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On s-semipermutable or ss-quasinormal subgroups of finite groups

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Abstract Suppose that G is a finite group and H is a subgroup of G. H is said to be s-semipermutable in G if $HG_p = G_pH$ for any Sylow p-subgroup G_p of G with $(p, |H|) = 1$; H is said to be an ss-quasinormal subgroup of G if there is a subgroup B of G such that $G = HB$ and H permutes with every Sylow subgroup of B. We will study finite groups G saisfying the following: for each noncyclic Sylow subgroup P of G, there exists a subgroup D of P such that $1 < |D| < |P|$ and every subgroup H of P with order $|D|$ is s-semipermutable or ss-quasinormal in G. Some recent results are generalized and unified.

Keywords s-Semipermutable subgroup · ss-Quasinormal subgroup · Saturated formation

Mathematics Subject Classification 20D10 · 20D20

1 Introduction

All groups considered in this paper are finite. G always means a group, $|G|$ is the order of G, $\pi(G)$ denotes the set of all primes dividing $|G|$ and G_p is a Sylow p-subgroup of G for some $p \in \pi(G)$.

Let $\mathscr{F}$ be a class of groups. We call $\mathscr{F}$ a formation, provided that (1) if $G \in \mathscr{F}$ and $H \trianglelefteq G$, then $G/H \in \mathscr{F}$, and (2) if G/M and G/N are in $\mathscr{F}$, then $G/(M \cap N)$

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is in $\mathscr{F}$ for any normal subgroups M, N of G. A formation $\mathscr{F}$ is said to be saturated if $G/\Phi(G) \in \mathscr{F}$ implies that $G \in \mathscr{F}$. $\mathscr{U}$ will denote the class of all supersolvable groups. Clearly, $\mathscr{U}$ is a saturated formation.

A subgroup H of G is called s-quasinormal (or s-permutable, π-quasinormal) in G provided H permutes with all Sylow subgroups of G, i.e, $HP=PH$ for any Sylow subgroup P of G. This concept was introduced by Kegel in [5] and has been studied extensively by Deskins [1] and Schmidt [11]. More recently, Zhang and Wang [14] generalized s-quasinormal subgroups to s-semipermutable subgroups. A subgroup H is said to be s-semipermutable in G if $HG_p = G_pH$ for any Sylow p-subgroup G_p of G with $(p, |H|) = 1$. Clearly, every s-quasinormal subgroup of G is an s-semipermutable subgroup of G, but the converse does not hold. Many authors consider minimal or maximal subgroups of a Sylow subgroup of a group when investigating the structure of G, such as in [1] and [4–14], etc. For example, Han in [4] proves the following result.

Theorem 1.1 *Let p be a prime dividing the order of a group G satisfying $(|G|, p-1) = 1$ and P a Sylow p-subgroup of G. Suppose there exists a nontrivial subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a nonabelian 2-group and $|P : D| > 2$) is s-semipermutable in G, then G is p-nilpotent.*

As another generalization of the s-quasinormality, Li et al. [7] introduce the following concept: A subgroup H of G is called an ss-quasinormal subgroup of G if there is a subgroup B of G such that $G = HB$ and H permutes with every Sylow subgroup of B. Many authors consider minimal or maximal subgroups of a Sylow subgroup of a group when investigating the structure of G, such as in [6–9], [12], etc. In [13], Wei and Guo provide a result as follows.

Theorem 1.2 *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$ and E be a normal subgroup of a group G such that $G/E \in \mathscr{F}$. Then $G \in \mathscr{F}$ if only if for every non-cyclic Sylow subgroup P of $F^*(E)$, there is a subgroup D of P with $1 < |D| < |P|$ such that every subgroup H of P with order $|D|$ or $2|D|$ whenever $|D| = 2$ is ss-quasinormal in G.*

The aim of this article is to unify and improve above Theorems using s-semipermutable and ss-quasinormal subgroups. Our main theorems are as follows:

Theorem 1.3 (i.e., Theorem 3.5) *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$, the class of all supersolvable groups and G a group with E as a normal subgroup of G such that $G/E \in \mathscr{F}$. Suppose that every non-cyclic Sylow subgroup P of $F^*(E)$ has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a nonabelian 2-group and $|P : D| > 2$) is either s-semipermutable or ss-quasinormal in G, where $F^*(E)$ is the generalized Fitting subgroup of E. Then $G \in \mathscr{F}$.*

2 Basic definitions and preliminary results

In this section, we collect some known results that are useful later.

Lemma 2.1 *Suppose that H is an s-semipermutable subgroup of G. Then the following assertions hold.*

(i) *If $H \le K \le G$, then H is s-semipermutable in K;*
(ii) *Let N be a normal subgroup of G. If H is a p-group for some prime $p \in \pi(G)$, then HN/N is s-semipermutable in G/N;*
(iii) *If $H \le O_p(G)$, then H is s-permutable in G*

Proof (i) is [14, Property 1], (ii) is [14, Property 2], and (iii) is [14, Lemma 3]. □

Lemma 2.2 *Let H be an ss-quasinormal subgroup of a group G.*

(i) *If $H \le L \le G$, then H is ss-quasinormal in L;*
(ii) *If N is normal in G, then HN/N is ss-quasinormal in G/N;*
(iii) *If $H \le F(G)$, then H is s-quasinormal in G;*
(iv) *If H is a p-subgroup(p a prime), then H permutes with every Sylow q-subgroup of G with $q \ne p$.*

Proof (i) and (ii) are [7, Lemma 2.1], (iii) is [7, Lemma 2.2], and (iv) is [7, Lemma 2.5]. □

Lemma 2.3 ([12]) *Let G be a group and P a Sylow p-subgroup of G, where p is the smallest prime dividing $|G|$. If every maximal subgroup of P is s-semipermutable in G, then G is p-nilpotent.*

Lemma 2.4 ([2] III, 5.2 and IV, 5.4). *Suppose that p is a prime and G is a minimal non-p-nilpotent group, i.e., G is not a p-nilpotent group but whose proper subgroups are all p-nilpotent.*

(i) *G has a normal Sylow p-subgroup P for some prime p and $G = PQ$, where Q is a non-normal cyclic q-subgroup for some prime $q \ne p$.*
(ii) *$P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$.*
(iii) *The exponent of P is p or 4.*

Lemma 2.5 ([4]) *Let N be an elementary abelian normal p-subgroup of a group G. If there exists a subgroup D in N such that $1 < |D| < |N|$ and every subgroup H of N with $|H| = |D|$ is s-semipermutable in G, then there exists a maximal subgroup M of N such that M is normal in G.*

Lemma 2.6 ([2], VI, 4.10) *Assume that A and B are two subgroups of a group G and $G \ne AB$. If $AB^g = B^g A$ holds for any $g \in G$, then either A or B is contained in a nontrivial normal subgroup of G.*

The generalized Fitting subgroup $F^*(G)$ of G is the unique maximal normal quasinilpotent subgroup of G. Its definition and important properties can be found in [3, X, 13]. We would like to give the following basic facts we will use in our proof.

Lemma 2.7 ([3], X, [13]) *Let G be a group and M a subgroup of G.*

(i) *If M is normal in G, then $F^*(M) \le F^*(G)$;*

(ii) $F^*(G) \neq 1$ *if* $G \neq 1$*; in fact,* $F^*(G)/F(G) = Soc(F(G)C_G(F(G))/F(G))$*;*
(iii) $F^*(F^*(G)) = F^*(G) \geq F(G)$*; if* $F^*(G)$ *is solvable, then* $F^*(G) = F(G)$.

Lemma 2.8 ([10]) *Let* $\mathscr{F}$ *be a saturated formation containing* $\mathscr{U}$*, the class of all supersolvable groups and* G *a group with* E *as a normal subgroup of* G *such that* $G/E \in \mathscr{F}$*. Suppose that every non-cyclic Sylow subgroup* P *of* $F^*(E)$ *has a subgroup* D *such that* $1 < |D| < |P|$ *and every subgroup* H *of* P *with order* $|H| = |D|$ *or with order* $2|D|$ *(if* P *is a nonabelian 2-group and* $|P : D| > 2$*) is weakly* s*-permutable in* G*, where* $F^*(E)$ *is the generalized Fitting subgroup of* E*. Then* $G \in \mathscr{F}$*.*

3 Main results

In this section, we will prove our main results.

Theorem 3.1 *Let* p *be the smallest prime dividing the order of a group* G *and* P *be a Sylow* p*-subgroup of* G*. If every maximal subgroup of* P *is either* s*-semipermutable or* ss*-quasinormal in* G*. Then* G *is* p*-nilpotent.*

Proof Assume that the theorem is not true and let G be a counterexample of minimal order. We derive a contradiction in several steps. □

By Lemmas 2.1 and 2.2, the following two steps are obvious.

Step 1. $O_{p'}(G) = 1$.

Step 2. G has a unique minimal normal subgroup N and G/N is p-nilpotent. Moreover, $\Phi(G) = 1$.

Step 3. $O_p(G) = 1$.

If $O_p(G) \neq 1$, then step 2 yields $N \leq O_p(G)$ and $\Phi(O_p(G)) \leq \Phi(G) = 1$. Therefore, G has a maximal subgroup M such that $G = MN$ and $G/N \cong M$ is p-nilpotent. Since $O_p(G) \cap M$ is normalized by N and M, we conclude that $O_p(G) \cap M$ is normal in G. The uniqueness of N yields $N = O_p(G)$. Clearly, $P = N(P \cap M)$. Furthermore, $P \cap M < P$, and, thus there exists a maximal subgroup P_1 of P such that $P \cap M \leq P_1$. Hence, $P = NP_1$. By hypothesis, P_1 is s-semipermutable or ss-quasinormal in G. In either case, we have that $P_1 M_q$ is a group for $q \neq p$. Hence

$$P_1 \langle M_p, M_q | q \in \pi(M), q \neq p \rangle = P_1 M$$

is a group. Then $P_1 M = M$ or G by maximality of M. If $P_1 M = G$, then

$$P = P \cap P_1 M = P_1(P \cap M) = P_1,$$

a contradiction. If $P_1 M = M$, then $P_1 \leq M$. Therefore, $P_1 \cap N = 1$ and N is of prime order. Then the p-nilpotency of G/N implies the p-nilpotency of G, a contradiction. Therefore, $O_p(G) = 1$.

Step 4. The final contradiction.

Assume that P has a maximal subgroup P_1 which is ss-quasinormal in G, then there exists a subgroup B of G such that $G = P_1 B$ and P_1 permutes with every Sylow subgroup of B. Noticing that $H \cap B$ is s-quasinormal in B, we have $P_1 \cap B \leq O_p(B)$. Since p is the smallest prime dividing the order of G and $H \cap B$ is a maximal subgroup of a Sylow p-subgroup of B, we see that $B/O_p(B)$ is p-nilpotent and so B is p-solvable. By [2, Chapter VI, Hauptsatz 1.7], we may assume that K is a Hall p'-subgroup of B, $\pi(K) = \{p_2, \ldots, p_s\}$ and $P_i \in \mathrm{Syl}_{p_i}(K)$. Since P_1 permutes with every P_i, $P_1 K = K P_1$ and therefore $P_1 K$ is normal in G. Now let $\{H_j : j = 1, \ldots, t\}$ be the set of all maximal subgroup of P. Then, by the above, there is K_j such that $H_j K_j$ is normal in G for $j = 1, \ldots, t$, respectively. Let $N = \bigcap\{H_j K_j : j = 1, \ldots, t\}$. Then $N \trianglelefteq G$ and G/N is a p-group. Thus we have $P \cap N = \bigcap_{j=1}^{t}((H_j K_j) \cap P) = \bigcap_{j=1}^{t}((P \cap K_j)H_j) = \bigcap_{j=1}^{t} H_j \leq \Phi(P)$. By Tate′s theorem [2, Chapter IV, Satz 4.7], N is p-nilpotent and hence G is p-nilpotent, a contradiction. Now we may assume that all maximal subgroups of P are s-semipermutable in G. Then G is p-nilpotent by Lemma 2.3, a contradiction.

Theorem 3.2 *Let p be the smallest prime dividing the order of a group G and P be a Sylow p-subgroup of G. If P has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a nonabelian 2-group and $|P : D| > 2$) is either s-semipermutable or ss-quasinormal in G. Then G is p-nilpotent.*

Proof Suppose that the theorem is false and let G be a counterexample of minimal order. We will derive a contradiction in several steps. □

Step 1. $O_{p'}(G) = 1$.

If $O_{p'}(G) \neq 1$, Lemma 2.1 (ii) and Lemma 2.2 (iii) guarantee that $G/O_{p'}(G)$ satisfies the hypotheses of the theorem. Thus $G/O_{p'}(G)$ is p-nilpotent by the choice of G. Then G is p-nilpotent, a contradiction.

Step 2. $|D| > p$.

Suppose that $|D| = p$. Since G is not p-nilpotent, G has a minimal non-p-nilpotent subgroup G_1. By Lemma 2.4 (i), $G_1 = [P_1]Q$, where $P_1 \in \mathrm{Syl}_p(G_1)$ and $Q \in \mathrm{Syl}_q(G_1)$, $p \neq q$. Let $X/\Phi(P_1)$ be a subgroup of $P_1/\Phi(P_1)$ of order p, $x \in X \backslash \Phi(P_1)$ and $L = \langle x \rangle$. Then L is of order p or 4 by Lemma 2.4 (iii). By the hypotheses, L is either s-semipermutable or ss-quasinormal in G, thus in G_1 by Lemmas 2.1 (i) and 2.2 (i). First, suppose that L is ss-quasinormal in G_1, then L is s-quasinormal in G_1 by Lemma 2.2 (iii). Thus $LQ \leq G_1$. Therefore, $LQ = L \times Q$. Then $G_1 = P_1 \times Q$. Hence G_1 is p-nilpotent, contrary to the choice of G_1. Therefore, $L = \langle x \rangle$ is s-semipermutable in G_1 for every element $x \in P_1$. Thus $LQ \leq G_1$. Therefore, $LQ = L \times Q$. Then $G_1 = P_1 \times Q$, a contradiction.

Step 3. $|P : D| > p$.

By Theorem 3.1.

Step 4. P has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a nonabelian 2-group and $|P : D| > 2$) is s-semipermutable in G. Assume that $H \leq P$ such that $|H| = |D|$ and H is ss-quasinormal in G. Then we may assume G has a normal subgroup M such that $|G : M| = p$ and $G = HM$. Since $|P : D| > p$ by Step 3, M satisfies the hypotheses of the theorem. The choice of G yields that M is p-nilpotent. It is easy to see that G is p-nilpotent, contrary to the choice of G.

Step 5. If $N \leq P$ and N is minimal normal in G, then $|N| \leq |D|$.

Suppose that $|N| > |D|$. Since $N \leq O_p(G)$, N is elementary abelian. By Lemma 2.5, N has a maximal subgroup which is normal in G, contrary to the minimality of N.

Step 6. Suppose that $N \leq P$ and N is minimal normal in G. Then G/N is p-nilpotent.

If $|N| < |D|$, G/N satisfies the hypotheses of the theorem by Lemma 2.1 (ii). Thus G/N is p-nilpotent by the minimal choice of G. So we may suppose that $|N| = |D|$ by Step 5. We will show that every cyclic subgroup of P/N of order p or order 4 (when P/N is a non-abelian 2-group) is s-semipermutable in G/N. Let $K \leq P$ and $|K/N| = p$. By Step 2, N is non-cyclic, so are all subgroups containing N. Hence there is a maximal subgroup $L \neq N$ of K such that $K = NL$. Of course, $|N| = |D| = |L|$. Since L is s-semipermutable in G by the hypotheses, $K/N = LN/N$ is s-semipermutable in G/N by Lemma 2.1 (ii). If $p = 2$ and P/N is non-abelian, take a cyclic subgroup X/N of P/N of order 4. Let K/N be maximal in X/N. Then K is maximal in X and $|K/N|$=2. Since X is non-cyclic and X/N is cyclic, there is a maximal subgroup L of X such that N is not contained in L. Thus $X = LN$ and $|L| = |K| = 2|D|$. By the hypotheses, L is s-semipermutable in G. By Lemma 2.1 (ii), $X/N = LN/N$ is s-semipermutable in G/N. Hence G/N satisfies the hypotheses. By the minimal choice of G, G/N is p-nilpotent.

Step 7. $O_p(G) = 1$.

Suppose that $O_p(G) \neq 1$. Take a minimal normal subgroup N of G contained in $O_p(G)$. By Step 6, G/N is p-nilpotent. It is easy to see that N is the unique minimal normal subgroup of G contained in $O_p(G)$. Furthermore, $O_p(G) \cap \Phi(G) = 1$. Hence $O_p(G)$ is an elementary abelian p-group. On the other hand, G has a maximal subgroup M such that $G = MN$ and $M \cap N = 1$. It is easy to deduce that $O_p(G) \cap M = 1$, $N = O_p(G)$ and $M \cong G/N$ is p-nilpotent. Then G can be written as $G = N(M \cap P)M_{p'}$, where $M_{p'}$ is the normal p-complement of M. Pick a maximal subgroup S of $M_p = P \cap M$. Then $NSM_{p'}$ is a subgroup of G with index p. Since p is the minimal prime in $\pi(G)$, we know that $NSM_{p'}$ is normal in G. Now by Step 3 and the induction, we have $NSM_{p'}$ is p-nilpotent. Therefore, G is p-nilpotent, a contradiction.

Step 8. The minimal normal subgroup L of G is not p-nilpotent.

If L is p-nilpotent, then it follows from the fact that $L_{p'}$ char $L \lhd G$ that $L_{p'} \leq O_{p'}(G) = 1$. Thus L is a p-group. Whence $L \leq O_p(G) = 1$ by Step 7, a contradiction.

Step 9. G is a non-abelian simple group.

Suppose that G is not a simple group. Take a minimal normal subgroup L of G. Then $L < G$. If $|L|_p > |D|$, then L is p-nilpotent by the minimal choice of G, contrary to Step 8. If $|L|_p \leq |D|$. Take $P_* \geq L \cap P$ such that $|P_*| = p|D|$. Hence P_* is a Sylow p-subgroup of P_*L. Since every maximal subgroup of P_* is of order $|D|$, every maximal subgroup of P_* is s-semipermutable in G by hypotheses, thus in P_*L by Lemma 2.1 (i). Now applying Theorem 3.1, we get P_*L is p-nilpotent. Therefore, L is p-nilpotent, contrary to Step 8.

Step 10. The final contradiction.

Suppose that H is a subgroup of P with $|H| = |D|$ and Q is a Sylow q-subgroup with $q \neq p$. Then $HQ^g = Q^gH$ for any $g \in G$ by the hypotheses that H is s-

semipermutable in G. Since G is simple by Step 9, $G = HQ$ from Lemma 2.6, the final contradiction.

The following corollary is immediate from Theorem 3.2.

Corollary 3.3 *Suppose that G is a group. If every non-cyclic Sylow subgroup of G has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a nonabelian 2-group and $|P : D| > 2$) is either s-semipermutable or ss-quasinormal in G, then G has a Sylow tower of supersolvable type.*

Theorem 3.4 *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$, the class of all supersolvable groups and G a group with E as a normal subgroup of G such that $G/E \in \mathscr{F}$. Suppose that every non-cyclic Sylow subgroup of E has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a nonabelian 2-group and $|P : D| > 2$) is either s-semipermutable or ss-quasinormal in G. Then $G \in \mathscr{F}$.*

Proof Suppose that P is a non-cyclic Sylow p-subgroup of E, $\forall p \in \pi(E)$. Since P has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a nonabelian 2-group and $|P : D| > 2$) is either s-semipermutable or ss-quasinormal in G by hypotheses, thus in E by Lemma 2.1 (i) and Lemma 2.2 (i). Applying Corollary 3.3, we conclude that E has a Sylow tower of supersolvable type. Let q be the maximal prime divisor of $|E|$ and $Q \in \mathrm{Syl}_q(E)$. Then $Q \trianglelefteq G$. Since $(G/Q, E/Q)$ satisfies the hypotheses of the theorem, by induction, $G/Q \in \mathscr{F}$. For any subgroup H of Q with $|H| = |D|$, since $Q \leq O_q(G)$, H is s-permutable in G by Lemmas 2.1 (iii) and 2.2 (iii). Since s-permutable implies weakly s-permutable and $F^*(Q) = Q$ by Lemma 2.7, we get $G \in \mathscr{F}$ by applying Lemma 2.8. □

Theorem 3.5 *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$, the class of all supersolvable groups and G a group with E as a normal subgroup of G such that $G/E \in \mathscr{F}$. Suppose that every non-cyclic Sylow subgroup of $F^*(E)$ has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a nonabelian 2-group and $|P : D| > 2$) is either s-semipermutable or ss-quasinormal in G. Then $G \in \mathscr{F}$.*

Proof We distinguish two cases:

Case 1. $\mathscr{F} = \mathscr{U}$.

Let G be a minimal counter-example.

Step 1. Every proper normal subgroup N of G containing $F^*(E)$ (if it exists) is supersolvable. □

If N is a proper normal subgroup of G containing $F^*(E)$, then $N/N \cap E \cong NE/E$ is supersolvable. By Lemma 2.7 (iii), $F^*(E) = F^*(F^*(E)) \leq F^*(E \cap N) \leq F^*(E)$, so $F^*(E \cap N) = F^*(E)$. For any Sylow subgroup P of $F^*(E \cap N) = F^*(E)$, P has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a nonabelian 2-group and $|P : D| > 2$) is either s-semipermutable or ss-quasinormal in G by hypotheses, thus in N by Lemma 2.1

(i) and Lemma 2.2 (i). So N and $N \cap H$ satisfy the hypotheses of the theorem, the minimal choice of G implies that N is supersolvable.

Step 2. $E = G$.

If $E < G$, then $E \in \mathscr{U}$ by Step 1. Hence $F^*(E) = F(E)$ by Lemma 2.7. It follows that every Sylow subgroup of $F^*(E)$ is normal in G. By Lemmas 2.1 (iii) and 2.2 (iii), every non-cyclic Sylow subgroup of $F^*(E)$ has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a nonabelian 2-group and $|P : D| > 2$) is s-permutable in G. Applying Lemma 2.8 for the special case $\mathscr{F} = \mathscr{U}$, $G \in \mathscr{U}$, a contradiction.

Step 3. $F^*(G) = F(G) < G$.

If $F^*(G) = G$, then $G \in \mathscr{F}$ by Theorem 3.4, contrary to the choice of G. So $F^*(G) < G$. By Step 1, $F^*(G) \in \mathscr{U}$ and $F^*(G) = F(G)$ by Lemma 2.7.

Step 4. The final contradiction.

Since $F^*(G) = F(G)$, each non-cyclic Sylow subgroup of $F^*(G)$ has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a nonabelian 2-group and $|P : D| > 2$) is s-permutable in G by Lemmas 2.1 (iii) and 2.2 (iii). Applying Lemma 2.8, $G \in \mathscr{U}$, a contradiction.

Case 2. $\mathscr{F} \neq \mathscr{U}$.

By hypotheses, every non-cyclic Sylow subgroup of $F^*(E)$ has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a nonabelian 2-group and $|P : D| > 2$) is either s-semipermutable or ss-quasinormal in G, thus in E Lemma 2.1 (i) and Lemma 2.2 (i). Applying Case 1, $E \in \mathscr{U}$. Then $F^*(E) = F(E)$ by Lemma 2.7. It follows that each Sylow subgroup of $F^*(E)$ is normal in G. By Lemmas 2.1 (iii) and 2.2 (iii), each non-cyclic Sylow subgroup of $F^*(E)$ has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a nonabelian 2-group and $|P : D| > 2$) is s-permutable in G. Applying Lemma 2.8, $G \in \mathscr{F}$. These complete the proof of the theorem.

The following corollaries are immediate from Theorem 3.5.

Corollary 3.6 *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$. Suppose that G is a group with a normal subgroup E such that $G/E \in \mathscr{F}$. Then $G \in \mathscr{F}$ if and only if every maximal subgroup of any Sylow subgroup of $F^*(E)$ is s-semipermutable in G.*

Corollary 3.7 *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$. Suppose that G is a group with a normal subgroup E such that $G/E \in \mathscr{F}$. Then $G \in \mathscr{F}$ if and only if every cyclic subgroup of any Sylow subgroup of $F^*(E)$ of prime order or order 4 is s-semipermutable in G.*

Corollary 3.8 ([6], Theorem 3.3) *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$. Suppose that G is a group with a normal subgroup H such that $G/H \in \mathscr{F}$. Then $G \in \mathscr{F}$ if and only if every maximal subgroup of any Sylow subgroup of $F^*(H)$ is ss-quasinormal in G.*

Corollary 3.9 ([6], Theorem 3.7) Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$. Suppose that G is a group with a normal subgroup H such that $G/H \in \mathscr{F}$. Then $G \in \mathscr{F}$ if and only if every cyclic subgroup of any Sylow subgroup of $F^*(H)$ of prime order or order 4 is ss-quasinormal in G.

Corollary 3.10 ([8], Theorem 3.4) Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$. Suppose that G is a group with a normal subgroup E such that $G/E \in \mathscr{F}$. Then $G \in \mathscr{F}$ if and only if every maximal subgroup of any Sylow subgroup of $F^*(E)$ is s-quasinormal in G.

Corollary 3.11 ([9], Theorem 3.3) Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$. Suppose that G is a group with a normal subgroup E such that $G/E \in \mathscr{F}$. Then $G \in \mathscr{F}$ if and only if every cyclic subgroup of any Sylow subgroup of $F^*(E)$ of prime order or order 4 is s-quasinormal in G.

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