\times[T_1,T_2])$ when $s_0-T_1>0$. Here, as in Sect. 1, $D_+(\Gamma_2\times[T_1,T_2])$ is the forward domain of influence of $\Gamma_2\times[T_1,T_2])$.

Let $\beta^{(i)}(s,\tau,\hat{y}')=(\beta_1^{(i)},\beta_2^{(i)},\dots,\beta_{n-1}^{(i)})$ be the solution of the system (cf. [11])

$$\frac{\partial\beta_j^{(i)}(s,\tau,\hat{y}')}{\partial\tau}=-g_{i0}^{0j}(s,\tau,\beta^{(i)}(s,\tau,\hat{y}')),\ 1\leq j\leq n-1,\ y_n>0, \tag{5.7}$$

$$\beta^{(i)}(s,\tau,\hat{y}')\big|_{\tau=\tau_0}=\hat{y}_i',\quad i=1,2,\quad \tau_0=T_2-T_1-s, \tag{5.8}$$

where $\hat{y}'=(\hat{y}_1,\dots,\hat{y}_{n-1})\in\Gamma_2$, s is a parameter in (5.7).

Let

$$\hat{s}=s,\hat{\tau}=\tau,\ \hat{y}'=\alpha^{(i)}(s,\tau,y'),\quad \alpha^{(i)}=(\alpha_1^{(i)},\dots,\alpha_{n-1}^{(i)}) \tag{5.9}$$

be the inverse to the map

$$s=\hat{s},\quad \tau=\hat{\tau},\quad y'=\beta^{(i)}(\hat{s},\hat{\tau},\hat{y}'), \tag{5.10}$$

i.e.

$$\alpha_j^{(i)}(s,\tau,\beta^{(i)}(s,\tau,\hat{y}'))=\hat{y}_j,\ \ 1\leq j\leq n-1. \tag{5.11}$$

Note that $\alpha_j^{(i)}(s,\tau,y')$, $1\leq j\leq n-1$, satisfy the equation

$$\frac{\partial\alpha_j^{(i)}(s,\tau,y')}{\partial\tau}-\sum_{k=0}^{n-1}g_{i0}^{k0}(s,\tau,y')\frac{\partial\alpha_j^{(i)}}{\partial y_k}=0,\quad \alpha_j^{(i)}\big|_{\tau=\tau_0}=y_j,\quad 1\leq j\leq n-1. \tag{5.12}$$

Let $\hat{a}_0^{(i)}(s,\tau,\hat{y}') = a_0^{(i)}(s,\tau,y')$, where $y' = \beta^{(i)}(s,\tau,\hat{y}')$. Then using (5.7) and (5.2) we get

$$\begin{aligned}\frac{\partial \hat{a}_0^{(i)}}{\partial \tau} &= \frac{\partial a_0^{(i)}}{\partial \tau} + \sum_{j=1}^{n-1} \frac{\partial a_0^{(i)}}{\partial y_j} \frac{\partial \beta_j^{(i)}}{\partial \tau} \\ &= \frac{\partial a_0^{(i)}}{\partial t} - \sum_{j=1}^{n} g_{i0}^{0j} \frac{\partial a_0^{(i)}}{\partial y_j} = \hat{B}^{(i)}(s,\tau,\hat{y}')\hat{a}_0^{(i)}(s,\tau,\hat{y}'),\end{aligned} \tag{5.13}$$

where $B^{(i)}(s,\tau,y') = -iA'_- - i\sum_{j=1}^{n-1} g_{i0}^{0j} A'_j + \frac{1}{2}\sum_{j=1}^{n-1} \frac{\partial g_{0i}^{0j}}{\partial y_j}$, $\hat{B}^{(i)}(s,\tau,\hat{y}') = B^{(i)}(s,\tau,\beta^{(i)}(s,\tau,\hat{y}'))$, $\hat{a}_0^{(i)}(s,\tau,\hat{y}')\big|_{\tau=\tau_0} = \chi_1(s)\chi_2(\hat{y}')$.

Therefore

$$a_0^{(i)}(s,\tau,y') = \chi_1(s)\chi_2(\alpha^{(i)}(s,\tau,y'))e^{b^{(i)}(s,\tau,\alpha^{(i)})} \tag{5.14}$$

where $b^{(i)}(s,\tau,\hat{y}') = \int_{\tau_0}^{\tau} \hat{B}^{(i)}(s,\hat{\tau},\hat{y}')d\hat{\tau}$. Substituting $u = u_N^{(i)} + u_i^{(N+1)}$ into (4.29) instead of u_i^f, integrating by parts in s and taking the limit when $k \to \infty$, we get

$$\begin{aligned}&\int\limits_{\mathbb{R}^{n-1}} e^{b^{(1)}(s_0',0,\alpha^{(1)})}\chi_2(\alpha^{(1)}(s_0',0,y'))\overline{v_1^g(s_0',0,y')}dy' \\ &\quad = \int\limits_{\mathbb{R}^{n-1}} e^{b^{(2)}(s_0',0,\alpha^{(2)})}\chi_2(\alpha^{(2)}(s_0',0,y'))\overline{v_2^g(s_0',0,y')}dy'.\end{aligned} \tag{5.15}$$

Note that $\tau = 0$ on $Y_{2T_1}^{(i)}$, $i = 1, 2$. In (5.15) $s_0 \in (T_1, T_2]$ is arbitrary, $s_0' = s_0 - T_1$.

Denote by $Y_{2T_1}^{(i)}(\tau')$ the intersection of the plane $\tau = \tau'$ with $X_{2T_1}^{(i)}$. Let $R_{2T_1}^{(i)}(\tau') \subset Y_{2T_1}^{(i)}(\tau')$ be the rectangle $\{\tau = \tau', 0 \le s \le T_2 - T_1 - \tau', y' \in \Gamma_2\}$. Note that $Y_{2T_1}^{(i)}(0) = Y_{2T_1}^{(i)}$ and $R_{2T_1}^{(i)}(0) = R_{2T_1}^{(i)}$.

Repeating the proof of Theorem 4.5 with $Y_{2T_1}^{(2)}$, $R_{2T_1}^{(2)}$ replaced by $Y_{2T_1}^{(2)}(\tau')$, $R_{2T_1}^{(2)}(\tau')$, $0 \le \tau' \le T_2 - T_1$, we get again, using the geometric optics construction (5.1), that (5.15) holds for any $(s,\tau) \in \Sigma$, where $\Sigma = \{(s,\tau), s \ge 0, \tau \ge 0, s + \tau \le T_2 - T_1\}$. Thus, we have

$$\begin{aligned}&\int\limits_{\mathbb{R}^{n-1}} e^{b^{(1)}}\chi_2(\alpha^{(1)}(s,\tau,y'))\overline{v_1^g(s,\tau,y')}dy' \\ &\quad = \int\limits_{\mathbb{R}^{n-1}} e^{b^{(2)}}\chi_2(\alpha^{(2)}(s,\tau,y'))\overline{v_2^g(s,\tau,y')}dy'.\end{aligned} \tag{5.16}$$

for $(s,\tau,y') \in X_{2T_1}^{(i)}$.

Let $\beta^{(i)}(\Sigma\times\overline{\Gamma}_2)$ be the image of $\Sigma\times\overline{\Gamma}_2$ under the map (5.10). Note that the support of geometric optics solution $u_N^{(i)}+u_i^{(N+1)}$ is contained in $D(\Gamma_2\times[T_1,T_2])$. Also we have that the curve $y'=\beta^{(i)}(s,\hat{\tau},\hat{y}')$ for $\tau_0\leq\tau\hat{\lesssim}\tau$, is contained in $X_{2T_1}^{(i)}$. Therefore $\beta^{(i)}(\Sigma\times\overline{\Gamma}_2)\subset X_{2T_1}^{(i)}$. Denote by $X_{\Gamma_2}^{(i)}$ the intersection of $\beta^{(i)}(\Sigma\times\overline{\Gamma}_2)$ with $\Sigma\times\overline{\Gamma}_2$. Note that $\Sigma\times\overline{\Gamma}_2=\bigcup_{0\leq\tau'\leq T_2-T_1}R_{2T_1}^{(i)}(\tau')$.

Finally, denote by $\tilde{X}_{\Gamma_2}^{(i)}$ the image of $X_{\Gamma_2}^{(i)}$ under the inverse map (5.9). Note that $\tilde{X}_{\Gamma_2}^{(i)}\subset\Sigma\times\overline{\Gamma}_2$.

Making the change of variables (5.10) in (5.16) we get

$$\begin{aligned}&\int\limits_{\Gamma_2}e^{b^{(1)}(s,\tau,\hat{y}')}\chi_1(\hat{y}')\overline{v_1^g(s,\tau,\beta^{(1)}(s,\tau,\hat{y}'))}J_1(s,\tau,\hat{y}')d\hat{y}'\\&\quad=\int\limits_{\Gamma_2}e^{b^{(2)}(s,\tau,\hat{y}')}\chi_2(\hat{y}')\overline{v_2^g(s,\tau,\beta^{(2)}(s,\tau,\hat{y}'))}J_2(s,\tau,\hat{y}')d\hat{y}',\end{aligned}\tag{5.17}$$

where J_i is the Jacobian of the map (5.10), $(s,\tau,\hat{y}')\in\Sigma\times\Gamma_2$.

Let $b^{(i)}=b_1^{(i)}+ib_2^{(i)}$, where $b_1^{(i)},b_2^{(i)}$ are real.

Since $\chi_2(y')\in C_0^\infty(\Gamma_2)$ is arbitrary, we have

$$e^{b_1^{(1)}-ib_2^{(1)}}v_1^g(s,\tau,\beta^{(1)})J_1=e^{b_1^{(2)}-ib_2^{(2)}}v_2^g(s,\tau,\beta^{(2)})J_2.\tag{5.18}$$

Let

$$\begin{aligned}&w_i^g(s,\tau,\hat{y}')=v_i^g(s,\tau,\beta^{(i)}(s,\tau,\hat{y}')),\quad \hat{y}'\in\Gamma_2,\\&\tilde{w}_i^g(s,\tau,\hat{y}')=w_i^g(s,\tau,\hat{y}')e^{-b^{(i)}(s,\tau,\hat{y}')}.\end{aligned}\tag{5.19}$$

Our strategy will be to show that $w_1^g(s,\tau,\hat{y}')=w_2^g(s,\tau,\hat{y}')$ in $\tilde{X}_{\Gamma_2}^{(1)}$ and then to show that the equations $\tilde{L}_1^{(1)}w_1^g=0$ and $\tilde{L}_1^{(2)}w_1^g=0$ have the same coefficients in $\tilde{X}_{\Gamma_2}^{(1)}$. Here $\tilde{L}_1^{(i)}$ is obtained from $L_1^{(i)}$ by the change of variables (5.10), $i=1,2$.

We shall show first that $e^{2b_1^{(1)}}J_1(s,\tau,\hat{y}')=e^{2b_1^{(2)}}J_2(s,\tau,\hat{y}')$. Consider the geometric optics solutions $v_{i,k}^g$ of the form (5.1), where $g=\chi_1(s)\chi_3(y')$, $\chi_3(y')\in C_0^\infty(\Gamma_2)$ is arbitrary. Substituting $v_{i,k}^g$ into (5.16), integrating by parts and passing to the limit when $k\to\infty$, we get

$$\begin{aligned}&\int\limits_{\mathbb{R}^{n-1}}e^{2b_1^{(1)}}\chi_2(\alpha^{(1)}(s_0',\tau,y'))\overline{\chi_3(\alpha^{(1)}(s_0',\tau,y'))}dy'\\&\quad=\int\limits_{\mathbb{R}^{n-1}}e^{2b_1^{(2)}}\chi_2((\alpha^{(2)}(s_0',\tau,y'))\overline{\chi_3(\alpha^{(2)}(s_0',\tau,y'))}dy',\end{aligned}\tag{5.20}$$

where $s_0'=s_0-T_1$.

Note that $e^{b^{(i)}}\overline{e^{b^{(i)}}} = e^{2b_1^{(i)}}$.

Making the change of variables $y' = \beta^{(i)}(s_0, \tau, \hat{y}')$ and using that χ_2 and χ_3 are arbitrary we get

$$e^{2b_1^{(1)}} J_1(s_0', \tau, \hat{y}') = e^{2b_1^{(2)}} J_2(s_0', \tau, \hat{y}'). \tag{5.21}$$

Therefore, (5.18) and (5.21) imply

$$e^{-b^{(1)}(s,\tau,\hat{y}')} v_1^g(s, \tau, \beta^{(1)}(s, \tau, \hat{y}')) = e^{-b^{(2)}(s,\tau,\hat{y}')} v_2^g(s, \tau, \beta^{(2)}(s, \tau, \hat{y}')) \quad \text{in } \Sigma \times \Gamma_2,$$
$$\text{i.e.} \quad \tilde{w}_1^g(s, \tau, \hat{y}') = \tilde{w}_2^g(s, \tau, \hat{y}'). \tag{5.22}$$

□

As in (4.12) the integration by parts gives

$$\int\limits_{Y_{3T_1}} (u_s^f \overline{v^g} - u^f \overline{v_s^g}) ds dy' = -2 \int\limits_{Y_{3T_1}} u^f \overline{v_s^g} ds dy' + \int\limits_{\partial Y_{3T_1} \cap \{y_n=0\}} u^f \big|_{y_n=0} \overline{v^g}\big|_{y_n=0} dy'.$$

Therefore $\int\limits_{Y_{3T_1}} u^f \overline{v_s^g} ds dy'$ is determined by the boundary data since $u^f\big|_{y_n=0} = f(T_2, y')$, $\overline{v^g}\big|_{y_n=0} = \overline{g}(T_2, y')$, i.e. the roles of u^f and v^g are reversed in comparison with (4.12). Therefore we get, as in (4.28),

$$\int\limits_{Y_{2s_0}^{(1)} \cap \{s \le s_0'\}} u_1^f \overline{v_{1s}^g} ds dy' = \int\limits_{Y_{2s_0}^{(2)} \cap \{s \le s_0'\}} u_2^f \overline{v_{2s}^g} ds dy'. \tag{5.23}$$

Substituting in (5.23) the geometric optics solution (5.1), integrating by parts in s, multiplying by ik and, finally, taking the limit when $k \to \infty$, we get (5.16) with v_i^g replaced by v_{is}^g. Note that we assumed that $v_i^g \in H_0^2(Y_{2T_1})$ when integrating by parts in (5.23). This can be achieved by requiring that $g \in H_0^2(\Gamma_{2T_1})$ and using the regularity results for hyperbolic initial–boundary value problems (cf. [13,15]). Therefore we get (5.18), with v_i^g replaced by v_{is}^g:

$$e^{b_1^{(1)} - ib_2^{(1)}} v_{1s}^g(s, \tau, \beta^{(1)}(s, \tau, \hat{y}')) J_1 = e^{b_1^{(2)} - ib_2^{(2)}} v_{2s}^g(s, \tau, \beta^{(2)}(s, \tau, \hat{y}')) J_2.$$

Using (5.21) we get

$$e^{-b^{(1)}} v_{1s}^g(s, \tau, \beta^{(1)}) = e^{-b^{(2)}} v_{2s}^g(s, \tau, \beta^{(2)}). \tag{5.24}$$

We shall need the following lemma:

Lemma 5.1 *The equalities*

$$\alpha_{js}^{(1)}(s, \tau, \beta^{(1)}(s, \tau, \hat{y}')) = \alpha_{js}^{(2)}(s, \tau, \beta^{(2)}(s, \tau, \hat{y}')), \quad 1 \le j \le n-1, \tag{5.25}$$

$$b^{(1)}(s,\tau,\hat{y}') = b^{(2)}(s,\tau,\hat{y}') \tag{5.26}$$

hold on $\tilde{X}^{(1)}_{\Gamma_2}$.

Proof Making the change of variables $\hat{y}' = \alpha^{(i)}(s,\tau,y')$ in (5.19), we get

$$e^{-b^{(i)}(s,\tau,\alpha^{(i)}(s,\tau,y'))} v_i^g(s,\tau,y') = \tilde{w}_i^g(s,\tau,\alpha^{(i)}(s,\tau,y')). \tag{5.27}$$

Differentiating in s we have

$$\begin{aligned}&\Big(-\frac{d}{ds}b^{(i)}(s,\tau,\alpha^{(i)}(s,\tau,\hat{y}'))\Big)e^{-b^{(i)}}v_i^g(s,\tau,y') + e^{-b^{(i)}}v_{is}^g(s,\tau,y')\\ &\quad= \frac{\partial\tilde{w}_i^g(s,\tau,\alpha^{(i)})}{\partial s} + \sum_{j=1}^{n-1}\frac{\partial\tilde{w}_i^g(s,\tau,\alpha^{(i)})}{\partial\hat{y}_j}\alpha_{js}^{(i)}(s,\tau,y').\end{aligned} \tag{5.28}$$

Returning back in (5.28) to $y' = \beta^{(1)}(s,\tau,\hat{y}')$ coordinates we get

$$\begin{aligned}&\frac{\partial\tilde{w}_i^g(s,\tau,\hat{y}')}{\partial s} + \sum_{j=1}^{n-1}\frac{\partial\tilde{w}_i^g(s,\tau,\hat{y}')}{\partial\hat{y}_j}\alpha_{js}^{(i)}(s,\tau,\beta^{(i)}(s,\tau,\hat{y}'))\\ &\quad= e^{-b^{(i)}(s,\tau,\hat{y}')}v_{is}^g(s,\tau,\beta^{(i)}(s,\tau,\hat{y}')) - \frac{d}{ds}b^{(i)}(s,\tau,\alpha(s,\tau,y'))\big|_{y'=\beta^{(i)}}\tilde{w}_i^g(s,\tau,\hat{y}').\end{aligned} \tag{5.29}$$

Subtracting (5.29) for $i=1$ from (5.29) for $i=2$ and taking into account (5.24) and (5.22) we get

$$\begin{aligned}&\sum_{j=1}^{n-1}\Big(\alpha_{js}^{(1)}(s,\tau,\beta^{(1)}(s,\tau,\hat{y}')) - \alpha_{js}^{(2)}(s,\tau,\beta^{(2)}(s,\tau,\hat{y}'))\Big)\frac{\partial\tilde{w}_1^g(s,\tau,\hat{y}')}{\partial\hat{y}_i}\\ &\quad+\Big(\frac{d}{ds}b^{(1)}(s,\tau,\alpha^{(1)}(s,\tau,y')\Big|_{y'=\beta^{(1)}} - \frac{d}{ds}b^{(2)}(s,\tau,\alpha^{(2)}(s,\tau,y')\Big|_{y'=\beta^{(2)}}\Big)\\ &\quad\tilde{w}_1^g(s,\tau,\hat{y}')) = 0\end{aligned} \tag{5.30}$$

for all $\tilde{w}_1^g(s,\tau,\hat{y}')$ where $(s,\tau,\hat{y}')\in\Sigma\times\Gamma_2$.

Fix $\tau=\tau'$, $0\le\tau'<T_0-T_1$. By the Density Lemma 3.1 $\{v_i^g(s,\tau',y')\}$ are dense in $H_0^1(R_{2T_1}^{(i)}(\tau'))$, where $g\in H_0^1(\Gamma_2\times\{T_1\le y_0\le T_2-\tau'\})$.

Let $\tilde{R}_{2T_1}^{(i)}(\tau')$ be the image of $R_{2T_1}^{(i)}(\tau')\cap\beta^{(i)}(\Sigma\times\overline{\Gamma}_2)$ under the map (5.9). Since $\tilde{w}_i^g = e^{-b^{(i)}}v_i^g(s,\tau,\beta^{(i)}(s,\tau,\hat{y}'))$ we have that $\tilde{w}_i^g(s,\tau',\hat{y}')$ are dense in $H_0^1(\tilde{R}_{2T}^{(i)}(\tau'))$. □

The following lemma is similar to arguments in [10, pp. 1749–1750].

Lemma 5.2 *Since* $\{w_1^g(s,\tau',\hat{y}'), g \in H_0^1(\overline{\Gamma}_2 \times \{T_1 \le y_0 \le T_2-\tau'\})\}$ *are dense in* $\tilde{R}_{2T_1}^{(1)}(\tau')$ *we have*

$$\alpha_{js}^{(1)}(s,\tau',\beta^{(1)}(s,\tau',\hat{y}')) = \alpha_{js}^{(2)}(s,\tau',\beta^{(2)}(s,\tau',\hat{y}')) \quad on \quad \tilde{R}_{2T_1}^{(1)}(\tau'), \tag{5.31}$$

$$\begin{aligned}&\frac{d}{ds}b^{(1)}(s,\tau',\alpha^{(1)}(s,\tau',y'))\Big|_{y'=\beta^{(1)}} \\ &\quad = \frac{d}{ds}b^{(2)}(s,\tau',\alpha^{(2)}(s,\tau',y'))\Big|_{y'=\beta^{(2)}} \quad on \quad \tilde{R}_{2T_1}^{(1)}(\tau').\end{aligned} \tag{5.32}$$

Proof Let $\gamma(s,\tau',\hat{y}') \in C_0^\infty(\tilde{R}_{2T_1}^{(1)}(\tau'))$. There exists a sequence $\tilde{w}_1^{g_k}(s,\tau',\hat{y}')$ convergent to $\gamma(s,\tau',\hat{y}')$ in $H_0^1(\tilde{R}_{2T_1}^{(1)}(\tau'))$. Therefore $\tilde{w}_1^{g_k}$ converges weakly to $\gamma(s,\tau',\tilde{y}')$. Passing in (5.30) to the limit when $k \to \infty$ we get

$$\begin{aligned}&\sum_{j=1}^{n-1}\Big(\alpha_{js}^{(1)}(s,\tau',\beta^{(1)}(s,\tau',\hat{y}')) - \alpha_{js}^{(2)}(s,\tau',\beta^{(2)}(s,\tau',\hat{y}'))\Big)\frac{\partial\gamma}{\partial\hat{y}_j} \\ &\quad + \Big(\frac{d}{ds}b^{(1)}(s,\tau',\alpha^{(1)}(s,\tau',y'))\Big|_{y'=\beta^{(1)}} \\ &\quad - \frac{d}{ds}b^{(2)}(s,\tau',\alpha^{(2)}(s,\tau',y'))\Big|_{y'=\beta^{(2)}}\Big)\gamma(s,\tau',\hat{y}') = 0.\end{aligned} \tag{5.33}$$

For any point $(s,\hat{y}') \in \tilde{R}_{2T_1}^{(1)}(\tau')$ we can find n $C_0^\infty(\tilde{R}_{2T_1}^{(1)})$ functions $\gamma_1(s,\hat{y}'),\ldots,$ $\gamma_n(s,\hat{y}')$ such that the determinant of $n\times n$ matrix

$$\begin{bmatrix} \frac{\partial\gamma_1}{\partial\hat{y}_1} & \cdots & \frac{\partial\gamma_1}{\partial\hat{y}_{n-1}} & \gamma_1 \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial\gamma_n}{\partial\hat{y}_1} & \cdots & \frac{\partial\gamma_n}{\partial\hat{y}_{n-1}} & \gamma_n \end{bmatrix}$$

is not equal to zero at the point $(s,\hat{y}')$. Therefore (5.31), (5.32) hold. □

Repeating the same arguments for any $0 \le \tau' \le T_2 - T_1$ we get that (5.31), (5.32) hold for any τ', i.e. it hold on $\tilde{X}_{\Gamma_2}^{(1)} = \bigcup_{0\le\tau'\le T_2-T_1}\tilde{R}_{2T_1}^{(1)}(\tau')$, since $\bigcup_{0\le\tau'\le T_2-T_1} R_{2T_1}^{(1)}(\tau') = \Sigma\times\Gamma_2$ and $\tilde{X}_{\Gamma_2}$ is the image of $(\Sigma\times\overline{\Gamma}_2)\cap\beta^{(1)}(\Sigma\times\overline{\Gamma}_2)$ under the map (5.9). This proves (5.25). To prove (5.26) we note that

$$\frac{d}{ds}b^{(i)}(s,\tau,\alpha^{(i)}(s,\tau,y'))\Big|_{y'=\beta^{(i)}} = \frac{\partial b^{(i)}}{ds} + \sum_{j=1}^{n-1}\frac{\partial b^{(i)}}{\partial\hat{y}_j}\alpha_{js}^{(i)}(s,\tau,\beta^{(i)}(s,\tau,\hat{y}').$$

Since (5.31), (5.32) hold, we have

$$\frac{\partial}{\partial s}(b^{(1)}-b^{(2)}) + \sum_{j=1}^{n-1}\frac{\partial}{\partial\hat{y}_j}(b^{(1)}-b^{(2)})\alpha_{js}^{(1)}(s,\tau,\beta^{(1)}(s,\tau,\hat{y}')) = 0. \tag{5.34}$$

Equation (5.34) is a linear homogeneous equation for $b^{(1)}(s,\tau,\hat{y}')-b^{(2)}(s,\tau,\hat{y}')$ on $\hat{X}^{(1)}_{\Gamma_2}$. Since $b^{(1)}=b^{(2)}=0$ when $y_n=0$, we get

$$b^{(1)}(s,\tau,\hat{y}')=b^{(2)}(s,\tau,\hat{y}') \quad \text{on} \quad \tilde{X}^{(1)}_{\Gamma_2}. \tag{5.35}$$

It follows from (5.35) and (5.22) that

$$w_1^g(s,\tau,\hat{y}')=w_2^g(s,\tau,\hat{y}') \quad \text{on} \quad \tilde{X}^{(1)}_{\Gamma_2}, \tag{5.36}$$

where $w_i^g=v_i^g(s,\tau,\beta^{(i)}(s,\tau,\hat{y}'))$.

6 The conclusion of the local step

We shall prove the following theorem.

Theorem 6.1 *Let $L_1^{(i)}v_i^g=0, i=1,2$. Make change of variables*

$$\hat{s}=s, \quad \hat{\tau}=\tau, \quad \hat{y}'=\alpha^{(i)}(s,\tau,y'), \quad i=1,2. \tag{6.1}$$

Let $\tilde{L}_1^{(i)}w^g=0$ be the operator $L_1^{(i)}$ in the new coordinates. Then the coefficients of $\tilde{L}_1^{(1)}$ and $\tilde{L}_1^{(2)}$ are equal on $\tilde{X}^{(1)}_{\Gamma_2}$.

Proof Equations $L_1^{(i)}v_i^g=0$ have the following form in (s,τ,y') coordinates (cf. (2.27)):

$$\begin{aligned} L_1^{(i)}v_i^g &= -2\frac{\partial}{\partial s}\Big(\frac{\partial}{\partial\tau}+iA_-^{(i)}\Big)v_i^g-2\Big(\frac{\partial}{\partial\tau}+iA_-^{(i)}\Big)\frac{\partial}{\partial s}v_i^g \\ &\quad +\sum_{j=1}^{n-1}2\Big(\frac{\partial}{\partial y_j}-iA_j^{(i)}\Big)g_{i0}^{+,j}\frac{\partial}{\partial s}v_i^g+\sum_{j=1}^{n-1}2\frac{\partial}{\partial s}\Big(g_{i0}^{+,j}\Big(\frac{\partial}{\partial y_j}-iA_j^{(i)}\Big)\Big)v_i^g \\ &\quad +\sum_{j,k=1}^{n-1}\Big(\frac{\partial}{\partial y_j}-iA_j^{(i)}\Big)g_{i0}^{jk}\Big(\frac{\partial}{\partial y_k}-iA_k^{(i)}\Big)v_i^g+V_1^{(i)}v_i^g=0, \end{aligned} \tag{6.2}$$

where $i=1,2,\ g_{i0}^{+,j}=g_{i0}^{0j},\ V_1^{(i)}$ is the same as in (2.27).

Making the change of variables (6.1) in (6.2) we get:

$$\begin{aligned} \tilde{L}_1^{(i)}w_i^g(s,\tau,\hat{y}') &= -2J_i^{-1}(s,\tau,\hat{y}')\Big(\frac{\partial}{\partial s}+i\tilde{A}_+^{(i)}\Big)J_i\Big(\frac{\partial}{\partial\tau}+i\tilde{A}_-^{(i)}\Big)w_i^g \\ &\quad -2J_i^{-1}\Big(\frac{\partial}{\partial\tau}+i\tilde{A}_-^{(i)}\Big)J_i\Big(\frac{\partial}{\partial s}+i\tilde{A}_+^{(i)}\Big)w_i^g \\ &\quad -\sum_{j=1}^{n-1}2J_i^{-1}\Big(\frac{\partial}{\partial\tau}+i\tilde{A}_-^{(i)}\Big)J_i\alpha_{js}^{(i)}(s,\tau,\beta^{(i)})\Big(\frac{\partial}{\partial y_j}-i\tilde{A}_j^{(i)}\Big)w_i^g \end{aligned}$$

$$
\begin{aligned}
&-\sum_{j=1}^{n-1} 2J_i^{-1}\Big(\frac{\partial}{\partial y_j} - i\tilde{A}_j^{(i)}\Big) J_i \alpha_{js}^{(i)}(s,\tau,\beta^{(i)})\Big(\frac{\partial}{\partial \tau} + i\tilde{A}_-^{(i)}\Big) w_i^g \\
&+\sum_{j,k=1}^{n-1} J_i^{-1}\Big(\frac{\partial}{\partial y_j} - i\tilde{A}_j^{(i)}\Big) J_i \tilde{g}_{i0}^{jk}\Big(\frac{\partial}{\partial y_k} - i\tilde{A}_k^{(i)}\Big) w_i^g \\
&+ V_1^{(i)}(s,\tau,\beta^{(i)}(s,\tau,\hat{y}'))w_i^g(s,\tau,\hat{y}') = 0,
\end{aligned}
\tag{6.3}
$$

where $w_i^g(s,\tau,\hat{y}') = v_i^g(s,\tau,\beta^{(i)}(s,\tau,\hat{y}'))$,

$$
\begin{aligned}
\tilde{g}_{i0}^{jk}(s,\tau,\hat{y}') &= \sum_{p,r=1}^{n-1} g_{i0}^{pr}(s,\tau,\beta^{(i)})\alpha_{jy_p}^{(i)}(s,\tau,\beta^{(i)})\alpha_{ky_r}^{(i)}(s,\tau,\beta^{(i)}) \\
&\quad - 2\alpha_{js}^{(i)}\alpha_{k\tau}^{(i)} - 2\alpha_{j\tau}^{(i)}\alpha_{ks}^{(i)} + 2\sum_{p=1}^{n-1} g_{i0}^{+,p}(\alpha_{js}\alpha_{ky_p} + \alpha_{jy_p}\alpha_{ks}).
\end{aligned}
\tag{6.4}
$$

We used in (6.3) that (see (5.12))

$$
\begin{aligned}
\tilde{g}_{i0}^{+,j}(s,\tau,\hat{y}') &= \sum_{k=1}^{n-1} g_{i0}^{+,k}(s,\tau,\beta^{(i)}(s,\tau^{(i)},\hat{y}'))\alpha_{jy_k}^{(i)}(s,\tau,\beta^{(i)}) - \alpha_{j\tau}^{(i)}(s,\tau,\beta^{(i)}) \\
&= 0.
\end{aligned}
\tag{6.5}
$$

Also we have

$$
\begin{aligned}
\tilde{g}_{i0}^{-,j}(s,\tau,\hat{y}') &= \sum_{k=1}^{n-1} g_{i0}^{-,k}(s,\tau,\beta^{(i)}(s,\tau,\hat{y}'))\alpha_{jy_k}^{(i)}(s,\tau,\beta^{(i)}) - \alpha_{js}^{(i)}(s,\tau,\beta^{(i)}) \\
&= -\alpha_{js}^{(i)}(s,\tau,\beta^{(i)}),
\end{aligned}
\tag{6.6}
$$

since $g_{i0}^{-,k} = 0$ (cf. (6.2)).

Note that $A_+^{(i)}, A_-^{(i)}, A_j^{(i)}$ and $\tilde{A}_+^{(i)}, \tilde{A}_-^{(i)}, \tilde{A}_j^{(i)}$ are related by the equality

$$
A_+^{(i)}ds + A_-^{(i)}d\tau - \sum_{j=1}^{n-1} A_j^{(i)}dy_j = \tilde{A}_+^{(i)}d\hat{s} + \tilde{A}_-^{(i)}d\hat{\tau} - \sum_{j=1}^{n-1}\tilde{A}_j^{(i)}d\hat{y}_j,
\tag{6.7}
$$

where $A_+^{(i)} = 0, i = 1, 2,\ s = \hat{s},\ \tau = \hat{\tau},\ y_j = \beta_j(s,\tau,\hat{y}')$.

Note that (5.21), (5.26) imply

$$
J_1(s,\tau,\hat{y}') = J_2(s,\tau,\hat{y}') \quad \text{in } \tilde{X}_{\Gamma_2}^{(1)}.
\tag{6.8}
$$

The first order term containing $\frac{\partial}{\partial\tau}$ in (6.3) is equal to

$$
\begin{aligned}
&-2i\tilde{A}_+^{(i)}\Big(\frac{\partial}{\partial\tau}\Big)w_i^g - 2iJ_i^{-1}\Big(\frac{\partial}{\partial\tau}\Big)J_i\tilde{A}_+^{(i)}w_i^g + 2i\sum_{j=1}^{n-1}\tilde{A}_j^{(i)}\alpha_{js}^{(i)}\frac{\partial}{\partial\tau}w_i^g \\
&\quad + i\sum_{j=1}^{n-1}2J_i^{-1}\Big(\frac{\partial}{\partial\tau}\Big)J_i\alpha_{js}^{(i)}\tilde{A}_j^{(i)}w_i^g.
\end{aligned}
\tag{6.9}
$$

It follows from (6.7) that

$$
A_+^{(i)} = \tilde{A}_+^{(i)} - \sum_{j=1}^{n-1}\tilde{A}_j^{(i)}\alpha_{js}\left(s,\tau,\beta^{(i)}(s,\tau,\hat{y}')\right). \tag{6.10}
$$

Since $A_+^{(i)} = 0$ we have that (6.10) implies that (6.9) is equal to zero.

Taking into account that $\alpha_{js}^{(1)}(s,\tau,\beta^{(1)}) = \alpha_{js}^{(2)}(s,\tau,\beta^{(2)})$, $1 \le j \le n-1$, $J_1 = J_2$ and $w_1^g(s,\tau,\hat{y}') = w_2^g(s,\tau,\hat{y}')$ we get that $\tilde{L}_1^{(1)} - \tilde{L}_1^{(2)}$ is a differential operator in $\frac{\partial}{\partial s}, \frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_n}$. We have

$$
\left(\tilde{L}_1^{(1)} - \tilde{L}_1^{(2)}\right)w_1^g = 0. \tag{6.11}
$$

Since $\{w_1^g, g \in C_0^\infty(\Gamma^{(1)} \times [T_1, T_2 - \tau']\}$ are dense in $H_0^1(\tilde{R}_{1T_1}(\tau'))$ we get as in Lemma 5.2 (cf. [10]) that all coefficients of $\tilde{L}_1^{(1)}$ and $\tilde{L}_1^{(2)}$ are equal in $\tilde{R}_{1T_1}^{(1)}(\tau')$. Since $\tau' \in [0, T_2 - T_1]$ is arbitrary, we get that on $\hat{X}_{\Gamma_2}^{(1)}$:

$$
\tilde{g}_{10}^{jk}(s,\tau,\hat{y}') = \tilde{g}_{20}^{jk}(s,\tau,\hat{y}'), \quad 1 \le j,k \le n-1, \tag{6.12}
$$

$$
\tilde{A}_-^{(1)} = \tilde{A}_-^{(2)},\ \tilde{A}_j^{(1)} = \tilde{A}_j^{(2)}(s,\tau,\hat{y}), \quad 1 \le j \le n-1, \tag{6.13}
$$

$$
V_1^{(1)}(s,\tau,\beta^{(1)}(s,\tau,\hat{y}')) = V_1^{(2)}(s,\tau,\beta^{(2)}(s,\tau,\hat{y}')). \tag{6.14}
$$

Therefore $\tilde{L}_1^{(1)} = \tilde{L}_1^{(2)}$ in $\tilde{X}_{\Gamma_2}^{(1)}$. □

This completes the proof of Theorem 6.1. Let $L_i'v_i^g = 0$ be the equation of the form (2.24). Making the change of variables (5.10) we get the equation $\tilde{L}_i'w_i^g = 0$, $i = 1, 2$, on $\tilde{X}_{\Gamma_2}^{(1)}$.

Note that $w_1^g = w_2^g$ on $\tilde{X}_{\Gamma_2}^{(1)}$. We shall prove that $\tilde{L}_1' = \tilde{L}_2'$ on $\tilde{X}_{\Gamma_2}^{(1)}$.

Let $\hat{g}_i^{+,-}, \hat{g}_i^{+,j}, \hat{g}_j^{jk}$, $1 \le j \le n-1$, $1 \le k \le n-1$, be the inverse metric tensor of L_i'. Note that for $L_1^{(i)}$ we have (cf. (2.27))

$$
g_{i0}^{+,j} = \frac{\hat{g}_i^{+,j}}{\hat{g}_i^{+,-}}, \quad g_{i0}^{jk} = \frac{\hat{g}_i^{jk}}{\hat{g}_i^{+,-}}, \quad i = 1, 2.
$$

Therefore the equation $\tilde{L}_i' w_i^g = 0$ has the inverse metric tensor with elements (cf. (6.4), (6.5), (6.6))

$$\begin{aligned} \tilde{g}_i^{jk} &= \hat{g}_i^{+,-} g_{i0}^{jk}, \quad 1 \le j,k \le n-1, \\ \tilde{g}_i^{-,k} &= -\hat{g}_i^{+,-} \alpha_{ks}^{(i)}(s,\tau,\beta^{(i)}), \quad \tilde{g}_i^{+,k} = 0, \quad 1 \le k \le n-1, \quad i = 1,2. \end{aligned} \tag{6.15}$$

Since $\alpha_{ks}^{(1)} = \alpha_{ks}^{(2)}$ and $\tilde{g}_{10}^{jk} = \tilde{g}_{20}^{jk}$ (see (6.12)), we get that the metric tensors of $\tilde{L}_1'$ and $\tilde{L}_2'$ are equal if we can prove that

$$\tilde{g}_1^{+,-}(s,\tau,\beta^{(1)}(s,\tau,\hat{y}')) = \tilde{g}_2^{+,-}(s,\tau,\beta^{(2)}(s,\tau,\hat{y}')). \tag{6.16}$$

We shall prove first that

$$g_1^{(1)}(s,\tau,\beta^{(1)}) = g_1^{(2)}(s,\tau,\beta^{(2)}), \tag{6.17}$$

where $g_1^{(i)} = \left| \det[\hat{g}_i^{jk}]_{j,k=1}^{n-1} \right|^{-1}$ (see (2.22)).

Note that $V_1^{(i)}(s,\tau,y')$ has the form (2.25) for $i = 1,2$, where $A^{(i)} = \ln(g_1^{(i)})^{\frac{1}{4}}$. Making the change of variables (6.1) we get (cf. (6.14))

$$V_1^{(1)}\left(s,\tau,\beta^{(1)}(s,\tau,\hat{y}')\right) - V_2^{(2)}\left(s,\tau,\beta^{(2)}(s,\tau,\hat{y}')\right) = 0. \tag{6.18}$$

Note that the metric tensors for $\tilde{L}_1'$ and $\tilde{L}_2'$ are equal on $\tilde{X}_{\Gamma_2}$. Let $\tilde{A}^{(i)}(s,\tau,\hat{y}') = A^{(i)}(s,\tau,\beta^{(i)}(s,\tau,\hat{y}'))$.

Using the equality

$$\tilde{A}_{y_j}^{(1)}\tilde{A}_{y_k}^{(1)} - \tilde{A}_{y_j}^{(2)}\tilde{A}_{y_k}^{(2)} = \left(\tilde{A}_{y_j}^{(1)} - \tilde{A}_{y_j}^{(2)}\right)A_{y_k}^{(1)} + \left(\tilde{A}_{y_k}^{(1)} - \tilde{A}_{y_k}^{(2)}\right)\tilde{A}_{y_j}^{(2)}$$

and similar equality involving derivatives in s and τ we can represent (6.18) as a homogeneous second order hyperbolic equation in $\tilde{A}^{(1)} - \tilde{A}^{(2)}$ with the coefficients depending on $\tilde{A}^{(1)}$ and $\tilde{A}^{(2)}$. Since the Cauchy data for $\tilde{A}^{(1)} - \tilde{A}^{(2)} = 0$ at $y_n = 0$ (cf. Lemma 2.1) we get, by the uniqueness of the Cauchy problem (cf. [25,29]), that $\tilde{A}^{(1)} = \tilde{A}^{(2)}$ in $\tilde{X}_{\Gamma_2}^{(1)}$. Therefore (6.17) holds.

Note that $\tilde{g}_i^{jk} = \tilde{g}_i^{+,-}\tilde{g}_0^{jk}$. Therefore

$$g_1^{(i)} = \det\left[\tilde{g}_i^{jk}\right]_{j,k=1}^{n-1} = \left(\tilde{g}_i^{+,-}\right)^{n-1}\det\left[g_{0i}^{jk}\right]_{j,k=1}^{n-1}.$$

Since $\tilde{g}_{10}^{jk} = \tilde{g}_{20}^{jk}$ and (6.17) holds, we get

$$\left(\tilde{g}_1^{+,-}(s,\tau,\beta^{(1)})\right)^{n-1}\left(\tilde{g}_2^{+,-}(s,\tau,\beta^{(1)})\right)^{n-1}, \tag{6.19}$$

and this proves (6.16), since we assumed that $n > 1$. Therefore metric tensors of $\tilde{L}_1'$ and $\tilde{L}_2'$ are equal. Combining this with (6.13), (6.14) we get $\tilde{L}_1' = \tilde{L}_2'$ on $\tilde{X}_{\Gamma_2}^{(1)} \supset \Sigma \times \overline{\Gamma}_1$.

Remark 6.1 Change Γ_2 to Γ_1. We have $\beta^{(i)}(\Sigma\times\overline{\Gamma}_1)\subset\beta^{(i)}(\Sigma\times\overline{\Gamma}_2)$. Since $\beta^{(i)}(\Sigma\times\overline{\Gamma}_1)\subset X_{1T_1}^{(i)}$ and $X_{1T_1}^{(i)}\subset(\Sigma\times\overline{\Gamma}_2)$, we get $\beta^{(i)}(\Sigma\times\overline{\Gamma}_1)\subset(\Sigma\times\overline{\Gamma}_2)$. Therefore, $\beta^{(i)}(\Sigma\times\overline{\Gamma}_1)\subset(\Sigma\times\overline{\Gamma}_2)\cap\beta^{(i)}(\Sigma\times\overline{\Gamma}_2)=X_{\Gamma_2}^{(i)}$. Applying the map (5.9) to $\beta^{(i)}(\Sigma\times\overline{\Gamma}_1)\subset X_{\Gamma_2}^{(i)}$ we get $\Sigma\times\overline{\Gamma}_1\subset\tilde{X}_{\Gamma_2}^{(i)}$. Therefore, $\tilde{L}_1'=\tilde{L}_2'$ on $\Sigma\times\overline{\Gamma}_1$.

We shall summarize the results of Sects. 2–6.

Theorem 6.2 (Local step) *Consider two initial boundary value problems*

$$\begin{aligned}&L^{(i)}u_i=0\quad in\ \ D_0^{(i)}\times\mathbb{R},\\&u_i=0\quad for\ \ x_0\ll 0,\\&u_i\big|_{\partial D_0^{(i)}}=f,\quad i=1,2,\end{aligned}\tag{6.20}$$

where $L^{(i)}$ have the form (1.1). Suppose $\Gamma_0\subset\partial D_0^{(1)}\cap\partial D_0^{(2)}$ and suppose the BLR condition holds for $L^{(1)}$ on $[t_0,T_{t_0}]$. Suppose the corresponding DN operators $\Lambda^{(i)}$ are equal on $\Gamma_0\times(t_0,T_2)$, $T_2\geq T_{t_0}$, i.e. $\Lambda^{(1)}f=\Lambda^{(2)}f$ on $\Gamma_0\times(t_0,T_2)$ for all f with support in $\overline{\Gamma_0}\times[t_0,T_2]$. Let T_2-T_1 be small. Suppose coefficients of $L^{(1)}$ and $L^{(2)}$ are analytic in x_0.

Let $\varphi^{(i)}$ be the changes of variables (2.14) for $i=1,2$ and let $\beta^{(i)}$, $i=1,2$, be the changes of variables (5.10). Let c_i be the gauge transformation (2.20), (2.21) for $i=1,2$. Then

$$\beta^{(1)}\circ c_1\circ\varphi^{(1)}\circ L^{(1)}=\beta^{(2)}\circ c_2\circ\varphi^{(2)}\circ L^{(2)}\quad on\ \ \Sigma\times\overline{\Gamma}_1,\tag{6.21}$$

where

$\Sigma=\{(s,\tau),s\geq 0,\tau\geq 0,s+\tau\leq T_2-T_1\}=\{(y_0,y_n):0\leq y_n\leq\frac{T_2-T_1}{2},\ T_1+y_n<y_0<T_2-y_n\}$.

7 The global step

Let $L^iu_i=0$ in $D_i=D_0^{(i)}\times\mathbb{R}$, $i=1,2$, $u_i=0$ for $x_0\ll 0$, $\partial D_0^{(1)}\cap\partial D_0^{(2)}\supset\Gamma_0$ and $u_i\big|_{\partial D_0^{(i)}\times\mathbb{R}}=f$, $i=1,2$, f has a compact support in $\overline{\Gamma_0}\times\mathbb{R}$.

First we extend the Theorem 6.2 for a larger time interval.

Let $[t_1,t_2]$ be an arbitrary time interval. Let $[t_0,T_{t_0}]$ be such that $T_{t_0}\leq t_1$ and the BLR condition holds on $[t_0,T_{t_0}]$. Thus the BLR condition is satisfied on $[t_0,t]$ for any $t\in[t_1,t_2]$. Let Γ_1 be arbitrary connected part of Γ_0, $\overline{\Gamma_1}\subset\Gamma_0$. Note that we do not require $\overline{\Gamma_1}$ to be small.

Let $\psi_{0i}^{\pm}(x_0,x',x_n)$, $i=1,2$, be the solution of the form (2.4) in $[t_0-1,t_2+1]\times\overline{\Gamma'}\times[0,\varepsilon_{\pm}]$ where $\overline{\Gamma_1}\subset\Gamma'\subset\Gamma_0$.

We impose the following initial conditions on $\psi_{0i}^{\pm}$, $i=1,2$,

$$\psi_{0i}^{+}\big|_{x_n=0}=x_0,\quad \psi_{0i}^{-}\big|_{x_n=0}=-x_0.\tag{7.1}$$

Such solutions exist in $[t_0-1,t_2+1]\times\overline{\Gamma'}\times[0,\varepsilon_0]\subset D_0^{(i)}\times\mathbb{R}$, when ε_0 is small. We choose $\psi_{0i}^{\pm}$ such that (2.6) is satisfied and we choose $\varepsilon_1>0$ such that $\varepsilon_1\le\varepsilon_0$ and $\{0<x_n<\varepsilon_1, x'\in\overline{\Gamma'}, x_0\in[t_0-1,t_2+1]\}$ do not intersect $\partial D_0^{(i)}\times\mathbb{R}$.

Let $\varphi_{ji}(x_0',x',x_n)$, $1\le j\le n-1$, be the solutions of the linear equations (cf. (2.7))

$$\sum_{p,k=0}^{n} g_i^{pk}(x)\psi_{0ix_p}^{-}\varphi_{jix_k}=0 \ \text{ in } \ [t_0-1,t_2+1]\times\overline{\Gamma'}\times[0,\varepsilon_1] \tag{7.2}$$

with initial conditions

$$\varphi_{ji}\big|_{x_n=0}=x_j,\quad 1\le j\le n-1. \tag{7.3}$$

Similarly to (2.14) consider the map $(y_0^{(i)}(x),y_i'(x),y_n^{(i)}(x))=(\varphi_0^{(i)},\varphi_i',\varphi_n^{(1)})$, $x\in[t_0-1,t_2+1]\times\overline{\Gamma'}\times[0,\varepsilon_1]$, where

$$\begin{aligned} y_0^{(i)}(x)&=\frac{\psi_{0i}^{+}-\psi_{0i}^{-}}{2},\\ y_j^{(i)}(x)&=\varphi_{ji}(x),\\ y_n^{(i)}(x)&=-\frac{\psi_{0i}^{+}+\psi_{0i}^{-}}{2}. \end{aligned} \tag{7.4}$$

As in (2.15) we have that the map $(x_0,x',x_n)\to(y_0,y',y_n)$ is the identity when $x_n=0$:

$$y_0^{(i)}\big|_{x_n=0}=x_0,\quad y_j^{(i)}\big|_{x_n=0}=x_j,\quad 1\le j\le n-1,\quad y_n^{(i)}\big|_{x_n=0}=0. \tag{7.5}$$

Let $u_s=\frac{1}{2}(u_{y_0}-u_{y_n})$, $u_\tau=-\frac{1}{2}(u_{y_0}+u_{y_n})$. Making the change of variables (7.4) in $L^iu_i=0$, the gauge transformation (2.18), (2.21) and the change of unknown function (2.26), we get in $t_0\le y_0\le t_2$, $0\le y_n\le T_0$, $y'\in\overline{\Gamma'}$, T_0 is small, the equation of the form

$$L_1^{(i)}u_1^{(i)}=0,\quad y\in\hat{\Omega}_0,$$

where $L_1^{(i)}$ has the form (2.28). Here

$$\overline{\Gamma_1}\subset\Gamma'\subset\Gamma_0,\ \hat{\Omega}_0\overset{def}{=}[t_0,t_2]\times\overline{\Gamma'}\times[0,T_0]. \tag{7.6}$$

We assume that $u_1^{(i)}$ satisfy the zero initial conditions

$$u_1^{(i)}=\frac{\partial u_1^{(i)}}{\partial y_0}=0\ \text{ when }\ y_0=t_0$$

and

$$u_1^{(i)}\big|_{y_n=0} = f, \quad i = 1, 2.$$

We also assume that DN operators for L^i and subsequently for $L_1^{(i)}$ are equal on $[t_0, t_2) \times \overline{\Gamma'}$.

Note that the change of variables

$$\hat{y}_n = y_n, \quad \hat{y}_0 = y_0, \quad \hat{y}_j' = \alpha_j^{(i)}(y_0, y', y_n), \quad 1 \le j \le n-1, \tag{7.7}$$

where $\alpha^{(i)}$ are the same as in (5.9), (5.11), are also defined globally on $\hat{\Omega}_0$.

Let $[T_1, T_2] \subset [t_1, t_2]$ be arbitrary such that $T_2 - T_1 = 2T_0$.

Applying Theorem 6.1 to the interval $[T_1, T_2]$ we get that the coefficients of $\tilde{L}_1^{(1)}$ and $\tilde{L}_1^{(2)}$ and the coefficients of L_1' and L_2' are equal on $\Sigma_{T_1T_2} \times \overline{\Gamma_1}$ where $\Sigma_{T_1T_2} = \{0 \le y_n \le T_0, T_1 + y_n \le y_0 \le T_2 - y_n\}$. We assume that $\Gamma' \supset \overline{\Gamma_1}$ is such that $\overline{\Gamma_2} \subset \overline{\Gamma_3} \subset \Gamma_0'$ for all $[T_1, T_2] \subset [t_1, t_2]$. Here Γ_2, Γ_3 are defined as in Sect. 3. Note that Γ_2, Γ_3 may depend on $[T_1, T_2]$.

If two intervals $[T_1, T_2]$ and $[T_1', T_2']$ intersect, then the coefficients of $\tilde{L}_1^{(1)}$ and $\tilde{L}_1^{(2)}$ coincide in $(\Sigma_{T_1T_2} \bigcup \Sigma_{T1'T_1'}) \times \Gamma_1$.

Therefore coefficients of $\tilde{L}_1^{(1)}$ and $\tilde{L}_1^{(2)}$ and consequently the coefficients of L_1' and L_2' (cf. (6.2), (6.3)) coincide for $0 \le y_n \le T_0,\ y' \in \overline{\Gamma_1},\ t_1 + T_0 < y_0 < t_2 - T_0$.

Therefore we proved

Lemma 7.1 *Suppose* $[t_1, t_2]$ *is arbitrary large,* $T_0 > 0$ *is small,* t_0 *is such that the BLR condition is satisfied on* $[t_0, t_1]$. *Let* $\Omega_0 = \{y_0 \in [t_0 + T_0, t_2 - T_0],\ y' \in \overline{\Gamma_1}, y_n \in [0, \frac{T_0}{2}]\}$. *Assume that the coefficients of* $L^{(i)}$ *are analytic in* x_0, $i = 1, 2$. *Then*

$$\beta^{(1)} \circ c_1 \circ \varphi^{(1)} \circ L^{(1)} = \beta^{(2)} \circ c_2 \circ \varphi^{(2)} \circ L^{(2)} \quad on \ \ \Omega_0.$$

□

Let $\Omega_i = (\beta^{(i)}\varphi^{(i)})^{-1}\Omega_0,\ i = 1, 2$. Note that $\Omega_i \subset D_0^{(i)} \times [t_0 - 1, t_2 + 1]$ since T_0 is small. We have that $\Phi_2 = (\beta^{(1)}\varphi^{(1)})^{-1}\beta_2\varphi^{(2)}$ maps Ω_2 onto Ω_1. Note that $\partial\Omega_1 \cap \partial\Omega_2 \supset \Gamma_1 \times [t_0, t_2]$ and $\Phi_2 = I$ on $\Gamma_1 \times [t_0 + T_0, t_2 - T_0]$. Note also that $\beta^{(i)} \circ c_i$ can be represented as $c_i' \circ \beta^{(i)}$ where c_i' is the gauge transformation in $(y_0, \hat{y}', y_n)$ coordinates. Analogously, $(\beta^{(1)} \circ c_1 \circ \varphi^{(1)})^{-1}\beta^{(2)} \circ c_2 \circ \varphi = c_3 \circ \Phi_2$, where c_3 is the gauge transformation. Therefore

$$c_3 \circ \Phi_2 \circ L^{(2)} = L^{(1)} \quad \text{in} \ \ \Omega_1. \tag{7.8}$$

Let B be a smooth domain in $D_0^{(1)}$ such that $\partial B \cap \partial D_0^{(1)} = \gamma_1 \subset \Gamma_0$. Suppose B is small and such that $B \times [t_1 + 1, t_2 - 1] \subset \Omega_1$.

Let $S_2 = \Phi_2^{-1}(B \times [t_1 + 1, t_2 - 1] \subset D_0^{(2)} \times \mathbb{R}$ and let $S_2^+ = \Phi_2^{-1}(B \times \{x_0 = t_2 - 1\}),\ S_2^- = \Phi_2^{-1}(B \times \{x_0 = t_1 + 1\})$. Let $\tilde{S}_2^{\pm}$ be space-like surfaces in $D_0^{(2)} \times \mathbb{R}$ such that $\tilde{S}_2^+$ is the extension of S_2^+ and $\tilde{S}_2^-$ is the extension of S_2^-.

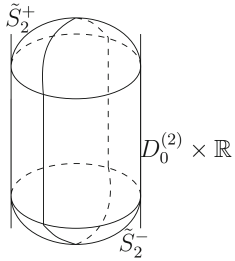


Fig. 8 The almost cylindrical domain $D_1^{(2)}$ is the part of $D_0^{(2)} \times \mathbb{R}$ bounded from above and from below by space-like surfaces $\tilde{S}_2^+$ and $\tilde{S}_2^-$

We assume that the projections of $\tilde{S}_2^{\pm}$ on $D_0^{(2)}$ is $D_0^{(2)}$. Let $D_1^{(2)}$ be the domain in $D_0^{(2)} \times \mathbb{R}$ bounded by $\tilde{S}_2^+$ and $\tilde{S}_2^-$ (cf. Fig. 8).

It follows from [16, Chapter 8], that there exists an extension $\tilde{\Phi}_2$ of Φ_2 from $S_2 \subset D_1^{(2)}$ to $D_1^{(2)}$ such that $\tilde{\Phi}_2\big|_{\Gamma_0\times[t_1+1,t_2-1]} = I$.

Define $\overline{D}_1^{(3)} = \tilde{\Phi}_2(\overline{D}_1^{(2)})$. There exists also an extension $\tilde{c}_3$ of the gauge c_3 from S_2 to $D_1^{(2)}$ such that $\tilde{c}_3 = 1$ on $\Gamma_0\times[t_1+1, t_2-1]$. Let $L^{(3)} = \tilde{c}_3\circ\tilde{\Phi}_2\circ L^{(2)}$, $L^{(3)}$ is defined on $D_1^{(3)}$. Thus $L^{(3)} = L^{(1)}$ on $B\times[t_1+1, t_2-1]$. Note that $D_1^{(3)}\cap(D_0^{(1)}\times[t_1+1, t_2-1]) \supset B \times [t_1 + 1, t_2 - 1]$, $\partial' D_1^{(3)} \cap (\partial D_0^{(1)} \times [t_1 + 1, t_2 - 1]) \supset \Gamma_0 \times [t_1 + 1, t_2 - 1]$. We denote by $\partial' D_1^{(3)}$ the lateral (time-like) part of $\partial D_1^{(3)}$ and by $\partial_\pm D_1^{(3)}$ the top and the bottom space-like parts of $\partial D_1^{(3)}$, i.e. $\partial D_1^{(3)} = \partial' D_1^{(3)} \cup \partial_+ D_1^{(3)} \cup \partial_- D_1^{(3)}$.

The following lemma is the key lemma of this section. It allows to reduce the solution of the inverse problem to an inverse problem in a smaller domain.

Lemma 7.2 *Consider two initial–boundary value problem* $L^{(1)}u_1 = 0$ *in* $D_0^{(1)} \times [t_1, t_2]$ *and* $L^{(3)}u_3 = 0$ *in* $D_1^{(3)}$,

$$
\begin{aligned}
&u_1\big|_{x_0=t_1} = \frac{\partial u_1}{\partial x_0}\Big|_{x_0=t_1} = 0, \quad x \in D_0^{(1)},\\
&u_3\big|_{\partial_- D_1^{(3)}} = \frac{\partial u_3}{\partial x_0}\Big|_{\partial_- D_1^{(3)}} = 0, \quad u_1\big|_{\partial D_0^{(1)}\times[t_1,t_2)} = f_1,\ u_3\big|_{\partial' D_1^{(3)}} = f_3.
\end{aligned}
$$

We assume that $(\partial D_0^{(1)} \times [t_1, t_2]) \cap \partial D_1^{(3)} \supset \Gamma_0 \times [t_1, t_2]$. *Assume that* $L^{(1)} = L^{(3)}$ *in a smooth domain* $B \times [t_1, t_2]$ *where* $B \times [t_1, t_2] \subset (D_0^{(1)} \times [t_1, t_2]) \cap D_1^{(3)}$, $\gamma_1 = \partial D_0^{(1)} \setminus \Gamma_0$, $\tilde{\Gamma}_3 = \partial' D_1^{(3)} \setminus (\Gamma_0 \times (t_1, t_2))$, $\partial B = \gamma_0 \cup \gamma_0'$, *where* γ_0, γ_0' *are smooth,* $\gamma_0 \subset \Gamma_0$, *(cf. Fig. 9).*

Suppose $\Lambda_1 = \Lambda_3$ *on* $\Gamma_0 \times [t_1, t_2]$.

Consider $L^{(1)}u_1 = 0$ *and* $L^{(3)}u_3 = 0$ *in smaller domains* $(D_0^{(1)}\setminus B)\times(t_1+\delta, t_2-\delta)$ *and* $(D_1^{(3)}\cap(t_1+\delta, t_2-\delta))\setminus(B\times(t_1+\delta, t_2-\delta))$. *Note that* $\partial(D_0^{(1)}\setminus B) \supset (\Gamma_0\setminus\gamma_0)\cup\gamma_0')$. *Then* $\Lambda_1' = \Lambda_3'$ *are equal on* $((\Gamma_0 \setminus \gamma_0) \cup \gamma_0') \times (t_1 + \delta, t_2 - \delta)$ *for some* $\delta > 0$. *Here* Λ_1', Λ_3' *are DN operators for the initial–boundary value problem*

$$
\begin{aligned}
&L^{(1)}u_i' = 0 \ \ in \ \ (D_0^{(1)} \setminus B) \times (t_1+\delta, t_2-\delta),\\
&L^{(3)}u_3' = 0 \ \ in \ \ (D_1^{(3)} \cap (t_1+\delta, t_2-\delta)) \setminus (B \times (t_1+\delta, t_2-\delta)),\\
&u_1'\big|_{x_0=t_1+\delta} = \frac{\partial u_1'}{\partial x_0}\Big|_{x_0=t_2+\delta} = 0,\\
&u_3'\big|_{\partial_-(D_1^{(3)}\cap(t_1+\delta,,t_2-\delta))} = \frac{\partial u_3'}{\partial x_0}\Big|_{\partial_-(D_1^{(3)}\cap(t_1+\delta,t_2-\delta))} = 0,\\
&u_1'\big|_{((\Gamma_0\setminus\gamma_0)\cup\gamma_0')\times(t_1+\delta,t_2-\delta)} = f, \quad u_1'\big|_{(\partial D_0^{(1)}\setminus\Gamma_0)\times(t_1+\delta,t_2-\delta)} = 0,\\
&u_3'\big|_{((\Gamma_0\setminus\gamma_0)\cup\gamma_0')\times(t_1+\delta,t_2-\delta)} = f, \quad u_3'\big|_{((\partial' D_1^{(3)}\cap(t_1+\delta,t_2-\delta))\setminus(\Gamma_0\times(t_1+\delta,t_2-\delta))} = 0,\\
&supp\ f \subset (((\Gamma_0 \setminus \gamma_0) \cup \gamma_0') \times (t_1+\delta, t_2-\delta)).
\end{aligned}
$$

To prove Lemma 7.2 we will need the following version of the Runge theorem about the approximation of solutions of the equation in a smaller domain by solutions of the same equation in a larger domain.

Lemma 7.3 *Denote by D_ε the domain bounded by Γ_0 and γ_ε such that $\gamma_\varepsilon \cup \gamma_1$ is smooth. Let W_0 be the space of $v \in H_s((D_0^{(1)} \setminus B) \times (t_1, t_2))$, $s \geq 1$, such that*

$$
\begin{aligned}
&v\big|_{\gamma_1} = 0, \quad v\big|_{x_o=t_1} = \frac{\partial v}{\partial x_0}\Big|_{x_0=t_1} = 0, \quad x \in (D_0^{(1)} \setminus B),\\
&L^{(1)}v = 0 \ \ in \ \ (D_0^{(1)} \setminus B) \times (t_1, t_2),
\end{aligned}
\tag{7.9}
$$

where $\gamma_1 = \partial D_0^{(1)} \setminus \Gamma_0$.

Denote by K the closure of W_0 in $L_2((D_0^{(1)} \setminus B) \times (t_1, t_2))$. Consider the space W of $u(x) \in H_s((D_0^{(1)} \cup D_\varepsilon) \times (t_1, t_2))$, $s \geq 1$ such that

$$
\begin{aligned}
&L^{(1)}u = 0 \ \ in \ \ (D_0^{(1)} \cup D_\varepsilon) \times (t_1, t_2), \quad u\big|_{(\gamma_1\cup\gamma_\varepsilon)\times(t_1,t_2)} = 0,\\
&u\big|_{x_0=t_1} = \frac{\partial u}{\partial x_0}\Big|_{x_0=t_1} = 0, \quad x \in D_0^{(1)} cup D_\varepsilon.
\end{aligned}
\tag{7.10}
$$

Then the closure of the restrictions of W to $L_2((D_0^{(1)} \setminus B) \times (t_1, t_2))$ is also equal to K. Thus any function $v \in W_0$ in $(D_0^{(1)} \setminus B) \times (t_1, t_2)$ can be approximated in $L_2((D_0^{(1)} \setminus B) \times (t_1, t_2))$ norm by the functions in W.

Proof Let $K^\perp$ be the orthogonal complement of K in $L_2((D_0^{(1)} \setminus B) \times (t_1, t_2))$. Take any $g \in K^\perp$ and denote by g_0 the extension of g by zero outside $(D_0^{(1)} \setminus B) \times (t_1, t_2)$. Let w be the solution of the initial–boundary value problem

$$
\begin{aligned}
&L_1^* w = g_0, \quad x \in (D_0^{(1)} \cup D_\varepsilon) \times (t_1, t_2),\\
&w\big|_{x_0=t_2} = \frac{\partial w}{\partial x_0}\Big|_{x_0=t_2} = 0, \quad x \in D_0^{(1)} \cup D_\varepsilon,\\
&w\big|_{\partial(D_0^{(1)}\cup D_\varepsilon)\times(t_1,t_2)} = 0,
\end{aligned}
\tag{7.11}
$$

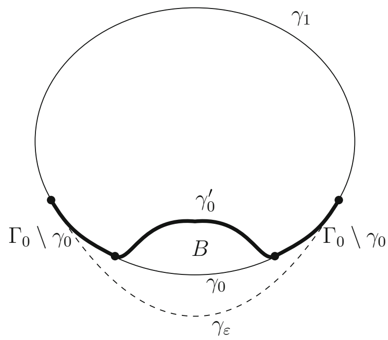


Fig. 9 The boundary of B is $\gamma_0 \cup \gamma_0'$. The boundary of D_ε is $\gamma_\varepsilon \cup \Gamma_0,\ \partial D_0^{(1)} = \Gamma_0 \cup \gamma_1$

where L_1^* is the formally adjoint to $L^{(1)}$. Note that $\partial(D_0^{(1)} \cup D_{\varepsilon 0}) = \gamma_1 \cup \gamma_\varepsilon$ (see Fig. 9).

By Hormander [15] and Eskin [13] (see also Lemma 3.3) such $w(x)$ exists and belongs to $H_1((D_0^{(1)} \cup D_\varepsilon)) \times (t_1, t_2))$. We shall show that $w = 0$ in $(B \cup D_\varepsilon) \times (t_1, t_2)$. Let $\varphi \in C_0^\infty((B \cup D_\varepsilon) \times (t_1, t_2))$ and let $u(x)$ be the solution of

$$L^{(1)}u = \varphi, \quad x \in (D_0^{(1)} \cup D_\varepsilon) \times (t_1, t_2)$$
$$u\big|_{x_0 = t_1} = \frac{\partial u}{\partial x_0}\Big|_{x_0 = t_1} = 0, \quad u\big|_{\partial(D_0^{(1)} \cup D_\varepsilon) \times (t_1, t_2)} = 0, \tag{7.12}$$

(cf. [13,15] and Lemma 3.3), i.e. $u \in W_0$ since $\varphi = 0$ in $(D_0^{(1)} \setminus B) \times (t_1, t_2)$.

Consider the L_2 inner product (φ, w) in $(D_0^{1)} \cup D_\varepsilon) \times (t_1, t_2)$. Since $\varphi = L^{(1)}u$ we get $(\varphi, w) = (L^{(1)}u, w)$. Integrating by parts we have $(L_1 u, w) = (u, L_1^* w) = (u, g_0) = 0$ since $u \in W_0, g_0 \in K^\perp$. Therefore $(\varphi, w) = 0,\ \forall \varphi$. Thus $w = 0$ in $(B \cup D_\varepsilon) \times (t_1, t_2)$.

Let now $\tilde{w}$ be any function in W. We have $(\tilde{w}, g_0)_0 = (\tilde{w}, L_1^* w)_0$, where $(\ )_0$ means that we integrate over $(D_0^{(1)} \setminus B) \times (t_1, t_2)$. Since $w = 0$ in $(B \cup D_\varepsilon) \times (t_1, t_2)$, we have that

$$w\big|_{(\Gamma_0 \setminus \gamma_0) \cup \gamma_0') \times (t_1, t_2)} = \frac{\partial w}{\partial \nu}\Big|_{(\Gamma_0 \setminus \gamma_0) \cup \gamma_0') \times (t_1, t_2)} = 0, \tag{7.13}$$

where $\frac{\partial}{\partial \nu}$ is the normal derivative.

Note that $(\Gamma_0 \setminus \gamma_0) \cup \gamma_0' = \partial(D_\varepsilon \cup B) \setminus \gamma_\varepsilon$. Since w satisfies the homogenous equation $L_1^* w = 0$ in $D_\varepsilon \cup B$ the restrictions of w and all derivatives on $\partial(D_\varepsilon \cup B)$ exists by the partial hypoellipticity (see, for example, [12]). Note that $\tilde{w}$ and w have zero values on γ_1. Therefore, integrating by parts, we have

$$(\tilde{w}, L_1^* w)_0 = (L^{(1)}\tilde{w}, w)_0 = 0,$$

since $L^{(1)}\tilde{w} = 0$ in $(D_0^{(1)} \setminus B) \times (t_1, t_2)$. Therefore $(\tilde{w}, g_0)_0 = 0,\ \ \forall g_0 \in K^\perp,$ i.e. $\tilde{w} \in \overline{K}$.

To make the integration by parts rigorous we approximate $\gamma_0' \cup (\Gamma_0 \setminus \gamma_0)$ by γ_{ε_1}', similar to $\gamma_\varepsilon,\ \gamma_{\varepsilon_1}' \subset D_\varepsilon \cup B$. Note that $w = 0$ in $D_\varepsilon \cup B$. Therefore integrating by parts over domain bounded by $\gamma_1 \cup \gamma_{\varepsilon_1}'$, and taking the limit when $\gamma_{e_1}' \to \gamma_0' \cup (\Gamma_0 \setminus \gamma_0)$ we get $(\tilde{w}, g_0) = 0$. □

Now we shall proof Lemma 7.2.

Let $\operatorname{supp} f \subset \Gamma_0' \times (t_1, t_2),\ \Gamma_0' = (\Gamma_0 \setminus \gamma_0) \cup \gamma_0'$. Let v_1 be the solutions of

$$
\begin{aligned}
&L^{(1)} v_1 = 0, \quad x \in (D_0^{(1)} \setminus B) \times (t_1, t_2),\\
&v_1\big|_{x_0 = t_1} = \frac{\partial v_1}{\partial x_0}\Big|_{x_0 = t_1} = 0,\\
&v_1\big|_{\partial (D_0^{(1)} \setminus B) \times (t_1, t_2)} = f_1,
\end{aligned}
\tag{7.14}
$$

where $\partial(D_0^{(1)} \setminus B) = \Gamma_0' \cup \gamma_1,\ f_1 = 0$ on $\gamma_1 \times (t_1, t_2),\ f_1 = f$ on $\Gamma_0' \times (t_1, t_2)$.

Let v_3 be solution of $L^{(3)} v_3 = 0$ in $D_1^{(3)} \setminus (B \times (t_1, t_2))$

$$
\begin{aligned}
&v_3\big|_{\partial_- D_1^{(3)}} = \frac{\partial v_3}{\partial x_0}\Big|_{\partial_- D_1^{(3)}} = 0, \quad v_3\big|_{(\partial' D_1^{(3)} \setminus (\Gamma_0 \times (t_1, t_2))} = 0,\\
&v_3\big|_{\Gamma_0' \times (t_1, t_2)} = f.
\end{aligned}
\tag{7.15}
$$

Let Λ_1' be the DN operator for (7.14) and Λ_3' be the DN operator for (7.15). Assuming that $\Lambda_1 = \Lambda_2$ on $\Gamma_0 \times (t_1, t_2)$ we shall prove that

$$
\Lambda_1' f\Big|_{\Gamma_0' \times (t_1 + \delta, t_2 - \delta)} = \Lambda_2' f\Big|_{\Gamma_0' \times (t_1 + \delta, t_2 - \delta)}
$$

for all f with supports in $\Gamma_0' \times (t_1 + \delta, t_2 - \delta)$. By Lemma 7.3 there exists a sequence of smooth solutions $w_{n1} \in W_0$ such that

$$
\|v_1 - w_{n1}\|_0 \to 0, \quad n \to \infty,
$$

where $\|v_1\|_0$ is the norm in $L_2((D_0^{(1)} \setminus B) \times (t_1, t_2))$. Note that $L^{(1)} w_{n1} = 0$ in $D_0^{(1)} \times (t_1, t_2),\ w_{n1}\big|_{\gamma_1 \times (t_1, t_2)} = 0,\ w_{n1}\big|_{x_0 = t_1} = \frac{\partial w_{n1}}{\partial x_0}\big|_{x_0 = t_1} = 0$, where $\gamma_1 = \partial D_0^{(1)} \setminus \Gamma_0$. Let $f_n = w_{n1}\big|_{\Gamma_0 \times (t_1, t_2)}$. Denote by w_{n3} the solution of

$$
\begin{aligned}
L^{(3)} w_{n3} = 0 \quad \text{in} \quad D_1^{(3)}, \quad w_{n3}\big|_{\partial' D_1^{(3)} \setminus (\Gamma_0 \times (t_1, t_2))} = 0, \quad w_{n3}\big|_{\Gamma_0 \times (t_1, t_2)} = f_n,&\\
w_{n3}\big|_{\partial_- D_1^{(3)}} = \frac{\partial w_{n3}}{\partial x_0}\Big|_{\partial_- D_1^{(3)}} = 0.&
\end{aligned}
\tag{7.16}
$$

Since $\Lambda_1 = \Lambda_2$ on $\Gamma_0 \times (t_1, t_2)$, we have

$$
\frac{\partial w_{n1}}{\partial \nu}\Big|_{\Gamma_0 \times (t_1, t_2)} = \frac{\partial w_{n3}}{\partial \nu}\Big|_{\Gamma_0 \times (t_1, t_2)}
\tag{7.17}
$$

Since $\gamma_o \subset \Gamma_0$, the equality (7.17) implies

$$w_{n1}\big|_{\gamma_0\times(t_1,t_2)} = w_{n3}\big|_{\gamma_0\times(t_1,t_2)}, \quad \frac{\partial w_{n1}}{\partial\nu}\Big|_{\gamma_0\times(t_1,t_2)} = \frac{\partial w_{n3}}{\partial\nu}\Big|_{\gamma_0\times(t_1,t_2)}.$$

We have $L^{(1)} = L^{(3)}$ on $B\times(t_1,t_2)$. Using the uniqueness theorem of [25] and [29], we get

$$w_{n1} = w_{n3} \ \text{ in } \ B\times(t_1+\delta, t_2-\delta), \tag{7.18}$$

where $\delta > 0$ is determined by the metric and by the domain B (cf. Fig. 9). In particular,

$$\begin{aligned} &w_{n1}\big|_{\gamma_0'\times(t_1+\delta,t_2-\delta)} = w_{n3}\big|_{\gamma_0'\times(t_1+\delta,t_2-\delta)}, \\ &\frac{\partial w_{n1}}{\partial\nu}\Big|_{\gamma_0'\times(t_1+\delta,t_2-\delta)} = \frac{\partial w_{n3}}{\partial\nu}\Big|_{\gamma_0'\times(t_1+\delta,t_2-\delta)}. \end{aligned} \tag{7.19}$$

Therefore

$$\begin{aligned} &w_{n1}\big|_{\Gamma_0'\times(t_1+\delta,t_2-\delta)} = w_{n3}\big|_{\Gamma_0'\times(t_1+\delta,t_2-\delta)}, \\ &\frac{\partial w_{n1}}{\partial\nu}\Big|_{\Gamma_0'\times(t_1+\delta,t_2-\delta)} = \frac{\partial w_{n3}}{\partial\nu}\Big|_{\Gamma_0'\times(t_1+\delta,t_2-\delta)}, \end{aligned} \tag{7.20}$$

where $\Gamma_0' = (\Gamma_0\setminus\gamma_0)\cup\gamma_0'$, i.e. $\Lambda_1' f_n' = \Lambda_2' f_n'$ on $\Gamma_0'\times(t_1+\delta, t_2-\delta)$, where $f_n' = w_{n1}\big|_{\Gamma_0'\times(t_1+\delta,t_2-\delta)} = w_{n3}\big|_{\Gamma_0'\times(t_1+\delta,t_2-\delta)}$. We have

$$\begin{aligned} \|f - f_n'\|_{-\frac12,\Gamma_0'\times(t_1+\delta,t_2-\delta)} &= \|f - f_n'\|_{-\frac12,\partial(D_0^{(1)}\setminus B)\times(t_1+\delta,t_2-\delta)} \\ &\le \|v_1 - w_{n1}\|_{0,(D_0^{(1)}\setminus B)\times(t_1+\delta_1,t_2-\delta)}, \end{aligned} \tag{7.21}$$

since $\partial(D_0^{(1)}\setminus B) = \Gamma_0'\cup\gamma_1$ and $f = f_n' = 0$ on $\gamma_1\times(t_1+\delta, t_2-\delta)$.

In (7.21) we again use the partial hypoellipticity property that restrictions of solutions of $L^{(1)}u = 0$ to the noncharacteristic boundary exists for any Sobolev's space H_s (cf. [12]). The same is true for all normal derivatives of u_1, and the same estimates hold as in the case of positive $s > 0$ (cf. [8] and [12]).

By Lemma 3.3 (see [13,15]) we have

$$\Big\|\frac{\partial v_1}{\partial\nu} - \frac{\partial w_{n1}}{\partial\nu}\Big\|_{-\frac32,\partial(D_0^{(1)}\setminus B)\times(t_1+\delta,t_2)-\delta} \le \big\|f - f_n'\big\|_{-\frac12,\partial(D_0^{(1)}\setminus B)\times(t_1+\delta,t_2-\delta)} \tag{7.22}$$

Analogously we have

$$\begin{aligned} &\Big\|\frac{\partial v_3}{\partial\nu} - \frac{\partial w_{n3}}{\partial\nu}\Big\|_{-\frac32,\partial'(D_1^{(3)}\cap(t_1+\delta,t_2-\delta)\setminus(B\times(t_1+\delta,t_2-\delta))} \\ &\quad\le \big\|f - f_n'\big\|_{-\frac12,\partial'(D_1^{(3)}\cap(t_1+\delta,t_2-\delta)\setminus(B\times(t_1+\delta,t_2-\delta))} \end{aligned} \tag{7.23}$$

Note that

$$\|f - f_n'\|_{-\frac{1}{2},\partial(D_0^{(1)}\setminus B)\times(t_1+\delta,t_2-\delta)} = \|f - f_n'\|_{-\frac{1}{2},\partial'(D_1^{(3)}\cap(t_1+\delta,t_2-\delta)\setminus(B\times(t_1+\delta,t_2-\delta))}.$$

Therefore, taking the limit as $n \to \infty$, we get, using (7.20), that

$$\frac{\partial v_1}{\partial \nu}\Big|_{\Gamma_0'\times(t_1+\delta,t_2-\delta)} = \frac{\partial v_3}{\partial \nu}\Big|_{\Gamma_0'\times(t_1+\delta,t_2-\delta)}. \tag{7.24}$$

Thus we proved that

$$\Lambda_1' f = \Lambda_2' f \quad \text{on} \;\; \Gamma_0' \times (t_1 + \delta, t_2 - \delta)$$

for any f with supp $f \subset \Gamma_0' \times (t_1 + \delta, t_2 - \delta)$. $\square$

Using Lemma 7.2 we reduce the inverse problem in $D_0^{(1)} \times (t_1, t_2)$ to the inverse problem in smaller domains $(D_0^{(1)} \setminus B) \times (t_1 + \delta, t_2 - \delta)$ and we can continue this process starting from $(D_0^{(1)} \setminus B) \times (t_1 + \delta, t_2 - \delta)$ instead of $D_0^{(1)} \times (t_1, t_2)$.

In all lemmas below we assume that DN operators for $L^{(1)}$ and $L^{(2)}$ are equal on $\Gamma_0 \times [t_1, t_2]$ and that the time interval $[t_1, t_2]$ is large enough. We shall continue to call the coordinates $(y_0, \hat{y}', y_n)$, given by the map $\beta^{(i)}\varphi^{(i)}$, the Goursat coordinates for $\tilde{L}_1^{(i)}$, $i = 1, 2$.

Lemma 7.4 *Let $\Gamma_1 \subset \Gamma_0$ and let $\tilde{\Gamma}_1 \subset \Gamma_0$ be such that $\overline{\Gamma}_1 \subset \tilde{\Gamma}_1$. We assume that $\tilde{\Gamma}_1 \subset \tilde{\Gamma}_2 \subset \tilde{\Gamma}_3 \subset \Gamma_0$ where $\tilde{\Gamma}_j$, $1 \le j \le 3$, are the same as Γ_j, $1 \le j \le 3$, in Sect. 3. Suppose the Goursat coordinates for $L^{(1)}$ exists in $\Omega_1 = (t_1, t_2) \times \tilde{\Gamma}_3 \times [0, \varepsilon_0]$, i.e. $L^{(1)}$ has the form $\tilde{L}_1^{(1)}$ in these coordinates (we include the gauge transformation (2.21) in $\tilde{L}_1^{(i)}$). Suppose the Goursat coordinates for $L^{(2)}$ exist in $(t_1, t_2) \times (\overline{\tilde{\Gamma}}_3 \setminus \Gamma_1) \times [0, \varepsilon_0]$. Let $\tilde{\Omega}_2 = (t_1, t_2) \times (\tilde{\Gamma}_1 \setminus \Gamma_1) \times [0, \varepsilon_0]$ and suppose $\tilde{L}_1^{(2)} = \tilde{L}_1^{(1)}$ in $\tilde{\Omega}_2$. Then $L^{(2)}$ has also Goursat coordinates in $\Omega_3 = (t_1, t_2) \times \Gamma_1 \times [0, \varepsilon_0]$, and $\tilde{L}_1^{(1)} = \tilde{L}_1^{(2)}$ in Ω_3.*

Proof Let $y^{(i)} = \psi^{(i)}(x)$ be the transformation to the Goursat coordinates, and let $\frac{D\psi^{(i)}(x)}{Dx}$ be the Jacobi matrix of this transformation. We have

$$[\hat{g}_i^{jk}(y)] = \frac{D\psi^{(i)}(x)}{D(x)}[g_i^{jk}(x)]\Big(\frac{D\psi^{(i)}}{Dx}\Big)^T, \quad i = 1, 2,$$

where $[\hat{g}_i^{jk}]^{-1}$ is the metric tensor in the Goursat coordinates. The Goursat coordinates degenerate at point $y_i^{(0)} = \psi^{(i)}(x^{(0)})$, when det $\frac{D\psi^{(i)}(x)}{Dx} \to \infty$ for $y \to y^{(0)}$ (or $x \to x^{(0)}$) (cf. (2.11)). We call such point a focal point. We shall prove that there is no focal points for $L^{(2)}$ in Ω_3.

We have

$$\det\big[\hat{g}_i^{jk}(y)\big] = \det\big[g_i^{jk}(x)\big]\Big(\det\frac{D\psi^{(i)}}{Dx}\Big)^2.$$

Suppose there exists the focal point $y^{(0)} = (y_0^{(0)}, y_0', y_n^{(0)})$, $y_n^{(0)} < \varepsilon_0$, $y_0' \in \Gamma_1$ such that there is no focal points for $L^{(2)}$ when $y_n < y_n^{(0)}$ for all $y_0 \in [t_1, t_2]$, $y' \in \overline{\Gamma}_1$.

Since $L^{(1)}$ and $L^{(2)}$ have Goursat coordinates for $y_n < y_n^{(0)}$ we get, by Lemma 7.1, that $\tilde{L}_1^{(1)} = \tilde{L}_1^{(2)}$ in $(t_1, t_2) \times \overline{\Gamma}_1 \times [0, y_n^{(0)} - \varepsilon]$, $\forall \varepsilon > 0$, and hence $[\hat{g}_1^{jk}] = [\hat{g}_2^{jk}]$ for $\varepsilon > 0$. Since $\det[\hat{g}_2^{jk}] = \det[\hat{g}_1^{jk}]$ for $y_n < y_n^{(0)} - \varepsilon$, we have that $\left(\det \frac{D\psi^{(2)}}{Dx}\right)^2 = \frac{\det[\hat{g}_1^{jk}]}{\det[g_2^{jk}]}$ is bounded when $\varepsilon \to 0$. Therefore $y^{(0)} = (y_0^{(0)}, y_0', y_n^{(0)})$ is not a focal point for $L^{(2)}$. Thus $L^{(2)}$ has no focal points in Ω_3 and then, by Lemma 7.1, we have $\tilde{L}_1^{(1)} = \tilde{L}_1^{(2)}$ in Ω_3 (cf. [9]). □

Lemma 7.5 *Assume that DN operators for $L^{(1)}$ and $L^{(2)}$ are equal on $\Gamma_0 \times [t_1, t_2]$. Let $\overline{\Gamma}_1 \subset \Gamma_0$. Assume that the Goursat coordinates for $L^{(1)}$ exists in $(t_1, t_2) \times \overline{\Gamma}_1 \times [0, \frac{T_0}{2}]$. Then the Goursat coordinates for $L^{(2)}$ also exists in $\Omega_1 = (t_1 + \delta, t_2 - \delta) \times \overline{\Gamma}_1 \times [0, \frac{T_0}{2}]$ for some $\delta > 0$ and $\tilde{L}_1^{(1)} = \tilde{L}_1^{(2)}$ in $\overline{\Omega}_1$, where $\tilde{L}_1^{(i)}$ are the operators $L^{(i)}$ in the Goursat coordinates.*

Proof Let $\overline{\Gamma}_1 \subset \tilde{\Gamma}_1$, $\overline{\tilde{\Gamma}}_1 \subset \Gamma_0$. If $0 \le y_n \le \varepsilon$, where $\varepsilon > 0$ is small enough, then $\tilde{\Gamma}_1 \subset \tilde{\Gamma}_2 \subset \tilde{\Gamma}_3 \subset \Gamma_0$, where $\tilde{\Gamma}_j$, $j = 1, 2, 3$, are the same as in Lemma 7.4. Applying Lemma 7.1 we get that the Goursat coordinates for $\tilde{L}_1^{(1)}$ and $\tilde{L}_1^{(2)}$ exist in $\Omega_{1\varepsilon} = [t_1, t_2] \times \tilde{\Gamma}_1 \times [0, \varepsilon]$ and

$$\tilde{L}_1^{(1)} = \tilde{L}_1^{(2)} \quad \text{in } \overline{\Omega}_{1\varepsilon}.$$

Let Σ_1 be the surface in (y', y_n) space such that $y_n = 0$ on $\tilde{\Gamma}_1 \setminus \Gamma_1$, $0 \le y_n \le \varepsilon$ on $\partial\Gamma_1$, $y_n = \varepsilon$ on Γ_1. Note that Σ_1 is not smooth since it has edges when $y_n = 0$, $y' \in \partial\Gamma_1$ and when $y_n = \varepsilon$, $y' \in \partial\Gamma_1$. We shall smooth Σ_1 by replacing it by smooth surface $\tilde{\Sigma}_1$, where $\tilde{\Sigma}_1$ differs from Σ_1 in a neighborhood of edges having the size $O(\varepsilon)$. Let Σ_2 be the surface, where $y_n = \varepsilon$ when $y' \in \tilde{\Gamma}_1 \setminus \Gamma_1(\varepsilon)$, $\Gamma_1(\varepsilon)$ is the ε-neighborhood of Γ_1, $\varepsilon \le y_n \le 2\varepsilon$, when $y' \in \partial\Gamma_1(\varepsilon)$, $y_n = 2\varepsilon$ when $y' \in \Gamma_1(\varepsilon)$ (cf. Fig. 10). □

Let $\tilde{\Sigma}_2$ be the smoothing of Σ_2. Denote by $\tilde{S}_1$ the domain between $\tilde{\Sigma}_1$ and $\tilde{\Sigma}_2$ when $y' \in \tilde{\Gamma}_1$. Since $\tilde{L}_1^{(1)} = \tilde{L}_1^{(2)}$ for $0 \le y_n \le \varepsilon$, we have, by Lemma 7.2, that DN operators for $L^{(1)}$ and $L^{(2)}$ are equal on $\tilde{\Sigma}_1 \times (t_1 + \delta_1, t_2 - \delta_1)$ for some $\delta_1 > 0$.

Suppose $\varepsilon > 0$ is small and such that we can introduce the Goursat coordinates for $L^{(1)}$ in $\tilde{S}_1 \times [t_1 + \delta_1, t_2 - \delta_1]$.

Note that ε and δ_1 are determined by $L^{(1)}$ only and are independent of $L^{(2)}$. It follows from Lemma 7.4 that Goursat coordinates for $L^{(2)}$ also hold on $\tilde{S}_1 \times (t_1 + \delta_1, t_2 - \delta_1)$ and $\tilde{L}_1^{(1)} = \tilde{L}_1^{(2)}$ in $\tilde{S}_1 \times (t_1 + \delta_1, t_2 - \delta_1)$.

By Lemma 7.2 DN operators for $L^{(1)}$ and $L^{(2)}$ are equal on $\tilde{\Sigma}_2 \times (t_2 + \delta_1, t_2 - \delta_1)$.

Let Σ_2' be the surface in (y', y_n) space such that $y_n = 0$ on $\tilde{\Gamma}_1 \setminus \Gamma_1$, $0 \le y_n \le 2\varepsilon$ on $\partial\Gamma_1$, $y_n = 2\varepsilon$ on Γ_1 and let Σ_3 be the surface where $y_n = \varepsilon$ on $\tilde{\Gamma}_1 \setminus \Gamma_1(\varepsilon)$, $\varepsilon \le y_n \le 3\varepsilon$ on $\partial\Gamma_1(\varepsilon)$, $y_n = 3\varepsilon$ on $\Gamma_1(\varepsilon)$. Let $\tilde{\Sigma}_2'$ and $\tilde{\Sigma}_3$ be the smoothing of Σ_2', Σ_3 and let $\tilde{S}_2$ be the domain between $\tilde{\Sigma}_2'$ and $\tilde{\Sigma}_3$ when $y' \in \tilde{\Gamma}_1$. Since DN operators for $L^{(1)}$ and $L^{(2)}$

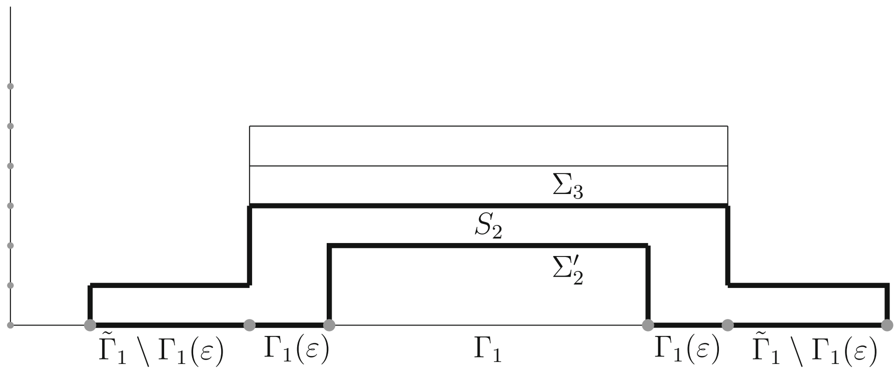


Fig. 10 Σ_2' is the surface $\{y_n = 0$ when $y' \in \tilde{\Gamma}' \setminus \Gamma_1,\ 0 \le y_n \le 2\varepsilon$ when $y' \in \partial\Gamma_1$, $y_n = 2\varepsilon$ when $y' \in \Gamma_1\}$, Σ_3 is the surface $\{y_n = \varepsilon$ when $y' \in \tilde{\Gamma}_1 \setminus \Gamma_1(\varepsilon),\ \varepsilon \le y_n \le 3\varepsilon$ when $y' \in \partial\Gamma_1(\varepsilon)$, $y_n = 3\varepsilon$ when $y' \in \Gamma_1(\varepsilon)\}$, S_2 is the region between Σ_3 and Σ_2'

are equal on $\tilde{\Sigma}_2' \times (t_1+\delta_1, t_2-\delta_1)$ and since the Goursat coordinates for $L^{(1)}$ hold on $\tilde{S}_2 \times (t_1+\delta_1, t_2-\delta_1)$, Lemma 7.4 implies that the Goursat coordinates hold for $L^{(2)}$ in $S_2 \times (t_1+\delta_1, t_2-\delta_1)$, $L_1^{(1)} = L_1^{(2)}$ in $S_2 \times (t_1+\delta_1+\delta_2, t_2-\delta_1-\delta_2)$ for some $\delta_2 > 0$, and DN operators for $L^{(1)}$ and $L^{(2)}$ are equal on $\tilde{\Sigma}_3 \times (t_1+\delta_1+\delta_2, t_2-\delta_1-\delta_2)$.

Analogously, for $k > 2$ denote by Σ_k' the surface such that $y_n = 0$ on $\tilde{\Gamma}_1 \setminus \Gamma_1$, $0 \le y_n \le k\varepsilon$ on $\partial\Gamma_1$ and $y_n = k\varepsilon$ on Γ_1. Let Σ_{k+1} be the surface, where $y_n = \varepsilon$ for $\tilde{\Gamma}_1 \setminus \Gamma_1(\varepsilon)$, $\varepsilon \le y_n \le (k+1)\varepsilon$ on $\partial\Gamma_1(\varepsilon)$ and $y_n = (k+1)\varepsilon$ on $\Gamma_1(\varepsilon)$.

Denote by $\tilde{\Sigma}_k'$ and $\tilde{\Sigma}_{k+1}$ the smoothing of Σ_k', Σ_{k+1}. Let $\tilde{S}_k$ be the domain between $\tilde{\Sigma}_k'$ and $\tilde{\Sigma}_{k+1}$. Applying successively the same arguments for $k = 3, \dots, m$, we prove as above that $\tilde{L}_1^{(2)} = \tilde{L}_1^{(1)}$ in $\tilde{S}_k \times [t_1 + \sum_{j=1}^k \delta_j, t_2 - \sum_{j=1}^k \delta_j]$, $k = 3, \dots, m$.

Let m be such that $(m+1)\varepsilon \ge \frac{T_0}{2}$. Then we get that $\tilde{L}_1^{(2)} = \tilde{L}_1^{(2)}$ in $(t_1 - \delta, t_2 + \delta) \times \overline{\Gamma}_1 \times [0, \frac{T_0}{2}]$, where $\delta = \sum_{j=1}^m \delta_j$. Note that we assume that $[t_1, t_2]$ is large. Thus $t_2 - t_1 \gg \delta$. $\square$

Suppose that after several applications of Lemma 7.2 we have

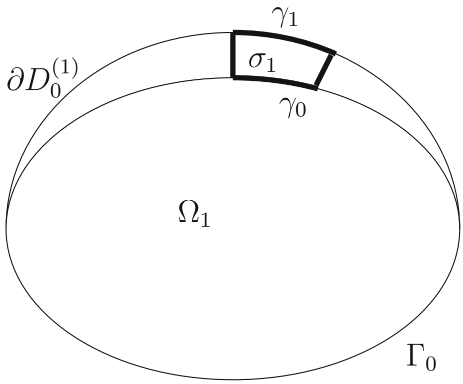
$$L^{(1)} u_1 = 0 \quad \text{in} \quad D_0^{(1)} \times [t_1, t_2],$$
$$L^{(m)} u_2 = 0 \quad \text{in} \quad D_1^{(m)},$$

where we are considering the interval $[t_1, t_2]$ instead of $[t_1+\delta, t_2-\delta]$ for the simplicity of notations. We assume that

$$u_1\big|_{x_0=t_1} = \frac{\partial u_1}{\partial x_0}\Big|_{x_0=t_1} = 0, \quad u_2\big|_{\partial_- D_1^{(m)}} = \frac{\partial u_2}{\partial x_0}\Big|_{\partial_- D_1^{(m)}} = 0.$$

We also assume that $\Gamma_0 \times (t_1, t_2) \subset \partial' D_1^{(m)} \cap (\partial D_0^{(1)} \times (t_1, t_2))$ and $\Omega_1 \times (t_1, t_2) \subset (D_0^{(1)} \times (t_1, t_2)) \cap D_1^{(m)}$ (cf. Fig. 11).

Fig. 11 $\gamma_1 \subset \partial D_0^{(1)}$, $\gamma_0 \subset \Omega_1$, γ_1 and γ_0 are close



We assume that $L^{(1)} = L^{(m)}$ in $\Omega_1 \times (t_1, t_2)$ and that DN operators Λ_1 and Λ_2 are equal on $\partial\Omega_1 \times (t_1, t_2)$ and $\Gamma_0 \times (t_1, t_2)$. Here, as above, $\partial' D_1^{(m)}$ means the time-like part of the boundary of $D_1^{(m)}$. It follows from Lemmas 7.4, 7.5 that the enlargement of the domain Ω_1 depends only on $L^{(1)}$ and does not depend on $L^{(2)}$. Therefore, as in [9], we arrive to the situation when Ω_1 and $D_1^{(0)}$ are close. To apply Lemma 7.2 to the domain $(D_1^{(0)} \setminus \Omega_1) \times \mathbb{R}$ we need new tools.

When $\partial\Omega_1$ and $\partial D_0^{(1)}$ are close, there is a narrow domain $\sigma_1 \subset D_0^{(1)} \setminus \Omega_1$ such that $\gamma_1 \subset \partial D_0^{(1)}$, $\gamma_0 \subset \partial\Omega_1$ and the distance between γ_0 and γ_1 is small (cf. Fig. 11)

Introduce Goursat coordinates for $L^{(1)}$ and $L^{(m)}$ near $\gamma_0 \times (t_1, t_2)$. We assume that operators $L^{(1)}$ and $L^{(m)}$ are defined in domains slightly larger than $D_0^{(1)} \times (t_1, t_2)$ and $D_1^{(m)}$. Let $L_1^{(1)}$ and $L_1^{(m)}$ be the operators $L^{(1)}$ and $L^{(m)}$ in corresponding Goursat coordinates. Let $y = \varphi_1(x)$ be the transformation to the Goursat coordinates for $L^{(1)}$. Let $\sigma_0 = (t_1, t_2) \times \gamma_0 \times [0, \varepsilon_0]$ be the domain where the Goursat coordinates for $L^{(1)}$ hold. We assume that σ_1 is so small that $\varphi_1(\sigma_1 \times (t_1, t_2)) \subset \sigma_0$. Let $\tau_0 = \varphi_1(\gamma_1 \times (t_1, t_2))$, i.e. τ_0 is the image of part of the boundary $\partial D_0^{(1)} \times (t_1, t_2)$ in Goursat coordinates. Denote by σ_0^+ the part of σ_0 between $\gamma_0 \times (t_1, t_2)$ and τ_0, i.e. σ_0^+ is the image of $\sigma_1 \times (t_1, t_2)$ in Goursat coordinates.

We assume that the Goursat coordinates for $L^{(m)}$ also hold in $(t_1, t_2) \times \gamma_0 \times [0, \varepsilon_0]$. Moreover, applying Lemmas 7.4, 7.5 repeatedly we get that $\tilde{L}_1^{(m)} = \tilde{L}_1^{(1)}$ in $\hat{\sigma}_0^+$, where $\hat{\sigma}_0^+ = \sigma_0^+ \cap (t_1 + \delta, t_2 - \delta)$. Here $\tilde{L}_1^{(1)}, \tilde{L}_1^{(m)}$ are operators $L^{(1)}, L^{(m)}$ in Goursat coordinates.

Consider the initial–boundary value problem for $\tilde{L}_1^{(1)}$ in Goursat coordinates

$$\begin{aligned} &\tilde{L}_1^{(1)}\tilde{u}_1 = 0 \quad \text{in } \hat{\sigma}_0^+, \\ &\tilde{u}_1\big|_{x_0 = t_1+\delta} = \frac{\partial \tilde{u}_1}{\partial x_0}\Big|_{x_0=t_1+\delta} = 0, \\ &\tilde{u}_1\big|_{(t_1+\delta, t_2-\delta)\times\gamma_0} = f, \quad \tilde{u}_1\big|_{\partial\hat{\sigma}_0^+} = 0, \end{aligned} \tag{7.25}$$

where supp $f \subset (t_1 + \delta, t_2 - \delta) \times \gamma_0$.

Consider now the equation $\tilde{L}_1^{(m)}\tilde{u}_2 = 0$ in $\hat{\sigma}_0^+$.

$$\begin{aligned}
&\tilde{L}_1^{(m)}\tilde{u}_2 = 0 \quad \text{in} \quad \hat{\sigma}_0^+,\\
&\tilde{u}_2\big|_{x_0=t_1+\delta} = \frac{\partial \tilde{u}_2}{\partial x_0}\Big|_{x_0=t_1+\delta} = 0,\\
&\tilde{u}_2\big|_{(t_1+\delta,t_2-\delta)\times\gamma_0} = f,
\end{aligned}$$

where f is the same as in (7.25).

Since $\tilde{L}_1^{(1)} = \tilde{L}_1^{(m)}$ in $\hat{\sigma}_0^+$ and since $\Lambda_1^{(1)} f = \Lambda_1^{(2)} f$ on $(t_1+\delta, t_2-\delta)\gamma_{0\,0}$ we have, by the unique continuation theorem of [25] and [29] that $\tilde{u}_1 = \tilde{u}_2$ in $\hat{\sigma}_1^+ \cap (t_1+2\delta, t_2-2\delta)$. Therefore by the continuity $\tilde{u}_2\big|_{\partial\hat{\sigma}_1^+\cap(t_1+2\delta,t_2-2\delta)} = 0$.

Let $\sigma_2^+ = \varphi_2^{-1}(\hat{\sigma}_0^+)$, $\tau_2 = \varphi_2^{-1}(\partial\hat{\sigma}_0^+)$, where $y = \varphi_2(x)$ is the transformation to the Goursat coordinates for $L^{(m)}$.

We shall show that τ_2 is a part of the boundary of $D_1^{(m)}$.

Construct the geometric optic solution $v_1(y)$ for $\tilde{L}_1^{(1)} u_1 = 0$ in Goursat coordinates as in (5.1). Since τ_0 is the boundary of the domain σ_0^+ and since the zero Dirichlet boundary condition holds on $\hat{\tau}_0 = \tau_0 \cap (t_1+\delta, t_2-\delta)$ this solution must reflect at $\hat{\tau}_0$ (cf. [14]).

Consider now the geometric optics solution $v_2(y)$ for $\tilde{L}_1^{(m)} u_2 = 0$ with the same initial condition. Since $\tilde{L}_1^{(1)} v_1 = \tilde{L}_1^{(m)} v_2$ in $\hat{\sigma}_0^+$ we have that $v_1(y) = v_2(y)$ before the reflection at $\hat{\tau}_0$. If $\tau_2 = \varphi_2^{-1}(\hat{\tau}_0)$ is not a part of the boundary of $\partial D_1^{(m)}$ there will be no reflection for $v_2(y)$ at $\hat{\tau}_0$. Thus, the solutions $v_1(y)$ and $v_2(y)$ will be different in $\hat{\sigma}_0^+$ near $\hat{\tau}_0$. This contradicts the fact that $v_1(y) = v_2(y)$ in $\hat{\sigma}_0^+$.

Therefore $\varphi_m = \varphi_2^{-1}\varphi_1$ maps the boundary $\gamma_1 \times (t_1, t_2)$ of $\partial D_0^{(1)} \times (t_1, t_2)$ on the part of boundary of $\partial D_1^{(m)}$. Let $\sigma_2 \subset D_1^{(m)}$ be the image of $\varphi_m = \varphi_2^{-1}\varphi_1(\sigma_1 \times (t_1 + 2\delta, t_2 - 2\delta))$. Let $\partial_\pm((\overline{\Omega}_1 \times [t_1+2\delta, t_2-2\delta]) \cup \overline{\sigma}_2)$ be the space-like parts of the boundary of $(\overline{\Omega}_1 \times [t_1+2\delta, t_2-2\delta]) \cup \overline{\sigma}_2$. Extend $\partial_+((\overline{\Omega}_1 \times [t_1+2\delta, t_2-2\delta]) \cup \overline{\sigma}_2)$ and $\partial_-((\overline{\Omega}_1 \times [t_1+2\delta, t_2-2\delta]) \cup \overline{\sigma}_2)$ as space-like surfaces S_m^+ and S_m^- to the whole domain $D_1^{(m)}$. Let $\tilde{D}_1^{(m)}$ be part of $D_1^{(m)}$ bounded from below and above by S_m^- and S_m^+, respectively. Note that $\partial' D_1^{(m)} \supset \Gamma_0 \times (t_1+2\delta, t_2-2\delta)$. Define $\tilde{\varphi}_m = I$ on $\Omega_1 \times (t_1+2\delta, t_2-2\delta)$, $\tilde{\varphi}_m = I$ on $\Gamma_0 \times (t_1+2\delta, t_2-2\delta)$, $\tilde{\varphi}_m = \varphi_m$ on $\sigma_1 \times (t_1+2\delta, t_2-2\delta)$. Let Φ_{m+1} be the extension of $\tilde{\varphi}_m$ (cf. [16]) to the whole domain $\tilde{D}_1^{(m)}$. Denote by $D_1^{(m+1)}$ the image of $\tilde{D}_1^{(m+1)}$ under the map Φ_{m+1}. If c_m is a gauge transformation on $\overline{\sigma}_1 \times [t_1+2\delta, t_2-2\delta]$ we denote by $\tilde{c}_{m+1}$ the extension of c_m to $D_1^{(m+1)}$ such that $\tilde{c}_{m+1} = 1$ on $\Omega_1 \times [t_1+2\delta, t_2-2\delta]$, $\tilde{c}_{m+1} = 1$ on $\Gamma_0 \times (t_1+2\delta, t_2-2\delta)$.

We just proved the following lemma:

Lemma 7.6 *Let $L^{(1)}$ and $L^{(m+1)} = \tilde{c}_{m+1} \circ \Phi_{m+1} \circ L^{(m)}$ be operators in $D_0^{(1)} \times [t_1+2\delta, t_2-2\delta]$ and $D_1^{(m+1)}$, respectively. Then $\Omega_2 \times (t_1+2\delta, t_2-2\delta) \subset D_1^{(m+1)} \cap (D_0^{(1)} \times (t_1+2\delta, t_2-2\delta))$ where $\overline{\Omega}_2 = \overline{\Omega}_1 \cup \overline{\sigma}_1$ and $L^{(1)} = L^{(m+1)}$ in $\Omega_2 \times (t_1+2\delta, t_2-2\delta)$.*

We shall proceed with the enlargement of the domain Ω_2 using Lemmas 7.2 and 7.6. Therefore after finite number of steps (cf. [9]) we get a domain $D_1^{(N)}$, operator $L^{(N)}$ on $D_1^{(N)}$ and the map Φ_N of $D_1^{(N)}$ onto $D_0^{(1)} \times (t_1 + \delta_N, t_2 - \delta_N)$ such that $c_N \circ \Phi_N \circ L^{(N)} = L^{(1)}$ in $D_0^{(1)} \times (t_1 + \delta_N, t_2 - \delta_N)$ for some $\delta_N > 0$. Here c_N is the gauge transformation. Remind that $\tilde{\Phi}_2$ is the diffeomorphism of $D_1^{(2)}$ onto $D_1^{(3)}$, Φ_3 is the diffeomorphism of $\tilde{D}_1^{(3)} \subset D_1^{(3)}$ onto $D_1^{(4)}$, etc.... $\tilde{\Phi}_{N-1}$ is the map of $\tilde{D}_1^{(N-1)}) \subset D_1^{(N-1)}$ onto $D_1^{(N)}$ and Φ_N is the map of $D_1^{(N)}$ onto $D_0^{(1)} \times (t_1+\delta_N, t_2-\delta_N)$.

Therefore, the diffeomorphism $\Phi^{-1} = \Phi_1^{-1}\Phi_3^{-1}...\Phi_N^{-1}$ maps $D_0^{(1)} \times [t_0+\delta_N, t_2-\delta_N]$ onto $D_1^{(2)}$. Thus Φ maps $D_1^{(2)}$ onto $D_0^{(1)} \times [t_1 + \delta_N, t_2 - \delta_N]$.

Note that $D_1^{(2)}$ is an almost cylindrical domain in $D_0^{(2)} \times \mathbb{R}$, i.e. $D_1^{(2)} = D_0^{(2)} \times \{S^-(x_1, \dots, x_n) \le x_0 \le S^+(x_1, \dots, x_n)\}$, where $x_0 = S^{\pm}(x_1, \dots, x_n)$ are space-like surfaces, $(x_1, \dots, x_n) \in D_0^{(2)}$.

Note that $[t_1, t_2]$ is arbitrary large and therefore $[t_1', t_2'] = [t_1 + \delta, t_2 - \delta]$ is also arbitrary large. Therefore we obtained the following theorem:

Theorem 7.7 *Let $L^{(1)}$ and $L^{(2)}$ be two operators in $D_0^{(1)} \times \mathbb{R}$ and $D_0^{(2)} \times \mathbb{R}$, respectively. Suppose $\Gamma_0 \subset \partial D_0^{(1)} \cap \partial D_0^{(2)}$ and the DN operators, corresponding to $L^{(i)}$, are equal on $\Gamma_0 \times \mathbb{R}$ for all f that have a compact support in $\overline{\Gamma}_0 \times \mathbb{R}$. Suppose that the conditions (1.2), (1.6) hold for $L^{(i)}$, $i = 1, 2$, and the coefficients of $L^{(1)}$ and $L^{(2)}$ are analytic in x_0 in $D_0^{(i)} \times \mathbb{R}$, $i = 1, 2$. Suppose for each $t_0 \in R$ there exists T_{t_0} such that the BLR condition is satisfied for $L^{(1)}$ on $[t_0, T_{t_0}]$. Let $[t_1', t_2']$ be an arbitrary sufficiently large time interval. Then there exists a diffeomorphism Φ^{-1} of $\overline{D}_0^{(1)} \times [t_1', t_2']$ on an almost cylindrical domain $\overline{D}_1^{(2)} \subset \overline{D}_0^{(2)} \times \mathbb{R}$, $\Phi = I$ on $\Gamma_0 \times [t_1', t_2']$ and there exists a gauge transformation $c(y)$ on $D_1^{(2)}$, $|c(y)| = 1$ on $D_1^{(2)}$, $c(y) = 1$ on $\Gamma_0 \times [t_1', t_2']$ such that*

$$c \circ \Phi^{-1} \circ L^{(2)} = L^{(1)} \quad on \;\; D_0^{(1)} \times [t_1', t_2'].$$

Now we shall use Theorem 7.7 to prove Theorem 1.2.

Proof of Theorem 1.2 Let $L^{(i)}$ be two operators in $D_0^{(i)} \times \mathbb{R}$, $i = 1, 2$, $\Gamma_0 \subset \partial D_0^{(1)} \cap \partial D_0^{(2)}$ and all conditions of Theorem 7.7 are satisfied.

Let (t_{j1}, t_{j2}) be an interval as in Theorem 7.7 and $\bigcup_{j=-\infty}^{\infty}(t_{j1}, t_{j2}) = \mathbb{R}$. We have $\overline{D}_0^{(1)} \times \mathbb{R} \subset \bigcup_{j=-\infty}^{\infty} \overline{D}_0^{(1)} \times [t_{j1}, t_{j2}]$. It follows from Theorem 7.7 that for each $j \in \mathbb{Z}$ there exists a diffeomorphism Ψ_j on $D_j^{(1)} \times [t_{j1}, t_{j2}]$ and a gauge transformation c_j such that $\Psi_j = I$ and $c_j = 1$ on $\overline{\Gamma}_0 \times [t_{j1}, t_{j2}]$, and

$$c_j \circ \Psi_j^{-1} \circ L^{(2)} = L^{(1)} \;\text{ in }\; \overline{D}_0^{(1)} \times [t_{j1}, t_{j2}]. \tag{7.26}$$

In (7.26) Ψ_j is a diffeomorphism of $\overline{D}_0^{(1)} \times [t_{j1}, t_{j2}]$ onto an almost cylindrical domain $\overline{D}_0^{(2)} \times \{S_j^-(x_1, \dots, x_n) \le x_0 \le S_j^+(x_1, \dots, x_n)\}$, where $x_0 = S_j^{\pm}(x_1, \dots, x_n)$ are space-like surfaces, $\Psi_j = I$ on $\overline{\Gamma}_0 \times [t_{j1}, t_{j2}]$, $|c_j(x)| = 1$ for all $x \in \overline{D}_0^{(1)} \times [t_{j1}, t_{j2}]$, $c_j = 1$ on $\Gamma_0 \times [t_{j1}, t_{j2}]$.

We shall show that

$$\Psi_j = \Psi_{j+1}, \quad c_j = c_{j+1} \tag{7.27}$$

on $\overline{D}_0^{(1)} \times [t_{j+1,1}, t_{j2}]$ where $[t_{j+1,1}, t_{j2}]$ is the intersection of $[t_{j1}, t_{j2}]$ and $[t_{j+1,1}, t_{j+1,2}]$.

Let $[t_{j1}, t_{j+1,2}] = [t_{j1}, t_{j2}] \cup [t_{j+1,1}, t_{j+1,2}]$. Note that the proof of Theorem 7.7 consists of several steps, each step is dealing with the arbitrary large time interval. Consider, for example, Lemma 7.1.

The changes of variables (7.4), (7.5) and (7.7) are defined on arbitrary time interval $[t', t'']$. Thus they are defined on $[t_{j1}, t_{j+1,2}]$. Therefore the maps (7.4), (7.5), (7.7) are defined on $[t_{j1}, t_{j2}]$ and $[t_{j+1,1}, t_{j+1,2}]$ and they coincide on $[t_{j+1,1}, t_{j2}] = [t_{j1}, t_{j2}] \cap [t_{j+1,1}, t_{j+1,2}]$.

Also using the extension theorems of [16, §8], we can find the extension $\tilde{\Phi}_2 \cap [t_{j1}, t_{j+1,2}]$ of $\Phi_2 \cap [t_{j1}, t_{j+1,2}]$ (cf. (7.8)) in two steps: First extend $\tilde{\Phi}_2 \cap [t_{j1}, t_{j+1,1}]$ and then extend continuously $\Phi_2 \cap [t_{j+1,1}, t_{j+1,2}]$ to get a continuous extension of $\Phi_2 \cap [t_{j1}, t_{j+1,2}]$. This way we show that (7.27) holds for some $j \in \mathbb{Z}$. Therefore starting with Ψ_0 on $[t_{01}, t_{02}]$ we can construct Ψ_1 on $[t_{11}, t_{12}]$ such that $\Psi_0 = \Psi_1$ on $[t_{01}, t_{02}] \cap [t_{11}, t_{12}]$. Continuing this construction we get (7.27) for any $j \in \mathbb{Z}$.

Let $\Psi = \Psi_j, c = c_j$ on $[t_{j1}, t_{j2}], \forall j \in \mathbb{Z}$. Then Ψ is a proper diffeomorphism of $\overline{D}_0^{(1)} \times \mathbb{R}$ onto $\overline{D}_0^{(2)} \times \mathbb{R}$, $\Psi = I$ on $\Gamma \times \mathbb{R}$ and

$$c \circ \Psi^{-1} \circ L^{(2)} = L^{(1)} \quad \text{on } \overline{D}_0^{(1)} \times \mathbb{R}.$$

□

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