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Regularity and geometric character of solution of a degenerate parabolic equation

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Abstract This work studies the regularity and the geometric significance of solution of the Cauchy problem for a degenerate parabolic equation $u_t = \Delta u^m$. Our main objective is to improve the Hölder estimate obtained by pioneers and then, to show the geometric characteristic of free boundary of degenerate parabolic equation. To be exact, for the weak solution $u(x, t)$, the present work will show that:

1. The function $\phi = (u(x, t))^\beta \in C^1(\mathbb{R}^n)$ for given $t > 0$ if β is large sufficiently;
2. The surface $\phi = \phi(x, t)$ is tangent to $\mathbb{R}^n$ at the boundary of the positivity set of $u(x, t)$;
3. The function $\phi(x, t)$ is a classical solution to another degenerate parabolic equation.

Moreover, some explicit derivative estimates and expressions about the speed of propagation of $u(x, t)$ and the continuous dependence on the nonlinearity of the equation are obtained.

Keywords Degenerate parabolic equation · Regularity · Geometric character · Hölder Estimate

Mathematics Subject Classification 35K15 · 35K55 · 35K65 · 53C25

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1 Introduction

Consider the Cauchy problem of nonlinear parabolic equation

$$\begin{cases} u_t = \Delta u^m & \text{in } Q, \\ u(x,0) = u_0(x) & \text{on } \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $Q = \mathbb{R}^n \times \mathbb{R}^+, m > 1, n \geq 1$ and

$$0 \leq u_0(x) \leq M, 0 < \int_{\mathbb{R}^n} u_0(x)dx < \infty. \tag{1.2}$$

The Eq. in (1.1) is an example of nonlinear evolution equations and many interesting results, such as the existence, uniqueness, regularity, continuous dependence on the nonlinearity of the equation and large time behavior (see [11,19,34] and therein) are obtained during the past several decades. By a weak solution of (1.1), (1.2) in Q, we mean a nonnegative function $u(x,t)$ such that, for any given $T > 0$,

$$\int_{Q_T} \left(u^2 + |\nabla u^m|^2\right) dxdt < \infty$$

and

$$\int_{Q_T} \left(\nabla u^m \cdot \nabla f - u f_t\right) dxdt = \int_{\mathbb{R}^n} u_0(x) f(x,0)dx$$

for any continuously differentiable function $f(x,t)$ with compact support in Q_T, where, $Q_T = \mathbb{R}^n \times (0,T)$.

We know that (see [5,6,23,31,32]) the Cauchy problem (1.1), (1.2) permits a unique weak solution $u(x,t)$ which has the following properties:

$$0 \leq u(x,t) \leq M, \tag{1.3}$$

$$\int_{\mathbb{R}^n} u(x,t)dx = \int_{\mathbb{R}^n} u_0(x)dx, \tag{1.4}$$

$$\frac{\partial u}{\partial t} \geq \frac{-u}{(m-1)t}, \tag{1.5}$$

$$\Delta\left(\frac{m}{m-1}u^{m-1}\right) \geq \frac{-n}{n(m-1)+2} \cdot \frac{1}{t}, \tag{1.6}$$

$$\|u - v\|_{L^2(Q_T)} \leq C \max_{s\in[0,M]} \left|s^{\frac{1}{j}} - s^{\frac{m}{j}}\right|, \tag{1.7}$$

where $j = 1, 2, 3\ldots$ and v is the solution to the Cauchy problem of linear heat equation with the same initial value,

$$\begin{cases} v_t = \Delta v \quad \text{in} \quad Q, \\ v(x,0) = u_0(x) \quad \text{on} \quad \mathbb{R}^n, \end{cases} \tag{1.8}$$

$C = O(T^\gamma)$ for large T, and

$$\gamma = 1 - \frac{(j-2)n}{j(2+n)m}.$$

In particular, the solution $u(x,t)$ can be obtained (see [12,33]) as a limit of solutions $u_\eta(\eta \longrightarrow 0^+)$ of the Cauchy problem

$$\begin{cases} u_t = \Delta u^m & \text{in } Q, \\ u(x,0) = u_0(x) + \eta & \text{on } \mathbb{R}^n, \end{cases} \tag{1.9}$$

and the solutions $u_\eta(x,t)$ are taken in the classical sense, but $u_\eta(x,t)$ are not in $L^1(\mathbb{R}^n)$ due to $u_\eta(x,t) \geq \eta$ in Q. We know that Aronson and Benilan (see Theorem 2, p. 104 in [6]) claimed that: if u is the weak solution to the Cauchy problem (1.1) with the initial value (1.2), then $u \in C(Q)$ and $u \geq 0$; Vazquez (see Proposition 6 in Ch. 2 of [34]) proved $u \in C^\infty(Q_+)$, where

$$Q_+ = \{(x,t) \in Q : \ u(x,t) > 0\}.$$

Before this, the same conclusion was established by Friedman (see Theorem 11 and Corollary 2 in Chapter 3 [27]). Moreover, employing so called *bootstrap argument*, Aronson et al. (see [3,4,29] also claimed $u \in C^\infty(Q_+)$.

Therefore, we can divide the space-time $Q = \mathbb{R}^n \times \mathbb{R}^+$ into two parts: $Q = Q_+ \cup Q_0$, where

$$Q_0 = \{(x,t) \in Q : \ u(x,t) = 0\}.$$

Furthermore, if Q_0 contains an open set, say, Q_1, we can also obtain $u(x,t) \in C^\infty(Q_1)$ owing to $u(x,t) \equiv 0$ in Q_1. Thereby, we may suspect that the solution of degenerate parabolic equation is actually smooth in Q except a set of measure 0. In order to improve the regularity of $u(x,t)$, many authors have made hard effort in this direction. The earliest contribution to the subject was made, maybe, by Aronson and Gilding and Peleiter (see [3,29]). They proved that the solution to the Cauchy problem

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u^m}{\partial x^2} & \text{in } \mathbb{R}^1 \times \mathbb{R}^+, \ m > 1, \\ u(x,0) = u_0(x) & \text{on } \mathbb{R}^1 \end{cases}$$

is continuous in $\mathbb{R}^1 \times (0,+\infty)$ if the nonnegative initial value satisfies a good condition. Moreover, if the initial value $0 \leq u_0 \leq M$ and u_0^m is Lip-continuous, then $u(x,t)$ can be continuous on $\mathbb{R}^1 \times [0,+\infty)$ (see [29]). As to the case of $n \geq 1$, Caffarelli and Friedman (see [12]) proved that the solution $u(x,t)$ to the Cauchy problem (1.1), (1.2) is Hölder continuous on $\mathbb{R}^n \times [\delta_0, \infty)$:

$$|u(x,t) - u(x_0,t_0)| \leq C \left(|x-x_0|^\alpha + |t-t_0|^{\frac{\alpha}{2}} \right)$$
$$\text{for some } \alpha \in (0,1), \ \text{ where } C \text{ depends on } \delta_0. \tag{1.10}$$

Moreover, Gilding et al. also discussed the Dirichlet problem (see [21,30]) and obtained a similar conclusion to (1.10) and a gradient expression (see [24])

$$|\nabla u(x,t) - \nabla u(x_0,t_0)| \leq C\left(|x-x_0|^\alpha + |t-t_0|^{\frac{\alpha}{2}}\right)$$

As to the general equation $u_t = \nabla \cdot (u\nabla p)$, $p = \kappa(u)$, Caffarelli and Vazquez ([14]) established the property of finite propagation and the persistence of positivity, where κ may be a general operator. To study this problem more precisely, Aronson et al. (see [2,7]) constructed a interesting radially symmetric solution $u(r,t)$ to the focusing problem for the equation of (1.1). Denoting the porous medium pressure $V = \frac{m}{m-1}u^{m-1}$, they claimed $V = Cr^\delta$ at the fusing time, where $0 < \delta < 1$, C is a positive constant. Furthermore,

$$\lim_{r\downarrow 0} \frac{V(r,\eta r^\alpha)}{r^{2-\alpha}} = \frac{\varphi(c^*\eta)}{-\eta}.$$

Because $u \in C^\infty(Q_+)$ and also, $u \in C^\infty(Q_1)$, where Q_+ and Q_1 is mentioned as before, many authors (see [8,15,16,18,20,22,26]) discussed the Harnark inequalities in Q_+ and the smoothness of *free boundary*

$$\Gamma = \partial\overline{H_u(t)} \qquad t > 0.$$

where $H_u(t)$ is the positivity set of $u(x,t)$:

$$H_u(t) = \{x \in \mathbb{R}^n : u(x,t) > 0\} \qquad t > 0.$$

For example, Aronson et al. (see [8]) proved the existence of corner point on interface at some time t^* for the case of $n = 1$, Daskalopoulos and Hamilton (see [13,17,18,20]) discussed the C^∞ smoothness of the interface of the equation (1.1), DiBenedetto et al. (see [22,25]) discussed the regularity and intrinsic Harnack type inequalities in Q_+.

To study the regularity of the weak solution of (1.1), (1.2) more precisely, the present work intends to show the dependency between α and m in (1.10). To be exact, for every given $m > 1$, we will prove

$$\begin{cases} \alpha = 1 & \text{if } 1 < m < 2, \\ \alpha \in (\frac{1}{m}, \frac{1}{m-1}) & \text{if } m \geq 2. \end{cases} \tag{1.11}$$

In (1.11), we see that the range of α is $(0, 1]$ rather than $(0, 1)$ as in (1.10). Unfortunately, our priori estimate (see below) is only right for the case of $n = 1$ owing to technical difficulties, so (1.11) is true only for $n = 1$ in this paper. As to the case of $n > 1$, we all know that $u \in C^\infty(Q_+)$ and some authors discuss the regularity (see [10] and therein) more deeply, but whether (1.11) is right or not in the whole area Q_+, we feel it might be an interesting topic.

Employing (1.11) (for the case of $n = 1$) and (1.10) (for the case of $n > 1$), the present work will show that ∇u^β are continuous if β is large sufficiently. In particular, we will prove that the function $\phi(x,t) = (u(x,t))^\beta$ satisfies the degenerate parabolic equation

$$\frac{\partial \phi}{\partial t} = m\left[\phi^{\frac{m-1}{\beta}}\Delta\phi + \frac{m-\beta}{\beta}\phi^{\frac{m-\beta-1}{\beta}}|\nabla\phi|^2\right]$$

in the classical sense. Therefore we can speak roughly, the weak solution to the Cauchy problem (1.1) is in fact a classical one.

For every fixed $t > 0$, we define a n-dimensional surface $S(t)$, which floats in the space $\mathbb{R}^{n+1}$ with the time t:

$$S(t):\begin{cases} x_i = x_i, & i = 1, 2, 3, \ldots, n, \\ x_{n+1} = \phi(x,t), & \end{cases}$$

where the function $\phi(x,t) = (u(x,t))^\beta$. Let

$$\begin{cases} g_1 = (1, 0, \ldots, \frac{\partial\phi}{\partial x_1}), \\ g_2 = (0, 1, \ldots, \frac{\partial\phi}{\partial x_2}), \\ \quad\cdots \\ g_n = (0, 0, \ldots 1, \frac{\partial\phi}{\partial x_n}). \end{cases} \tag{1.12}$$

Define the Riemannian metric on $S(t)$:

$$(ds)^2 = \sum_{i,j=1}^{n} g_{ij}dx_i dx_j,$$

where $g_{ij} = g_i \cdot g_j$. Clearly,

$$\begin{aligned}(ds)^2 &= \sum_{i=1}^{n}(1+\phi_{x_i}^2)(dx_i)^2 + \sum_{i\neq j, i,j=1}^{n}\phi_{x_i}\phi_{x_j}dx_i dx_j \\ &= \sum_{i=1}^{n}(dx_i)^2 + \left(\sum_{i=1}^{n}\phi_{x_i}dx_i\right)^2.\end{aligned}$$

Recalling $\sum_{i=1}^{n}\phi_{x_i}dx_i = d\phi = dx_{n+1}$ for fixed $t > 0$, we get

$$(ds)^2 = \sum_{i=1}^{n+1}(dx_i)^2 = \sum_{i=1}^{n}(dx_i)^2 + (d\phi)^2.$$

If the derivatives $\frac{\partial\phi}{\partial x_i}$ are bounded for $i = 1, 2, \ldots, n$ and for $x \in \mathbb{R}^n$ uniformly, then we can get a positive constant C, such that $(\sum_{i=1}^{n}\phi_{x_i}dx_i)^2 \le C(\sum_{i=1}^{n}dx_i)^2$. Denoting $(d\rho)^2 = \sum_{i=1}^{n}(dx_i)^2$, which is the Euclidean metric on $\mathbb{R}^n$, we get

$$(d\rho)^2 \leq (ds)^2 \leq (1+C)(d\rho)^2. \tag{1.13}$$

As a consequence of (1.13), we see that the completeness of $\mathbb{R}^n$ yields the completeness of $S(t)$ and therefore, $S(t)$ is a complete Riemannian manifold. On the other hand, if we can obtain

$$\nabla\,(\phi(x,t))\,|_{\partial H_u(t)} = 0 \tag{1.14}$$

for every fixed $t > 0$, where $H_u(t)$ is mentioned as before, then (1.14) encourage us to prove that: the manifold $S(t)$ is tangent to $\mathbb{R}^n$. However, we need to point out that: we can only give an explicit dependency relationship between β and m for the case of $n = 1$.

It is well-known that the function

$$v(x,t) = \left(\frac{1}{2\sqrt{\pi t}}\right)^n \int_{\mathbb{R}^n} u_0(\xi) e^{-\frac{(x-\xi)^2}{4t}} d\xi$$

is the solution of the Cauchy problem of the linear heat Eq. (1.8) and $v(x,t) > 0$ in Q everywhere if only the initial value u_0 satisfies (1.2). This fact shows that the speed of propagation of $v(x,t)$ is infinite, that is to say,

$$\sup_{x\in H_v(t)} |x| = \infty. \tag{1.15}$$

However, the degeneracy of the equation in (1.1) causes an important phenomenon to occur, i.e. finite speed of propagation of disturbance. We have observed this phenomenon on the $source - type$ solution $B(x,t;C)$ (see [9]), where

$$B(x,t,C) = t^{-\lambda}\left(C - \kappa\frac{|x|^2}{t^{2\mu}}\right)_+^{\frac{1}{m-1}} \tag{1.16}$$

is the equation in (1.1) with a initial mass $M\delta(x)$, and

$$\lambda = \frac{n}{n(m-1)+2}, \quad \mu = \frac{\lambda}{n}, \quad \kappa = \frac{\lambda(m-1)}{2mn}.$$

We see that the function $B(x,t;C)$ has compact support in space for every fixed time. More precisely, if $u(x,t)$ is the solution of (1.1), (1.2), then

$$\sup_{x\in H_u(t)} |x| = O\left(t^{\frac{1}{n(m-1)+2}}\right) \tag{1.17}$$

when t is large enough (see Proposition 17 in [34]). Comparing (1.15) and (1.17) and recalling the mass conservation $\int_{\mathbb{R}^n} u(x,t)dx = \int_{\mathbb{R}^n} u_0(x)dx = \int_{\mathbb{R}^n} v(x,t)dx$, the present work will prove that the solution continuously depends on the nonlinearity of the Eq. (1.1):

$$\|u(\cdot,t)-v(\cdot,t)\|^2_{L^2(|x|\le k)} \le C\left[(m-1)+\frac{1}{k}\right].$$

We read the main conclusions of the present work as follows:

Theorem 1 *Assume $u(x,t)$ be the weak solution to (1.1), (1.2). Then $u(x,t)\in C(Q)$ and moreover,*

(1) *If $n=1$, for every given $\tau>0$, $N>0$, there exists a positive ν such that*

$$|u(x_1,t_1)-u(x_2,t_2)|\le \nu\left(|x_1-x_2|^{\frac{1}{h}}+|t_1-t_2|^{\frac{1}{2h}}\right)$$

where $|x_i|\le N, t_i\ge\tau,\ i=1,2,$

$$h=\begin{cases}1 & \text{if } 1<m<2,\\ h\in(m-1,m) & \text{if } m\ge 2;\end{cases} \tag{1.18}$$

(2) *The function $\phi=(u(x,t))^\beta\in C^1(\mathbb{R}^n)$ and the surface $\phi=\phi(x,t)$ is tangent to $\mathbb{R}^n$ on $\partial H_u(t)$ for every fixed $t>0$, where $\beta>h$ and h is defined by (1.18) for the case of $n=1$; $\beta>\frac{1}{\alpha}$ and α is stated in (1.10) for the case of $n>1$;*

(3) *If $\beta>2h$ (for $n=1$) or $\beta>\frac{2}{\alpha}$ (for $n>1$), the function $\phi(x,t)$ satisfies the degenerate parabolic equation*

$$\frac{\partial\phi}{\partial t}=m\left(\phi^{\frac{m-1}{\beta}}\Delta\phi+\frac{m-\beta}{\beta}\phi^{\frac{m-\beta-1}{\beta}}|\nabla\phi|^2\right)$$

in the classical sense in Q.

Theorem 2 *Assume $n\ge 1$ and $u(x,t)$ be the weak solution to (1.1), (1.2), $B_\delta=\{x\in\mathbb{R}^n:\ |x|<\delta\}$ for some $\delta>0$. If supp $u_0\subset B_\delta$, then for every given $t>0$,*

$$\sup_{x\in H_u(t)}|x|\ge\chi(t), \tag{1.19}$$

where,

$$\chi(t)=\left[(m-1)\pi^{\frac{(1-m)n}{2}}\Gamma(1+\frac{n}{2})^{m-1}\cdot\left(\int_{\mathbb{R}^1}u_0dx\right)^{m-1}t\right]^{\frac{1}{2+(m-1)n}}.$$

Moreover, for every given $T>0$, there is a positive $C_*=C_*(T)$ such that

$$\int_{|x|\le k}[v(x,t)-u(x,t)]^2\,dx\le C_*\left[(m-1)+\frac{1}{k}\right] \tag{1.20}$$

with respect to $t\in(0,T)$ uniformly, where $v(x,t)$ is the solution of (1.8).

We see that $S(t)$ and $\mathbb{R}^n$ are two manifolds in $\mathbb{R}^{n+1}$ and the Cauchy problem (1.1), (1.2) can be regarded as a mapping $\Phi(t):\ \mathbb{R}^n \longrightarrow S(t)$. Thus, besides the theorems mentioned above, we will give an example to show the intrinsic properties about the manifold $S(t)$.

Remark 1 Although we may obtain the regularity for the weak solution $u(x,t)$ for all $m>1$ in (2) and (3) of Theorem 1 for all $n\geq 1$, and particularly, (3) tells us that the weak solution $u(x,t)$ is in fact a classical one, we see that we can only show the explicit relationship between β and m for the case of $n=1$. As to the case of $n>1$, we can not get such relationship owing to we can not know how α depend on m in (1.10).

2 The proof of Theorem 1

Lemma 1 *If $n=1$ and $u(x,t)$ is the weak solution to (1.1), (1.2) in Q. Then for every given $m>1$, there is a $h\in(m-1,m)$ and $C_0=C_0(m,M,T)$ such that*

$$\left|\frac{\partial u^h}{\partial x}\right|^2 \leq \frac{C_0}{t} \tag{2.1}$$

in the sense of distributions in Q.

Proof We first prove (2.1) for the classical solutions $u_\eta(x,t)$. Set

$$V^q = u_\eta^m \quad \text{for } q\in(1,\frac{m}{m-1}).$$

Then

$$V_t = mV^{q-\frac{q}{m}}\frac{\partial^2 V}{\partial x^2} + m(q-1)V^{q-1-\frac{q}{m}}\left|\frac{\partial V}{\partial x}\right|^2. \tag{2.2}$$

We differentiate the Eq. (2.2) with respect to x and multiply though by $\frac{\partial V}{\partial x}$, and let $H=\frac{\partial V}{\partial x}$, we get

$$\begin{aligned}
(H^2)_t &= 2mV^{q-\frac{q}{m}}H\frac{\partial^2 H}{\partial x^2} + 2m\left(q-\frac{q}{m}\right)V^{q-1-\frac{q}{m}}H^2H_x\\
&\quad + 2m(q-1)\left(q-1-\frac{q}{m}\right)V^{q-2-\frac{q}{m}}H^4 + 4m(q-1)V^{q-1-\frac{q}{m}}H^2H_x\\
&= mV^{q-\frac{q}{m}}\frac{\partial^2 H^2}{\partial x^2} + 2m\left(3q-2-\frac{q}{m}\right)V^{q-1-\frac{q}{m}}H^2H_x\\
&\quad + 2m(q-1)\left(q-1-\frac{q}{m}\right)V^{q-2-\frac{q}{m}}H^4 - 2mV^{q-\frac{q}{m}}(H_x)^2.
\end{aligned}$$

Moreover, it follows from $q \in (1, \frac{m}{m-1})$ that $(q-1-\frac{q}{m}) < 0$, hence

$$2m(q-1)(q-1-\frac{q}{m})V^{q-2-\frac{q}{m}} \leq -C_1, \tag{2.3}$$

where, $C_1 = 2m|(q-1)\left(q-1-\frac{q}{m}\right)|(M+\eta)^{m-\frac{2m}{q}-1}$. Setting

$$L(H^2) = mV^{q-\frac{q}{m}}\frac{\partial^2 H^2}{\partial x^2} + 2m\left(3q-2-\frac{q}{m}\right)V^{q-1-\frac{q}{m}}H^2 H_x$$
$$-C_1 H^4 - mV^{q-2\frac{q}{m}}(H_x)^2,$$

we get

$$H_t^2 \leq L(H^2) \quad \text{in } Q. \tag{2.4}$$

Clearly, the function $Z^2 = \frac{1}{C_1 t}$ satisfies the equation $\frac{\partial}{\partial t}Z^2 = L(Z^2)$ and $Z^2(0) = +\infty$. By comparison theorem, we get

$$H^2 \leq \frac{1}{C_1 t} \quad \text{in } Q,$$

or we rewrite this inequality as

$$|\nabla u_\eta^h|^2 \leq \frac{1}{C_1 t} \tag{2.5}$$

with $h = \frac{m}{q}$ for all $1 < q < \frac{m}{m-1}$. Of course, we can take $q = m$ if $1 < m < 2$; $1 < q < \frac{m}{m-1}$ if $m \geq 2$. Therefor, (2.5) is right for

$$\begin{cases} h = 1 & \text{if } 1 < m < 2, \\ h \in (m-1, m) & \text{if } m \geq 2. \end{cases}$$

Letting $\eta \longrightarrow 0$ in (2.5), we obtain the conclusion of our lemma with a constant $C_0 = \frac{1}{C_1}$ and (2.1) follows. □

To prove Theorem 1, we need to show an ordinary inequality firstly:

$$|a-b|^\beta \leq |a^\beta - b^\beta| \quad \text{for } a, b \geq 0, \ \beta > 1. \tag{2.6}$$

In fact, (2.6) is right for $a = b$. If $a > b$, we can easily get the following inequalities:

$$\left(1-\frac{b}{a}\right)^\beta \leq 1-\frac{b}{a} \quad \text{and} \quad 1-\left(\frac{b}{a}\right)^\beta \geq 1-\frac{b}{a}$$

thanks to $0 \leq \frac{b}{a} < 1$. Thereby, $\left(1-\frac{b}{a}\right)^{\beta} \leq 1-\left(\frac{b}{a}\right)^{\beta}$. This inequality gives

$$|a-b|^{\beta} = a^{\beta}\left|1-\frac{b}{a}\right|^{\beta} \leq a^{\beta}-b^{\beta}.$$

So (2.6) holds for $a > b \geq 0$. Certainly, (2.6) is also right if $0 \leq a < b$.

We are now in the position to establish Theorem 1.

To prove (1) of Theorem 1 It follows from (2.1) that

$$\left|u^{h}(x_1,t)-u^{h}(x_2,t)\right| \leq \left(\frac{C_0}{t}\right)^{-\frac{1}{2}}|x_1-x_2| \tag{2.7}$$

for every $(x_1,t),(x_2,t) \in Q, h \in (m-1,m)$. If $1 < m < 2$, we take $h = 1$, and therefore, (2.7) yields

$$|u(x_1,t)-u(x_2,t)| \leq \left(\frac{C_0}{t}\right)^{-\frac{1}{2}}|x_1-x_2|.$$

If $m \geq 2$, we take $h \in (m-1,m)$. In this case, we use (2.6) in (2.7) and obtain

$$|u(x_1,t)-u(x_2,t)| \leq \left(\frac{C_0}{t}\right)^{-\frac{1}{2h}}|x_1-x_2|^{\frac{1}{h}}.$$

Therefore, for every given $m > 1$, there always exists a suitable positive number$h \geq 1$ such that

$$|u(x_1,t)-u(x_2,t)| \leq \left(\frac{C_0}{\tau}\right)^{-\frac{1}{2h}}|x_1-x_2|^{\frac{1}{h}} \tag{2.8}$$

for every $(x_1,t),(x_2,t) \in \mathbb{R}^1 \times [\tau,\infty)$ with given $\tau > 0$, where

$$\begin{cases} h = 1 & \text{for } 1 < m < 2, \\ h \in (m-1,m) & \text{for } m \geq 2. \end{cases}$$

Employing the well-known theorem on the Hölder continuity with respect to the time variable (see [28]), we obtain

$$|u(x,t_1)-u(x,t_2)| \leq K|t_1-t_2|^{\frac{1}{2h}} \tag{2.9}$$

for $|x| \leq N,\ t_1,t_2 > \tau$ and $|t_1-t_2|$ small sufficiently, where K depends on $(\frac{C_0}{\tau})^{-\frac{1}{2h}}$, N is any fixed positive constant. Combining (2.8) and (2.9) we get another positive constant ν such that

$$|u(x_1,t_1)-u(x_2,t_2)| \leq \nu\left[|x_1-x_2|^{\frac{1}{h}}+|t_1-t_2|^{\frac{1}{2h}}\right] \tag{2.10}$$

for all $|x_i| \le N, t_i \ge \tau,\ i = 1, 2$, where

$$h = \begin{cases} 1 & \text{if } 1 < m < 2, \\ h \in (m-1, m) & \text{if } m \ge 2. \end{cases}$$

Certainly, (2.10) gives $u \in C(Q)$. □

To prove (2) of Theorem 1 We show the proof for the case of $n = 1$ first. Denote $\beta = h + \varepsilon$ and set

$$\phi(x, t) = u^{\beta}$$

for every $\varepsilon > 0$. It follows from (2.1) that

$$\left| \frac{\partial}{\partial x} \phi(x, t) \right| \le C_2 (u(x, t)^{\varepsilon}) t^{-\frac{1}{2}} \quad \text{in } Q \tag{2.11}$$

for some $C_2 > 0$. To prove the function $\phi(x, t) \in C^1(\mathbb{R}^1)$ for fixed $t > 0$, we first see $u \in C^{\infty}(Q_+)$; second, (2.11) implies $\frac{\partial}{\partial x}\phi(x_*, t_*) = 0$ for every $(x_*, t_*) \in Q_0$. Therefore, (2.11) gives

$$\left| \frac{\partial}{\partial x} \phi(x, t) - \frac{\partial}{\partial x} \phi(x_*, t_*) \right| \le C_2 t^{-\frac{1}{2}} (u(x, t))^{\varepsilon}.$$

This inequality tells us $\frac{\partial}{\partial x}\phi(x, t) \in C(\mathbb{R}^1)$, that is to say, $\phi(x, t) \in C^1(\mathbb{R}^1)$. Furthermore, for every fixed $t > 0$, the continuity of $u(x, t)$ implies $H_u(t)$ is an open set. Thus $\phi(x_*, t) = \frac{\partial}{\partial x}\phi(x_*, t) = 0$ for $(x_*, t) \in \partial H_u(t)$. This fact claims that the manifold $S(t)$ touches $\mathbb{R}^1$ at $\partial H_u(t)$. In other words, $\mathbb{R}^1$ is just the tangent of $S(t)$ at $\partial H_u(t)$.

As to the case of $n > 1$, on the one hand, for every given $t > 0$, $\phi(x_0, t) \equiv 0$ if $(x_0, t) \in Q_0$. Thereby, $\nabla\phi(x, t) \longrightarrow 0$ when (x, t) converges to (x_0, t) along any direction inside Q_0. On the other hand, employing (1.10) we get

$$|u(x, t) - u(x_0, t)| \le C|x - x_0|^{\alpha}$$

for every $(x, t), (x_0, t) \in Q$. In particular, if $(x_0, t) \in Q_0$, the above inequality can be rewritten as

$$u(x, t) \le C|x - x_0|^{\alpha}.$$

Therefore, if we denote $\phi(x, t) = u^{\beta}$ with $\beta = \frac{1}{\alpha} + \varepsilon$ for any $\varepsilon > 0$, we get

$$\left| \frac{\phi(x, t) - \phi(x_0, t)}{x - x_0} \right| \le C|x - x_0|^{\alpha\varepsilon} \tag{2.12}$$

(2.12) yields $\nabla\phi(x, t) \longrightarrow 0$ when (x, t) converges to (x_0, t) from outside Q_0. Thus we know that $\nabla\phi(x, t)$ is continuous at (x_0, t). Because we have know $\nabla\phi \in C^{\infty}(Q_+)$, so $\phi(x, t) \in C^1(\mathbb{R}^n)$ for every given $t > 0$, in particular,

$$\phi(x,t)=\nabla\phi(x,t)=0 \quad \text{when } (x,t)\in\partial H_u(t).$$

Therefore, the surface $\phi=\phi(x,t)$ is tangent to $\mathbb{R}^n$ on $\partial H_u(t)$ for every fixed $t>0$. □

To prove (3) of Theorem 1 We also prove the result for the case of $n=1$ first. Recalling $u(x,t)=\lim_{\eta\longrightarrow 0^+} u_\eta$ and $u_\eta\geq\eta$, u_η are the classical solutions to the Cauchy problem (1.9), we can make

$$\phi_\eta=u_\eta^\beta$$

for $\beta=h+\varepsilon$ and $\varepsilon>h$. Then the function ϕ_η satisfies the degenerate parabolic equation

$$\frac{\partial\phi_\eta}{\partial t}=m\left[\phi_\eta^{\frac{m-1}{\beta}}\frac{\partial\phi_\eta^2}{\partial x^2}+\frac{m-\beta}{\beta}\phi_\eta^{\frac{m-\beta-1}{\beta}}\left|\frac{\partial\phi_\eta}{\partial x}\right|^2\right] \quad \text{in } Q.$$

Recalling $\lim_{\eta\longrightarrow 0^+}\phi_\eta=\phi$ and $u\in C^\infty(Q_+)$, we see that $\phi(x,t)\in C^\infty(Q_+)$ and ϕ satisfies the equation

$$\frac{\partial\phi}{\partial t}=m\left[\phi^{\frac{m-1}{\beta}}\frac{\partial^2\phi}{\partial x^2}+\frac{m-\beta}{\beta}\phi^{\frac{m-\beta-1}{\beta}}\left|\frac{\partial\phi}{\partial x}\right|^2\right] \tag{2.13}$$

in Q_+. We next prove that (2.13) is also right in Q. In fact, if $(x,t_*)\in Q_0$, then $\phi(x,t)\equiv 0$ for all $0\leq t<t_*$ [see (3.2) of this paper]. This yields $\frac{\partial}{\partial t}\phi(x,t_*-0)=0$. Moreover, (2.10) yields

$$u(x,t)-u(x,t_*)=u(x,t)\leq\nu\,(t-t_*)^{\frac{1}{2h}} \quad \text{for } t\geq t_*.$$

Thus, $\phi(x,t)-\phi(x,t_*)=\phi(x,t)\leq\nu^\beta(t-t_*)^{\frac{\beta}{2h}}$, thereby,

$$\frac{\phi(x,t)-\phi(x,t_*)}{t-t_*}\leq\nu^\beta\,(t-t_*)^{\frac{\beta-2h}{2h}}\,. \tag{2.14}$$

By (2.14), we get $\frac{\partial}{\partial t}\phi(x,t_*+0)=0$ owing to $\beta>2h$. Now we see that $\frac{\partial}{\partial t}\phi(x,t_*+0)=\frac{\partial}{\partial t}\phi(x,t_*-0)=0$, so we obtain the continuity of the function $\frac{\partial\phi}{\partial t}$, specially, $\frac{\partial\phi}{\partial t}|_{(x,t_*)}=0$. On the other hand, by (2.10) and (2.11), we have

$$\begin{aligned}\left|\frac{\partial\phi(x,t)}{\partial x}-\frac{\partial\phi(x_*,t)}{\partial x}\right|=\left|\frac{\partial\phi(x,t)}{\partial x}\right|&\leq C_2t^{-\frac{1}{2}}(u(x,t))^\varepsilon\\&\leq C_2t^{-\frac{1}{2}}(u(x,t))^{\varepsilon-h}\nu^h|x-x_*|\end{aligned}$$

for $(x_*,t)\in Q_0$. This yields

$$\frac{1}{|x-x_*|}\left|\frac{\partial\phi(x,t)}{\partial x}-\frac{\partial\phi(x_*,t)}{\partial x}\right|\leq C_2t^{-\frac{1}{2}}(u(x,t))^{\varepsilon-h}\nu^h.$$

This gives the continuity of the function $\frac{\partial^2\phi}{\partial x^2}$, specially,

$$\frac{\partial^2\phi}{\partial x^2} = 0 \quad \text{on } Q_0$$

thanks to $\varepsilon > h$. It follows from $m > 1$ that $\phi^{\frac{m-1}{\beta}}\frac{\partial^2\phi}{\partial x^2} = 0$ on Q_0. Similarly, the function $\frac{\partial\phi}{\partial x}$ is continuous and $\phi^{\frac{m-\beta-1}{\beta}}|\frac{\partial\phi}{\partial x}|^2 = 0$ on Q_0. Combining the argument mentioned above, we deduce that the function $\phi(x,t)$ satisfies (2.13) in Q. Similarly, we can also get the same conclusion if $n > 1$. □

As an application of our Theorem 1, here we give an example to show the large time behavior on the intrinsic properties of the manifold $S(t)$.

Example (the intrinsic properties of $S(t)$) Here we will discuss the relationship between $S(t)$ and $\mathbb{R}^n$ in this example for $n = 1$.

In fact, by (1.13) and (2.11), we get

$$\begin{aligned}(ds)^2 &= \left(1+\phi_x^2\right)(d\rho)^2 \\ &\le \left[1+(C_2u^{\varepsilon})^2t^{-1}\right](d\rho)^2.\end{aligned}$$

Furthermore, recalling $u(x,t) \le Ct^{\frac{-1}{m+1}}$ for some positive C (see Theorem 9 in Ch. III of [34]), we get

$$\frac{(ds)^2}{(d\rho)^2} = 1 + O\left(t^{-\frac{2\varepsilon}{m+1}-1}\right) \tag{2.15}$$

when t is large sufficiently.

3 The proof of Theorem 2

It is well-known that if λ_1 is the minimum positive eigenvalue and ψ_1 is the corresponding eigenfunction of the Dirichlet problem

$$\begin{cases}\Delta u = -\lambda u & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega,\end{cases}$$

then $\lambda_1\|\psi_1\|^2_{L^2(\Omega)} = \|\nabla\psi_1\|^2_{L^2(\Omega)}$, where Ω is a boundary domain in $\mathbb{R}^n$. Moreover, if $\psi \in H_0^1(\Omega)$, then Poincaré inequality claims that there exists a positive constant k such that $k\|\psi\|^2_{L^2(\Omega)} \le \|\nabla\psi\|^2_{L^2(\Omega)}$. In particular, if Ω is a sphere of $\mathbb{R}^n$

$$\Omega = \{x \in \mathbb{R}^n : |x - x_0| < \rho\}$$

for $x_0 \in \mathbb{R}^n$ and $\rho > 0$, then the constant k can be written as $k \le \rho^{-2}$ (see [1,35]). To be exact, we have

Lemma 2 *If Ω is a sphere mentioned above, $u \in H_0^1(\Omega)$. Then*

$$\|u\|_{L^2(\Omega)} \le \rho \|\nabla u\|_{L^2(\Omega)}. \tag{3.1}$$

Now we are in the position to prove our Theorem 2.

To prove (1.19) Assume $u(x,t)$ be the weak solution of (1.1), (1.2). Integrating (1.5) from t_1 to t_2 yields $\ln u(x,t_2) - \ln u(x,t_1) \ge -\frac{1}{m-1}(\ln t_2 - \ln t_1)$ for $t_2 > t_1$. This means$u(x,t_2) \cdot t_2^{\frac{1}{m-1}} \ge u(x,t_1) \cdot t_1^{\frac{1}{m-1}}$ for $t_2 > t_1 \ge 0$. Therefore,

$$\begin{cases} if\ u(x_0,t_0) = 0, & then\ u(x_0,t) = 0 \quad for\ every\ \ 0 \le t < t_0; \\ if\ u(x_0,t_0) > 0, & then\ u(x_0,t) > 0 \ \ for\ every\ \ t > t_0. \end{cases} \tag{3.2}$$

By (3.2), we see that

$$H_u(t) \supset B_\delta \quad t > 0.$$

Clearly, the proof is finished if $\sup_{x\in H_u(t)} |x| = \infty$, otherwise, we set

$$K' = \gamma + \sup_{x\in H_u(t)} |x|$$

for $\gamma > 0$. Thus,

$$u(x,t) = 0 \quad \text{for } x \in \mathbb{R}^n - B_{K'}. \tag{3.3}$$

It follows from (1.5) that

$$\int_{B_{K'}} u^m \Delta u^m dx \ge -\frac{1}{(m-1)t} \int_{B_{K'}} u^{1+m} dx,$$

so that

$$\int_{B_{K'}} |\nabla u^m|^2 dx \le \frac{1}{(m-1)t} \int_{B_{K'}} u^{1+m} dx.$$

Using (3.1) in this inequality, we obtain

$$\int_{B_{K'}} u^{2m} dx \le \frac{K'^2}{(m-1)t} \int_{B_{K'}} u^{1+m} dx. \tag{3.4}$$

Employing the Hölder inverse inequality (see Sect. 2.6 of Ch. 2 in [1]), we have

$$\int_{B_{K'}} u^{2m} dx \ge \left(\int_{B_{K'}} u^{1+m} dx \right)^{\frac{2m}{1+m}} |B_{K'}|^{\frac{1-m}{1+m}},$$

where $|B_{K'}|$ is the volume of $B_{K'}$ and $|B_{K'}| = \pi^{\frac{n}{2}}\Gamma(1+\frac{n}{2})^{-1}K'^n$. Using this inequality in (3.4) yields

$$\left(\int_{B_{K'}} u^{1+m}dx\right)^{\frac{m-1}{1+m}} |B_{K'}|^{\frac{1-m}{1+m}} \leq \frac{K'^2}{(m-1)t}. \tag{3.5}$$

Using the Hölder inverse inequality again, we have

$$\int_{B_{K'}} u^{1+m}dx \geq \left(\int_{B_{K'}} udx\right)^{1+m} |B_{K'}|^{-m}. \tag{3.6}$$

It follows from (1.4) and 3.3 that $\int_{\mathbb{R}^n} u(x,t)dx = \int_{B_{K'}} u(x,t)dx = \int_{\mathbb{R}^n} u_0(x)dx$. Combining 3.5 and 3.6 yields

$$\left(\int_{\mathbb{R}^n} u_0dx\right)^{m-1} |B_{K'}|^{1-m} \leq \frac{K'^2}{(m-1)t}. \tag{3.7}$$

Now we get

$$(m-1)\left(\int_{\mathbb{R}^n} u_0dx\right)^{m-1} t \leq K'^{2+(m-1)n} \cdot \pi^{\frac{(m-1)n}{2}} \cdot \left(\Gamma(1+\frac{n}{2})\right)^{1-m}.$$

Letting $\gamma \longrightarrow 0$ gives

$$\sup_{x\in H_u(t)} |x| \geq \chi(t) \quad t > 0.$$

□

To prove (*1.20*) Assume $u(x,t)$ and $v(x,t)$ be the solutions to (1.1) and (1.8) respectively. Employing the well-known result (see [34]), we have

$$\frac{1}{1+m}\int_{\mathbb{R}^n} u^{1+m}(x,T)dx + \int_{Q_T} |\nabla u^m|^2 dxdt \leq \frac{1}{1+m}\int_{\mathbb{R}^n} u_0^{1+m}dx \tag{3.8}$$

and

$$\frac{1}{2}\int_{\mathbb{R}^n} v^2(x,T)dx + \int_{Q_T} |\nabla v|^2 dxdt \leq \frac{1}{2}\int_{\mathbb{R}^n} u_0^2dx \tag{3.9}$$

for every given $T > 0$ and $Q_T = \mathbb{R}^n \times (0,T)$. Let

$$G = v - u_\eta^m,$$
$$\psi = \int_T^t Gd\tau \quad 0 < t < T,$$

where u_η are the solutions of (1.9). Let $\{\zeta_k\}_{k>1}$ be a smooth cutoff sequence with the following properties: $\zeta_k(x) \in C_0^\infty(\mathbb{R}^1)$ and

$$\zeta_k(x) = \begin{cases} 1 & |x| \leq k, \\ 0 < \zeta_k(x) < 1 & k < |x| < 2k, \\ 0 & |x| \geq 2k. \end{cases}$$

Clearly, there is a positive constant γ such that

$$|\nabla \zeta_k| \leq \frac{\gamma}{k} \quad \text{and} \quad |\Delta \zeta_k| \leq \frac{\gamma}{k^2}. \tag{3.10}$$

Recalling $(v - u_\eta)_t = \Delta G$ in Q_T, we multiply the equation by $\psi \zeta_k$ and integrate by parts in Q_T, we obtain

$$\int_{Q_T} (\zeta_k \nabla G \cdot \nabla \psi + \psi \nabla G \cdot \nabla \zeta_k) dx dt = \int_{Q_T} (v - u_\eta) G \zeta_k dx dt. \tag{3.11}$$

Because $\zeta_k \in C_0^\infty(\mathbb{R}^1)$ and the functions v, u_η are bounded and are classical solutions to (1.8) and (1.9) respectively, so we can differentiate (3.11) with respect to T, we get

$$\begin{aligned} \int_{\mathbb{R}^n} (v - u_\eta) G \zeta_k dx &= -\int_{Q_T} \left(\zeta_k |\nabla G|^2 + G \nabla G \cdot \nabla \zeta_k \right) dx dt \\ &\leq -\frac{1}{2} \int_{Q_T} \nabla G^2 \cdot \nabla \zeta_k dx dt. \end{aligned} \tag{3.12}$$

Letting $\eta \longrightarrow 0$ in (3.12) yields

$$\int_{\mathbb{R}^n} (v - u)(v - u^m) \zeta_k dx \leq \frac{\varphi}{k} \tag{3.13}$$

for some positive $\varphi = \varphi(T)$ thanks to (3.8), (3.9) and (3.10). On the other hand,

$$\int_{\mathbb{R}^n} (v - u)(v - u^m) \zeta_k dx = \int_{\mathbb{R}^n} (v - u)^2 \zeta_k dx + \int_{\mathbb{R}^n} (v - u)(u - u^m) \zeta_k dx dt.$$

Now we conclude that

$$\begin{aligned} \int_{\mathbb{R}^n} (v - u)^2 \zeta_k dx &\leq \int_{\mathbb{R}^n} |v - u| \cdot |u - u^m| \zeta_k dx + \frac{\varphi}{k} \\ &\leq \frac{1}{2} \int_{\mathbb{R}^n} \left[(v - u)^2 + (u - u^m)^2 \right] \zeta_k dx + \frac{\varphi}{k}. \end{aligned}$$

This implies

$$\begin{aligned}\int_{\mathbb{R}^n}(v-u)^2\zeta_k dx &\le \int_{\mathbb{R}^n}(u-u^m)^2\zeta_k dx+\frac{2\varphi}{k}\\ &\le (m-1)\int_{\mathbb{R}^n}\xi^{m-1}|u-u^m|\zeta_k dx+\frac{2\varphi}{k}\\ &\le (m-1)M^{m-1}\int_{\mathbb{R}^n}|u-u^m|\zeta_k dx+\frac{2\varphi}{k}.\end{aligned}$$

Recalling the definition of ζ_k, we see $\int_{\mathbb{R}^n}(v-u)^2\zeta_k dx \ge \int_{|x|\le k}(v-u)^2dx$. Moreover, $\int_{\mathbb{R}^n}|u-u^m|\zeta_k dx$ is bounded thanks to (1.3) and (1.4). Hence, we can get a positive constant $C_*=C_*(T)$ such that

$$\int_{|x|\le k}[v(x,t)-u(x,t)]^2\,dx \le C_*\left[(m-1)+\frac{1}{k}\right]$$

with respect to $t\in(0,T)$ uniformly. □

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